AN INDEX THEOREM FOR FAMILIES ELLIPTIC OPERATORS INVARIANT WITH RESPECT TO A BUNDLE OF LIE GROUPS

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ABSTRACT. We define the equivariant family index of a family of elliptic operators invariant with respect to the free action of a bundle $\mathcal{G}$ of Lie groups. In this paper we concentrate on the issues specific to the case when $\mathcal{G}$ is trivial, so the action reduces to the action of a Lie group G. For G simply-connected solvable, we then compute the Chern character of the (equivariant family) index, the result being given by an Atiyah-Singer type formula. We also study traces on the corresponding algebras of pseudodifferential operators and obtain a local index formula for families of invariant operators, if the bundles are trivial. We discuss then two applications, one to higher-eta invariants, which are morphisms $K_n(\Psi^{\infty}_{\text{inv}}(Y)) \to \mathbb{C}$, and the other one to Fredholm boundary conditions on a simplex. As ann application of our formalism with traces, we obtain also new proofs of the regularity at $s = 0$ of $\eta(D_0, s)$, the eta function of D_0, and of the relation $\eta(D_0, s) = \pi^{-1}\overline{\text{Tr}}_1(D^{-1}D')$ (here $D = D_0 + \partial_t$, $D' = [D, t]$). The algebras of invariant pseudodifferential operators that we study, $\psi^{\infty}_{\text{inv}}(Y)$ and $\Psi^{\infty}_{\text{inv}}(Y)$, are generalizations of "parameter dependent" algebras of pseudodifferential operators (with parameter in $\mathbb{R}^q$), so our results provide also an index theorem for elliptic, parameter dependent pseudodifferential operators.

CONTENTS

INTRODUCTION

Families of Dirac operators invariant with respect to a bundle of Lie groups appear in the analysis of the Dirac operator on certain non-compact manifolds. They arise, for example, in the analysis of the Dirac operator on an S^1-manifold M, if we desingularize the action of S^1 by replacing the original metric g with $\phi^{-2}g$,

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where ϕ is the length of the infinitesimal generator X of the S^1-action. In this way, X becomes of length one in the new metric. The main result of [25] states that the kernel of the new Dirac operator on the open manifold $M \smallsetminus M^{S^1}$ is naturally isomorphic to the kernel of the original Dirac operator.

It turns out that the Fredholm property of the resulting Dirac operator (obtained by the above procedure on the non-compact manifold $M \smallsetminus M^{S^1}$) is controled by the invertibility of a family of operators invariant with respect to the action of a bundle of solvable Lie groups. This follows from the results of [16] and it will be discussed in greater detail in a future paper. In general, neither the bundle $Y \to B$, on which these operators act, nor the bundle of Lie groups $\mathcal{G} \to B$, acting on Y, are trivial. A natural problem then is to study the invertibility of these invariant families of operators, and more generally, their index.

We define the (equivariant family) index of a family of invariant, elliptic operators as it is customary, using K-theory. It turns out that the index of such an elliptic family is the obstruction to finding an invertible perturbation of the original family by invariant regularizing operators, if we exclude the degenerate case $\dim Y = \dim \mathcal{G}$. This shows the relevance of computing the index to the problem of determining the invertibility of a given family.

In this paper, we study the index and certain non-local invariants of families of elliptic operators invariant with respect to a bundle of *solvable* Lie groups $\mathcal{G}$. Actually, for most of our results, we assume that the bundle of Lie groups $\mathcal{G}$ is trivial, $\mathcal{G} = B \times G$, so invariance with respect to $\mathcal{G}$ reduces to invariance with respect to G. The case $\mathcal{G} = B \times G$ is significant, non-trivial, and it suffices for many applications. It is resonable then to first study this case, which allows us to avoid some conceptual issues; moreover, the results are easier to state and grasp in this case.

For $\mathcal{G} = B \times G$, where G is a simply-conneced Lie group, we obtain a formula for the Chern character of the index bundle that is similar to the Atiyah-Singer index formula for families. Then, we turn to the local analysis of these families, for which we assume that $\mathcal{G}$ consists of abelian groups. Our analysis leads then to the construction of several traces on the algebras $\Psi^m_{\mathrm{inv}}(Y)$. We use these traces to obtain local index theorems.

In a remarkable paper [6], Bismut and Cheeger have generalized the Atiyah-Patodi-Singer index theorem [3] to families of Dirac operators on manifolds with boundary (see also [21]). Their results apply to operators whose indicial parts are invertible. The indicial parts are actually families of Dirac operators invariant with respect to a one-parameter group, so they fit into the framework of this paper (with $\mathcal{G} = B \times \mathbb{R}$). In addition to the usual ingredients of an index theorem – curvature and characteristic classes – their result was stated in terms of a new invariant, called the "eta-form," in analogy to the additional invariant appearing in the Atiyah-Patodi-Singer index formula for operators on manifolds with boundary. Thus, the results of this paper are relevant also to the problem of determining an explicit formula for the index of a family of pseudodifferential elliptic operators on a bundle of manifolds with boundary.

With an eye towards this problem, we also give a new proof of the regularity of the eta function at the origin and discuss some possible generalizations the eta invariant. Actually, we suggest two possible generalizations, one which is a direct generalization of a result of [19] and one using higher algebraic K-theory. The first possible generalization is to associate to a Dirac operator invariant with respect to $\mathbb{R}^q$ the quantitiy defined by the formula $\overline{\mathrm{Tr}}_q\big((D^{-1}dD)^{2k-1}\big)$. This possible generalization was considered before by Lesch and Pflaum [17], who proved that this formula does not lead to new invariants for Dirac operators and also that this formula is not additive for a product of two invertible operators, except for $k = 1$, when one recovers the usual eta invariant [19]. The second possible generalization, which has the advantage of being additive, is to define the higher eta invariant as a morphism on higher algebraic K-theory.

We now describe the contents of each section of this paper. In Section 1, we discuss the action of a bundle of Lie groups $\mathcal{G}$ on a fiber bundle Y and we introduce the algebras $\psi^\infty_{\mathrm{inv}}(Y)$ and $\Psi^\infty_{\mathrm{inv}}(Y)$, which will be our main object of study. (Both these algebras consist of invariant pseudodifferential operators.) Assuming that both $\mathcal{G}$ and Y are trivial with fixed trivializations, we prove that the group of gauge transformations of $\mathcal{G}$ acts on $\psi^\infty_{\mathrm{inv}}(Y)$ and $\Psi^\infty_{\mathrm{inv}}(Y)$. In Section 2, we define the index of a family of elliptic, invariant pseudodifferential operators A. We shall sometimes use the term "the equivariant family index" of an elliptic operator $A \in \psi^\infty_{\mathrm{inv}}(Y)$, for the index of such an operator, to stress that it involves both a vector bundle component and an equivariant component. We prove that the index of A is the obstruction to finding a regularizing operator R such that $A + R$, acting between suitable Sobolev spaces, is invertible in each fiber (excluding the degenerate case $\dim Y = \dim \mathcal{G}$). This generalizes the usual property of the Fredholm index of a Fredholm operator. Then, using the methods of [20], we obtain a formula for the Chern character of the index of an operator $A \in \psi^\infty_{\mathrm{inv}}(Y)$. In the third section, we develop the necessary facts about the assymptotics of the trace of the operators in $\Psi^\infty_{\mathrm{inv}}(Y)$. This allows us to define various regularized and residue traces. Using these traces, we obtain in Section 4 two local formulae for the (equivariant family) index of an elliptic $A \in \Psi^\infty_{\mathrm{inv}}(Y)$, if the bundles Y and $\mathcal{G}$ are trivial and $\mathcal{G}$ consists of abelian Lie groups.

More applications are contained in the last two sections. In Section 5, we use our formalism with traces to give a new proof of the regularity of the eta function of D_0, $\eta(D_0, s)$, at $s = 0$, and we discuss two possible generalizations of the eta invariant. We also obtain a new proof of the relation $\eta(D_0, s) = \overline{\mathrm{Tr}}_1(D^{-1}D')$, $D = D_0 + \partial_t$, $D' = [D, t]$, first proved in [19] using the local index theorem (see also [17]). Operators invariant with respect to $\mathbb{R}^q$, a particular case of our operators when the base is reduced to a point, appear in the formulation of elliptic (or Fredholm) boundary conditions for b-pseudodifferential operators on manifolds with corners. In the last section, we discuss how the equivariant index can be used to study this problem. Let Y be a smooth manifold without corners and T be an elliptic b-pseudodifferential operator on $\Delta_n \times Y$. More precisely, we give a necessary and sufficient condition for the existence of a perturbation of T by a smoothing operator in the same algebra that makes it Fredholm. Our results are complete if we exclude the case $\dim Y = 0$. These results are relevant for the problem of extending the Atiyah-Patodi-Singer boundary conditions to manifolds with corners.

The algebras of invariant pseudodifferential operators that we study, $\psi^\infty_{\mathrm{inv}}(Y)$ and $\Psi^\infty_{\mathrm{inv}}(Y)$, are generalizations of "parameter dependent" algebras of pseudodifferential operators considered by Agmon [1], Grubb and Seeley [13], Lesch and Pflaum [17], Melrose [19], Shubin [30], and others. Our index theorem, Theorem 2, hence gives a solution to the problem of determining the index of elliptic, parameter dependent families of pseudodifferential operators, parameterized by $\lambda \in \mathbb{R}^q$. We note that the concept of index of such a family requires a proper definition, and that the Fredholm index, "dimension of the kernel" - "dimension of the cokernel," is not appropriate for $q \geq 1$. Actually, our definition of the (equivariant, family) index is closer related to the definition of the L^2-index for covering spaces introduced in [2] and [29]. For the proof of our local index theorem, we use ideas of non-commutative geometry [10], more precisely, the general approach to index theorems using cyclic cohomology as developed in [26]. These computations are also an example of a computation of a bivariant Chern-Connes character [23]. We also expect our results to have applications to adiabatic limits of eta invariants [7, 31].

I would like to thank Richard Melrose for several useful discussions. Actually, this paper grew out of an initial joint project. Part of the results of this paper were presented in a talk at University of Illinois at Chicago in January 1997, and I would like to thank the Chairman of their Mathematics Department for his kind invitation. Also, I would like to thank for the hospitality of Fourier Institute, where part of this project was completed, and Louis Funar for his kind invitation to spend some time at his Institute.

All pseudodifferential operators considered in this paper are "classical," that is, one-step polyhomogeneous.

1. Invariant pseudodifferential operators

We now describe the setting in which we shall work. Let B be a smooth compact manifold and

$$d : \mathcal{G} \to B \quad \text{ and } \pi : Y \to B$$

be two smooth fiber bundles with fibers $\mathcal{G}_b := d^{-1}(b)$ and $Y_b := \pi^{-1}(b)$. We shall assume that $\mathcal{G}$ is a bundle of Lie groups acting smoothly on Y, and then we shall consider families of operators along the fibers of Y and invariant under the action of $\mathcal{G}$. For simplicity, we shall assume that B is connected. The index and local invariant of these operators will form our main object of study. We now make all these assumptions and concepts more precise.

Throughout this paper, $\mathcal{G}$ will denote a *bundle of Lie groups* on a manifold usually denoted B. Recall that this means that $\mathcal{G} \to B$ is a smooth fiber bundle, that each $\mathcal{G}_b$ is a Lie group, and that the multiplication and inverse depend differentiably on b. Hence the map

$$\mathcal{G} \times_B \mathcal{G} := \{(g', g) \in \mathcal{G} \times \mathcal{G}, d(g') = d(g)\} \ni (g', g) \longrightarrow g'g^{-1} \in \mathcal{G} \tag{1}$$

is differentiable. This implies, by standard arguments, that the map sending a point $b \in \mathcal{G}_b$ to e_b, the identity element of $\mathcal{G}_b$, is a diffeormorphism onto a smooth submanifold of $\mathcal{G}$. It also implies that the map $\mathcal{G} \in g \to g^{-1} \in \mathcal{G}$ is differentiable.

We also assume that $\mathcal{G}$ acts smoothly on Y. This means that there are given actions $\mathcal{G}_b \times Y_b \to Y_b$ of $\mathcal{G}_b$ on Y_b, for each b, such that the induced map,

$$\mathcal{G} \times_B Y := \{(g, y) \in \mathcal{G} \times Y, d(g) = \pi(y)\} \ni (g, y) \longrightarrow gy \in Y,$$

is differentiable. We shall also assume that the action of $\mathcal{G}$ on Y is *free*, that is, that the action of $\mathcal{G}_b$ on Y_b is free for each b.

On Y, we consider smooth families $A = (A_b)$, $b \in B$, of classical pseudodifferential operators acting on the fibers of $Y \to B$ such that each A_b is invariant with respect to the action of the group $\mathcal{G}_b$. Unless mentioned otherwise, we assume that these operators act on half densities along each fiber. The algebra that we are interested in consists of such invariant operators satisfying also a *support condition.* To state this support condition, first notice that a family $A = (A_b)$ defines a continuous map $\mathcal{C}_c^\infty(Y) \to \mathcal{C}^\infty(Y)$, and, as such, it has a distribution (or Schwartz) kernel, which is a distribution K_A on $Y \times_B Y \subset Y \times Y$. (We ignore the vector bundles in which this distribution takes its values.) Because the family $A = (A_b)$ is invariant, the distribution K_A is also invariant with respect to the action of $\mathcal{G}$. Consequently, K_A is the pull back of a distribution k_A on $(Y \times_B Y)/\mathcal{G}$. *We will require that k_A have compact support.* We shall sometimes call k_A the *convolution kernel* of A. This condition on the support of k_A ensures that each A_b is a properly supported pseudodifferential operators, and hence it maps compactly supported functions (or sections of a vector bundle, if we consider operators acting on sections of a smooth vector bundle) to compactly supported functions (or sections). This support condition is automatically satisfied if $Y/\mathcal{G}$ is compact and each A_b is a differential operator. The space of smooth, invariant families A of order m pseudodifferential operators acting on the fibers of $Y \to B$ such that k_A has compact support will be denoted by $\psi^m_{\rm inv}(Y)$. Then $\psi^\infty_{\rm inv}(Y)$ is an algebra, by classical results [14].

We now discuss the principal symbols of the invariant operators that we study. Let

$$T_{vt}Y := \ker TY \to TB$$

be the bundle of *vertical* tangent vectors to Y, and let $T^*_{vt}Y$ be its dual. We fix compatible metrics on $T_{vt}Y$ and $T^*_{vt}Y$, and define $S^*_{vt}Y$, the *cosphere bundle of the vertical tangent bundle* to Y, to be the set of vectors of length one of $T^*_{vt}Y$. Also, let

$$\sigma_m : \Psi^m(Y_b) \to \mathcal{C}^\infty(S^*_{vt} \cap T^*Y_b)$$

be the usual principal symbol map, defined on the space of pseudodifferential operators of order m on Y_b. The definition of σ_m depends on the choice of a trivialization of the bundle of homogeneous functions of order m on $T^*_{vt}Y$, regarded as a bundle over $S^*_{vt}Y$. The principal symbols $\sigma_m(A_b)$ of an element (or family) $A = (A_b) \in \psi^m_{\rm inv}(Y)$ then gives rise to a smooth function on $\mathcal{C}^\infty(S^*_{vt}Y)$, which is invariant with respect to $\mathcal{G}$, and hence descends to a smooth function on $S^*_{vt}Y$, which has compact support because of the support condition on the kernel of A. The resulting function,

$$\sigma_m(A) \in \mathcal{C}_c^\infty((S^*_{vt}Y)/\mathcal{G}), \tag{2}$$

will be referred to as the *principal symbol* of an element (or operator) in $\psi^m_{\rm inv}(Y)$.

In the particular case $Y = \mathcal{G}$, $\psi^\infty_{\rm inv}(\mathcal{G})$ identifies with convolution operators on each fiber $\mathcal{G}_b$ that have compactly supported kernels, are smooth outside the identity, and have only conormal singularities at the identity. In particular, $\psi^{-\infty}_{\rm inv}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G})$, with the fiberwise convolution product.

Suppose now that the quotient $Y/\mathcal{G}$ is compact, which implies that $(S^*_{vt}Y)\mathcal{G}$ is also compact. As it is customary, an operator $A \in \psi^m_{\rm inv}(Y)$ is called *elliptic* if, and only if, its principal symbol is everywhere invertible. The same definition applies to

$$A = [A_{ij}] \in M_N(\psi^m_{\rm inv}(Y)) :$$

the operator A, regarded as acting on sections of the trivial vector bundle $\mathbb{C}^N$, is elliptic if, and only if, its principal symbol

$$\sigma_m(A) := [\sigma_m(A_{ij})] \in M_n(\mathcal{C}^\infty(S^*_{vt}Y/\mathcal{G}))$$

is invertible.

Assume that there is given a $\mathcal{G}$-invariant metric on $T_{vt}Y$, the bundle of vertical tangent vectors, and a $\mathcal{G}$-equivariant bundle W of modules over the Clifford algebras of $T_{vt}Y$. Then a typical example of a family $D = (D_b) \in \psi^{\infty}_{\text{inv}}(Y)$ is that of the family of Dirac operators D_b acting on the fibers Y_b of $Y \to B$. (Each D_b acts on sections of $W|_{Y_b}$, the restriction of the given Clifford module W to that fiber.)

Before proceeding, in the next section, to define the equivariant family index of an elliptic family invariant with respect to a bundle of Lie groups, let us take a closer look at a particular case of the previous construction. Take

$$Y = B \times Y_0 \times \mathbb{R}^q,$$

with Y_0 a compact manifold and

$$\mathcal{G} = B \times \mathbb{R}^q,$$

with π and d being the projections onto the first components of each product. The action of $\mathcal{G}$ on Y is given by translation on the last component of Y. Then the $\mathcal{G}$-invariance condition becomes simply $\mathbb{R}^q$ invariance with respect to the resulting $\mathbb{R}^q$ action. If Y and $\mathcal{G}$ are as described here, then we call Y *a flat $\mathcal{G}$-space.*

One disadvantage of the algebras $\psi^{\infty}_{\text{inv}}(Y)$ is the following. It is possible to find families $A = (A_b) \in \psi^{0}_{\text{inv}}(Y)$ such that each A_b is invertible as a bounded operator, but the family (A_b^{-1}) is not in $\psi^{0}_{\text{inv}}(Y)$, although it consists of invariant, pseudodifferential operators. This pathology is due to the support condition. Neverthess, for $\mathcal{G}$ consisting of abelian groups, it is easy to remedy this pathology by enlarging the algebra $\psi^{-\infty}_{\text{inv}}(Y)$, as follows.

Since the enlargement of the algebra $\psi^{-\infty}_{\text{inv}}(Y)$ is done locally, we may assum that Y is a flat $\mathcal{G}$–space. The residual ideal of the algebra $\psi^{\infty}_{\text{inv}}(Y)$ is $\psi^{-\infty}_{\text{inv}}(Y)$ and consists of operators that are regularizing along each fiber. More precisely, it consists of those families of smoothing operators on $Y = B \times Y_0 \times \mathbb{R}^q$ that are translation-invariant under the action of $\mathbb{R}^q$ and have *compactly supported* convolution kernels. Thus

$$\psi^{-\infty}_{\text{inv}}(Y) \cong \mathcal{C}^{\infty}_{\text{c}}(B \times \mathbb{R}^q; \Psi^{-\infty}(Y_0)) \subset \mathcal{S}(B \times \mathbb{R}^q; \Psi^{-\infty}(Y_0)) \simeq \mathcal{S}(B \times Y_0 \times Y_0 \times \mathbb{R}^q).$$

(Here $\mathcal{S}$ is the generic notation for the space of Schwartz functions on a suitable space, in this case on $B \times \mathbb{R}^q$, and with values regularizing operators.) The second isomorphism above is obtained from the isomorphism

$$\Psi^{-\infty}(Y_0) \simeq \mathcal{C}^{\infty}(Y_0 \times Y_0)$$

defined by the choice of a nowhere vanishing density on Y_0. If we also endow $\Psi^{-\infty}(Y_0)$ with the locally convex topology induced by this isomorphism, then it becomes a nuclear locally convex space. We now enlarge the algebra $\psi^{\infty}_{\text{inv}}(Y)$ to include all invariant, regularizing operators whose kernels are in $\mathcal{S}(B \times Y_0 \times Y_0 \times \mathbb{R}^q)$:

$$\Psi^{m}_{\text{inv}}(Y) = \psi^{m}_{\text{inv}}(Y) + \mathcal{S}(B \times Y_0 \times Y_0 \times \mathbb{R}^q). \tag{3}$$

Explicitely, the action of $T \in \mathcal{S}(B \times Y_0 \times Y_0 \times \mathbb{R}^q)$ on a smooth function $f \in \mathcal{C}^{\infty}_{\text{c}}(B \times Y_0 \times \mathbb{R}^q)$ is given by

$$Tf(b, y_1, t) = \int_{Y_0 \times \mathbb{R}^q} T(b, y_1, y_0, t - s) f(b, y_0, s) dy_0 ds.$$

For B reduced to a point, the algebras $\Psi^{\infty}_{\text{inv}}(Y)$ were introduced in [18] as the range space of the indicial map for pseudodifferential operators on manifolds with corners

Still assuming that we are in the case of a flat $\mathcal{G}$-space, we notice that the Fourier transformation $\mathcal{F}_2$ in the translation-invariant directions gives a dual identification

(4)
$$\Psi_{\mathrm{inv}}^{-\infty}(Y) \ni T \to \hat{T} := \mathcal{F}_2 T \mathcal{F}_2^{-1} \in \mathcal{S}(B \times Y_0 \times Y_0 \times \mathbb{R}^q) = \mathcal{S}(B \times \mathbb{R}^q; \Psi^{-\infty}(Y_0))$$

with the space of Schwartz functions with values in the smoothing ideal of Y_0. The point of this identification is that the convolution product is transformed into the pointwise product. The Schwartz topology from (4) then gives $\Psi_{\mathrm{inv}}^{-\infty}(Y)$ the structure of a nuclear locally convex topological algebra. Following the same receipe, the Fourier transform also gives rise to an *indicial map*

$$\Phi : \Psi_{\mathrm{inv}}^{\infty}(Y) \ni T \to \hat{T} := \mathcal{F}_2 T \mathcal{F}_2^{-1} \in \mathcal{C}^{\infty}(B \times \mathbb{R}^q; \Psi^{\infty}(Y_0)). \tag{5}$$

We denote

$$\hat{T}(\tau) = \Phi(T)(\tau).$$

The map Φ is not an isomorphism since $\hat{A}(\tau)$ has joint symbolic properties in the variables of $\mathbb{R}^q$ and the fiber variables of T^*Y_0. Actually, the principal symbols of the operators $\hat{A}(\tau)$ is constant on the fibers of $T^*_{vt}Y \to B$.

Lemma 1. *Assume $Y = B \times Y_0 \times \mathbb{R}^q$ is a flat $\mathcal{G}$-space. Then the action of the group $GL_q(\mathbb{R})$ on the last factor of $Y = B \times Y_0 \times \mathbb{R}^q$ extends to an action by automorphisms of $\mathcal{C}^\infty(B, GL_q(\mathbb{R}))$ on $\psi_{\mathrm{inv}}^{\infty}(Y)$ and on $\Psi_{\mathrm{inv}}^{\infty}(Y)$.*

Proof. The vector representation of $GL_q(\mathbb{R})$ on the second component of $Y_0 \times \mathbb{R}^q$ defines an action of $GL_q(\mathbb{R})$ on $\Psi^\infty(Y_0 \times \mathbb{R}^q)$ that preserves the class of properly supported operators and the products of such operators. It also normalizes the group $\mathbb{R}^q$ of translations, and hence it maps $\mathbb{R}^q$-invariant operators to $\mathbb{R}^q$-invariant operators. This property extends right away to the action of $\mathcal{C}^\infty(B, GL_q(\mathbb{R}))$ on families of operators on $B \times Y_0 \times \mathbb{R}^q$, and hence $\mathcal{C}^\infty(B, GL_q(\mathbb{R}))$ maps $\psi_{\mathrm{inv}}^{m}(Y)$ isomorphically to itself. From the isomorphism (4), we see that $\mathcal{C}^\infty(B, GL_q(\mathbb{R}))$ also maps $\Psi_{\mathrm{inv}}^{-\infty}(Y)$ to itself. This gives an action by automorphisms of $\mathcal{C}^\infty(B, GL_q(\mathbb{R}))$ on $\Psi_{\mathrm{inv}}^{\infty}(Y)$, which is the sum of $\Psi_{\mathrm{inv}}^{-\infty}(Y)$ and $\psi_{\mathrm{inv}}^{\infty}(Y)$. □

Suppose the family of Lie groups $\mathcal{G}$ consists of abelian Lie groups, so that $\mathcal{G}$ is a vector bundle. By choosing a lift of $Y/\mathcal{G} \to Y$, which is possible because the fibers are contractible, we obtain that locally the bundle Y is isomorphic to a flat $\mathcal{G}$ space. Then the above lemma allows us to extend the previous definitions, including those of the algebras $\Psi_{\mathrm{inv}}^{\infty}(Y)$ and of the indicial family from the flat case to the case $\mathcal{G}$ abelian. The indicial family $\hat{A}$ of an operator $A \in \Psi_{\mathrm{inv}}^{\infty}(Y)$, will then be a family of pseudodifferential operators acting on the fibers of $Y \times_B \mathcal{G}^* \to \mathcal{G}^*$ (here $\mathcal{G}^*$ is the dual of the vector bundle $\mathcal{G}$):

$$\hat{A}(\tau) \in \Psi^*(Y_b/\mathcal{G}_b), \quad \text{if } \tau \in \mathcal{G}^*.$$

The considerations of this section extend immediately to operators acting between sections of a $\mathcal{G}$-equivariant vector bundle.

2. Homotopy invariants

We now define homotopy invariants of elliptic operators in $\psi_{\mathrm{inv}}^{\infty}(Y)$, the main invariant being the (equivariant family) index of such an invariant, elliptic family. For $\mathcal{G}$ consisting of simply connected, solvable Lie groups and $\dim Y > \dim \mathcal{G}$, we then show that the index gives the obstruction for an invariant, elliptic family A to have a perturbation by regularizing operators which consists of invertible operators (on suitable Hilbert spaces, see Proposition 1). For families of abelian Lie groups $\mathcal{G}$, we give an interpretation of the index of an elliptic operator in terms of its indicial family. This leads to an Atiyah-Singer index type formula for the Chern character of the index of a family of invariant, elliptic operators. If $\mathcal{G}$ is abelian (that is, if its fibers are abelian Lie groups), then we can consider the algebra $\Psi^{\infty}_{\text{inv}}(Y)$ instead of $\psi^{\infty}_{\text{inv}}(Y)$.

We now proceed to define the index of an elliptic family $A \in \psi^{m}_{\text{inv}}(Y)$. This will be done using the K-theory of Banach algebras. Let $C^*(\mathcal{G})$ be the closure of $\psi^{-\infty}_{\text{inv}}(\mathcal{G})$ with respect to the norm

$$\|A\| = \sup_{b \in B} \|A_b\|$$

each operator A_b acting on the Hilbert space of square integrable densities on the fiber Y_b. For example, if $\mathcal{G}$ is abelian, we have $C^*(\mathcal{G}) \simeq C_0(\mathcal{G}^*)$, the space of continuous functions vanishing at ∞ on $\mathcal{G}^*$.

Assumption. *From now on in this paper we shall assume that $\mathcal{G}$ consists of simply-connected solvable Lie groups.* By "simply-connected" we mean, as usual, "connected with trivial fundamental group."

We shall denote by $\mathfrak{g}$ the bundle of Lie algebras of the groups $\mathcal{G}_b$ and by

$$\exp : \mathfrak{g} \to \mathcal{G}.$$

the exponential map. In order to study the algebra $C^*(\mathcal{G})$, we shall deform it to a commutative algebra. This deformation is obtained as follows. Let $\mathcal{G}_{ad} = \{0\} \times \mathfrak{g} \cup (0, 1] \times \mathcal{G}$, $B_1 = [0, 1] \times B$, and $d : \mathcal{G}_{ad} \to B_1$ be the natural projection. On $\mathcal{G}_{ad}$ we put the smooth structure induced by the bijection

$$\phi : B_1 \times \mathfrak{g} \to \mathcal{G}_{ad}$$

$\phi(0, X) = (0, X)$ and $\phi(t, X) = (t, \exp(tY))$, for all $X \in \mathfrak{g}$ in a small neighborhood of the zero section. Then we endow $\mathcal{G}_{ad}$ with the Lie bundle structure induced by the pointwise product. Evaluation at $t \in [0, 1]$ induces algebra morphisms

$$e_t : \psi^{m}_{\text{inv}}(\mathcal{G}_{ad}) \to \psi^{m}_{\text{inv}}(\mathcal{G}), \quad t > 0,$$

and

$$e_0 : \psi^{m}_{\text{inv}}(\mathcal{G}_{ad}) \to \psi^{m}_{\text{inv}}(\mathfrak{g}), \quad t = 0.$$

Passing to completions, we obtain morphisms e_t from $C^*(\mathcal{G}_{ad})$ to $C^*(\mathcal{G})$, for $t > 0$, and to $C^*(d^{-1}(0)) \simeq C_0(\mathfrak{g}^*)$, for $t = 0$.

Lemma 2. *The morphisms $e_t : C^*(\mathcal{G}_{ad}) \to C^*(\mathcal{G})$, for $t > 0$, and $e_0 : C^*(\mathcal{G}_{ad}) \to C_0(\mathfrak{g}^*)$, for $t = 0$, induce isomorphisms in K-theory.*

Proof. Assume first that there exists a Lie algebra bundle morphism $\mathcal{G} \to B \times \mathbb{R}$. (In other words, there exists a map $\mathcal{G} \to \mathbb{R}$ that is a morphism on each fiber.) Let $\mathcal{G}'$ denote the kernel of this morphism and let $\mathcal{G}'_{ad}$ be obtained from $\mathcal{G}'$ by the same deformation construction by which $\mathcal{G}_{ad}$ was obtained from $\mathcal{G}$. Then we obtain a smooth map $\mathcal{G}_{ad} \to \mathbb{R}$ that is a group morphism on each fiber, and hence

$$C^*(\mathcal{G}_{ad}) \simeq C^*(\mathcal{G}'_{ad}) \rtimes \mathbb{R}, \;\; C^*(\mathcal{G}) \simeq C^*(\mathcal{G}') \rtimes \mathbb{R}, \;\; \text{and } C_0(\mathfrak{g}^*) \simeq C_0({\mathfrak{g}'}^*) \rtimes \mathbb{R}.$$

Moreover, all above isomorphisms are natural, and hence compatible with the morphisms e_t. Assuming now that the result was proved for all Lie group bundles of smaller dimension, we obtain the desired result for $\mathcal{G}$ from the Connes' Thom isomorphism in K-theory [9], which in this particular case gives:

$$K_i(C^*(\mathcal{G}_{ad})) \simeq K_{i+1}(C^*(\mathcal{G}'_{ad})),\ K_i(C^*(\mathcal{G})) \simeq K_{i+1}(C^*(\mathcal{G}')), \text{ and}$$

$$K_i(C_0(\mathfrak{g}^*)) \simeq K_{i+1}(C_0(\mathfrak{g}'^*)).$$

This will allow us to complete the result in the following way. Let U_k be the open subset of B consisting of those $b \in B$ such that $[\mathcal{G}_b, \mathcal{G}_b]$ has dimension $\geq k$. From the Five Lemma and the six term exact sequences in K-theory associated to the ideal $C^*(\mathcal{G}_{ad}|_{U_k \setminus U_{k+1}})$ of $C^*(\mathcal{G}_{ad}|_{U_{k-1} \setminus U_{k+1}})$, for each k, we see that it is enough to prove our result for $\mathcal{G}_{ad}|_{U_k \setminus U_{k+1}}$ for all k. Thus, by replacing B with $U_k \setminus U_{k+1}$, we may assume that the rank of the abelianization of $\mathcal{G}_b$ is independent of b. Consequently, the abelianizations of $\mathcal{G}_b$ form a vector bundle

$$\mathcal{A} := \cup \mathcal{G}_b / [\mathcal{G}_b, \mathcal{G}_b]$$

on B.

A similar argument, using the Meyer-Vietoris exact sequence in K-theory and the compatibility of K-theory with inductive limits [8] shows that we may also assume the vector bundle $\mathcal{A}$ of abelianizations to be trivial. Then the argument at the beginning of the proof applies, and the result is proved. □

From the above lemma we immediately obtain the following corollary:

Corollary 1. *Let $\mathcal{G}$ be a bundle of simply connected, solvable Lie groups. Then*

$$K_i(C^*(\mathcal{G})) \simeq K_i(C_0(\mathfrak{g}^*)) = K^i(\mathfrak{g}^*).$$

Define $C^*(Y, \mathcal{G})$ to be the closure of $\psi^{-\infty}_{\text{inv}}(Y)$ in the sup norm of operators acting on $L^2(Y_b)$, for each b.

We denote below by $\widehat{\otimes}$ the minimal tensor product of algebras. This minimal tensor product is defined to be (isomorphic to) the completion of the image of $\pi_1 \otimes \pi_2$, the tensor product of two injective representations π_1 and π_2. For the cases we are interested, the minimal and the maximal tensor product coincide [28].

Lemma 3. *Assume* $\dim Y > \dim \mathcal{G}$. *Also, let $\mathcal{K} = \mathcal{K}(Y_b/\mathcal{G}_b)$ denote the algebra of compact operators on one of the fibers $Y_b/\mathcal{G}_b$, for some fixed, but arbitrary, $b \in B$. Then*

$$C^*(Y, \mathcal{G}) \simeq C^*(\mathcal{G}) \widehat{\otimes} \mathcal{K}.$$

Consequently, $K_i(C^*(Y, \mathcal{G})) \simeq K_i(C^*(\mathcal{G})) \simeq K^i(\mathfrak{g}^*)$.

Proof. If Y is a flat $\mathcal{G}$-space, then this follows, for example, from the results of [16]. Our assumptions imply that $\mathcal{K}$ is infinite dimensional, and hence its group of automorphisms is contractible, see [11]. This implies that there is no obstruction to trivialize the bundle of algebras $\mathcal{K}(Y_b/\mathcal{G}_b)$. Let $\mathfrak{A}$ be the space of sections of this bundle of algebras, then $\mathfrak{A} \simeq C_0(B) \widehat{\otimes} \mathcal{K}$, and hence

$$C^*(Y, \mathcal{G}) \simeq C^*(\mathcal{G}) \widehat{\otimes}_{C_0(B)} \mathfrak{A} \simeq C^*(\mathcal{G}) \widehat{\otimes} \mathcal{K},$$

as desired.

The last part of lemma follows from the above results and from the natural isomorphism $K_i(A \widehat{\otimes} \mathcal{K}) \simeq K_i(A)$, valid for any C^*-algebra A. □

We proceed now to define the index of an elliptic, invariant family of operators

$$A = (A_b) \in M_N(\psi^m_{\mathrm{inv}}(Y)) = \psi^m_{\mathrm{inv}}(Y; \mathbb{C}^N).$$

We assume that $Y/\mathcal{G}$ is compact, for simplicity; otherwise, we need to use algebras with adjoint units. First, we observe that there exists an exact sequence

$$(6) \qquad 0 \to C^*(Y, \mathcal{G}) \to \mathcal{E} \to \mathcal{C}^\infty(S^*_{vt}Y) \to 0, \quad \mathcal{E} := \psi^0_{\mathrm{inv}}(Y) + C^*(Y, \mathcal{G}),$$

obtained using the results of [16]. The operator A (or, rather, the family of operators $A = (A_b)$) has an invertible principal symbol, and hence the family $T = (T_b)$,

$$T_b := (1 + A_b^* A_b)^{-1/2} A_b$$

consists of elliptic, operators. Moreover, $T \in \mathcal{E} = \psi^0_{\mathrm{inv}}(Y) + C^*(Y, \mathcal{G})$. It's principal symbol is still invertible, and hence defines a class in $K_1(\mathcal{C}^\infty(S^*_{vt}Y)) \simeq K^1(S^*_{vt}Y)$. Let

$$\partial : K_1^{alg}(S^*_{vt}Y)) \to K_0^{alg}(C^*(Y, \mathcal{G})) \simeq K_0(C^*(Y, \mathcal{G}))$$

be the boundary map in the K-theory exact sequence

$$K_1^{alg}(C^*(Y, \mathcal{G})) \to K_1^{alg}(\mathcal{E}) \to K_1^{alg}(S^*_{vt}Y)) \xrightarrow{\partial} K_0^{alg}(C^*(Y, \mathcal{G})) \\ \to K_0^{alg}(\mathcal{E}) \to K_0^{alg}(S^*_{vt}Y))$$

associated to the exact sequence (6). Because $K_0(C^*(Y, \mathcal{G})) \simeq K^0(\mathfrak{g}^*)$ by Corollary 1, this gives rise to a group morphism

$$(7) \qquad \mathrm{ind}_a : K_1^{\mathrm{alg}}(\mathcal{C}^\infty(S^*_{vt}Y)) \to K^0(\mathfrak{g}^*),$$

which we shall call *the analytic index morphism.* The image of A under the composition of the above maps will be denoted $\mathrm{ind}_a(A)$ and called *the analytic index of* A. The analytic index morphism descends in this case to a group morphism $K_1^{\mathrm{alg}}(\mathcal{C}^\infty(S^*_{vt}Y)) \to K^0(\mathfrak{g}^*)$ denoted in the same way. For $\mathcal{G}$ abelian, we can replace $\mathcal{E}$ with $\Psi^0_{\mathrm{inv}}(Y)$.

The main property of the analytic index of an operator A is that it gives the obstruction to the existence of invertible perturbations of A by lower order operators. We denote by $H^s(Y_b)$ the sth Sobolev space of 1/2-densities on Y_b, which is defined using the bounded geometry of Y_b for $Y/\mathcal{G}$ is compact.

Proposition 1. *Assume that Y and $\mathcal{G}$ are as above and that $Y/\mathcal{G}$ is compact of positive dimension. Let $A \in \psi^m_{\mathrm{inv}}(Y, \mathbb{C}^N)$ be an elliptic operator. Then we can find $R \in \psi^{m-1}_{\mathrm{inv}}(Y, \mathbb{C}^N)$ such that*

$$A_b + R_b : H^s(Y_b)^N \to H^{s-m}(Y_b)^N$$

is invertible for all $b \in B$ if, and only if, $\mathrm{ind}_a(A) = 0$.

Proof. It is clear from definition that if we can find R with the desired properties, then $\mathrm{ind}_a(A) = 0 \in K^0(\mathfrak{g}^*)$. Suppose now that $A \in \psi^m_{\mathrm{inv}}(Y, \mathbb{C}^N)$ is elliptic and has vanishing analytic index. Using the notation $T = (1 + A^*A)^{-1/2}$, we see that A_b is invertible between the indicated Sobolev spaces if, and only if, each T_b is invertible as a bounded operator on $L^2(Y_b)$. Because $C^*(Y, \mathcal{G})$ is a stable C^*-algebra for $\dim \mathcal{G} > \dim Y$ (that is, $C^*(Y, \mathcal{G}) \simeq C^*(Y, \mathcal{G}) \widehat{\otimes} \mathcal{K}$, by Lemma 3), the "Atiyah-Singer trick" ([4], Proposition (2.2)) can be used to prove that the vanishing of $\mathrm{ind}_a(B) = \mathrm{ind}_a(A)$ implies that T (and hence also A) has a perturbation by invariant, regularizing operators in $\psi^{-\infty}_{\mathrm{inv}}(Y, \mathbb{C}^N)$ that is invertible on each fiber (see [24] for details). ∎

We now give an interpretation of $\mathrm{ind}_a(A)$, for $\mathcal{G}$ abelian, using the properties of the indicial family $\hat{A}(\tau)$ of A.

We shall also use the following construction. Let X be a compact manifold with boundary. If $T(x)$ is a family of elliptic pseudodifferential operators acting on the fibers of some bundle $M \to X$ whose fibers are compact manifolds without corners, then we can realize the index of T as an element in the relative group $K^0(X, \partial X)$. This can be done using Kasparov's theory, or the "Atiyah-Singer trick" as follows. We assume for simplicity that the bundle on which B acts is trivial, and proceed as in [4], Proposition (2.2), to define a smooth family of maps $R(x) : \mathbb{C}^N \to \mathcal{C}^\infty(Y)$, such that the induced map $V := T \oplus R : \mathcal{C}^\infty(X)^N \oplus \mathcal{C}^\infty(X, L^2(Y)) \to \mathcal{C}^\infty(X, L^2(Y))$ is onto for each x. Since $T(x)$ is invertible for $x \in \partial X$, we can choose $R(x) = 0$ for $x \in \partial X$. Then $\ker(V)$ is a vector bundle on X, which is canonically trivial on the boundary ∂X. The general definition of the index of the family T in [4] is that of the difference of the kernel bundle $\ker(V)$ and the trivial bundle $X \times \mathbb{C}^N$. Since the bundle $\ker(V)$ is canonically trivial on the boundary of X, we obtain an element in the relative group $K^0(X, \partial X)$. We shall use this constructin for $X = B_R$, a large closed ball in $\mathcal{G}^*$.

Returning to our considerations, we continue to assume that $Y/\mathcal{G}$ is compact, and we fix a metric on $\mathcal{G}$ (which, we recall, is a vector bundle in these considerations). If $A \in \Psi^\infty_{\mathrm{inv}}(Y)$ is elliptic (in the sense that its principal symbol is invertible outside the zero section), then the indicial operators $\hat{A}(\tau)$ are invertible for $|\tau| \geq R$, $\tau \in \mathcal{G}^*$, and some large R. In particular, by restricting the family $\hat{A}$ to the ball

$$B_R := \{|\tau| \leq R\},$$

we obtain a family of elliptic operators that are invertible on the boundary of B_R, and hence $\hat{A}$ defines an element in the K-group of the ball of radius R, relative to its boundary, as explained above. We define the *degree*

$$\deg_{\mathcal{G}}(A) \in K^0(B_R, \partial B_R) \simeq K^0(\mathfrak{g}^*) \tag{8}$$

to be the class defined above.

For the proof of the following result we borrow terminology from algebraic topology: if $I_k \subset A_k$ are two-sided ideal of some algebras A_0 and A_1 and $\phi : A_0 \to A_1$ is an algebra morphism, we say that *ϕ induces a morphism of pairs $\phi : (A_0, I_0) \to (A_1, I_1)$* if, by definition, $\phi(I_0) \subset I_1$.

Theorem 1. *Let $\mathcal{G}$ be a bundle of abelian Lie groups and $A \in \Psi^m_{\mathrm{inv}}(Y; \mathbb{C}^N)$ be an elliptic operator. Then*

$$\mathrm{ind}_a(A) = \deg_{\mathcal{G}}(A).$$

Proof. Let $B_R = \{|\tau| \leq R\} \subset \mathcal{G}^*$ be as above. The algebra

$$\mathfrak{A}_R := \mathcal{C}^\infty(B_R, \Psi^\infty(Y_b))$$

of $\mathcal{C}^\infty$-families of pseudodifferential operators on B_R acting on fibers of $Y \times_B B_R \to B_R$, contains as an ideal $\mathfrak{I}_R = \mathcal{C}^\infty_0(B_R, \Psi^{-\infty}(Y_b))$, the space of familis of smoothing operators that vanish of infinite order at the boundary of B_R. If A is an elliptic family, as in the statement of the lemma, and R is large enough, then $\hat{A}$, the indicial family of A, defines by restriction an element of $M_N(\mathfrak{A})$ that is invertible modulo $M_N(\mathfrak{I})$.

Recall that the boundary map ∂_1 in algebraic K-theory associated to the ideal $\mathfrak{I}$ of the algebra $\mathfrak{A}$ gives $\partial_1[A] = \deg_{\mathcal{G}}(A)$, by definition. Also, the boundary map ∂_0

in algebraic K-theory associated to the ideal $\Psi_{\rm inv}^{-\infty}(Y)$ of the algebra $\Psi_{\rm inv}^{\infty}(Y)$ gives $\partial_0[A] = {\rm ind}_a(A)$. We want to prove that $\partial_1[A] = \partial_0[A]$. The desired equality will follow by a deformation argument, which involves constructing an algebra smoothly connecting the ideals $\mathfrak{I}$ and $\Psi_{\rm inv}^{-\infty}(Y)$.

Consider inside $\mathcal{C}^\infty([0, R^{-1}], \Psi_{\rm inv}^{-\infty}(Y))$ the subalgebra of families $T = (T_x)$ such that $\hat{T}_x(\tau) = 0$ for $|\tau| \geq x^{-1}$. (In other words, $T_x \in \mathfrak{I}_{x^{-1}}$ if $x \neq 0$ and T_0 is arbitrary.) Denote this subalgebra by $\mathfrak{I}_{R\infty}$. Also, let $\mathfrak{A}_{R\infty}$ be the set of families $A = (A_x)$, $x \in [0, R^{-1}]$, $A_x \in \mathfrak{A}_{x^{-1}}$, if $x \neq 0$, $A_0 \in \Psi_{\rm inv}^{\infty}(Y)$ such that the families $AT := (A_x T_x)$ and $TA =: (T_x A_x)$ are in $\mathfrak{I}_{R\infty}$, for all families $T = (T_x) \in \mathfrak{I}_{R\infty}$.

It follows that $\mathfrak{I}_{R\infty}$ is a two-sided ideal in $\mathfrak{A}_{R\infty}$ and that the natural restrictions of operators to $x = R^{-1}$ and, respectively, $x = 0$, give rise to morphisms of pairs

$$e_1 : (\mathfrak{A}_{R\infty}, \mathfrak{I}_{R\infty}) \to (\mathfrak{A}_R, \mathfrak{I}_R), \quad \text{and}$$

$$e_0 : (\mathfrak{A}_{R\infty}, \mathfrak{I}_{R\infty}) \to (\Psi_{\rm inv}^{\infty}(Y), \Psi_{\rm inv}^{-\infty}(Y)).$$

Moreover, the indicial family of the operator A gives rise, by restriction to larger and larger balls B_r, to an invertible element in $\mathfrak{A}_{R\infty}$, also denoted by A. Let ∂ be the boundary map in algebraic K-theory associated to the pair $(\mathfrak{A}_{R\infty}, \mathfrak{I}_{R\infty})$. Then $(e_0)_*\partial[A] = \partial_0[A]$ and $(e_1)_*\partial[A] = \partial_1[A]$. Since $(e_0)_* : K_0(\mathfrak{I}_{R\infty}) \to K_0(\Psi_{\rm inv}^{-\infty}(Y))$ and $(e_1)_* : K_0(\mathfrak{I}_{R\infty}) \to K_0(\mathfrak{I}_R)$ are natural isomorphisms, our result follows. □

We now want to compute the Chern character of the analytic index ${\rm ind}_a(A)$, for an elliptic family $A \in \Psi_{\rm inv}^m(Y)$, along the lines of the Atiyah-Singer index theorem for families. One difficulty that we encounter is that the space on which the functions defining the principal symbols live, that is $(S_{vt}^*Y)/\mathcal{G}$, is not orientable in general. However, S_{vt}^*Y is always orientable. There is no such problem when $\mathcal{G} = B \times G$ is trivial. For G simply-connected solvable, we obtain complete results.

We denote by $\mathcal{T}$ the Todd class of the vector bundle $(T_{vt}Y)/\mathcal{G} \otimes \mathbb{C}$ over $Y/\mathcal{G}$ and by π_* the integration along the fibers of $(S_{vt}^*Y)/\mathcal{G} \to B$. We assume B compact.

Theorem 2. *Assume $\mathcal{G} = B \times G$ is trivial, with G simply-connected, solvable. Let $A \in \Psi_{\rm inv}^m(Y, \mathbb{C}^N)$ be an elliptic, invariant family, and let $[\sigma_m(A)] \in K^1((S_{vt}^*Y)/\mathcal{G})$ be the class defined by the principal symbol $\sigma_m(A)$ of A. Then the Chern character of the analytic index of A is given by*

$$Ch({\rm ind}_a(A)) = (-1)^n \pi_*\big(Ch[\sigma_m(A)]\mathcal{T}\big) \in H^*(B) \simeq H_c^{*+q}(\mathfrak{g}^*),$$

where n is the dimension of the fibers of $(S_{vt}^*Y)/\mathcal{G} \to B$.

Proof. Note first that we can deform the bundle of Lie groups $\mathcal{G}$ to the bundle of *commutative* Lie groups $\mathfrak{g}$ as before, using $\mathcal{G}_{ad}$. Moreover, we can keep the principal symbol of A constant along this deformation. This shows that we may assume $\mathcal{G}$ to consist of commutative Lie groups. Thus we shall assume from now on that $G = \mathbb{R}^q$.

In [20], this theorem was proved in the case B reduced to a point. To obtain the proof in general, we repeat the arguments of that paper for families. Let us briefly review those arguments as modified in our setting.

First, we choose a trivialization of the principal $\mathbb{R}^q$ bundle $Y \to Y/\mathbb{R}^q$, so we may assume that $Y = Y_0 \times \mathbb{R}^q$. The idea of the proof is to prove the theorem by induction on q. For the inductive step, which relates the case q with the case $q-1$, we use the manifold with boundary $Z := Y_0 \times [0, 1) \times \mathbb{R}^{q-1}$ and the algebra $\mathfrak{A}(Z)$, the closure of the algebra of b-pseudodifferential operators on Z. The results of [16] show that the algebra $\mathfrak{A}(Z)$ fits into two exact sequences. The first exact sequence is associated to the face of codimention one and the second exact sequence is associated to the kernel of the principal symbol map.

The K-theory six term peridic exact sequences associated to the above two exact sequences of C^*-algebras provide us with the isomorphisms necessary for the inductive step of the argument. Then, for $q = 0$, the theorem is nothing but the Atiyah-Singer index theorem for families [4]. □

3. Regularized traces

Having in mind future applications and generalizations, we also want to give a local formula for the equivariant family index of an invariant, elliptic family of operators, as considered in the previous section. This will be done in terms of various residue type traces. In this section, we develop the analytic tools required to define these regularized traces.

Assumption. *Throughout the rest of this paper, we shall assume that $\mathcal{G}$ consist of abelian Lie groups, and hence that it is a vector bundle.*

We shall say that Y is a flat $\mathcal{G}$-bundle if $Y = B \times Y_0 \times \mathbb{R}^q$ and $\mathcal{G} = B \times \mathbb{R}^q$. The results we will establish are local in B, and hence we can reduce the general case to the flat case. Actually, it is easier to assume first that B is reduced to a point. We thus carry the analysis first in this case, and then we extend the results to the general case.

There is an action of $\mathbb{R}^q$ on $\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$, the action of $\xi \in \mathbb{R}^q$ is obtained by multiplying the convolution kernel of an operator $A \in \Psi^p_{\rm inv}(Y_0 \times \mathbb{R}^q)$ by $\exp(it \cdot \xi)$, where $t \in \mathbb{R}^q$ are coordinates for the second component in $Y_0 \times \mathbb{R}^q$. In terms of the Fourier transform representation of these operators, the action of ξ becomes translation by $\xi \in \mathbb{R}^q$.

We shall denote by Tr the usual (Fredholm) trace on the space of trace class operators on a given Hilbert space.

Lemma 4. *The space of $\mathbb{R}^q$–invariant traces on $\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$ is one-dimensional.*

Proof. Consider the map

$$\overline{\operatorname{Tr}}(A) = \int Tr(I(A,\tau))\, d\tau. \tag{9}$$

We need to show that this is the only invariant trace functional. In terms of indicial families, the infinitesimal generators of the $\mathbb{R}^q$–action correspond to the multiplication operators with the functions t_k. Let

$$\mathrm{HH}_0(\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)) := \Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)/[\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q), \Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)]$$

be the first homology group of $\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$. It remains to show that the subspace of $\mathrm{HH}_0(\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)) \simeq \mathcal{S}(\mathbb{R}^q)$ spaned by $t_k\, \mathrm{HH}_0(\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q))$ has codimension 1. Indeed, the kernel $\mathrm{HH}_0(\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)) \to \mathbb{C}$ of the evaluation at 0 is the span of $t_k\, \mathrm{HH}_0(\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q))$. This proves the lemma. □

Let x be the identity function on $[0,\infty)$ and l_s be a smooth function on $[0,\infty)$ such that $l_s'(x) = x^{s-1}$ for $x \geq 1$. (So that, in particular, $l_0(x) = \ln x$, for x large.) We define then the spaces of functions

$$\mathcal{M}_k = S^\infty([0,\infty)) + \mathbb{C}[x] l_0, \quad \text{for } k \in \mathbb{Z},$$

and

$$\mathcal{M}_s = S^\infty([0,\infty))l_s, \quad \text{for } s \in \mathbb{C} \setminus \mathbb{Z}.$$

Thus, $\mathcal{M}_0$ consists of smooth functions on $[0,\infty)$, that can be written, for any $M \in \mathbb{Z}_+$, as

$$f(x) = h_M(x) + \sum_{k=-M}^{-1} a_k x^k + \sum_{k=0}^{N} (b_k + c_k \log x) x^k, \quad \forall x \geq 1, \tag{10}$$

where a_k, b_k, c_k are complex parameters, $N \in \mathbb{Z}_+$, and $h_M \in S^{-M-1}([0,\infty))$. Similarly, the space $\mathcal{M}_s$, $s \notin \mathbb{Z}$, consists of smooth functions $f \in \mathcal{C}^\infty([0,\infty))$ that can be written, for any $M \in \mathbb{Z}_+$, as

$$f(x) = h_M(x) x^s + \sum_{k=-M}^{N} \alpha_k x^{k+s}, \quad \forall x \geq 1. \tag{11}$$

for some constants $\alpha_k \in \mathbb{C}$, $N \in \mathbb{Z}$, and $h_M \in S^{-M-1}([0,\infty))$. Fix $M > \max\{s, 0\}$ and $R \geq 1$, and define

$$I(f)(x) = \int_0^x f(t)dt \ , \tag{12}$$

$$\text{pv--}\!\!\int f = \int_0^R f(t)dt + \int_R^\infty h_M(t)dt - \sum_{k=-M}^{-2} \frac{a_k R^{k+1}}{k+1} - a_{-1} \log R \tag{13}$$

$$- \sum_{k=0}^{N} \frac{R^{k+1}}{k+1} \big(b_k - \frac{c_k}{(k+1)} + c_k \log R\big), \quad \text{if } f \in \mathcal{M}_l, \ l \in \mathbb{Z} \tag{14}$$

and

(15)

$$\text{pv--}\!\!\int f = \int_0^R f(t)dt + \int_R^\infty h_M(t) t^s dt - \sum_{k=-M}^{N} \frac{\alpha_k R^{k+s+1}}{k+s+1}, \quad \text{if } f \in \mathcal{M}_s, \ s \notin \mathbb{Z}.$$

It is easy to see that I maps $\mathcal{M}_0$ to itself and that the definition of $\text{pv--}\!\int f$ is independent of M and R, for $f \in \mathcal{M}_s$, $s \in \mathbb{C}$. Moreover,

$$\text{pv--}\!\!\int I^k(f) = \text{pv--}\!\!\int I^{k+1}(f'), \tag{16}$$

if $f \in \mathcal{M}_0$, because $I^k(f) - I^{k+1}(f')$ is a polynomial of degree at most k and $\text{pv--}\!\int$ vanishes on polynomials. If $s \notin \mathbb{Z}$, then I maps $\mathcal{M}_s$ to $\mathcal{M}_s + \mathbb{C}[x] \simeq \mathcal{M}_s \oplus \mathbb{C}[x]$, and since this is a direct sum decomposition, we can extend $\text{pv--}\!\int$ to $\mathcal{M}_s + \mathbb{C}[x]$ to vanish on polynomials, which guaranties that the Equation (16) is still satisfied. Moreover, if $f \in \mathcal{M}_s$ is as in Equation (11), $P(x) = \sum_{k=0}^N b_k x^k$, and $g = f + P$, then the original definition of $\text{pv--}\!\int$ is still valid in this case, with the obvious changes:

$$\text{pv--}\!\!\int g = \int_0^R g(t)dt + \int_R^\infty h_M(t) t^s dt - \sum_{k=-M}^{N} \frac{\alpha_k R^{k+s+1}}{k+s+1} - \sum_{k=0}^{N} \frac{b_k R^{k+1}}{k+1}. \tag{17}$$

It not difficult to check that $\text{pv--}\!\int$ extends the integral on $[0,\infty)$, that is

$$\text{pv--}\!\!\int f = \int_0^\infty f(t)dt, \quad \text{if } f \in \mathcal{S}^{-2}([0,\infty)), \tag{18}$$

and that it is homogeneous of degree -1 for $s \notin \mathbb{Z}$. Indeed, this follows from the definition. If $f \in \mathcal{C}^\infty(\mathbb{R})$ is such that $f_+, f_- \in \mathcal{M}_s + \mathbb{C}[X]$, for some s, where $f_+ = f|_{[0,\infty)}$ and $f_-(\tau) = f(-\tau)$, $\tau \geq 0$, then we define

$$\text{pv-}\!\!\int f = \text{pv-}\!\!\int f_+ + \text{pv-}\!\!\int f_+.$$

Then equation (16) has to be changed to $\text{pv-}\!\int I^{2k}\partial_{2k} f = \text{pv-}\!\int f$.

Fix from now on an invertible positive operator $D \in \Psi^1_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. Let $\mathbb{C} \ni z \to A(z) \in \Psi^m_{\text{inv}}(Y_0 \times \mathbb{R}^q)$ be an entire function. Then

$$f_z(\tau) = Tr\big(\hat{D}(\tau)^{-z}|\tau|^k \hat{A}(\tau)\big), \quad k \in \mathbb{Z}_+ \tag{19}$$

is defined and holomorphic for any $\text{Re}(z) > m + \dim Y_0 = m + d$ and any fixed $\tau \in \mathbb{R}^q$. Moreover, by classical results (see [12], for example), the function $z \to f_z(\tau)$ has a meromorphic extension to $\mathbb{C}$, for each fixed τ, with at most simple poles at integers. Let

$$\Omega = (\mathbb{C} \setminus \mathbb{Z}) \cup \{z, \text{Re}(z) > m + d\},$$

$d = \dim Y_0$.

Lemma 5. *Let $A(z) \in \Psi^m_{\text{inv}}(Y_0 \times \mathbb{R}^q)$ be an entire function. Also, let $f_z(\tau)$, defined for $z \in \Omega$, be as above*

(i) The function $f_z(\tau)$ is in $\mathcal{C}^\infty(\Omega \times \mathbb{R}^q)$ and the map $z \to f_z(\tau)$ is holomorphic on Ω, for each fixed $\tau \in \mathbb{R}^q$.

(ii) There is a holomorphic map $g : \Omega \to S^{m+d}([0,\infty)) \subset \mathcal{M}_0$ such that $f_z(\tau) = g_z(\tau)|\tau|^{k-z}$, for all $\tau \geq 1$, and hence $f_z \in \mathcal{M}_{m+d+k-z}$, $d = \dim Y_0$.

(iii) The function $z \to \text{pv-}\!\int f_z$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, with at most simple poles at integers.

Proof. The proof for $k \neq 0$ is the same as for $k = 0$, so we will assume $k = 0$ in Equation (19).

We first prove the lemma for $m = -\infty$, that is for $A(z) \in \Psi^{-\infty}_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. Denote by $\mathcal{K}$ the algebra of compact operators acting on $L^2(Y_0)$ and by $\mathcal{C}_1 \subset \mathcal{K}$ the normed ideal of trace class operators. For any $M \in \mathbb{Z}_+$ the product $\hat{D}^M(\tau)\hat{A}(z,\tau)$ is in $\mathcal{S}(\mathbb{R}, \mathcal{C}_1) = \mathcal{S}(\mathbb{R})\widehat{\otimes}\mathcal{C}_1$ (here $\widehat{\otimes}$ denotes the completed projective tensor product). Also, because D is invertible and positive, the function $(z,\tau) \to \hat{D}(\tau)^{-z} \in \mathcal{K}$ is differentiable, with bounded derivatives, and holomorphic in z, for $\text{Re}(z) \geq 1$. Since $\text{Tr} : \mathcal{K}\widehat{\otimes}\mathcal{C}_1 \to \mathbb{C}$ is continuous, it follows that the function

$$(z,\tau) \to Tr(\hat{D}(\tau)^{-z-M-1}\hat{D}(\tau)^{M+1}\hat{A}(z,\tau)) \in \mathbb{C}$$

is differentiable, with bounded derivatives, and holomorphic in z for $\text{Re}(z) \geq -M$. Since M is arbitrary, this proves (i) and (ii) for $A(z) \in \Psi^{-\infty}_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. The last statement is an immediate consequence of (ii), because $z \to g_z \in \mathcal{S}(\mathbb{R}^q)$ is holomorphic.

Using now the fact that the lemma is true for $A(z)$ in the residual ideal, we may assume, using a partition of unity, that $Y_0 = \mathbb{R}^d$ and that the Schwartz convolution kernels of $\hat{A}(z,\tau)$ are contained in a fixed compact set.

Let $\Delta_0, \Delta_1 \geq 0$ be the constant coefficient Laplacians on $Y_0 = \mathbb{R}^d$ and $\mathbb{R}^q$, respectively. We define $D_0 = (1 + \Delta_0 + \Delta_1)^{1/2} \in \Psi^1_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. To prove the lemma for $A(z) \in \Psi^m_{\text{inv}}(Y_0 \times \mathbb{R}^q)$, $m > \infty$, we shall first assume that $D = D_0$.

Clearly, $D_0 \in \Psi^1_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. If $A(z) = a(z, x, D_x, D_t)$, for a symbol $a(z, \cdot, \cdot, \cdot) \in S^m(T^*Y_0 \times \mathbb{R}^q) = S^m(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^q)$, then

$$\hat{A}(z,\tau) = a(z, x, D_x, \tau) \quad \text{and} \quad \hat{D}_0(\tau)^{-z} = (1 + \Delta_0 + |\tau|^2)^{-z}.$$

This gives, by the standard calculus, that $\hat{A}(z,\tau)\hat{D}_0(\tau)^{-z} = a_1(z, x, D_x, \tau)$ for

$$a_1(z, x, \xi, \tau) = a(z, x, \xi, \tau)(1 + |\tau|^2 + |\xi|^2)^{-z/2}. \tag{20}$$

From the above relation, we obtain

$$\begin{aligned} f_z(\tau) &= Tr\big(\hat{A}(\tau)\hat{D}(\tau)^{-z}\big) \\ &= (2\pi)^{-d} \int_{T^*Y_0} a(z, x, \xi, \tau)(1 + |\tau|^2 + |\xi|^2)^{-z} d\xi dx, \quad \text{for } \tau \geq 1. \end{aligned}$$

Using the substitution $\xi \to |\tau|\xi$ and the assymptotic expansion of a_2 in homogeneous functions in (ξ, τ), we obtain (i) and (ii) for this particular choice of $D = D_0$. The proof of (iii) is similar, except that one integrates with respect to $(x, \xi, \tau) \in T^*Y_0 \times \mathbb{R}^q$ and that one separates the integrals as the sum of two other integrals, one for $|\tau| \leq 1$ and one for $|\tau| \geq 1$.

The case D arbitrary follows by writing $D^{-z} = D_0^{-z}\big(D_0^z D^{-z}\big)$ and observing that $\mathbb{C} \ni z \to D_0^z D^{-z} \in \Psi^0_{\text{inv}}(Y_0 \times \mathbb{R}^q)$ is an entire function. □

Assume now that $q = 1$ (and hence that $Y = Y_0 \times \mathbb{R}$). Using the above lemma and the functionals pv-$\textstyle\int$ and I, we obtain, as in [19], a functional $\overline{\text{Tr}}_1$ on $\Psi^\infty_{\text{inv}}(Y_0 \times \mathbb{R})[|\tau|]$, by the formula

$$\overline{\text{Tr}}_1(A) = \text{pv-}\!\!\int I^k(f_+ + f_-), \tag{21}$$

where $k > m + \dim Y_0 + 1$, $f_+(\tau) = \partial_\tau^k \hat{A}(\tau)$, $f_-(\tau) = \partial_\tau^k(\hat{A}(-\tau))$, and $\tau \leq 0$. From equation (16), we see that this definition is independent on k. The tracial property of $\overline{\text{Tr}}_1$ follows from

$$\partial_\tau[\hat{A}(\tau), \hat{B}(\tau)] = [\partial_\tau \hat{A}(\tau), \hat{B}(\tau)] + [\hat{A}(\tau), \partial_\tau \hat{B}(\tau)].$$

We continue to assume that B is reduced to a point.

Lemma 6. *Restriction of the indicial family $\hat{A}$ to $\mathbb{R}x$, $x \in \mathbb{S}^{q-1}$ defines an $O(q)$-equivariant family of algebra morphism $r_x : \Psi^\infty_{\text{inv}}(Y_0 \times \mathbb{R}^q) \to \Psi^\infty_{\text{inv}}(Y_0 \times \mathbb{R})$. Each r_x restricts to a degree preserving isomorphism $\Psi^\infty_{\text{inv}}(Y_0 \times \mathbb{R}^q)^{O(q)} \simeq \Psi^\infty_{\text{inv}}(Y_0 \times \mathbb{R})^{\mathbb{Z}_2}$ independent of x.*

Proof. It is clear that the restriction of $\hat{A}$ to a line $\mathbb{R}x$ is in $\mathcal{S}(\mathbb{R}x, \Psi^{-\infty}(Y_0))$, provided that A is in $\Psi^{-\infty}_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. Moreover, this gives isomorphisms

$$\Psi^{-\infty}_{\text{inv}}(Y_0 \times \mathbb{R}^q)^{O(q)} \simeq \mathcal{S}(\mathbb{R}^q, \Psi^{-\infty}(Y_0))^{O(q)} \simeq \mathcal{S}(\mathbb{R}, \Psi^{-\infty}(Y_0))^{\mathbb{Z}_2} \simeq \Psi^{-\infty}_{\text{inv}}(Y_0 \times \mathbb{R})^{\mathbb{Z}_2}.$$

These isomorphisms allow us to assume, using a partition of unity argument, that $Y_0 = \mathbb{R}^l$. Using the fact that a symbol $a \in S^m(T^*Y_0 \times \mathbb{R}^q)$ restricts to a symbol in $S^m(T^*Y_0 \times \mathbb{R}x)$, $x \in \mathbb{S}^{p-1}$, and the relation

$$\hat{A}(\tau) = a(x, D_x, \tau),$$

if $A = a(x, D_x, D_\tau)$, we see that the restriction of $\hat{A}$ to $\mathbb{R}x$ is the indicial family of an operator in $\Psi^m_{\text{inv}}(Y_0 \times \mathbb{R})$, denoted $r_x(A)$. Since

$$S^m(T^*Y_0 \times \mathbb{R}^q)^{O(q)} \simeq S^m(T^*Y_0 \times \mathbb{R})^{\mathbb{Z}_2},$$

the isomorphism $\Psi^{\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)^{O(q)} \simeq \Psi^{\infty}_{\rm inv}(Y_0 \times \mathbb{R})^{\mathbb{Z}_2}$ follows. □

Let $A \in \Psi^{\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$ and denote by $A_1 = \int_{O(q)} v(A)dv$ its average over $O(q)$, which we identify with an element of $\Psi^{\infty}_{\rm inv}(Y_0 \times \mathbb{R})^{\mathbb{Z}_2}$, thanks to Lemma 6. Define

$$\overline{\rm Tr}_q(A) = \overline{\rm Tr}_1(|\tau|^{q-1}A_1).$$

Lemma 7. *The functional $\overline{\rm Tr}_q$ is an $O(q)$–invariant trace on $\Psi^{\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$, which extends the trace $\overline{\rm Tr}$ defined on $\Psi^{-\infty}_{\rm inv}(Y_0 \times \mathbb{R}^q)$ by Equation (9).*

Proof. The map $\overline{\rm Tr}_q$ is obviously well defined and $O(q)$-invariant, in view of the above lemma. In order to check the tracial property, we use the definition. Fix $x \in \mathbb{R}^q$ of length one, arbitrarily, then

$$\overline{\rm Tr}_q(A) = \int_{O(q)} \overline{\rm Tr}_1(|\tau|^{q-1}r_{vx}(A))dv$$

Since r_x is a morphism, $|\tau|$ is central, and $\overline{\rm Tr}_1$ is a trace [19], the tracial property of $\overline{\rm Tr}_q$ follows.

To complete the proof, we need only prove that $\overline{\rm Tr}_q$ extends $\overline{\rm Tr}$, and this follows by integration in polar coordinates using also Equation (18). □

It is not essential in the above statements that A have integral order. Both the formula for $r_x(A)$ and the definition of $\overline{\rm Tr}_q(A)$ make sense for $A \in \Psi^{s}_{\rm inv}(Y)$, with s not necessarily integral. We shall use this for operators of the form $D^{-z}A$ in the following proposition. Actually, more is true of the $\overline{\rm Tr}_q$-traces of elements of non-integral order than for elements of integral order: let $s \notin \mathbb{Z}$, then the action of $GL_q(\mathbb{R})$ by automorphisms on $\Psi^{s}_{\rm inv}(Y_0 \times \mathbb{R}^q)$ has the property

$$\overline{\rm Tr}_q(T(A)) = |\det(T)|^{-1}\overline{\rm Tr}_q(A). \tag{22}$$

Proposition 2. *For any self-adjoint, invertible, positive element $D \in \Psi^{1}_{\rm inv}(Y_0 \times \mathbb{R}^q)$ and any holomorphic function $A : \mathbb{C} \to \Psi^{m}_{\rm inv}(Y_0 \times \mathbb{R}^q)$, the function*

$$F_D(A;z) = \overline{\rm Tr}_q(D^{-z}A(z))$$

is holomorphic in $\operatorname{Re} z > m + q + \dim Y_0$ and extends to a meromorphic function with a simple pole at $z = 0$. The residue of this holomorphic function depends only on $A(0)$ and will be denoted by $\operatorname{Tr}_{\rm R}(A(0))$. Moreover, $\operatorname{Tr}_{\rm R}(A(0))$ vanishes on regularizing elements, is independent of D, and defines a trace on $\Psi^{m}_{\rm inv}(Y_0 \times \mathbb{R}^q)$.

Proof. The properties of F_D stated in the first part of this proposition follow from Lemma 5(iii) and the definition of $\overline{\rm Tr}_q$ in terms of $\overline{\rm Tr}_1$, see Proposition 7. For the rest of the proof, it is enough to assume that $A(z)$ is independent of z. We set then $A = A(z) = A(0)$.

The proof of the fact that $\operatorname{Tr}_{\rm R}$ is a trace and that it is independent of the choice of D is obtained from a standard reasoning, as follows. We first write

$$\overline{\rm Tr}_q(D^{-z}[A,B]) = \overline{\rm Tr}_q(D^{-z}[D^{-z}, D^{z}A]B)$$

and observe that $[D^{-z}, D^{z}A]B$ is a holomorphic function vanishing at 0. This shows that $\overline{\rm Tr}_q$ vanishes on commutators. The independence of $\overline{\rm Tr}_q$ on D is a consequence of

$$\overline{\rm Tr}_q(D^{-z}A) - \overline{\rm Tr}_q(D_1^{-z}A) = \overline{\rm Tr}_q(D^{-z}(\operatorname{Id} - D^{z}D_1^{-z})A),$$

using that $(\operatorname{Id} - D^{z}D_1^{-z})A$ is a holomorphic function vanishing at 0. □

Proposition 3. *Let $D \geq 0$ be as above, let $A \in \Psi^m_{\rm inv}(Y_0 \times \mathbb{R}^q)$. We denote $B_z = D^{-z}A \in \Psi^{m-z}_{\rm inv}(Y_0 \times \mathbb{R}^q)$, and write the assymptotic expansion*

$$Tr(\hat{B}_z(\tau)) \sim \sum_k \alpha_k(z, \tau/|\tau|)|\tau|^{-z},$$

with α_k homogeneous of order k in τ, for $|\tau| \geq 1$. Then

$$\mathrm{Tr}_{\mathrm{R}}(A) = \int_{|\tau|=1} \alpha_{-1}(0, \tau),$$

and

$$\lim_{z\to 0}(\overline{\mathrm{Tr}}_q(D^{-z}A) - z^{-1}\,\mathrm{Tr}_{\mathrm{R}}(A)) = \overline{\mathrm{Tr}}_q(A) + \int_{|\tau|=1} \partial_z \alpha_{-1}(z, \tau)|_{z=0}.$$

Proof. By the definition of $\overline{\mathrm{Tr}}_q$ using integration with respect to the orthogonal group, it is enough to prove the result for $q = 1$ and $\mathbb{Z}_2$ invariant operators. The $\overline{\mathrm{Tr}}_q$ trace of these operators is completely determined by $f_+(\tau) = \partial_\tau^k \hat{A}(\tau)$, and $f_-(\tau) = \partial_\tau^k(\hat{A}(-\tau))$, for $\tau \leq 0$. This reduces our analysis to a lemma about integrals of functions in $\mathcal{M}_s$.

Fix $\epsilon > 0$, and define $\mathcal{C}_\epsilon$ to be the space of functions $f(s, x)$, $-\epsilon < \mathrm{Re}(s) < \epsilon$, $x \geq 0$, with the following properties:

1. *f is smooth in (s, x) and holomorphic in s for each fixed x;*
2. *For any $M \in \mathbb{N}$ there exist $R > 0$, $c_k \in \mathbb{C}$ and complex valued functions $\alpha_k(s), b_k(s)$, c_k, and $h_M(s, x)$ satisfying*

$$f(0, x) = h_M(0, x) + \sum_{k=-M}^{-1} \alpha_k x^k + \sum_{k=0}^{N} \big(\alpha_k(0) + b_k(0) + c_k \log x\big)x^k,$$

   for $s = 0$, $x \geq R$, and

$$f(s, x) = h_M(s, x)x^{-s} + \sum_{k=-M}^{N} \alpha_k x^{k-s} + \sum_{k=0}^{N} \big(b_k(s) + c_k s^{-1}(1 - x^{-s})\big)x^k,$$

   for $s \neq 0$, $x \geq R$, and $-\epsilon < \mathrm{Re}(s) < \epsilon$.
3. *$a_k(s)$, $b_k(s)$ are holomorphic in s and $h_M(s, x)$ is holomorphic in s, for each fixed x, and $h_M(s, \cdot) \in \mathcal{S}^{-M-1}([0, \infty))$, for each s in the strip $-\epsilon < \mathrm{Re}(s) < \epsilon$.*

Of course the choice of $R > 0$ is not important. It is easy to see that if $f \in \mathcal{C}_\epsilon$, then $I(f) \in \mathcal{C}_\epsilon$. Moreover, for any $f \in \mathcal{C}_\epsilon$,

$$\lim_{s\to 0}\ \mathrm{pv}\!\!-\!\!\!\int f(s, x)dx - \alpha_{-1}(s)s^{-1}R^{-s} = \mathrm{pv}\!\!-\!\!\!\int f(0, x)dx + \alpha_{-1}(0)\log R,$$

from definition. This gives

$$\lim_{s\to 0}\big(\ \mathrm{pv}\!\!-\!\!\!\int f(s, x)dx - \alpha_{-1}(0)s^{-1}\big) = \mathrm{Tr}_{\mathrm{R}}(f(0, \cdot)) + \alpha'_{-1}(0).$$

In view of our first remark, this completes the proof. □

We now drop the assumption that B be reduced to a point. To extend the above results to the general case, we proceed to a large extent as we did when B was reduced to a point.

Fix an invertible positive operator $D \in \Psi^1_{\mathrm{inv}}(Y)$, and let $\mathbb{C} \ni z \to A(z) \in \Psi^m_{\mathrm{inv}}(Y)$ be an entire function. Then

$$f_z(\tau) = Tr\big(\hat{D}(\tau)^{-z}|\tau|^k \hat{A}(\tau)\big), \quad k \in \mathbb{Z}_+, \tag{23}$$

is defined and holomorphic for any $\mathrm{Re}(z) > m + d = m + \dim Y_0$ and any fixed $\tau \in \mathcal{G}^*$, and the function $z \to f_z(\tau)$ has a meromorphic extension to $\mathbb{C}$, for each fixed τ, with at most simple poles at integers. Let $\Omega = (\mathbb{C}\setminus\mathbb{Z}) \cup \{z, \mathrm{Re}(z) > m + \dim Y_0\}$.

Lemma 8. *Let $A(z) \in \Psi^m_{\mathrm{inv}}(Y)$ be an entire function. Also, let $f_z(\tau)$ be as above, with $z \in \Omega$ and $\tau \in \mathcal{G}^*$.*

(i) The function $f_z(\tau)$ is in $\mathcal{C}^\infty(\Omega \times \mathcal{G}^*)$ and the map $z \to f_z(\tau)$ is holomorphic on Ω, for each fixed $\tau \in \mathcal{G}^*$.

(ii) There is $g \in \mathcal{C}^\infty(\Omega \times \mathcal{G}^*)$, $g(s,\cdot) \in S^{m+d}([0,\infty)) \subset \mathcal{M}_0$, for each s, $g(s,\tau)$ holomorphic in s, for each fixed τ, such that $f_z(\tau) = g_z(\tau)|\tau|^{k-z}$, for all $\tau \geq 1$, and hence $f_z \in \mathcal{M}_{m+d+k-z}$, $d = \dim Y - \dim \mathcal{G}$.

(iii) The function $z \to \mathrm{pv}\text{-}\!\!\int f_z$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$, with at most simple poles at integers.

Proof. Since all the statements of the above theorem are statements about the local behaviour of certain functions, we may assume that Y is a flat $\mathcal{G}$-space. This means, we recall, that $Y = B \times Y_0 \times \mathbb{R}^q$ and $\mathcal{G} = B \times \mathbb{R}^q$. Then we just repeat the proof of Lemma 5 including an extra parameter $b \in B$, with respect to which all functions involved are smooth. □

We now extend the definition of the various traces and functionals we considered above when B was reduced to a point. This is not completely canonical, because we need to fix a metric on $\mathcal{G}^*$ in order to obtain a volume form on the fibers of $\mathcal{G}^* \to B$. The choice of the metric defines the group $O(\mathcal{G})$ of fiberwise orthogonal isomorphisms of $\mathcal{G}$. We also fix a lifting $Y/\mathcal{G} \to Y$, which gives an isomorphism

$$Y \simeq Y/\mathcal{G} \times_B \mathcal{G}.$$

This isomorphism and the metric on $\mathcal{G}$ give an action of $O(\mathcal{G})$ on Y, which normalizes the structural action of $\mathcal{G}$ by translations on Y. Consequently, the group $O(\mathcal{G})$ acts by isomorphisms on the algebra $\psi^\infty_{\mathrm{inv}}(Y)$. By Lemma 1, the group $O(\mathcal{G})$ also acts by isomorphisms on $\Psi^\infty_{\mathrm{inv}}(Y)$.

Lemma 9. *Fix a lifting $Y/\mathcal{G} \to Y$, which gives an isomorphism $Y \simeq Y/\mathcal{G} \times_B \mathcal{G}$ and an action by automorphisms of the group $O(\mathcal{G})$ on $\Psi^\infty_{\mathrm{inv}}(Y)$, as above. Then there exists an $O(\mathcal{G})$-linear map*

$$E_Y : \Psi^\infty_{\mathrm{inv}}(Y) \to \Psi^\infty_{\mathrm{inv}}(Y)^{O(\mathcal{G})}$$

such that $E_Y(AB) = AE_Y(B)$ and $E_Y(BA) = E_Y(B)A$, for all $A \in \Psi^\infty_{\mathrm{inv}}(Y)^{O(\mathcal{G})}$ and $B \in \Psi^\infty_{\mathrm{inv}}(Y)$. Moroever, $\Psi^\infty_{\mathrm{inv}}(Y)^{O(\mathcal{G})} \simeq \Psi^\infty_{\mathrm{inv}}(Y/\mathcal{G} \times \mathbb{R})^{\mathbb{Z}/2\mathbb{Z}}$, and hence the isomorphism class of the algebra $\Psi^\infty_{\mathrm{inv}}(Y)^{O(\mathcal{G})}$ depends only on $Y/\mathcal{G}$.

Proof. If Y is a flat $\mathcal{G}$ space, then this result follows right away from Lemma 6. In general, we can choose trivilizations of Y such that the transition functions preserve the metric on $\mathcal{G}$, and hence the transition functions are in $O(\mathcal{G})$. Because the isomorphisms of Lemma 6 commute with the action of the orthogonal group, the result follows. □

We now consider $\mathcal{C}^\infty(B)$–linear traces on $\Psi^m_{\text{inv}}(Y)$, for a general $\mathcal{G}$ space Y. That is, we consider $\mathcal{C}^\infty(B)$–linear maps $T : \Psi^m_{\text{inv}}(Y) \to \mathcal{C}^\infty(B)$ such that

$$T(fA) = fT(A), \quad \text{for } f \in \mathcal{C}^\infty(B) \text{ and } A \in \Psi^m_{\text{inv}}(()Y),$$

and

$$T([A, B]) = 0, \quad \text{for } A, B \in \Psi^m_{\text{inv}}(Y).$$

If we fix a metric on $\mathcal{G}$, then we obtain a $\mathcal{C}^\infty(B)$–linear trace $\overline{\text{Tr}}_Y$ that generalizes the $\overline{\text{Tr}}_q$–traces as follows. Suppose $Y = B \times Y_0 \times \mathbb{R}^q$ and $A = (A_b) \in \Psi^m_{\text{inv}}(Y)$, then we set

$$\overline{\text{Tr}}_Y(A)(b) = \overline{\text{Tr}}_q(A_b). \tag{24}$$

Because trace $\overline{\text{Tr}}_Y$, for $Y = B \times Y_0 \times \mathbb{R}^q$, is invariant with respect to the action of the orthogonal group, the choice of an isomorphism $\underline{Y/\mathcal{G}} \times \mathcal{G} \simeq \mathcal{G}$ of $\mathcal{G}$-spaces and of a metric on $\mathcal{G}$ allow us to extend the definition of $\overline{\text{Tr}}_Y$ to arbitrary Y. We stress that this trace depends on the choices we made. This new trace satisfies

$$\overline{\text{Tr}}_Y(A) = \overline{\text{Tr}}_{Y/\mathcal{G}\times\mathbb{R}}(E_Y(A)). \tag{25}$$

We then have the following immediate generalization of Proposition 2 above:

Proposition 4. *For any self-adjoint, invertible, positive element $D \in \Psi^1_{\text{inv}}(Y)$ and any holomorphic function $A : \mathbb{C} \to \Psi^m_{\text{inv}}(Y)$, the function*

$$F_D(A; z) = \overline{\text{Tr}}_Y(D^{-z}A(z))$$

is holomorphic in $\operatorname{Re} z > -m + p + \dim Y - \dim \mathcal{G}$ and extends to a meromorphic function with a simple pole at $z = 0$. The residue of this holomorphic function depends only on $A(0)$ and will be denoted by $\text{Tr}_{\text{R}Y}(A(0))$. Moreover, $\text{Tr}_{\text{R}Y}(A(0))$ vanishes on regularizing elements, is independent of D, and defines a $\mathcal{C}^\infty(B)$-linear trace on $\Psi^m_{\text{inv}}(Y_0 \times \mathbb{R}^q)$. This trace is independent of the choice of the trivializations.

Proof. Everything in this proposition follows from the case when B is reduced to a point, except the independence of trivialization. For this we also use Equation (22). □

Proposition 3 extends virtually without change to families (that is, to the case B-nontrivial).

The traces $\overline{\text{Tr}}_q$ and $\overline{\text{Tr}}_Y$ extend to matrix algebras by taking the sum of the traces of the entries on the main diagonal.

4. Local index formulae

We now return to the the study of the index of a family of invariant, elliptic operators. More precisely, we want local formulae for $Ch(\text{ind}_a(A))$, and, to this end, we shall use regularized traces and their properties developed in the previous section. If A is a family of Dirac operators and $\mathcal{G}$ is trivial, local formulae for $Ch(\text{ind}_a(A))$ were obtained using heat kernels by Bismut in a remarkable paper, [5]. Our results are a step towards a similar result for arbitrary families of pseudodifferential operators invariant with respect to a bundle of Lie groups.

Fix a group morphism $\chi : H^m_c(\mathfrak{g}^*) \to \mathbb{C}$. We want to obtain local formulae for $\chi(Ch(\text{ind}_a(A)))$. For simplicity, we shall assume in this section that Y is a flat $\mathcal{G}$-space, that is $Y = B \times Y_0 \times \mathbb{R}^q$ and $\mathcal{G} = B \times \mathbb{R}^q$, so $\mathfrak{g}^*$ also identifies with $B \times \mathbb{R}^q$. We shall also assume in this section that B and Y_0 are compact. The case of a general bundle is more technical and will be treated in a future paper.

Our approach is based on an interpretation of $\chi(Ch(\mathrm{ind}_a(A)))$ using the Fedosov (or $\star$) product. We begin by recalling the definition of the Fedosov product and by making some general remarks on traces and their pairing with the K-theory of the algebras we consider.

Let $\mathfrak{A} = \oplus_{k=0}^N \mathfrak{A}_k$ be a graded algebra endowed with a graded derivation $d : \mathfrak{A}_k \to \mathfrak{A}_{k+1}$, the *Fedosov product* is defined by

$$a \star b = ab + (-1)^{\deg a}(da)(db).$$

(The name is due to Cuntz and Quillen who have thoroughly studied the Fedosov product in connection to their approach to Non-commutative de Rham cohomology.) We shall denote by $Q\mathfrak{A}$ the algebra $\mathfrak{A}$ with the Fedosov (or $\star$) product and by $Q_{ev}\mathfrak{A} \subset Q\mathfrak{A}$ the subalgebra of even elements.

Since we shall work with non-unital algebras also, it is sometimes necessary to adjoin a unit "1" to $Q\mathfrak{A}$. The resulting algebra will be simply denoted by $Q^+\mathfrak{A} \simeq Q\mathfrak{A} \oplus \mathbb{C}$. Similarly, $Q^+_{ev}\mathfrak{A} := Q_{ev}\mathfrak{A}$.

A graded trace τ on $Q^+\mathfrak{A}$ restricts to an ordinary trace on $Q^+_{ev}\mathfrak{A}$, and hence it gives rise to a morphism

$$\tau_* : K_0^{\mathrm{alg}}(Q^+_{ev}\mathfrak{A}) \longrightarrow \mathbb{C}, \quad \tau_*[e] = \sum_j \tau(e_{jj}),$$

for any idempotent $e = [e_{ij}] \in M_N(Q^+_{ev}\mathfrak{A})$. If $\pi_* : K_0^{\mathrm{alg}}(Q\mathfrak{A}) \to K_0^{\mathrm{alg}}(\mathfrak{A}_0)$ is the natural morphism induced by $\pi : Q^+_{ev}\mathfrak{A} \to \mathfrak{A}_0 \oplus \mathbb{C}$, then π_* is an isomorphism, by standard algebra results. Consequently, the trace τ also gives rise to a morphism

$$\widetilde{\tau} := \tau_* \circ \pi_*^{-1} : K_0^{\mathrm{alg}}(\mathfrak{A}_0 \oplus \mathbb{C}) \longrightarrow \mathbb{C}, \tag{26}$$

see [10, 27].

The explicit form of the morphism $\widetilde{\tau}$ is not difficult to determine. Let $e \in \mathfrak{A}_0 \oplus \mathbb{C}$ be an idempotent, then

$$\bar{e} = \sum_{k\geq 0}^{N} (-1)^k \frac{(2k)!}{k!} e(de)^{2k} \tag{27}$$

is an idempotent in $Q^+_{ev}\mathfrak{A}$. Assume the trace τ is concentrated on $\mathfrak{A}_{2k}$, $k \in \mathbb{N}$. Then the explicit formula for $\widetilde{\tau}([e])$ is

$$\widetilde{\tau}([e]) = (-1)^k \frac{(2k)!}{k!} \tau((edede)^k), \tag{28}$$

(we used $e(de)^k = (edede)^k$, valid for all e satisfying $e^2 = e$).

Traces on $Q^+\mathfrak{A}$ are easy to obtain. Indeed, if τ is an even graded trace on $\mathfrak{A}$ satisfying $\tau(\mathfrak{A}_j) = 0$, if $j \neq p$, and $\tau(d\mathfrak{A}) = 0$, then

$$\tau(a \star b - (-1)^{ij} b \star a) = 0,$$

for any $a \in \mathfrak{A}_i$ and $b \in \mathfrak{A}_j$, and hence τ defines a graded trace on $Q\mathfrak{A}$. The trace τ defined above then extends to a trace on $Q^+_{ev}\mathfrak{A}$ by setting $\tau(1) = 0$.

We now define the algebras to which we shall apply the above considerations. Let $\Omega^*(B)$ be the space of smooth forms on B, and consider the differential graded algebra

$$\mathfrak{A} := \Psi^\infty_{\mathrm{inv}}(Y; \mathbb{C}^N) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \simeq M_N(\Psi^\infty_{\mathrm{inv}}(Y)) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q,$$

Let d_B the the de Rham differential in the B variables (here we use the assumption that Y is a flat $\mathcal{G}$-space). The differential of $\mathfrak{A}$ is then the usual de Rham differential d_{DR}:

$$d_{DR}(A) = \sum [t_i, A] d\tau_i + d_B(A), \quad \text{and} \quad d(A\omega) = d(A)\xi,$$

if $A \in \Psi^{\infty}_{\mathrm{inv}}(Y; \mathbb{C}^N) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \simeq M_N(\Psi^{\infty}_{\mathrm{inv}}(Y)) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B)$ and ξ is a product of some of the "constant" forms $d\tau_1, \dots, d\tau_p$. Thus the differential d_{DR} is with respect to the $B \times \mathbb{R}^q$ variables.

Let $\pi^*\Lambda^* T^* B$ be the pull back to Y of the exterior algebra of the cotangent bundle of B and $F = \pi^*\Lambda^* T^* B \otimes \Lambda^* \mathbb{R}^q$. Then

$$\mathfrak{A} \simeq \Psi^{\infty}_{\mathrm{inv}}(Y; F \otimes \mathbb{C}^N)$$

Inside $\mathfrak{A}$ we have the ideal of regularizing operators

$$\mathfrak{I} := \Psi^{-\infty}_{\mathrm{inv}}(Y; \mathbb{C}^N) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \simeq \Psi^{-\infty}_{\mathrm{inv}}(Y; F \otimes \mathbb{C}^N)$$

with quotient algebra

$$\mathfrak{B} := \mathfrak{A}/\mathfrak{I} = M_N(\Psi^{\infty}_{\mathrm{inv}}(Y)/\Psi^{-\infty}_{\mathrm{inv}}(Y)) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q \\ \simeq \Psi^{\infty}_{\mathrm{inv}}(Y; F \otimes \mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{inv}}(Y; F \otimes \mathbb{C}^N).$$

We consequently obtain the exact sequence of algebras

$$0 \to Q_{ev}\mathfrak{I} \to Q_{ev}\mathfrak{A} \to Q_{ev}\mathfrak{B} \to 0, \tag{29}$$

which gives rise to the boundary map ∂_Q,

$$\partial_Q : K_1^{alg}(Q_{ev}\mathfrak{B}) \to K_0^{alg}(Q_{ev}\mathfrak{I}_0) \subset K_0^{alg}(Q_{ev}^+\mathfrak{I}_0),$$

on algebraic K-theory.

We now define the traces we shall consider on the algebras $\mathfrak{A}, \mathfrak{B}, \dots, Q\mathfrak{I}$. Let $\omega : \Omega^k(B) \to \mathbb{C}$ be a closed current, that is, a continuous map such that $\omega(d\eta) = 0$ for any form η. Then ω defines a morphism $\chi_\omega : H^*_c(\mathfrak{g}^*) \to \mathbb{C}$. Also let

$$A \otimes \zeta \otimes \xi \in \mathfrak{A} := \Psi^{\infty}_{\mathrm{inv}}(Y; \mathbb{C}^N) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q,$$

and define

$$\tau_\omega(A \otimes \zeta \otimes \xi) = \frac{1}{(2k)!} \overline{\mathrm{Tr}}_q(A)\omega(\zeta) \int_{\mathbb{R}^q} \xi. \tag{30}$$

Then τ_ω is a trace on $\mathfrak{A}$ satisfying $\tau_\omega(d(a)) = 0$ if $a \in \mathfrak{I}$.

If $e \in M_N(\Psi^{-\infty}_{\mathrm{inv}}(Y) + \mathbb{C})$ is an idempotent, then e defines a class $x_e \in K^0(\mathfrak{g}^*)$. It is useful to review the definition of this class. First, we can replace e by an equivalent projection such that there exists another projection p satisfying $dp = 0$ and $pe = ep = p$. The projection p is required to consist of a smooth family of projections on $\mathcal{G}^*$, acting on the fibers of $\mathcal{G}^* \times_B Y \to \mathcal{G}^*$, with values in $M_N(\Psi^{-\infty}(Y_b) + \mathbb{C})$, and constant along the fibers of $\mathcal{G}^* \to B$. This projection p has the property that $p\mathfrak{I}_0 \subset \mathfrak{I}_0$. Then we define the vector bundle V_e on $\mathfrak{g}^*$ such that its fiber at τ is the range of the indicial operator $\hat{e}(\tau) - p$. Finally, the class x_e defined by e, which we are looking for, is $[V_e] - r[1]$, where r is the rank of V_e.

It is interesting to compare the Chern character of the bundle V_e to the pairing $\widetilde{\tau}_\omega[e]$. First, the vector bundle V_e is trivial at infinity. The curvature of the Grasmannian connection $\nabla^e = e \circ d$ is then $R^e := (e \circ d)^2 = edede$, by a standard computation. If $k+q=2p>0$ is even, then $\tilde{\tau}_\omega$ is an even, graded trace, and hence it pairs with $[e]$. Using the explicit formula from Equation (28), we obtain that

$$\tilde{\tau}_\omega[e] = \omega\big((-R^e)^k/k!\big) \tag{31}$$

recovers the pairing of $Ch(V_e)$, the Chern character of V_e, with the cohomology class of the form ω. In the notation introduced above, we have

$$\tilde{\tau}_\omega[e] = \chi_\omega(Ch(V_e)) = \chi_\omega(Ch(x_e)).$$

See [10]. It also follows that all morphisms $K_0^{alg}(\Psi_{\rm inv}^{-\infty}(Y)) \simeq K_0(\Psi_{\rm inv}^{-\infty}(Y)) \to \mathbb{C}$ are of the form $\tilde{\tau}_\omega$, for a suitable ω.

Recall now that we defined the analytic index ind_a to be the composite map

$$\mathrm{ind}_a : K_1^{alg}(\mathfrak{B}_0) \xrightarrow{\partial} K_0^{alg}(\mathfrak{I}_0) \simeq K^0(\mathfrak{g}^*),$$

where $\mathfrak{I}_0 = M_N(\Psi_{\rm inv}^{-\infty}(Y))$ and ∂ is the boundary map in algebraic K-theory associated to the exact sequence $0 \to \mathfrak{I}_0 \to \mathfrak{A}_0 \to \mathfrak{B}_0 \to 0$. As we explained at the beginning of the section, we are interested in understanding the morphism

$$\chi \circ Ch \circ \mathrm{ind}_a : K_1^{alg}(\Psi_{\rm inv}^{\infty}(Y;\mathbb{C}^N)/\Psi_{\rm inv}^{-\infty}(Y;\mathbb{C}^N)) = K_1^{alg}(\mathfrak{B}_0) \to \mathbb{C},$$

where $\chi : H_c^*(\mathfrak{g}^*) \to \mathbb{C}$ is an arbitrary group morphism. By the above discussion and linearity, we may assume that $\chi = \chi_\omega$, for some closed current $\omega : \Omega^k(B) \to \mathbb{C}$. Then a preliminary formula for the composition $\chi_\omega \circ Ch \circ \mathrm{ind}_a$ is given in the following lemma.

Lemma 10. *Let $\omega : \Omega^k(B) \to \mathbb{C}$ be a closed current such that $k+q=2p>0$ is even. Denote by $\chi_\omega : H_c^*(\mathfrak{g}^*) \to \mathbb{C}$ the morphism defined by ω. Then*

$$\chi_\omega \circ Ch \circ \mathrm{ind}_a = \widetilde{\tau}_\omega \circ \partial : K_1^{alg}(\mathfrak{B}_0) \to \mathbb{C}.$$

Proof. This follows by applying the above constructions to $\mathrm{ind}_a(A)$, A elliptic. ☐

We now turn to the computation of $\tilde{\tau}_\omega \circ \mathrm{ind}_a$. We shall use the generic notation π for all the quotient morphisms $Q\mathfrak{A} \to \mathfrak{A}_0$ and $Q\mathfrak{B} \to \mathfrak{B}_0$, and $Q\mathfrak{I} \to \mathfrak{I}_0$. Also, we shall denote by

$$[A,B]_\star = A \star B - (-1)^{ij} B \star A,$$

for $A \in \mathfrak{A}_i$, $B \in \mathfrak{A}_j$, the graded commutator in $Q\mathfrak{A}$ with respect to the $\star$-product.

Lemma 11. *Let $u \in \mathfrak{B}_0$ be an invertible element with inverse v. Choose liftings A and B of u and, respectively, v. Also, let τ be a closed graded trace on $\mathfrak{I}$ and $B' := \sum_{k=0}^{\infty}(-1)^k B(dAdB)^k$. Then*

$$\widetilde{\tau} \circ \partial[u] = \tau\big([A,B']_\star\big).$$

Proof. The map

$$\pi_* : K_1^{\rm alg}(Q\mathfrak{B}) \to K_1^{\rm alg}(\mathfrak{B}_0)$$

is onto because if $u \in \mathfrak{B}_0 = \Psi_{\rm inv}^{\infty}(Y;\mathbb{C}^N)/\Psi_{\rm inv}^{-\infty}(Y;\mathbb{C}^N)$ is invertible in $\mathfrak{B}_0$ with inverse v, then its image u' in $Q\mathfrak{B}$ is also invertible with inverse

$$v' := \sum_{k=0}^{\infty}(-1)^k v(dudv)^k.$$

(The sum is actually finite for our algebras.) The relations

$$u' \star v' = 1 = v' \star u'$$

are easily checked. From the naturality of the boundary map in algebraic K–theory, we obtain that

$$\pi_* \circ \partial_Q = \partial \circ \pi_* : K_1^{\rm alg}(Q\mathfrak{B}) \to K_0^{\rm alg}(\mathfrak{I}_0),$$

and hence

$$\tilde{\tau}_\omega \circ \partial[u] = \tau_* \circ \partial_Q[u']. \tag{32}$$

This simple relation will play an important role in what follows because it reduces the computation of $\tilde{\tau}_\omega \circ \partial$ to the computation of $\tau_* \circ \partial_Q$.

Lift $u \in M_N(\mathfrak{B}_0)$ to an element $A \in M_N(\mathfrak{A}_0) = M_N(\Psi^\infty_{\rm inv}(Y))$ and its inverse v to an element $B \in M_N(\Psi^\infty_{\rm inv}(Y))$, as in the statement of the lemma. This gives for v' (the inverse of the image u' of u in $Q\mathfrak{B}$ with respect to the $\star$ product) the explicit lift

$$B' := \sum_{k=0}^{\infty} (-1)^k B(dAdB)^k.$$

From a direct computation (see for example [26])

$$\tau_* \circ \partial_Q([u']) = \tau(2[A, B']_\star - [A, B'AB']_\star) = \tau(A \star B' - B' \star A).$$

From this the result of the lemma follows. □

Let $D \in \Psi^1_{\rm inv}(Y)$ be the operator used to define $\overline{\rm Tr}_q$ and ${\rm Tr}_{\rm R}$ in the previous section. Also, let $\iota : \Lambda^q\mathbb{R}^q \to \mathbb{C}$ be the isomorphism given by contraction with the (dual) of the top form on $\mathbb{R}^q$. This gives rise to maps

$$\overline{\rm Tr}_Y \otimes \iota, {\rm Tr}_{{\rm R}Y} \otimes \iota : Q\Psi^\infty_{\rm inv}(Y) \to \Omega^*(B), \tag{33}$$

which vanish on forms of degree less than q in $d\tau_1, \dots, d\tau_q$. Because $\overline{\rm Tr}_Y$ and ${\rm Tr}_{{\rm R}Y}$ are $\mathcal{C}^\infty(B)$-linear graded traces on $\mathfrak{A}_0$, $\overline{\rm Tr}_Y \otimes \iota, {\rm Tr}_{{\rm R}Y} \otimes \iota$ are $\Omega^*(B)$-linear (graded) traces. If $\omega : \Omega^*(B) \to \mathbb{C}$ is a closed current, we denote

$$\rho_\omega(A) = \langle \omega, {\rm Tr}_{{\rm R}Y} \otimes \iota(A) \rangle. \tag{34}$$

and note that ρ_ω is a closed graded trace. Also, note that $\tau_\omega(A) = \langle \omega, \overline{\rm Tr}_Y \otimes \iota(A) \rangle$, which we shall use to extend τ_ω to operators of non-integral orders, still preserving the tracial property. Moreover, $\tau_\omega(dA) = 0$, if A has non-integer order. Consequently, $\tau_\omega([A, B]_\star) = 0$ if ${\rm ord}\, A + {\rm ord}\, B$ is not an integer. This is seen by noticind that $\tau_\omega(d(D^{-z}A) = 0$ for ${\rm Re}(z)$ large first, and then for z such that $z\, {\rm ord}\, D + {\rm ord}\, A$ not an integer, by analytic continuation.

We then have

Lemma 12. *If $A(z) \in Q\mathfrak{A}$ is holomorphic in a neighborhood of $0 \in \mathbb{C} \setminus \mathbb{Z}^*$, then the function $\tau_\omega(D^{-z} \star A(z))$ holomorphic except possibly at 0, where it has a simple pole with residue $\rho_\omega(A(0))$.*

Proof. We have that

$$D^{-z} \star A(z) = D^{-z}A(z) + dD^{-z}dA(z).$$

Now we observe that

$$\lim_{z \to 0} z^{-1} dD^{-z} = \lim_{z \to 0} z^{-1} \sum [t_j, D^{-z}] d\tau_j = -\sum [t_j, \log D] d\tau_j = -d \log D$$

Consequently, $D^{-z} \star A(z) = D^{-z}B(z)$ for some holomorphic function B such that $B(0) = A(0)$. The result then is an immediate consequence of Proposition 4. □

Let $\partial : K_1^{alg}(\mathfrak{B}_0) = K_1^{\rm alg}(\Psi_{\rm inv}^{\infty}(Y)/\Psi_{\rm inv}^{-\infty}(Y))) \to K_0^{\rm alg}(\Psi_{\rm inv}^{-\infty}(Y)) = K_0^{alg}(\mathfrak{I}_0)$ be the boundary map in algebraic K-theory, as above. We continue to assume that $Y = B \times Y_0 \times \mathbb{R}^q$ is a flat $\mathcal{G} = B \times \mathbb{R}^q$-space and that B and Y_0 are compact.

Recall that the map $\tilde{\tau}_\omega : K_0^{\rm alg}(\mathfrak{I}_0) \to \mathbb{C}$ is given by the Equations (26) and (31) and that ρ_ω is given by the Equation (34).

Theorem 3. *Fix $u \in M_N(\Psi_{\rm inv}^{\infty}(Y)/\Psi_{\rm inv}^{-\infty}(Y))$. Assume u to be invertible, and choose $A, B \in M_N(\Psi_{\rm inv}^{\infty}(Y))$ such that A maps to u and B maps to u^{-1}. If $\omega : \Omega^k(B) \to \mathbb{C}$ is a closed current such the $k+q = 2p$ is even, then*

$$\begin{aligned}\tilde{\tau}_\omega \circ \partial[u] = -2(-1)^p \tau_\omega((dAdB)^p) = 2(-1)^p \tau_\omega \otimes \omega((dBdA)^p)\\ = -\rho_\omega\big(u^{-1}[\log D, u](u^{-1}du)^{2p}\big),\end{aligned}$$

Proof. We shall denote

$$\mathfrak{A} = M_N(\Psi_{\rm inv}^{\infty}(Y) \otimes_{\mathcal{C}^\infty(B)} \Omega^*(B) \otimes \Lambda^* \mathbb{R}^q),$$

as before. Moreover, $\mathfrak{B}$ and $\mathfrak{I}$ will have the meaning they had before.

We shall use Lemma 11. Let $B' = \sum_{k=0}^{\infty}(-1)^k B(dAdB)^k$ be as in that lemma and evaluate the commutator $[A, B']_\star$ (with respect to the $\star$ product). We obtain

$$\begin{aligned}[A, B']_\star = \sum_{l=0}^{\infty}(-1)^l AB(dAdB)^l - \sum_{l=0}^{\infty}(-1)^l B(dAdB)^l A\\ + \sum_{l=0}^{\infty}(-1)^l (dAdB)^{l+1} - \sum_{l=0}^{\infty}(-1)^l (dBdA)^{l+1},\end{aligned}$$

the sum being of course finite.

We next observe that $\tau_\omega(AB(dAdB)^l) = \tau_\omega(B(dAdB)^l A)$ and $\tau_\omega((dAdB)^l) = -\tau_\omega((dBdA)^l)$ because τ_ω is a graded trace on $\mathfrak{A}$ (with the usual product). Using this we obtain from Lemma 11 that

$$\tilde{\tau}_\omega \circ \partial[u] = \tau_\omega(A \star B' - B' \star A) = 2(-1)^{p-1}\tau_\omega((dAdB)^p). \tag{35}$$

This proves the first half of our formula.

The commutator $[A, B']\star = A \star B' - B' \star A$ maps to $u \star v - v \star u = 0$ in $Q\mathfrak{B}$, and hence $[A, B']_\star$ is in $Q\mathfrak{I}$, and hence $\tau_\omega([A, B]_\star) = \lim_{z\to 0} \tau_\omega(D^{-z} \star [A, B]_\star)$. Next we use that $\lim_{z\to 0} z^{-1} dD^{-z} = -d \log D$ and hence $d \log D$ is actualy in $\Psi_{\rm inv}^{-1}(Y)$, in spite of the fact that $\log D$ is not in the algebra $\Psi_{\rm inv}^{-1}(Y)$. Moreover, $z^{-1} dD^{-z}$ is holomorphic at 0.

Using that $\tau_\omega([A, D^{-z}B']_\star)$ for all z such that $-z + \operatorname{ord} A + \operatorname{ord} B$ not an integer, we finally obtain

$$\begin{aligned}\tau_\omega([A, B']) = \lim_{z\to 0} \tau_\omega(D^{-z} \star [A, B']_\star)\\ = \lim_{z\to 0} \tau_\omega([D^{-z}, A]_\star \star B') = \lim_{z\to 0} z\tau_\omega(D^{-z}F(z)),\end{aligned}$$

where $F(z) = z^{-1}[D^{-z}, D^z A]_\star \star B'$. Since F is a holomorphic function in a neighborhood of 0 with

$$F(0) = -[\log D, A]B' + d[logD, A]dB',$$

it further follows that

$$\lim_{z\to 0} z\tau_\omega(D^{-z}F(z)) = \rho_\omega(F(0)) = -\rho_\omega\big(u^{-1}[\log D, u](u^{-1}du)^{2p}\big)$$

Putting toghether the above formulae, we obtain the desired result. ∎

5. Higher eta invariants in algebraic K-theory

We consider the same setting as in the previous section. In particular, $\omega : \Omega^k(B) \to \mathbb{C}$ is a closed current and τ_ω is the associated trace on $\mathfrak{A}$. Then a direct computation gives that

$$\phi_\omega(a_0, a_1, \dots, a_l) = \tau_\omega(a_0 da_1 \dots da_l)/l!, \quad l = q + k$$

is a l-Hochschild cocycle on $\Psi^\infty_{\text{inv}}(Y)$. The Dennis trace map [15]

$$K_l^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \to \mathrm{HH}_l(\Psi^\infty_{\text{inv}}(Y))$$

and the morphism $\mathrm{HH}_l(\Psi^\infty_{\text{inv}}(Y)) \to \mathbb{C}$ defined by ϕ_ω give rise to a morphism

$$\eta_\omega : K_l^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \to \mathbb{C}. \tag{36}$$

Because the restriction of ϕ_ω to $\Psi^{-\infty}_{\text{inv}}(Y)$ is cyclic (so it defines a cyclic cocycle), the composition

$$K_l^{\text{alg}}(\Psi^{-\infty}_{\text{inv}}(Y)) \longrightarrow K_l^{\text{alg}}(\Psi^\infty_{\text{inv}}(Y)) \xrightarrow{\eta_\omega} \mathbb{C}$$

factors as

$$K_l^{\text{alg}}(\Psi^{-\infty}_{\text{inv}}(Y)) \longrightarrow K_l^{\text{top}}(\Psi^{-\infty}_{\text{inv}}(Y)) \xrightarrow{(\phi_\omega)_*} \mathbb{C},$$

where $(\phi_\omega)_*$ is the pairing of cyclic homology with topological K-theory. In particular, η_ω is non-zero if ω is not exact.

The morphism η_ω does not factor through topological K-Theory though, as a matter of fact

$$K_l^{\text{top}}(\Psi^{-\infty}_{\text{inv}}(Y)) \longrightarrow K_l^{\text{top}}(\Psi^\infty_{\text{inv}}(Y))$$

vanishes for any p, as proved in [20]. Moreover, for B reduced to a point, $\mathcal{G} = \mathbb{R}$, $k = 1$, $\omega(f dt) = \int_{\mathbb{R}} f(t) dt$, and

$$D = D_0 + \partial_t \in M_N(\Psi^\infty_{\text{inv}}(Y)),$$

the indicial map of an admissible (chiral) Dirac operator on $Y \times \mathbb{R}$, the main result of [19] states that $\eta_\omega(D) = \eta(D_0)/2$, where $\eta(D_0)$ is the "eta"–invariant introduced by Atiyah, Patodi, and Singer in [3]. This is not difficult to see as follows.

Consider a first order, elliptic, self-adjoing operator D_0 acting on sections of a vector bundle E on a smooth, compact manifold M. Let λ_n, $n \in \mathbb{Z}$ be the eigenvalues of D_0, counted with multiplicity. Assume that D_0 is invertible, for simplicity (so $\lambda_n \neq 0$, for all n). Then $\eta(D_0, s)$, the "eta-function of" D_0 is

$$\eta(D_0, s) = \sum \frac{\lambda_n}{|\lambda_n|} |\lambda_n|^{-s}$$

for $\operatorname{Re} s > n$. This function was considered in connection with elliptic operators by Atiyah, Patodi, and Singer in a cellebrated paper, [3].

Now let $Y = M \times \mathbb{R}$, $\mathcal{G} = \mathbb{R}$, so B is reduced to a point. Let t be the coordinate on $\mathbb{R}$ and consider $D = D_0 + \partial_t \in \Psi^1_{\text{inv}}(Y)$. Note that D is in $\Psi^\infty_{\text{inv}}(Y)$ because D_0 is a differential operator. If D_0 is the Dirac operator on M associated to some $Spin^c$–structure on M and M is odd dimensional, then D is the *Chiral* Dirac operator on the even dimensional manifold $M \times \mathbb{R}$. (The full Dirac operator has D as one of its corners.) Then $\hat{D}(\tau) = D_0 + i\tau$. Consider the functions

$$f(s) = \int_{\mathbb{R}} (1 + x^2)^{-s/2-1} dx, \quad \text{and}$$

$$g(s) = \frac{1}{2\pi i}\overline{\mathrm{Tr}}_1 \otimes \iota((D^*D)^{-s/2}D^{-1}dD) = \frac{1}{2\pi}\int_{\mathbb{R}} Tr((D_0^2+\tau^2)^{-s/2-1}(D_0 - i\tau))d\tau$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} Tr((D_0^2+\tau^2)^{-s/2-1}D_0)d\tau, \quad \mathrm{Re}\, s > \dim M.$$

Then f is defined and holomorphic for $\mathrm{Re}\, s > -1$ and g is defined and holomorphic for $\mathrm{Re}\, s > \dim M$. Moreover, we know that $g(s)$ has a meromorphic extension to $\mathbb{C}$, whose only possible poles are simple poles at integers. A simple computation then gives

$$g(s) = f(s)\eta(D_0, s), \tag{37}$$

on the common domain of definition. This shows that $\eta(s)$ has a meromorphic extension to $\mathrm{Re}\, s > -1$ with the same poles as g. (Actually, it follows that η has a meromorphic extension to $\mathbb{C}$, because f extens meromorphically to $\mathbb{C}$ also, but this is not important for us.)

Now we use that

$$g(s) = \mathrm{pv}\text{–}\!\!\int I^{2p} Tr\big(\partial_\tau^{2p}(D_0^2+\tau^2)^{-s/2-1}D_0\big)d\tau$$
$$= \mathrm{pv}\text{–}\!\!\int I^{2p} Tr\big((D_0^2+\tau^2)^{-s/2-1-p}P(\tau, D_0)\big)d\tau,$$

with P an *even* polynomial in τ, with coefficients differential operators on M (more precisely, constant coefficient polynomials in D_0). Now, as expected,

$$Tr\big((D_0^2+\tau^2)^{-s/2-1-p}P(\tau, D_0)\big)$$

has an expansion in a neighborhood of $\pm\infty$ (τ real) *with only even powers* of τ. This is a particular case of Theorem 2.7, page 503, of [13]. (Their theorem is much more general, we use it for $m = 2$, $A = D_0^2$, $\lambda = \tau^2$, Q one of the differential operators appearing as coefficients of P, and p large.) By integrating an even number of times, the property that the coefficients of odd, negative powers of $|\tau|$ vanish is preserved, and we get the following result.

Lemma 13. *Let $g(\tau) = I^{2p}Tr(\partial_\tau^{2p}(D_0^2+\tau^2)^{-s/2-1}D_0)$. Then*

$$g(\tau) \sim \sum_{k=-\infty}^{N} \alpha_k^{\pm}(s)|\tau|^k$$

as $\tau \to \pm\infty$ and $\alpha_k^{\pm} = 0$ for k odd, $k < 0$.

Note that the above arguments also give $\alpha^+ = \alpha^-$, but this makes no difference in the next applications.

From the above lemma, using Proposition 3 we obtain the following result which combines results from [3, 12, 19]. See also [7, 22, 32] for generalizations. We use the notation that $D' = [D, t]$, t being the parameter of the $\mathbb{R}$-action.

Theorem 4. *Let D_0 be a first order, self-adjoint, differential operator. Then the function $\eta(D_0, s)$ is regular at $s = 0$ and*

$$\eta(D_0, s) = \frac{1}{\pi}\overline{\mathrm{Tr}}_1(D^{-1}D'), \quad D = D_0 + \partial_t.$$

Proof. By Equation (37), the residue of $\pi\eta(D_0, s)$ at 0 is the same as that of the function

$$\overline{\mathrm{Tr}}_1\big((D^*D)^{-s/2}D^{-1}D'\big) = \overline{\mathrm{Tr}}_1\big((D^*D)^{-s/2-1}D_0\big).$$

Let $\alpha_k^{\pm}$ be as in Lemma 13 above, for some large $p \in \mathbb{N}$. The first part of Proposition 3 gives that the residue of $\overline{\mathrm{Tr}}_1\big((D^*D)^{-s/2}D_0\big)$ at $s = 0$ is $\alpha_{-1}^{+}(0) + \alpha_{-1}^{-}(0)$. By Lemma 13 above, we obtain that $\alpha_{-1}^{+}(0) + \alpha_{-1}^{-}(0) = 0$. This proves the regularity of the function $\eta(D_0, s)$ at $s = 0$.

The second relation of Proposition 3 gives

$$\overline{\mathrm{Tr}}_1(D^{-1}D') - \lim_{s\to 0}\big[\overline{\mathrm{Tr}}_1\big((D^*D)^{-s/2}D^{-1}D'\big) - Res_{s=0}\overline{\mathrm{Tr}}_1\big((D^*D)^{-s/2}D^{-1}D'\big)\big]$$
$$= -\partial_s\alpha_{-1}^{+}(0) - \partial_s\alpha_{-1}^{-}(0).$$

Since $Res_{s=0}\overline{\mathrm{Tr}}_1((D^*D)^{-s/2}D^{-1}D') = 0$ and $\partial_s\alpha_{-1}^{+}(0) + \partial_s\alpha_{-1}^{-}(0) = 0$, we obtain

$$\overline{\mathrm{Tr}}_1(D^{-1}D') = \lim_{s\to 0} g(s) = \pi\eta(D_0, 0),$$

which proves the second part of the theorem. □

Put differently, the second part of the above theorem states that the eta invariant of D_0 is the value at D of a group morphism $K_1(\Psi_{\mathrm{inv}}^{\infty}(M \times \mathbb{R})) \to \mathbb{C}$. This group morphism coincides with η_ω, $\omega(f) = \int_{\mathbb{R}} f$, in the above notation.

It is tempting to try to define a higher eta invariant on $\Psi_{\mathrm{inv}}^{q}(Y)$, $q = 2k - 1$, by the formula $\eta_k(D_0) = \overline{\mathrm{Tr}}_q((D^{-1}dD)^{2k-1})$, $D = D_0 + c(\tau)$; however, as it was proved by Lesch and Pflaum, this is not multiplicative, and besides, it coincides with usual eta invariant of D_0 (up to a multiple depending only on k).

6. Index theory on a simplex

In this section, we discuss an application of the computation of the degree in Section 4 to the question of formulating (*i.e.,* the existence of) Fredholm boundary conditions for b-pseudodifferential operators on the simplex

$$\Delta_n = \{(x_0, x_1, \dots, x_n), x_i \geq 0, \sum x_i = 1\}.$$

More generally, we shall consider the same question for the manifold with corners $\Delta_n \times Y$, where Y is a smooth, compact manifold without corners. We obtain complete results for $\dim Y > 0$. The answer to this existence question is "local," that is, it can be given in terms of the principal symbol. The problem of computing the index of the resulting Fredholm operators, if any, is a non-local' problem and will be addressed in a future paper.

In addition to the results of the previous sections, we shall also use computations from [20]. All the definitions not included in this section can be found in that paper.

We formalize the above question as the following natural problem:

Problem $(\mathcal{F}(T))$. *Suppose an elliptic operator $T \in \Psi_b^m(\Delta_n \times Y; E)$ is given. For what T we can find a perturbation $T + R$ of T, by a regularizing operator $R \in \Psi_b^{-\infty}(\Delta_n)$, such that $T + R : H^s(\Delta_n \times Y; E) \to H^{s-m}(\Delta_n \times Y; E)$ is Fredholm?*

We can reduce the general case to the case $m = 0$ by replacing T with $D^{-m}T$, where D is an elliptic strictly positive operator in $\Psi_b^1(\Delta_n \times Y; E)$. (The resulting operator $D^{-m}T$ is then in the norm completion of $\Psi_b^0(\Delta_n \times Y; \mathrm{End}(E))$, but this makes no difference to us.) For $m = 0$, we then obtain an operator acting on the Hilbert space $L^2(\Delta_n)$.

The answer to the above Problem is independent on the metric chosen on $\Delta_n \times Y$, although the choice of R may depend on the metric.

Let $\overline{\Psi}_b^0(\Delta_n \times Y)$ be the norm closure of $\Psi_b^0(\Delta_n \times Y)$. By the results of [20], the algebra $\overline{\Psi}_b^0(\Delta_n \times Y)$ contains the ideal of compact operators $\mathcal{K}$ on $L^2(\Delta_n \times Y)$. (See also [16].) Denote by Q be the algebra of "joint symbols" of $\Psi_b^0(\Delta_n \times Y)$:

$$Q := \overline{\Psi}_b^m(\Delta_n \times Y)/\mathcal{K},$$

The principal symbol morphism then extends to a surjective morphism $\sigma_0 : Q \to C({}^bS^*\Delta_n \times Y)$ with kernel J.

A first, tautological observation is that Problem $(\mathcal{F}(T))$ has a solution if, and only if, the invertible element $\sigma_0(T) \in C({}^bS^*\Delta_n \times Y; \mathrm{End}(E))$ has a lifting to an invertible element in Q (the lifting is $T+R$). Since there is a necessary condition for this to happen in terms of K-theory, and the K-groups involved do not depend on E, up to isomorphism, we obtain the following necessary condition for our problem to have a solution: Problem $(\mathcal{F}(T))$ has a solution only if the class $[\sigma_0(T)] \in K_1(C({}^bS^*\Delta_n \times Y))$ is in the image of the morphism $K_1(Q) \to K_1(C({}^bS^*\Delta_n \times Y))$.

Now the standard six term K-theory exact sequence associated to the short exact sequence of C^*-algebras

$$0 \to J \to Q \to C({}^bS^*\Delta_n \times Y) \to 0$$

tells us that the class $[\sigma_0(T)] \in K_1(C({}^bS^*\Delta_n \times Y))$ is in the image of the morphism $K_1(Q) \to K_1(C({}^bS^*\Delta_n \times Y))$ if, and only if, it is in the kernel of the boundary map $\partial : K_1(C({}^bS^*(\Delta_n \times Y))) \to K_0(J)$ associated to the above exact sequence of algebras.

Lemma 14. *If Problem ($\mathcal{F}(T)$ has a solution, then $\partial[\sigma_m(T)] = 0 \in K_0(J)$. This condition is also sufficient if* $\dim Y > 0$

Proof. Only the second part was not yet proved. Now, for $\dim Y > 0$, the ideal I is stable (*i.e.,* $J \otimes \mathcal{K} \simeq J$, where $\mathcal{K}$ is the algebra of compact operators on an separable, infinite Hilbert space). This is enough to prove that the existence of a stable lifting for an invertible element T in Q implies the existence of an invertible lifting of T. ☐

Denote by $I = \overline{\Psi}_b^{-1}(\Delta_n \times Y)$ the norm closure of $\Psi_b^{-1}(\Delta_n \times Y)$ (the closure is in the norm topology of bounded operators on $L^2(\Delta_n \times Y)$). Then $J = I/\mathcal{K}$. The results of [20] give a spectral sequence converging to the K–theory groups of I (and, with some obvious changes, a spectral sequence converging to the K–theory of J). The E^1 complex of this spectral sequence is independent of Y and is (dual to) the combinatorial simplicial complex of Δ_n, more precisely, it is a direct sum of complexes isomorphic to

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{\frac{n(n-1)}{2}} \longrightarrow \ldots \mathbb{Z}^n \longrightarrow \mathbb{Z} \longrightarrow 0.$$

The direct sum commes from the fact that K-theory is indexed by $\mathbb{Z}/2\mathbb{Z}$ and not by $\mathbb{Z}$. The resulting spectral sequence, which is indexed by $\mathbb{Z} \times \mathbb{Z}$, will be periodic of period 2 in each variable.

It follows that the spectral sequence converging to $K_n(I)$ degenerates at E^2, which shows that $K_n(I) \simeq \mathbb{Z}$ and $K_{n-1}(I) = 0$ (the same n as in Δ_n). To obtain the K–theory of J one proceeds similarly, the only difference being that one drops the last copy of $\mathbb{Z}$ in the above complex. Finaly, this gives $K_0(J) \simeq K_0(I)$ and $K_1(J) \simeq K_1(I) \oplus \mathbb{Z}$.

Denote by

$$\mathrm{In}_k : \Psi_b^\infty(\Delta_n \times Y) \to \Psi_{\mathrm{inv}}^\infty(Y),$$

$k = 0, \dots, n$, the indicial maps corresponding to the corners of $\Delta_n \times Y$. If B is reduced to a point and $\mathcal{G} = \mathbb{R}^n$, we shall denote $\deg_{\mathcal{G}} = \deg_n$. We are now ready to formulate and prove the main result of this section.

Theorem 5. *Let $T \in \Psi_b^m(\Delta_n \times Y)$, $m > 0$, be an elliptic operator. Assume $\dim Y > 0$. Then, for n odd, Problem $(\mathcal{F}(T))$ always has a solution. For n even, Problem $(\mathcal{F}(T))$ has a solution if, and only if, $\deg_n(\mathrm{In}_k(T)) = 0$ for some (equivalently, for all) k.*

Proof. Using the results of [20], we first observe that, in order for $T + R$ to be Fredholm, it is necessary that $\mathrm{In}_k(T + R)$ be invertible for all k, so

$$\deg_n(\mathrm{In}_k(T)) = \deg_n(\mathrm{In}_k(T + R)) = 0, \quad \text{for all } k.$$

This condition is automatically satisfied for n odd. The vanishing of all $\deg_n(\mathrm{In}_k(T))$ is hence necessary for Problem $(\mathcal{F}(T))$ to have a solution,

To prove that the vanishing of any of $\deg_n(\mathrm{In}_k(T))$ is enough for Problem $(\mathcal{F}(T))$ to have a solution, we shall show that all $\deg_n(\mathrm{In}_k(T))$ are equal and that $\partial[\sigma_0(T)] = 0$ if, and only if, $\deg_n(\mathrm{In}_k(T)) = 0$. The result will follow then from Lemma 14.

As discussed above it is enough to consider the case $m = 0$. Now

$$K_0(J) = E^2_{0n} = \ker(\mathbb{Z}^n \to \mathbb{Z}^{n(n-1)/2}),$$

so $K_0(J) = \{(p, p, \dots, p)\} \subset \mathbb{Z}^n$. Because

$$\partial[\sigma_0(T)] = (\deg_n(\mathrm{In}_0(T)), \deg_n(\mathrm{In}_1(T)), \dots, \deg_n(\mathrm{In}_n(T))) \in E^0_{0n},$$

we obtain that the degrees $\deg_n(\mathrm{In}_k(T)))$ are all equal and that the vanishing of $\partial[\sigma_0(T)]$ is equivalent to the vanishing of any of the degrees $\deg_n(\mathrm{In}_k(T)))$. As observed above, an application of Lemma 14 is now enough to complete the proof. □

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