ON THE GENERALIZED FUTAKI INVARIANT

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ABSTRACT. We study the algebraic properties of the generalized Futaki invariant of an almost Fano variety and prove that it is in fact a pushforward to a point of an appropriate equivariant Chow class of the variety. This allows us to use Bott-type formulae for calculating the invariant. We show this use on some examples.

1. INTRODUCTION

Let M be a compact complex manifold with positive first Chern class. Such manifolds are called Fano manifolds. The manifold M is called Einstein-Kähler if there exists a Kähler metric g on it, such that the Kähler form ω_g and the Ricci form ρ_g of g satisfy the equation : $\omega_g = \rho_g$. In 1957 Matsushima [20] has proved that the existence of Einstein-Kähler metrics on M implies that the algebra $\mathfrak{A}(M)$ of holomorphic vector fields on M is reductive. In 1983 A. Futaki [11] introduced a character F of $\mathfrak{A}(M)$ which depends only on the complex structure of M , and vanishes identically if M is Einstein-Kähler. Futaki also showed an example of a Fano manifold with reductive algebra $\mathfrak{A}(M)$ and with nontrivial character F . F is called the Futaki invariant of the Fano manifold M .

In spite of its analytic definition, F has very algebraic in nature properties. For example, Mabuchi [18] has proved that F vanishes on all nilpotent elements of $\mathfrak{A}(M)$. Secondly, Futaki and Morita [13] have shown that F can be defined by using appropriate polynomials invariant under the action of $\mathfrak{A}(M)$ (we call these Futaki-Morita polynomials), and have proved that there is a relation between F and the equivariant cohomologies of M .

In 1992 Ding and Tian [6] defined a generalization of the Futaki invariant for almost Fano varieties X (i.e., normal complete $\mathbf{Q}$-Gorenstein varieties with ample degrees of their anticanonical sheaf). They proved that, if one considers special degenerations of a Fano manifold M (such degenerations have almost Fano central fibres), then the existence of Einstein-Kähler metrics on M depends on the behaviour of the generalized Futaki invariants of the central fibres of these degenerations. This generalized Futaki invariant proved to be a very efficient tool in investigating Fano manifolds. By using a refined version of Ding-Tian's theorem, Tian [22] found a counterexample to the long-standing hypothesis saying that each Fano manifold with a discrete group of biholomorphisms is Einstein-Kähler. The significance of the generalized Futaki invariant stems also from the fact that it is connected, via the notion of weakly K-stability of a Fano manifold introduced by Tian, with the Chow-Mumford stability w.r.t. the very ample degrees of the anticanonical sheaf of X [22]. In this paper we study the properties of the generalized Futaki invariant in the spirit of Futaki-Morita and Mabuchi. Establishing the connection of F with the equivariant cohomologies of X , we found it very convenient to express F in terms of Edidin-Graham's equivariant Chow cohomology groups. In particular. we use the definition of equivariant Chern classes as given in [7], and avoid using of invariant connections to define them. When the group acting is compact, these coincide with the equivariant characteristic classes as defined by Berline and Vergne [2] . As a result we prove that F can be regarded as a pushforward

to a point of an equivariant Chow cohomology class on X. The benefit from this result is two-fold. First of all, it gives a natural way of defining a Futaki invariant for the varieties other than normal almost Fano ones - this is useful when studying the degenerations of Fano manifolds. Secondly, it allows us to use the machinery of Bott-type residue formulas for calculating F. In this way we recover not only the known (Bott-type formulas) for F but also the one for complete intersections due to Zhiqin Lu [17].

We now explain the content of the paper.

In Section 2 we recall some basic properties of the Edidin-Graham's equivariant Chow groups of an algebraic scheme in the extent needed for our purposes. We emphasize here the behaviour of these groups when passing to a subgroup acting on the scheme. We find a way of attaching to each cohomology class α on X a map f_α from the Lie algebra of the group of biholomorphisms of X to the complex numbers, and show that f_α is actually a character of this Lie algebra for some classes α.

In Section 3 we recall the definition of the generalized Futaki invariant F of an almost Fano variety X, show that it is insensitive to the singularities of X and prove two useful integral representations. One of these, combined with an argument of Mabuchi [18], helps one to prove that F vanishes on the nilpotent elements of $Lie\, Aut(X)$. The other shows that F can be defined by using an appropriate Futaki-Morita polynomial.

Section 4 is devoted to a study of the Futaki-Morita polynomials for non-smooth algebraic varieties. In this section we find a necessary and sufficient condition for a Futaki-Morita polynomial to be represented by some f_α. In particular we find such α for the generalized Futaki invariant.

In section 5 we recall the essence of the Bott-type residue formulas, as dive-loped by Edidin-Graham, and give some examples of calculating F. We first, without appeal to Bott formulas, prove the Lu's formula for F when X is a complete intersection in a projective space. Then we use the Bott formula for calculating for some degenerations of the blow-up of the three dimensional projective space in a twisted cubic curve.

2. Basic facts from the theory of the Equivariant Chow groups

The purpose of this paper is to establish the algebraic nature (in terms of equivariant Chow groups) of both - the classical Futaki invariant [11] and the generalized Futaki invariant [6], and to give some examples of how to calculate them. In this section we present some definitions and recall some properties of the equivariant Chow groups to the extent and in form we will need them in the text. The aim is to define, for each element α of an operational equivariant Chow group of a normal algebraic variety, a polynomial f_α on the Lie algebra of the group of automorphisms of that variety. In section 3 we show that both the classical and the generalized Futaki invariants are special cases of this construction.

The main sources on the theory of Chow groups for us are [10], [7], and [4]. We work over the field of complex numbers $\mathbf{C}$.

2.1. Equivariant Chow Groups.

In our research we will use the definition of equivariant Chow groups due to Edidin and Graham [7]. By an algebraic group we always mean a linear algebraic group.

Let X be a scheme (over $\mathbf{C}$), $\dim X = n$, and let G be an algebraic group acting on X. Suppose V is a linear representation of G such that there is an open subset $U \subset V$ on which G acts freely, and let $U \to U/G$ be the principal quotient bundle. It is shown in [7] that for any (linear) algebraic group G and for any q there exists a linear representation V of G such that $codim_V(V - U) > n - q$ and the quotient space U/G is a scheme. The mixed space $X_G := X \times_G U$ exists as an algebraic space, but if X is normal, the case we are mainly interested in, X_G is in fact a scheme. We define the q-th G-equivariant Chow group of X by setting

$$A_q^G(X) := A_{q+\dim V - \dim G}(X_G),$$

where on the right-hand-side we use the usual Chow group. As is shown in [7] the definition above does not depend on the particular choice of the representation. Obviously, $A_q^G(X) = 0$ for all $q > \dim X$.

Without going into the theory of the equivariant Chow groups (for which we refer to [7] and [4]) we recall here the properties which govern their behaviour when passing to a closed subgroup.

1. Let $H < G$ be a closed subgroup of G, and let V be a representation of G which defines the group $A_q^G(X)$ as above. Hence, V defines also the group $A_q^H(X)$. There is a natural smooth morphism

$$f_{H,G} : X \times_H U \longrightarrow X \times_G U$$

with fibre G/H. The flat pull back morphism for the usual Chow groups

$$f_{H,G}^* : A_*(X \times_G U) \longrightarrow A_{*+\dim G/H}(X \times_H U).$$

can be interpreted as a morphism

$$\phi_*^{H,G} : A_*^G(X) \longrightarrow A_*^H(X)$$

of equivariant Chow groups.

In particular, if G/H is isomorphic to an affine space, then $\phi_*^{H,G}$ will be an isomorphism ([14]) due to the homotopy invariance of the usual Chow groups. As a consequence we get that $\phi_*^{H,G}$ is an isomorphism for any maximal reductive closed subgroup H of G. (We call such subgroups Levi subgroups of G).

Remark. In the notations above, X defines elements $[X]_G$ and $[X]_H$ in $A_n^G(X)$ and $A_n^H(X)$ respectively. We have that $\phi_*^{H,G}([X]_G) = [X]_H$.

2. (Proper push-forward) For any proper G-scheme X, the canonical map

$$\pi_X : X \longrightarrow pt,$$

viewed naturally as a G-equivariant map, gives rise to a morphism

$$\pi_{X*}^G : A_*^G(X) \longrightarrow A_*^G(pt).$$

This map can be seen as an integration along the fibres of the fibration

$$X \times_G U \longrightarrow pt \times_G U = U/G$$

with fibre X.

The commutative diagram of maps

$$X \times_H U \xrightarrow{f_{H,G}} X \times_G U$$
$$\downarrow \qquad\qquad \downarrow$$
$$pt \times_H U \longrightarrow pt \times_G U$$

gives rise to the commutative diagram

$$A_*^G(X) \xrightarrow{\phi_*^{H,G}} A_*^H(X)$$
$$\pi_{X*}^G \downarrow \qquad\qquad \downarrow \pi_{X*}^H$$
$$A_*^G(pt) \xrightarrow{\phi_*^{H,G}} A_*^H(pt),$$

i.e. $\pi_{X*}^H \circ \phi_*^{H,G} = \phi_*^{H,G} \circ \pi_{X*}^G$.

3. The q-th G-equivariant Borel-Moore homology group is defined as

$$H_{BM,q}^G(X) := H_{BM,q+2\dim V-2\dim G}(X_G)$$

notations and assumptions being as above. There is a natural "cycle" map (cf. [7])

$$cl_X^G : A_q^G(X) \longrightarrow H_{BM,2q}^G(X),$$

compatible with the operations on the equivariant Chow groups. As in the case of the Chow groups there are morphisms

$$\phi_*^{H,G} : H_{BM,q}^G(X) \longrightarrow H_{BM,q}^H(X),$$

such that the diagram

$$A_q^G(X) \xrightarrow{cl_X^G} H_{BM,2q}^G(X)$$
$$\phi_*^{H,G} \downarrow \qquad\qquad \downarrow \phi_*^{H,G}$$
$$A_q^H(X) \xrightarrow{cl_X^H} H_{BM,2q}^H(X)$$

commutes, i.e. $\phi_*^{H,G} \circ cl_X^G = cl_X^H \circ \phi_*^{H,G}$.

Remark. When X is smooth $H_{BM,q}^G(X) \cong H^{2n-q}(X_G) \cong H_G^{2n-q}(X)$ the last being the G-equivariant cohomology group of X.

2.2. **Equivariant Operational Chow groups.** We will use the equivariant operational Chow groups as defined in [7]. For $i \geq 0$ we set

$$A_G^i(X) = \left\{ c(f) : A_*^G(Y) \to A_{*-i}^G(Y) \,\middle|\, \begin{array}{l} f : Y \to X \text{ is a } G\text{-equivariant} \\ \text{map of normal varieties} \end{array} \right\}$$

where the morphisms $c(f)$ are compatible with the operations on the equivariant Chow groups (pull-back for l.c.i. morphisms, proper push-forwards, etc.). The composition of maps defines a graded ring structure on $A_G^*(X) := \bigoplus_{i \geq 0} A_G^i(X)$.

We list below some of the properties of the equivariant operational Chow groups that we will be using further.

1. If V is a linear representation of G, $U \subset V$ is an open subset on which G acts freely, and $\mathrm{codim}_V(V - U) > k$, then
$$A_G^k(X) = A^k(X_G).$$

Let $E \to X$ be a G-equivariant vector bundle. Then $E_G := U \times_G E$ is a vector bundle over X_G. The equivariant Chern class $c_i^G(E)$ of E is defined as the operational class $c_i(E_G)$.

2. (Cap-product) For any $\alpha \in A_G^q(X)$, the identity map $id : X \to X$ defines an operation $\alpha(id) : A_p^G(X) \to A_{p-q}^G(X)$ determining in this way the cap-product
$$\cap \; : \; A_G^q(X) \otimes A_p^G(X) \longrightarrow A_{p-q}^G(X)$$
$$\alpha \otimes a \longmapsto \alpha \cap a := \alpha(id)(a).$$
This cap-product makes $A_*^G(X)$ into a $A_G^*(X)$-module.

3. The canonical map $\pi_X^G : X \to pt$ defines a degree 0 morphism of graded rings
$$\pi_X^{G*} : A_G^*(pt) \longrightarrow A_G^*(X).$$
Under this morphism $A_*^G(X)$ becomes a $A_G^*(pt)$- module.

4. Let $H < G$ be a closed subgroup. The natural morphism, defined as above,
$$U \times_H X \longrightarrow U \times_G X$$
defines a morphism
$$A^*(X_G) \longrightarrow A^*(X_H)$$
which, when $\mathrm{codim}_V(V - U)$ is appropriate, gives rise to a morphism
$$\phi_{H,G}^* : A_G^*(X) \longrightarrow A_H^*(X).$$
Further, the morphisms $\phi_{H,G}^*$ and $\phi_*^{H,G}$ are compatible with the cap-product morphism - the diagram
$$\begin{array}{ccc} A_G^k(X) \otimes A_m^G(X) & \overset{\cap}{\longrightarrow} & A_{m-k}^G(X) \\ {\scriptstyle \phi_{H,G}^* \otimes \phi_*^{H,G}} \downarrow & & \downarrow {\scriptstyle \phi_*^{H,G}} \\ A_H^k(X) \otimes A_m^H(X) & \overset{\cap}{\longrightarrow} & A_{m-k}^H(X) \end{array}$$
commutes.

5. When X is smooth, the mapping
$$A_G^q(X) \ni c \longmapsto c \cap [X]_G \in A_{n-q}^G(X)$$
is an isomorphism, i.e. $A_G^*(X) \cong A_{n-*}^G(X)$. Due to this there is an induced "cycle class map"
$$cl_G^X : A_G^*(X) \longrightarrow H_G^{2*}(X)$$
which commutes with the maps $\phi_{H,G}^*$:
$$\phi_{H,G}^* \circ cl_G^X = cl_H^X \circ \phi_{H,G}^*.$$

6. When X is a complete G- scheme there is an integration map
$$\int_X^G := \pi_*^G \circ (\bullet \cap [X]_G) \; : \; A_G^q(X) \longrightarrow A_{n-q}^G(pt).$$

For any closed subgroup $H < G$, we have

$$\int_X^H \circ \phi_{H,G}^* = \phi_*^{H,G} \circ \int_X^G.$$

In what follows we consider only complete schemes X, such that the map $\int_X^G$ is always defined.

7. (Structure Theorem) Suppose G is a reductive algebraic group, $T < G$ a maximal torus, and $W := N(T)/T$ -the Weyl group of (G, T). Let $\chi(T)$ be the group of characters of T, and denote by $S_{\mathbf{Z}}(T) = \mathbf{Sym}[\chi(T)]$ and $S(T) = S_{\mathbf{Z}}(T) \otimes_{\mathbf{Z}} \mathbf{Q}$ the corresponding symmetric algebras. The Weyl group W acts naturally on $A_*^T(X)$ for any G-scheme X (cf. [7] and [4]). We have the following structure theorem

Theorem 2.2.1. *The graded ring $A_T^*(pt)$ is isomorphic to $S_{\mathbf{Z}}(T)$. The map*

$$\phi_{T,G}^* : A_G^*(pt) :\longrightarrow A_T^*(pt)$$

is an injection over $\mathbf{Q}$, and identifies $A_G^*(pt)_{\mathbf{Q}} := A_G^*(pt) \otimes_{\mathbf{Z}} \mathbf{Q}$ with the W-invariant subring $S(T)^W$ of $S(T)$.

For any G-scheme X, the map $\phi_*^{T,G}$ induces and isomorphism

$$A_*^G(X)_{\mathbf{Q}} \cong A_*^T(X)_{\mathbf{Q}}^W.$$

2.3. **Definition and basic properties of the map f_α.** Let $\xi \in \mathfrak{g}$ be an element of the Lie algebra $\mathfrak{g}$ of G. Denote by $G(\xi)$ the minimal algebraic subgroup of G containing $\exp(\xi)$. If $\xi = \xi_s + \xi_n$ is the Jordan decomposition of ξ, then $G(\xi) = T(\xi) \times G(\xi_n)$, where $T(\xi) = G(\xi_s)$ is a subtorus of G, uniquely determined by ξ, and $G(\xi_n) = \{\exp(t\xi_n)|t \in \mathbf{C}\}$ is a unipotent subgroup of G. (cf. [21].)

For any $\alpha \in A_G^q(X)$ the element

$$\int_X^{G(\xi)} \phi_{G(\xi),G}^*(\alpha) = \phi_*^{G(\xi),G} \circ \int_X^G \alpha$$

can be identified via $\phi_*^{T(\xi),G(\xi)}$ with the element

$$\int_X^{T(\xi)} \phi_{T(\xi),G}^*(\alpha) \in A_{n-q}^{T(\xi)}(pt) \cong A_{T(\xi)}^{q-n}(pt).$$

By the structure theorem above,

$$A_{n-q}^{T(\xi)}(pt) = \mathbf{Sym}[\chi(T(\xi))]_{q-n}$$

and $\int_X^{T(\xi)} \phi_{T(\xi),G}^*(\alpha)$ is a polynomial of degree $q - n$ on $Lie\, T(\xi)$. (It is zero for $q < n$.)

Define

$$f(\xi, \alpha) := (\int_X^{T(\xi)} \phi_{T(\xi),G}^*(\alpha))(\xi_s) \in \mathbf{C}.$$

The number $f(\xi, \alpha)$ depends only on ξ and α, and for any $g \in G$

$$f(ad(g)\xi, \alpha) = f(\xi, \alpha).$$

Another way of calculating $f(\xi, \alpha)$ is via maximal tori in G containing $T(\xi)$. Choose a maximal torus T of G such that $T(\xi) < T$. Then, $\xi_s \in Lie\, T(\xi) \subset Lie\, T$, and

$$f(\xi, \alpha) = (\phi_*^{T(\xi),T} \circ \phi_*^{T,G} \int_X^G \alpha)(\xi_s)$$

$$= (\phi_*^{T,G} \int_X^G \alpha)(\xi_s).$$

As is clear from their definition, the numbers $f(\xi, \alpha)$ depend linearly on the second argument. The dependence on the first argument is rather complicated in general. Nevertheless, there are some cases in which, for fixed second argument, $f_\alpha := f(\bullet, \alpha)$ is an $Ad(G)$-invariant polynomial on $\mathfrak{g}$. One such case, which is closely related to the generalized Futaki invariant, is described below.

Let $T < G$ be a maximal torus of G, let $H < G$ be a Levi subgroup (i.e. a maximal closed reductive subgroup of G) containing T. Let $W := W(H, T)$ be the corresponding Weyl group. Then,

$$G = H \ltimes Rad_u G,$$

$$H = Z(H).H',$$

$$T = (Z(H))_0 \times T',$$

where $Rad_u G$ is the unipotent radical of G, $H' = (H, H)$ is the commutator of H, $T' = T \cap H'$ is the maximal torus of H', corresponding to T, and $(Z(H))_0$ is the identity component of the centre $Z(H)$. The Weyl group W can be naturally identified with $W(H', T')$.

We already know by the structure theorem that (over $\mathbf{Q}$) the map $\phi_*^{T,G}$ is an embedding with image

$$\mathbf{Sym}[\chi(T)]^W = \mathbf{Sym}[\chi(Z(H))] \otimes_{\mathbf{Q}} \mathbf{Sym}[\chi(T')]^W.$$

Suppose now that α is such that

$$\phi_*^{T,G} \int_X^G \alpha \in \mathbf{Sym}[\chi(Z(H))]$$

and denote by $j : \mathfrak{g} \to Lie\, Z(H)$ the Lie algebra morphism corresponding to the canonical map $G \to H/H'$. Then α defines an $Ad\, G$-invariant polynomial f_α on $\mathfrak{g}$: for $\xi \in \mathfrak{g}$

$$f_\alpha(\xi) := (\phi_*^{T,G} \int_X^G \alpha)(j(\xi)).$$

The algebra $\mathbf{Sym}[\chi(T')]^W$ has no elements of degree one. Hence

$$(\mathbf{Sym}[\chi(Z(H))] \otimes_{\mathbf{Q}} \mathbf{Sym}[\chi(T')]^W)_1 = (\mathbf{Sym}[\chi(Z(H))])_1,$$

and we have thus proved the following

Proposition 2.3.1. *In the above notations if $\alpha \in A_G^{n+1}(X)$, then f_α is a Lie algebras morphism*

$$f_\alpha : \mathfrak{g} \longrightarrow \mathbf{C}$$

and $f_\alpha(\xi) = f(\xi, \alpha)$. This morphism vanishes on the nilpotent elements of $\mathfrak{g}$.

Remark A more intrinsic proof of the proposition above, which I owe to M.Brion, goes as follows. For any algebraic group G the group $A_G^1(pt)$ is naturally isomorphic to the Picard group $Pic^G(pt)$ of the G-equivariant line bundles over a point. The latter is the same as the character group of G: each G-equivariant line bundle $L \to pt$ corresponds to a homomorphism

$$\chi_L : G \to \mathbf{C}^*.$$

So, given $\alpha \in A_G^{n+1}(X)$ its push-forward $\int_X^G \alpha \in A_G^1(pt)$ can be identified with a character χ_L of G. The map f_α is just the differential $d\chi_L$ of that character.

3. The Generalized Futaki Invariant

In this section we recall the definition of the generalized Futaki invariant following [6], [22], and show that it has the properties of the map f_α from Proposition 2.3.1. In the case of the classical Futaki invariant this is shown by Futaki-Morita [13], and Mabuchi [18]. We follow very closely the approach in the cited papers with the appropriate changes needed when working with almost Fano varieties - the object of our interest. In the next section we'll show that in fact the properties from Proposition 2.3.1 are sufficient for the generalized Futaki invariant to be represented as f_α for an appropriate α.

Let X be a normal complete $\mathbf{Q}$-Gorenstein variety. Let $L \to X$ be the line bundle, such that $L^*_{|X_{reg}} = \omega^{\otimes k}_{X_{reg}}$, where $\omega_{X_{reg}}$ is the dualizing sheaf of the regular part X_{reg} of X. Suppose L is ample. Such varieties X are called **almost Fano varieties** ([6], [22]).

A Kähler form ω on X_{reg} is called **admissible** (on X) if there exist an embedding of X, defined by some power of L,

$$\phi_{L^m} \,:\, X \,\hookrightarrow\, \mathbf{P}^N,$$

and a Kähler form $\tilde{\omega}$ on $\mathbf{P}^N$ representing $2\pi c_1(\mathbf{P}^N)$, such that

$$\omega \,=\, \frac{1}{km}.\phi^*_{L^m}(\tilde{\omega}) \qquad on \qquad X_{reg}.$$

A vector field ξ on X_{reg} is called **admissible** (on X) if there exist an embedding as above, and a vector field $\tilde{\xi}$ on $\mathbf{P}^N$, such that $\tilde{\xi}$ is tangent to X along X_{reg} , and

$$\xi \,=\, \tilde{\xi}_{|X_{reg}}.$$

Suppose ω is an admissible form on X. Then on X_{reg}

$$Ric(\omega) \,-\, \omega \,=\, \sqrt{-1}\partial\bar{\partial}f_\omega,$$

where $Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\omega^n$, and $f_\omega \in C^\infty(X_{reg}, \mathbf{R})$. It is proved by Ding and Tian [6] (see also [24]) that, given an admissible vector field ξ, for any admissible form ω the number

$$F(\xi) \,=\, \frac{1}{(2\pi)^n} \int_{X_{reg}} \xi(f_\omega)\omega^n$$

is well defined, does not depend on the choice of ω, and defines a character of the algebra of the admissible vector fields on X. When X is smooth, i.e. X is a Fano manifold, the map F coincides with the classical Futaki invariant [11]. In the general case of almost Fano varieties the map F is called **the generalized Futaki invariant** of X. The significance of this invariant stems from the fact ([6] and [22]) that it is closely related to the existence of Einstein-Kähler metrics on Fano manifolds.

Now we want to show that F has an appropriate integral representation, by using which one can prove that it vanishes on the nilpotent admissible vector fields, and can relate it to the Futaki-Morita polynomials from the next section.

Suppose $\omega = (1/km)\phi^*_{L^m}\tilde{\omega}$, for some embedding $\phi_{L^m} : X \hookrightarrow \mathbf{P}^N$. Since the admissible vector fields on X correspond to the elements of $Lie\, G$, where $G = Aut\, X$, we can find a vector field $\tilde{\xi}$ on $\mathbf{P}^N$ which restricts to ξ on X_{reg}. There exists a Hermitian metric $\tilde{h}$ on $[H]$, the hyperplane line bundle on $\mathbf{P}^N$, such that $Ric\,\tilde{h} = \tilde{\omega}$. (Here we denote $(Ric\,\tilde{h})_{|U} = -\sqrt{-1}\,\partial\bar{\partial}\log(\tilde{h}_U)$ for some local expression $\tilde{h}_U$ of the metric $\tilde{h}$.) Hence, on X_{reg},

$$Ric\,\omega - \omega \,=\, Ric\,\omega - \frac{1}{km}\,\phi^*_{L^m}\,Ric\,\tilde{h} \,=\, Ric\,\omega - Ric\,h,$$

where $h = (\tilde{h})^{\frac{1}{km}}$ is a Hermitian metric on the anticanonical bundle of X_{reg}. So, on X_{reg} we have the following equalities:

$$Ric\,\omega - Ric\,h = \sqrt{-1}\,\partial\bar{\partial}\,\log\frac{h}{\omega^n} = \sqrt{-1}\,\partial\bar{\partial}\,f_\omega$$

$$\xi(f_\omega) = \xi(\log\frac{h}{\omega^n}) = -\frac{L_\xi(\omega^n)}{\omega^n} + h^{-1}\,L_\xi(h).$$

It is easy to check that

$$L_\xi(\omega^n) = L_\xi\left(\frac{1}{km}\,\phi_{L^m}^*\,\tilde{\omega}^n\right) = \phi_{L^m}^*\,L_{\tilde{\xi}}\left(\frac{\tilde{\omega}}{km}\right)^n.$$

Hence, on X_{reg},

$$L_\xi(\omega^n) = \phi_{L^m}^*\,d\,i_{\tilde{\xi}}\left(\frac{\tilde{\omega}}{km}\right)^n = d\left(\phi_{L^m}^*\,i_{\tilde{\xi}}\left(\frac{\tilde{\omega}}{km}\right)^n\right).$$

We apply the Stokes' theorem and get

$$F(\xi) = \frac{1}{(2\,\pi)^n}\int_{X_{reg}} h^{-1}\,L_\xi\,h\,(Ric\,h)^n.$$

On the other hand, let $P' \to X$ be the principal $\mathbf{C}^*$-bundle corresponding to L, and θ' be the connection $(1,0)$-form on $P'_{|X_{reg}}$ corresponding to the metric $h' = (\phi_{L^m}^*\,\tilde{h})^{\frac{1}{m}}$. Let further, e_U be a local frame of L over an open subset $U \subset X_{reg}$, and denote by $\phi_U : U \to P'_{|X_{reg}}$ the corresponding local section of P'. Then, on U, which we identify with its image in P' under ϕ_U,

$$h^{-1}\,L_\xi(h) = \frac{1}{k}\,(h')^{-1}\,L_\xi(h') = \frac{1}{k}\,\theta'_{|U}(\xi'),$$

where ξ' is the vector field on P' (over X_{reg}) induced by ξ. Hence

$$F_U(\xi) := \frac{1}{(2\pi)^n}\cdot\frac{1}{k^{n+1}}\int_U (h')^{-1}\,L_\xi(h')\,.(Ric\,h')^n$$

$$= \frac{1}{(2\pi)^n}\cdot\frac{1}{k^{n+1}}\int_U \theta'_{|U}(\xi')\,(Ric\,h')^n$$

$$= \frac{1}{n+1}\cdot\frac{1}{(2\pi)^{n+1}}\cdot\frac{1}{k^{n+1}}\int_U (\theta'_{|U}(\xi') + \sqrt{-1}\Theta'_{|U})^{n+1},$$

where Θ' is the curvature of θ'.

Suppose now that $\nu : \tilde{X} \to X$ is a G-equivariant resolution of singularities of X. Then ξ has a unique lift $\tilde{\xi}$ to a tangent vector field on $\tilde{X}$, and

$$\int_U (\theta'_{|U}(\xi') + Ric\,h')^{n+1} = \int_{\tilde{U}} (\tilde{\theta}'_{|\tilde{U}}(\tilde{\xi}') + Ric\,\tilde{h}')^{n+1},$$

where $\tilde{h}' = \nu^*(h')$, and $\tilde{\theta}' = \nu^*(\theta')$ are the pull-backs of the corresponding objects to $\tilde{L} := \nu^*L$ and $\tilde{P}' = \nu^*P'$ respectively. Thus we get the desired formulas for the generalized Futaki invariant:

$$F(\xi) = \left(\frac{1}{2\pi}\right)^n\cdot\left(\frac{1}{k}\right)^{n+1}\int_{\tilde{X}} (\tilde{h}')^{-1}\,L_{\tilde{\xi}}(\tilde{h}')\,(Ric\,\tilde{h}')^n$$

$$= \frac{1}{(n+1).k^{n+1}}\int_{\tilde{X}} (\tilde{\theta}'(\tilde{\xi}') + \frac{\sqrt{-1}}{2\pi}\tilde{\Theta}'_{|\tilde{U}})^{n+1}.$$

Here $\frac{1}{2\pi}Ric\,\tilde{h}' = \frac{\sqrt{-1}}{2\pi}\tilde{\Theta}'_{|\tilde{U}}$, where $\tilde{\Theta}'$ is the curvature of the metric $\tilde{h}'$.

One can apply the same argument as in [18] to the first of the above formulas, and to show that the generalized Futaki invariant vanishes on each nilpotent element of the algebra of admissible vector fields on X - the argument from the cited paper is valid also for line bundles which give a

birational morphism of a manifold to a projective space. We will use the second formula in the next section.

4. "FUTAKI-MORITA" FOR NON-SMOOTH VARIETIES

The purpose of this section is to show that the generalized Futaki invariant, although analytically defined, has purely algebraic description. This invariant appears as a special case of the Futaki-Morita polynomials (cf. [13]) defined as soon as we have a H-equivariant principal G-bundle over a (normal) variety (we work with affine algebraic groups only). Although we are mainly interested in the special case of the Futaki invariant, it is instructive to consider the general case of such polynomials.

4.1. Futaki-Morita polynomials. Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Let $I^*(G)$ denote the subalgebra $\mathbf{C}[\mathfrak{g}^*]^G$ of $\mathrm{Ad}\,G$-invariant polynomials over $\mathfrak{g}$. The Weil algebra morphism W_G is a map

$$W_G : I^*(G) \longrightarrow H^{2*}(BG, \mathbf{C}),$$

where BG is the classifying space (or its algebraic approximation) of the principal G-bundles. Given a principal G-bundle $P \to X$, there is a map $\langle P \rangle : X \to BG$ which defines a map in the cohomologies

$$\langle P \rangle^* : H^*(BG, \mathbf{Z}) \longrightarrow H^*(X, \mathbf{Z}).$$

The composition map $w_G(P) := \langle P \rangle^* \circ W_G$ is the Weil homomorphism corresponding to the principal bundle $P \to X$.

Suppose we are given a principal G-bundle $p : P \to X$ such that an algebraic group H acts on P and X commuting with the action of G on P and making the projection map p H-equivariant. By using the universal H-bundle (or an appropriate approximation) $EH \to BH$ we can construct the principal G-bundle

$$p_H : P_H := EH \times_H P \longrightarrow EH \times_H X,$$

and define the corresponding map

$$\langle P_H \rangle^* : H^*(BG, \mathbf{Z}) \longrightarrow H^*(EH \times_H X, \mathbf{Z}).$$

If X is proper (of dimension n), we can compose $\langle P_H \rangle^*$ with the Gyzin map

$$\gamma_*^{X,H} : H^*(EH \times_H X, \mathbf{Z}) \longrightarrow H^{*-2n}(BH, \mathbf{Z}),$$

corresponding to the H-equivariant map $X \to pt$.

Suppose X is smooth n-manifold. In this case Futaki and Morita define a map

$$F := F_P : I^*(G) \longrightarrow I^{*-n}(H)$$

as follows (cf. [13]). For an element $\eta \in Lie\,H$ and $\phi \in I^{n+k}(G)$

$$F(\phi)(\eta) := \int_X \phi(\Theta + \theta(\eta_*)),$$

where θ is a $(1,0)$-type G-connection on P, Θ is its curvature, and η_* is the vector field on P generated by η. Futaki and Morita [13] prove that F does not depend on the choice of the

connection θ , and fits in the following commutative diagram

$$
\begin{array}{ccc}
I^*(G) & \xrightarrow{\quad F \quad} & I^{*-n}(H) \\
{\scriptstyle W_G}\downarrow & & \downarrow{\scriptstyle W_H} \\
H_G^{2*}(pt, \mathbf{C}) \xrightarrow{\langle P_H\rangle^*} H_H^{2*}(X, \mathbf{C}) & \xrightarrow{\gamma_*^{X,H}} & H_H^{2(*-n)}(pt, \mathbf{C})
\end{array}
$$

Now we will show that the map F can be defined, and such a diagram can be drawn, in the case of nonsmooth varieties too. In what follows we consider only H-equivariant principal G-bundles, such that the actions of H and G on them commute.

Notice first that for any birational morphism $\phi : Y \to Z$ of proper manifolds, and principal G-bundle $Q \to Z$ the morphisms F_Q and F_{ϕ^*Q} coincide. On the other hand, for any two H-equivariant resolutions of singularities $\nu_i : \tilde{X}_i \to X$, $i = 1, 2$, there exists a H-resolution $\nu : \tilde{X} \to X$, and maps $\mu_i : \tilde{X} \to \tilde{X}_i$, such that $\nu_i \circ \mu_i = \nu$, $i = 1, 2$. It follows that $F_{\nu_1^* P} = F_{\nu_2^* P} = F_{\nu^* P}$, and we can define the map F_P by using any H-equivariant resolution of the singularities of X ([3]).

Let $\nu : \tilde{X} \to X$ be a H-equivariant resolution of singularities. We then have the commutative diagram

$$
\begin{array}{ccc}
H_G^{2*}(pt, \mathbf{C}) \xrightarrow{\langle P_H\rangle^*} H_H^{2*}(X, \mathbf{C}) & \xrightarrow{\gamma_*^{X,H}} & H_H^{2(*-n)}(pt, \mathbf{C}), \\
{\scriptstyle \langle \tilde{P}_H\rangle^*}\searrow \quad \downarrow{\scriptstyle \nu^*} \quad \nearrow{\scriptstyle \gamma_*^{\tilde{X},H}} & & \\
H_H^{2*}(\tilde{X}, \mathbf{C}) & &
\end{array}
$$

where $\tilde{P} := \nu^* P$.

By the Futaki-Morita theorem we have that

$$
W_H \circ F_{\tilde{P}} = \gamma_*^{\tilde{X},H} \circ \langle \tilde{P}_H\rangle^* \circ W_G,
$$

and by the diagram above we conclude that

$$
W_H \circ F_P = \gamma_*^{X,H} \circ \langle P_H\rangle^* \circ W_G.
$$

It will be more convenient for our purposes to consider the map $\tilde{F} := (\sqrt{-1}/2\pi)^n \cdot F$ instead of F. Futaki-Morita theorem can be rephrased for $\tilde{F}$ as follows:

$$
\gamma_*^{X,H} \circ \langle P_H\rangle^* \circ ((\sqrt{-1}/2\pi)^* W_G) = ((\sqrt{-1}/2\pi)^{*-n} W_H) \circ \tilde{F}.
$$

In the special case of an almost Fano variety X and a principal bundle $\tilde{P}'$ as in the preceding section, we have $H := Aut\, X$, and $I^*(G) = I^*(\mathbf{C}^*) = \mathbf{C}[t]$. The generalized Futaki invariant of X in this case is given by $\tilde{F}(\frac{1}{n+1} \cdot (\frac{t}{k})^{n+1})$.

Remark. It is well known that for a reductive group G with a maximal torus T and Weyl group W there is an isomorphism $I^*(G) \to I^*(T)^W$, induced by the natural restriction map. On the other hand, for $S_{\mathbf{Z}}(T) := \mathbf{Sym}[\chi(T)]$ we have the inclusion

$$
S_{\mathbf{Z}}(T)^W \subset S_{\mathbf{Z}}(T)^W \otimes \mathbf{C} = I^*(T)^W,
$$

and the coefficient before W_G is chosen so that, for $G = GL(n, \mathbf{C})$, the homogeneous generators $\sigma_1, \ldots, \sigma_n$ of $S_{\mathbf{Z}}(T)$ as an algebra map to the Chern classes of the principal G-bundle $P_H \to M_H$, i.e. to the H-equivariant Chern classes of P.

4.2. Universality of the Weil morphism. Let $G = G_1 \ltimes Rad_u G$ be a Levi decomposition of the algebraic group G. Since $Rad_u G$ is isomorphic to an affine space, any principal G-bundle can be reduced to a principal G_1-bundle (in C^∞ category). In other words, there exist a principal G_1-bundle $q : Q \to X$, and an injection of principal bundles $i : Q \hookrightarrow P$, such that the corresponding homomorphism $i : G_1 \to G$ coincides with the above embedding of G_1 into G.

By using a connection on P induced by a connection on Q, one can show that the Weil morphism W_G factors through the algebra $I^*(G_1)$
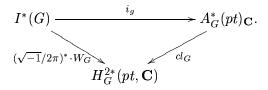
$$
\begin{array}{ccc}
I^*(G) & \xrightarrow{\quad res \quad} & I^*(G_1), \\
\end{array}
$$
$$
(\sqrt{-1}/2\pi)^* \cdot W_G \searrow \qquad \swarrow (\sqrt{-1}/2\pi)^* \cdot W_{G_1}
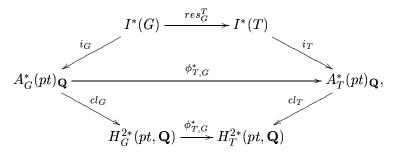$$
$$
H_G^{2*}(pt, \mathbf{C})
$$

where res is the restriction map corresponding to the embedding $i_* : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}$.

The general theory of equivariant cohomology and Chow groups says that $I^*(G_1)$ can be identified (over $\mathbf{C}$) both with $A^*_{G_1}(pt) \cong A^*_G(pt)$ and $H^{2*}_{G_1}(pt) \cong H^{2*}_G(pt)$. After identifying $I^*(G_1)$ with $A^*_G(pt)_{\mathbf{C}} := A^*_G(pt) \otimes_{\mathbf{Z}} \mathbf{C}$, denote by i_g the resulting map from $I^*(G)$ to $A^*_G(pt)_{\mathbf{C}}$.

Claim. We have the commutative diagram

$$
\begin{array}{ccc}
I^*(G) & \xrightarrow{\quad i_g \quad} & A^*_G(pt)_{\mathbf{C}}. \\
\end{array}
$$
$$
(\sqrt{-1}/2\pi)^* \cdot W_G \searrow \qquad \swarrow cl_G
$$
$$
H_G^{2*}(pt, \mathbf{C})
$$

Proof of the claim. Without a loss of generality we may assume that G is reductive itself. Let T be its maximal torus, and $W = N(T)/T$ be the corresponding Weyl group. We start with the commutative diagram

$$
\begin{array}{ccccc}
 & & I^*(G) & \xrightarrow{res^T_G} & I^*(T) \\
 & \nearrow{i_G} & & & \searrow{i_T} \\
A^*_G(pt)_{\mathbf{Q}} & & \xrightarrow{\phi^*_{T,G}} & & A^*_T(pt)_{\mathbf{Q}}, \\
 & \searrow{cl_G} & & & \swarrow{cl_T} \\
 & & H_G^{2*}(pt, \mathbf{Q}) & \xrightarrow{\phi^*_{T,G}} & H_T^{2*}(pt, \mathbf{Q})
\end{array}
$$

the horizontal arrows being injective with images the W-invariant subalgebras in the targets.

It is a straightforward calculation to check that $cl_T \circ i_T = (\sqrt{-1}/2\pi)^* \cdot W_T$. On the other hand, we will prove in a moment that

$$
\phi^*_{T,G} \circ W_G = W_T \circ res^T_G, \tag{$*$}
$$

which gives $(\sqrt{-1}/2\pi)^* \cdot W_G = cl_G \circ i_G$, and we will be done.

To prove (*), one may use the algebraic description of the Weil morphism due to Beilinson and Kazhdan. To make the exposition self-contained, we give here the idea of the proof, following the presentation of Beilinson-Kazhdan construction as given in [9].

Recall the Atiyah extension $\mathfrak{A}(P)$ associated to a principal G-bundle P over a smooth manifold Y (cf. [1]):

$$
\mathfrak{A}(P) : 0 \longrightarrow Ad_Y \mathfrak{g} \longrightarrow TP/G \longrightarrow TY \longrightarrow 0,
$$

where $Ad_Y \mathfrak{g} := P \times_\rho \mathfrak{g}$ (ρ is the adjoint representation of G in $\mathfrak{g}$), and TP/G is considered as a vector bundle on Y.

In their construction Beilinson and Kazhdan consider the iterated n-extension associated to $\mathfrak{A}(P)^*$ - the dual Atiyah extension :

$$\mathfrak{A}(P)_n^* \ : \ 0 \longrightarrow \Omega_Y^n \longrightarrow \wedge^n(T^*P/G) \longrightarrow \wedge^{n-1}(T^*P/G) \otimes Ad_Y\mathfrak{g}^* \longrightarrow \dots$$

$$\dots \longrightarrow \wedge^{n-i}(T^*P/G) \otimes S^i Ad_Y\mathfrak{g}^* \longrightarrow \cdots \longrightarrow S^n Ad_Y\mathfrak{g}^* \longrightarrow 0,$$

and show that the connecting morphism

$$H^0(Y, S^n Ad_Y\mathfrak{g}^*) \longrightarrow H^n(Y, \Omega_Y^n),$$

when evaluated on $EG = (G^{l+1})_l \to (G^{l+1}/G)_l = BG$ - the universal principal G-bundle, gives the Weil morphism

$$H^0(BG, S^n Ad_{BG}\mathfrak{g}^*) = (S^n \mathfrak{g}^*)^G \longrightarrow H^n(BG, \Omega_{BG}^n) = H_{DR}^{2n}(BG).$$

As usual, for the RHS equality one needs the reductivity of G.

Suppose we are given a principal H-bundle $Q \to X$, where H is a closed subgroup of G. We have the morphism of principal bundles on X : $\quad Q \to Q \times_H G$. By the functorial properties of the Atiyah extension [1] we have the morphism of the corresponding extensions $\mathfrak{A}(Q \times_H G)^* \to \mathfrak{A}(Q)^*$, which in turn determines a morphism of the iterated n-extensions associated. Finally we have the commutative diagram

$$H^0(X, S^n Ad_X\mathfrak{g}^*) \longrightarrow H^0(X, S^n Ad_X\mathfrak{h}^*)$$

$$H^n(X, \Omega_X^n)$$

where the horizontal morphism is induced by the canonical map $\mathfrak{g}^* \to \mathfrak{h}^*$.

Suppose finally that $X = BH = (H^{l+1}/H)_l$, and $Q = EH = (H^{l+1})_l$. Since $EH \times_H G$ is also a pull-back of EG, we have the commutative diagram

$$(S^n\mathfrak{g}^*)^G = H^0(BG, S^n Ad_{BG}\mathfrak{g}^*) \xrightarrow{(\frac{\sqrt{-1}}{2\pi})^n . W_G} H^n(BG, \Omega_{BG}^n) = H_{DR}^{2n}(BG)$$

with vertical maps res_G^H, and down to

$$H^0(BH, S^n Ad_{BH}\mathfrak{g}^*) \longrightarrow H^n(BH, \Omega_{BH}^n)$$

and vertical map $\phi_{H,G}^*$, and

$$(S^n\mathfrak{h}^*)^H = H^0(BH, S^n Ad_{BH}\mathfrak{h}^*) \xrightarrow{(\frac{\sqrt{-1}}{2\pi})^n . W_H} H^n(BH, \Omega_{BH}^n) = H_{DR}^{2n}(BH).$$

The right-hand-side equalities hold when G and H are reductive. Thus we get (*).

Another way of proving (*) goes as follows.

Let $EG \to BG$ be the universal G-bundle, and let B be a Borel subgroup of G. Consider the corresponding fibration

$$\varphi : EG/B \to BG$$

with fibre G/B. Then,

$$\varphi^*(EG) = EG \times_B G,$$

where B acts on G from the left, and the cohomology morphism

$$\varphi^* : H^*(BG, \mathbf{C}) \to H^*(EG/B, \mathbf{C})$$

coincides with the morphism

$$\phi_{B,G}^* : H_G^*(pt, \mathbf{C}) \to H_B^*(pt, \mathbf{C}).$$

By the construction, the structure group of the principal G-bundle $\varphi^*(EG)$ is reducible to the subgroup B, and since the map $\varphi^* \circ W_G = \omega_G(\varphi^*(EG))$ does not depend on the connection chosen, we take a B-connection on $EG \to EG/B$ and induce a G-connection on $\varphi^*(EG) \to EG/B$. This induced connection gives

$$\omega_G(\varphi^*(EG)) = W_B \circ res_G^B,$$

or in the other notations

$$\phi_{B,G}^* \circ W_G = W_B \circ res_G^B. \tag{1}$$

We may apply the arguments above also to the bundle $EG \to EG/B$ and to the maximal subtorus T of B. The principal B-bundle EG is reducible to a T-principal subbundle because the manifold B/T is isomorphic to an affine space, and therefore the fibration $EG/T \to EG/B$ has a section. Thus we get
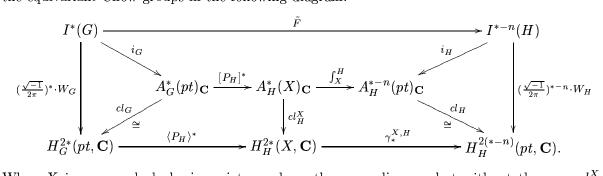
$$W_T \circ res^T B = \phi_{T,B}^* \circ W_B. \tag{2}$$

Now (*) easily follows from (1) and (2).

4.3. Inserting a row in the Futaki-Morita diagram.

We can apply the same reasoning as in the case of equivariant cohomology groups above to a H-equivariant principal G-bundle $p : P \to X$, with commuting actions of H and G on P. We obtain a map

$$[P_H]^* : A_G^*(pt)_{\mathbf{Q}} \longrightarrow A_H^*(X)_{\mathbf{Q}},$$

where $[P_H] : X_H \to BG$ is the classifying map corresponding to the principal G-bundle $p_H : P_H \to X_H$. When X is smooth and proper, we can combine the Futaki-Morita polynomials and the equivariant Chow groups in the following diagram:

When X is a normal algebraic variety we have the same diagram but without the map cl_H^X. This is easily seen by using H-desingularizations. (The map $\int_X^H \circ [P_H]^*$ does not depend on the desingularizations and, under the identifications in the diagram, coincides with $\gamma_*^{X,H} \circ \langle P_H \rangle^*$.)

Choose a Levi decomposition $H = H_1 \ltimes Rad_u\, H$ of the group H. According to this decomposition we have

$$I^*(H) \subset \mathbf{C}[\mathfrak{h}_1] \otimes \mathbf{C}[rad_u\mathfrak{h}] = \mathbf{C}[\mathfrak{h}_1][rad_u\mathfrak{h}].$$

By its definition, i_H is given by sending $\varphi \in I^*(H)$ to its degree 0 component in the latter algebra

$$\varphi_0 := i_H(\varphi) \in \mathbf{C}[\mathfrak{h}_1].$$

Since φ is $Ad\, H$-invariant, and since all Levi subgroups of H are conjugate, the property that the difference $\varphi - \varphi_0$ is 0 is well defined : if it holds for some Levi subgroup H_1 of H, then it holds for all such subgroups.

Theorem 4.3.1. *Let G and H be algebraic groups, $p : P \to X$ a H- equivariant principal G-bundle over a normal variety X. Let the actions of G and H on P commute. Given $\phi \in I^*(G)$, denote by α its image in $A_H^*(X)_{\mathbf{C}}$: $\alpha = [P_H]^* \circ i_G(\phi)$. Then,*

1. The polynomial $\tilde{F}(\phi)$ is **algebraic**, *i.e. for each $\eta \in \mathfrak{h} = Lie\,H$ $\tilde{F}(\phi)(\eta) = f_\alpha(\eta)$, iff $\tilde{F}(\phi) - \tilde{F}(\varphi)_0 = 0$ for some Levi decomposition $H = H_1 \ltimes Rad_u H$ of H ;*

2. In particular, if H is reductive, then $\tilde{F}(\phi)$ is algebraic for all $\phi \in I^*(G)$.

Note that for degree 1 elements $\varphi \in I^*(H)$ the property $\varphi - \varphi_0 = 0$ means exactly that φ vanishes on the nilpotent elements of $\mathfrak{h}$ .As we have already seen in the previous section, the generalized Futaki invariant of an almost Fano variety X vanishes on all nilpotent elements of $Lie\,H$, where $H = Aut(X)$. Combining this with the Remark in the subsection 3.1 it follows that

Corollary 4.3.2. *The generalized Futaki invariant is algebraic. It can be represented as $\frac{1}{n+1} f_\alpha$, where $\alpha = (c_1^H(X))^{n+1}$.*

Remarks.1. According to the remark in the end of Section 2, we have that the generalized Futaki invariant can be lifted to a group character. Another approach to see this is by using the Chern-Simons invariants. A deep study of this approach can be found in [13].

2. In the notations above let further T be a maximal torus of H which lies in H_1, and $T = T' \times T''$, where T' is a maximal torus in H_1' and $T'' = Z(H_1)_0$ - the connected component of the unit of the center of H_1. Let $W = W(H_1', T')$ be the corresponding Weyl group. By the structure theorem for the Chow cohomology groups we have

$$A_H^*(pt)_{\mathbf{C}} \cong A_T^*(pt)_{\mathbf{C}}^W = I^*(T)^W.$$

On the other hand, a theorem of Chevalley says that the restriction map $I^*(H_1) \to I^*(T)^W$ is an isomorphism of graded algebras. Hence, each element $\alpha \in A_H^*(pt)_{\mathbf{Q}}$ defines, implicitly, a polynomial from $I^*(H_1)$ and, by using the canonical morphism $\mathfrak{h} \to \mathfrak{h}_1$, an element of $I^*(H)$.

3. Consider the subalgebra $\mathfrak{z}(\mathfrak{h}_1) = \mathfrak{t}''$ of $\mathfrak{h}$. The corresponding restriction morphism

$$res_{\mathfrak{t}''}^{\mathfrak{h}} : I^*(H) \to I^*(T'') = \mathbf{C}[\mathfrak{t}'']$$

provides a correct definition of the property $\varphi_1 := \varphi - res_{\mathfrak{t}''}^{\mathfrak{h}} \varphi$ being zero. In the particular case of $\varphi \in I^1(H)$, this property is equivalent to φ being a character of $\mathfrak{h}$ which vanishes on its nilpotent elements. Finally, the canonical projection $\mathfrak{h} \to \mathfrak{z}(\mathfrak{h}_1) = \mathfrak{t}''$ determines an embedding

$$\mathbf{C}[\mathfrak{t}''] \hookrightarrow I^*(H),$$

which gives a way of finding explicitly the polynomials in $I^*(H)$ corresponding to the elements $f_\alpha \in \mathbf{Sym}[\chi(T'')]_{\mathbf{Q}}$. These polynomials are f_α themselves.

5. EXAMPLES.

We give in this section examples of calculating the generalized Futaki invariant F of some almost Fano varieties X. Since F is an equivariant Chow class over a point obtained by integration over X, the main tool we use, when it cannot be calculated directly, is the Bott-type residue formula proved by Edidin and Graham [8]. This formula can also be used for calculating the polynomials $\tilde{F}(\phi), \phi \in I^{n+k}(G)$, provided that they are algebraic in the sense of Theorem 4.3.1.

Suppose $\tilde{F}(\phi)$ is algebraic. As we have already proved, to see the behaviour of $\tilde{F}(\phi)$ on the elements of $\mathfrak{h}$ it is enough to fix a maximal torus T of H and find the element of $A_T^k(pt)$ corresponding to $\tilde{F}(\phi)$. As we have shown, when $k = 1$ this element defines an element of $\mathfrak{h}^*$ representing $\tilde{F}(\phi)$ as a functional on $\mathfrak{h}$.

If one has a good description of a H-desingularization $\tilde{X}$ of X (as it is in the case of toric varieties), he could calculate $\tilde{F}(\phi)$ on $\tilde{X}$. There are some cases though where, by using the Bott-type residue formula, it is easier to calculate $\tilde{F}(\phi)$ on X itself. For one such example with $\tilde{F}(\phi) = F$ see below.

Remark. In the general case of a smooth Kähler manifold M and a polynomial $\tilde{F}(\phi)$ there are formulas proved by Futaki-Morita [13] Futaki [12], and Tian [23] which calculate the value of $\tilde{F}(\phi)$ on nondegenerate holomorphic vector fields ξ on M. When ξ lies in a Lie subalgebra $\mathfrak{k}$ corresponding to a compact subgroup K of H, these are precisely the Bott-type residue formulas we use. In fact, one can prove that for any compact subgroup $K \subset H$ the restriction of $\tilde{F}(\phi)$ to $Lie\, K$ can be represented by a K-equivariant cohomology class of a point.

We recall now the essence of the Bott residue formula for algebraic varieties as presented in [8].

Let T be an algebraic torus which acts on an algebraic scheme X, and let M be a smooth algebraic T-manifold. Denote by X^T (resp. M^T) the set of connected components of the fixed points on X (resp. M) under the action of T. Suppose

$$f : X \longrightarrow M$$

is a T-equivariant embedding such that $X^T \subset M^T$ (meaning that if $F \in M^T$ and $F \cap X \neq \varnothing$, then $F \in X^T$). For each $F \in X^T$ denote by $i_F : F \to X$ (resp. $j_F : F \to M$) its embedding into X (resp. M). Hence, for each such F we have $j_f = f \circ i_F$. Denote finally by $\mathfrak{Q}$ the quotient ring of $\mathbf{Sym}[\chi(T)]$. Then, Proposition 6 of [8] says that the map

$$f_* : A_*(X) \otimes \mathfrak{Q} \longrightarrow A_*^T(M) \otimes \mathfrak{Q}$$

is an inclusion, and for $\alpha \in A_*^T(X) \otimes \mathfrak{Q}$ we have

$$\alpha = \sum_{F \in X^T} i_{F*} \frac{j_F^* \circ f_* \alpha}{c_{top}^T(N_F M)} \ ,$$

where $N_F M$ denotes the normal sheaf of F in M, and $c_{top}^T(N_F M)$ denotes the top Chern T-equivariant class of that sheaf. Recall that by a theorem of Iversen [15], the components of the fixed-point set of a torus action on a smooth algebraic variety are smooth submanifolds.

In particular,

$$[X]_T = \sum_{F \in X^T} i_{F*}\beta_F,$$

where

$$\beta_F = \frac{j_F^* \circ f_*[X]_T}{c_{top}^T(N_F M)} \ .$$

The algebraic Futaki-Morita polynomials $\tilde{F}(\phi)$ of X can be expressed as follows

$$\tilde{F}(\phi) = \int_X^T p^T(P) = \pi_{X*}^T \left(p^T(P) \cap [X]_T \right)$$

$$= \sum_{F \in X^T} \pi_{F*} \left(p^T(i_F^* P) \cap \beta_F \right),$$

and to find $\tilde{F}(\phi)$ one needs only to know β_F for each $F \in X^T$. Here $p^T(P) := [P]^\circ i_G(\phi)$. When $G = GL(r, \mathbf{C})$ $p^T(P)$ is just a polynomial of the Chern classes of the principal bundle P.

In the special case when i_F and f are regular embeddings, e.g. when $F \cap Sing\, X = \varnothing$, we have

$$j_F^* \circ f_*\, [X]_T = i_F^* \circ f^* \circ f_*\, [X]_T = i_F^*(c_{top}^T(N_X M) \cap [X]_T)$$

$$= c_{top}^T(i_F^* N_F M) \cap [F]_T \,,$$

as well as

$$c_{top}^T(N_F M) = c_{top}^T(N_F X) \cup c_{top}^T(i_F^* N_F M) \,.$$

Hence in this case

$$\beta_F = \frac{[F]_T}{c_{top}^T(N_F X)} \,.$$

Another important case in which one has an effective method of computing β_F is when F is an attractive isolated fixed point. Although it happens that all fixed points in our examples are attractive, we will not use this method, referring to [4] and [5] for details about it.

1. **Complete intersections.** As a first example we prove a formula for the generalized Futaki invariant F of complete intersections in $\mathbf{P}^N$. In this case, since all ingredients of the equivariant class representing F can be expressed in terms of equivariant classes of vector bundles on $\mathbf{P}^N$, it can be calculated directly without use of residue formulas. The formula we prove has already appeared in the recent preprint by Zhiqin Lu [17], where he proves it by using technique different from ours.

Let $X \hookrightarrow \mathbf{P}^N$ be a complete intersection given by the polynomials $f_1, \ldots, f_k$ of degree deg $f_i = d_i$, $i = 1, \ldots, k$. Denote by $H := Aut(X \subset \mathbf{P}^N) < Aut(\mathbf{P})^N$ the group of automorphisms of X in $\mathbf{P}^N$. Let $T < H$ be a maximal torus of H, and suppose it is m-dimensional. We choose coordinates $(X_0 : \cdots : X_N)$ of $\mathbf{P}^N$ such that the action of T on $\mathbf{P}^* N$ is given by

$$(u_1, \ldots, u_m) \circ (X_0 : \cdots : X_N) := (u^{\gamma_0} X_0 : \cdots : u^{\gamma_N} X_N) \,,$$

where $u := (u_1, \ldots, u_m) \in T$, for $i = 0, \ldots, N$ we denote $u^{\gamma_i} = \prod_{j=1}^m u_j^{\gamma_i^j}$, and $\gamma_0 + \cdots + \gamma_N = 0$.

Without a loss of generality we may assume that $f_1, \ldots, f_k$ are T-semi-invariants, i.e. that $u \circ f_i = u^{k^i} f_i$ for $k^i \in \mathbf{Z}^m$.

Let $e_1, \ldots, e_m$ be a basis of $\mathfrak{t} = Lie\, T$, and $e^1, \ldots, e^m$ be its dual basis of $\mathfrak{t}^*$. The Futaki invariant can be interpreted as an element of $\mathfrak{t}^*$ and in the basis chosen it has the form

$$F = a_1 e^1 + \cdots + a_m e^m \,.$$

Since the component $a_i e^i$ is the restriction of F to the i-th factor $\mathbf{C}\, e_i$ of $\mathfrak{t}$, we reduce the calculation of F to the case of 1-dimensional torus action $T = \mathbf{C}^*$. So, we have now a one-dimensional torus T acting on $\mathbf{P}^N$ via

$$u \circ (X_0 : \cdots : X_N) := (u^{\gamma_0^i} X_0 : \cdots : u^{\gamma_N^i} X_N) \,,$$

for some integer γ_j^i, $j = 1, \ldots, N$ and

$$u \circ f_j = u^{k_i^j} f_j \qquad j = 1, \ldots, k \,.$$

It is well known (see for example [7]) that the T-equivariant Chow ring of the projective space under chosen action is given by

$$A_T^*(\mathbf{P}^N) = \mathbf{Z}[h, t] / \prod_{j=0}^N (h + \gamma_j^i e^i) \,,$$

and that

$$[X]_T = \prod_{j=1}^k (d_j\, h + k_i^j\, e^i) \quad \in A_T^k(\mathbf{P}^N) \,.$$

On the other hand, since $\gamma_0^i + \cdots + \gamma_N^i = 0$ the equivariant Chern class $c_1^T(\mathbf{P}^N) = (N+1)\, h$, and by the adjunction formula we have

$$c_1^T(X) = c_1^T(\mathbf{P}^N) - c_k^T(N_X M)$$

$$= (N + 1)\, h - (d\, h + k_i\, e^i)\,,$$

where $d = d_1 + \cdots + d_k$, and $k_i = k_i^1 + \cdots + k_i^k$.

The generalized Futaki invariant is given now by

$$F = \frac{1}{N - k + 1} \int_X^T ((N + 1 - d)\, h - k_i\, e^i)^{N+1-k}$$

$$= \frac{1}{N - k + 1} \left[((N - d + 1)\, h - k_i\, e^i)^{N-k+1} \cdot \prod_{j=1}^k (d_j\, h + k_i^j\, e^i) \right]_{h^N}.$$

Thus we get that in the basis $(e^1, \ldots, e^m)$ the i-th coordinate of F is given by the formula

$$a_i = (N - d + 1)^{N-k} \cdot \prod_{j=1}^k d_j \sum_{j=1}^k \left(\frac{N - d + 1}{N - k + 1} \cdot \frac{1}{d_j} - 1 \right) k_i^j\,.$$

2. Toric examples. For at least three reasons the Bott-type formula is very convenient for computing the generalized Futaki invariant of toric varieties. First, all toric varieties have toric resolutions, which have very clear combinatorial explanation. Second, the torus action on a toric variety has only isolated fixed points. Third, the equivariant line bundles with respect to the torus action are well understood.

We give below two examples of almost Fano toric varieties and compute their Futaki invariants. The reason we have chosen these will be explain further on.

2.1. Let X be the blow up of $\mathbf{P}^3$ in a curve given by the ideal $(X_0\, X_2, X_0\, X_3, X_2\, X_3)$ ($(X_0 : X_1 : X_2 : X_3)$ are the coordinates of $\mathbf{P}^3$). Then X is a Gorenstein toric almost Fano variety with two terminal singular points. Denote by T the torus according to which X is toric. This variety has a small resolution $\tilde{X}$, and for its Futaki invariant F we have

$$F = \frac{1}{4} \int_X^T c_1^T(X)^4 = \frac{1}{4} \int_{\tilde{X}}^T (f^* c_1^T(X))^4$$

$$= \frac{1}{4} \int_{\tilde{X}}^T c_1^T(\tilde{X})^4 = \frac{1}{4} \sum_{P \in \tilde{X}^T} i_P^*(K_{\tilde{X}}^*)^4 .\beta_P\,.$$

So, we proceed further with computations on $\tilde{X}$.

A combinatorial description of $\tilde{X}$. If we start with the fan in $N_{\mathbf{R}} = \mathbf{R}^3$

$$\langle e_1, e_2, e_3 \rangle, \langle e_0, e_1, e_2 \rangle, \langle e_0, e_1, e_3 \rangle, \langle e_0, e_2, e_3 \rangle$$

for $\mathbf{P}^3$, where $e_0 + e_1 + e_2 + e_3 = 0$, and e_0, e_1, e_2, e_3 span $\mathbf{R}^3$, then $\tilde{X}$ has the fan

$$\sigma_1 = \langle e_1, e_2, e_2 + e_3 \rangle,\ \sigma_2 = \langle e_1, e_3, e_2 + e_3 \rangle,\ \sigma_3 = \langle e_3, -e_1 - e_2, e_2 + e_3 \rangle$$

$$\sigma_4 = \langle -e_1 - e_2 - e_3, e_2, -e_2 - e_3 \rangle,\ \sigma_5 = \langle e_1, e_2, -e_2 - e_3 \rangle,\ \sigma_6 = \langle -e_1 - e_2 - e_3, -e_1 - e_2, -e_2 - e_3 \rangle$$

$$\sigma_= \langle -e_1 - e_2 - e_3, e_2, -e_1 - e_2 \rangle,\ \sigma_8 = \langle e_2, -e_1 - e_2, e_2 + e_3 \rangle,\ \sigma_9 = \langle e_1, e_3, -e_2 - e_3 \rangle$$

$$\sigma_{10} = \langle e_3, -e_2 - e_3, -e_1 - e_2 \rangle\,.$$

The anticanonical line bundle of $\tilde{X}$ is given then by the vectors in $M = N^*$

$$m(1) = -e^1 - e^2, m(2) = -e^1 - e^3, m(3) = e^1 - e^3, m(4) = -e^2 + 2\, e^3, m(5) = -e^1 - e^2 + 2\, e^3,$$

$$m(6) = e^2, m(7) = 2\, e^1 - e^2, m(8) = 2\, e_1 - e^2, m(9) = -e^1 + 2\, e^2 - e^3, m(10) = -e^1 + 2\, e^2 - e^3.$$

$(m(i)$ is determined by $\sigma_i)$ Denote by $P_1, \ldots, P_{10}$ the fixed points of the action of T on $\tilde{X}$ corresponding to $\sigma_1, \ldots, \sigma_{10}$ respectively. An easy computation shows that, in the notations above and if we denote $\beta_i := \beta_{P_i}$,

$$\beta_1 = \frac{-1}{e^1 (e^2 - e^3) e^3} \ , \ \beta_2 = \frac{1}{e^1 e^2 (e^2 - e^3)} \ , \ \beta_3 = \frac{-1}{e^1 (e^1 - e^2) (e^1 - e^2 + e^3)} \ ,$$

$$\beta_4 = \frac{1}{e^1 (e^2 - e^3) e^1 - e^3)} \ , \ \beta_5 = \frac{1}{e^1 (e^2 - e^3) e^3} \ , \ \beta_6 = \frac{-1}{(e^1 - e^2) (e^2 - e^3) (e^1 - e^2 + e^3)} \ ,$$

$$\beta_7 = \frac{1}{(e^1 - e^2) (e^1 - e^3) e^3} \ , \ \beta_8 = \frac{-1}{e^1 e^3 (e^1 - e^2 + e^3)} \ , \ \beta_9 = \frac{-1}{e^1 e^2 (e^2 - e^3)} \ ,$$

$$\beta_{10} = \frac{1}{e^1 (e^1 - e^2) (e^1 - e^2 + e^3)} \ .$$

Now we are ready to compute the Futaki invariant of X:

$$F = \frac{1}{4} \sum_{i=1}^{10} m(i)^4 \cdot \beta_i = 4 \left(-e^1 - e^2 + e^3 \right).$$

2.2. The second example of an almost Fano toric variety X is the blow up of $\mathbf{P}^3$ in a curve with ideal $(X_0 X_1, X_0 X_2, X_1 X_2)$. This variety has three terminal singular points. It has also a small resolution $\tilde{X}$ with 12 fixed points. Starting with the same fan for $\mathbf{P}^3$ we get a fan for $\tilde{X}$

$\sigma_1 = \langle -e_1 - e_2 - e_3, e_1, e_1 + e_2 \rangle$, $\sigma_2 = \langle -e_1 - e_2 - e_3, e_2, e_1 + e_2 \rangle$, $\sigma_3 \langle -e_1 - e_2 - e_3, e_2, e_2 + e_3 \rangle$,

$\sigma_4 = \langle -e_1 - e_2 - e_3, e_3, e_2 + e_3 \rangle$, $\sigma_5 = \langle -e_1 - e_2 - e_3, e_3, e_1 + e_3 \rangle$, $\sigma_6 = \langle -e_1 - e_2 - e_3, e_1, e_1 + e_3 \rangle$,

$\sigma_7 = \langle e_1, e_1 + e_2, e_1 + e_2 + e_3 \rangle$, $\sigma_8 = \langle e_1, e_1 + e_3, e_1 + e_2 + e_3 \rangle$, $\sigma_9 = \langle e_2, e_1 + e_2, e_1 + e_2 + e_3 \rangle$,

$\sigma_{10} = \langle e_2, e_2 + e_3, e_1 + e_2 + e_3 \rangle$, $\sigma_{11} = \langle e_3, e_1 + e_3, e_1 + e_2 + e_3 \rangle$, $\sigma_{12} = \langle e_3, e_2 + e_3, e_1 + e_2 + e_3 \rangle$.

Proceeding now as in the previous example, we compute that

$$F = 4 (e^1 + e^2 + e^3).$$

Remark. The generalized Futaki invariant of an almost Fano toric variety has a very nice geometric interpretation. As an element of $M_{\mathbf{R}}$ F coincides with the barycentre of the polytope corresponding to the anticanonical sheaf of that variety, with respect to the Duistermaat-Heckman's measure on it.(See [19] for the smooth case, and [24] for the general case.) By using this interpretation the author [24] has computed F for the above toric examples getting the same result.

3.**Example.** As a last example consider the blow up X of $\mathbf{P}^3$ in the curve given by the ideal $(X_0 X_3, X_1 X_3, X_0 X_2 - X_1^2)$. Then X is an almost Fano variety with one singular point. We can naturally consider X as a subvariety of $M := \mathbf{P}^3 \times \mathbf{P}^2$. Denote by F the corresponding embedding. Let $(Y_0 : Y_1 : Y_2)$ be the coordinates of $\mathbf{P}^2$.

The group $H := Aut X$ has a two dimensional maximal torus

$$T := \{ \ (z_1, z_2) \mid z_i \in \mathbf{C}^* \ , i = 1, 2 \ \} .$$

If we let T act on M via

$$(z_1, z_2) \circ (X_0; X_1 : X_2 : X_3; Y_0 : Y_1 : Y_2)$$

$$:= (X_0 : z_1 X_1 : z_1^2 X_2 : z_2 X_3; z_2 Y_0 : z_1 z_2 Y_1 : z_1^2 Y_2),$$

then the embedding f will be T-equivariant. The action of T on X has seven fixed points. As points of M they have coordinates

$$P_1(1 : 0 : 0 : 0; 1 : 0 : 0) \ , \ P_2(1 : 0 : 0 : 0; 0 : 0 : 1) \ , \ P_3(0 : 1 : 0 : 0; 0 : 0 : 1)$$

$$P_4\,(0:0:1:0;0:1:0)\ ,\ P_5\,(0:0:1:0;0:0:1)\ ,\ P_6\,(0:0:0:1;1:0:0)$$
$$P_7\,(0:0:0:1;0:1:0)\ .$$

The point P_5 is the only singular point of X. Denote by $i_j : P_j \hookrightarrow X$ the corresponding embeddings. Since the action of T on M has only finite number of fixed points, we have an inclusion $X^T \subset M^T$ and can apply the Edidin-Graham theorem for computing F:

$$F = \frac{1}{4}\sum_{j=1}^{7} i_j^*(K_X^*)\,.\beta_j\,.$$

The calculation of $m(j) := i_j^*(K_X^*)$ and β_j for $j \neq 5$ is straightforward. It happens that P_5 is an attractive fixed point. So, β_5 could be calculated by using the method from [4]. We here avoid finding β_5 in computing F proceeding in the following way. By the Bott residue theorem we have

$$38 = (-K_X)^3 = \sum_{j=1}^{7} m(j)^3\,.\beta_j.$$

Hence, we can exclude the term β_5 from the formula for F:

$$F = \frac{1}{4}\left[\,38\,m(5) + \sum_{j\neq 5} m(j)^3\,.\beta_j\,\left(\,m(j)-m(5)\,\right)\,\right]\ .$$

Then, the term $m(5)$ is uniquely determined by the fact that the sum on the RHS is a linear polynomial of the generators e^1, e^2 of the character group $\chi(T)$ of T. Notice that we don't actually need to know β_5 to compute F.

In this way we get the input data:

$$\beta_1 = \frac{1}{e^1\,e_2\,(2\,e^1 - e^2)}\ ,\ \beta_2 = \frac{-1}{2\,(e^1)^2\,(2\,e^1 - e^2)}\ ,\ \beta_3 = \frac{1}{(e^1)^2\,(e^1 - e^2)}$$

$$\beta_4 = \frac{1}{e^1\,(e^1 - e^2)\,(2\,e^1 - e^2)}\ ,\ \beta_6 = \frac{-1}{e^1\,e^2\,(2\,e^1 - e^2)}\ ,\ \beta_7 = \frac{-1}{e^1\,(e^1 - e^2)\,2\,e^1 - e^2)}$$

$$m(1) = 3\,e^1\ ,\ m(2) = e^1 + e^2\ ,\ m(3) = -e^1 + e^2\ ,\ m(4) = -2\,e^1$$
$$m(5) = -3\,e^1 + e^2\ ,\ m(6) = 3\,e^1 - 2\,e^2\ ,\ m(7) = 2\,(e^1 - e^2)$$

and compute the output

$$F = 4\,(\,3e^1 - e^2\,).$$

We conclude this section by explaining why the last three examples are interesting to the author.

Recall ([23]) that a special degeneration of a Fano manifold M is called a morphism $\pi : W \to \Delta$, without multiple fibres, where Δ is the one dimensional unit disc, such that

1) the relative anticanonical sheaf of π is ample,

2) the family of dilatons $\phi(\lambda) : \Delta \to \Delta$, $t \mapsto \lambda t$, $0 < |\lambda| \leq 1$, extends to a group $\Phi(\lambda)$ of transformations of W,

3) the central fibre W_0 of π is a normal almost Fano variety, and all other fibres are isomorphic to M.

Since W_0 is $\Phi(\lambda)$-invariant, there is defined an admissible vector field $v_W := -\Phi'(1)$ on it. A deep theorem by Ding and Tian [6], strengthened by Tian [23], says that if M is Einstein-Kähler, then the real part of the generalized Futaki invariant F, evaluated on v_W is nonnegative.

The last three examples from this section can be realized as central fibres of special degenerations of the Fano manifold V_{38} - the blow-up of the projective three-dimensional space in a twisted

cubic curve [24]. One can show that in the all three cases the real parts of the corresponding evaluations are positive.

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