# SINGULAR FROBENIUS OPERATORS ON SIEGEL MODULAR FORMS WITH CHARACTERS AND ZETA FUNCTIONS

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**Abstract**

The technique of singular Hecke operators for groups $\Gamma_0^2(q)$ is extended to singular Frobenius operators acting on Siegel modular forms with Dirichlet characters for the groups of prime levels $q$. The question of simultaneous diagonalization of the operators with regular Hecke operators is considered. An application is given to Euler factorization of radial Dirichlet series associated to eigenfunctions of the operators.[1]

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## Introduction

**0.1. Modular forms with characters.** Let $k$, $q$ be natural numbers and $\chi$ a Dirichlet character modulo $q$. A complex-valued function $F$ on *the (Siegel) upper half-plane of genus* $n$,

$$\mathbb{H}_n = \{Z = X + iY \in \mathbb{C}_n^n; \quad {}^tZ = Z, Y > 0\},$$

is called *a (Siegel) modular form of weight* $k$ *and character* $\chi$ *for the group*

$$\Gamma_0^n(q) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}_{2n}^{2n}; \quad {}^tMJ_nM = J_n, \, C \equiv 0 \pmod q \right\},$$

[1]**Key words**: Frobenius operators, Hecke operators, Hecke–Shimura rings, Siegel modular forms, zeta functions of modular forms.
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where $J_n = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ with the zero matrix $0 = 0_n$ and the unit matrix $E = E_n$ of order $n$, if it is holomorphic on $\mathbb{H}_n$, including all cusps of the group, if $n = 1$, and satisfies the functional equation

$$F\left((AZ+B)(CZ+D)^{-1}\right) = \chi(\det D)\det(CZ+D)^k F(Z) \tag{0-1}$$

for each matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of the group $\Gamma_0^n(q)$. We shall denote by $\mathfrak{M}_k^n(q, \chi)$ the $\mathbb{C}$-linear space of all such functions. Every function $F$ of the space has *Fourier expansion* of the form

$$F(Z) = \sum_{A \in \mathbb{E}_n,\, A \geq 0} f(A)\exp(\pi i Tr(AZ)), \tag{0-2}$$

where $\mathbb{E}_n$ is the set of all integral symmetric matrices of order $n$ with even entries on the principal diagonal (*even matrices*), with constant *Fourier coefficients* $f(A)$ satisfying relations

$$f({}^tUAU) = \chi(\det U)(\det U)^k f(A) \qquad (A \in \mathbb{E}_n,\, U \in GL_n(\mathbb{Z})).$$

Modular forms $F$ whose Fourier coefficients together with Fourier coefficients of the functions $\det(CZ+D)^{-k}F((AZ+B)(CZ+D)^{-1})$ for all matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of *the (Siegel) modular group of genus n*,

$$\Gamma^n = \Gamma_0^n(1),$$

are equal to zero on all singular matrices are called *cusp forms of weight k and character $\chi$ for the group* $\Gamma_0^n(q)$. The subspace of all cusp forms of $\mathfrak{M}_k^n(q, \chi)$ will be denoted by $\mathfrak{N}_k^n(q, \chi)$.

The spaces $\mathfrak{M}_k^n(q, \chi)$ (resp.,$\mathfrak{N}_k^n(q, \chi)$) for all Dirichlet characters modulo $q$ can be joined together into *the space*

$$\mathfrak{M}_k^n(q, q) = \sum_{\chi \bmod q} \mathfrak{M}_k^n(q, \chi) \tag{0-3}$$

(resp.,

$$\mathfrak{N}_k^n(q, q) = \sum_{\chi \bmod q} \mathfrak{N}_k^n(q, \chi)) \tag{0-4}$$

*of modular forms* (resp.,*cusp forms*) *of weight k for the group*

$$\Gamma^n(q, q) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^n(q); \quad \det A \equiv 1 \pmod{q} \right\}. \tag{0-5}$$

The space $\mathfrak{M}_k^n(q,\,q)$ can be characterized as the set of all holomorphic functions on $\mathbb{H}_n$ including cusps of $\Gamma^n(q,\,q)$, if $n=1$, which satisfy

$$\text{(0-6)}\quad F((AZ+B)(CZ+D)^{-1})=\det(CZ+D)^kF(Z)\quad (\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\Gamma^n(q,\,q)).$$

Fourier coefficients $f(A)$ of functions of $\mathfrak{M}_k^n(q,\,q)$ satisfy the relations

$$\text{(0-7)}\qquad f({}^tUAU)=f(A)\qquad (\forall\, A\in\mathbb{E}_n,\, U\in SL_n(\mathbb{Z})).$$

The space $\mathfrak{M}_k^n(q,\,q)$ is finite-dimensional over $\mathbb{C}$. The subspace $\mathfrak{N}_k^n(q,\,q)$ of cusp forms have structure of Hilbert space with respect to *Petersson scalar product* defined by invariant integration on a fundamental domain of $\Gamma^n(q,\,q)$ on $\mathbb{H}_n$. The decomposition (0-4) is orthogonal with respect to the scalar product.

For details on modular forms see [An(87)],Ch.2, or [An-Z(90)],Ch.2.

**0.2. Hecke–Shimura rings.** Fourier coefficients of modular forms have important multiplicative properties, which reflect multiplicative relations in certain rings, whose basic definitions we shall now recall.

Let $\Delta$ be a multiplicative semigroup and $G$ a subgroup of $\Delta$ such that every double coset $GMG$ of $\Delta$ modulo $G$ is a finite union of left cosets $GM'$. Let us consider the vector space over the field $\mathbb{Q}$ of rational numbers consisting of all formal finite linear combinations with coefficients in $\mathbb{Q}$ of symbols $(GM)$ with $M\in\Delta$ being in one-to-one correspondence with left cosets $GM$ of the set $\Delta$ modulo $G$. The group $G$ naturally acts on the space by right multiplication defined on the symbols $(GM)$ by

$$(GM)\gamma=(GM\gamma)\qquad (M\in\Delta,\,\gamma\in G).$$

We denote by

$$\mathcal{H}(G,\,\Delta)=HS_{\mathbb{Q}}(G,\,\Delta)$$

the subspace of all $G$-invariant elements. The multiplication of elements of $\mathcal{H}(G,\,\Delta)$ given by the formula

$$\Big(\sum_i a_i(GM_i)\Big)\Big(\sum_j b_j(GN_j)\Big)=\sum_{i,j}a_ib_j(GM_iN_j)$$

does not depend on the choice of representatives $M_i\in GM_i$ and $N_j\in GN_j$, and turns $\mathcal{H}(G,\,\Delta)$ into an associative algebra over $\mathbb{Q}$ with the unity element $(G1_G)$, called *the Hecke–Shimura ring (HS–ring) of* $\Delta$ *relative to* $G$ *(over* $\mathbb{Q}$*).* Elements

$$\text{(0-8)}\qquad (M)=(M)_G=\sum_{M_i\in G\backslash GMG}(GM_i)\qquad (M\in\Delta)$$

being in one-to-one correspondence with double cosets of $\Delta$ modulo $G$ belong to $\mathcal{H}(G,\,\Delta)$ and form a basis of the ring over $\mathbb{Q}$. For brevity, the symbols $(GM)$ and $(M)$ will be referred as *left* and *double classes (of* $\Delta$ *modulo* $G$*)*, respectively.

In the situation of Siegel modular forms of genus $n$ one needs mostly $HS$-rings of semigroups $\Delta$ contained in the semigroup

$$\Sigma^n = \{M \in \mathbb{Z}^{2n}_{2n};\quad {}^tMJ_nM = \mu(M)J_n,\, \mu(M) > 0\} \tag{0-9}$$

of all integral symplectic matrices of order $2n$ with positive *multipliers* $\mu(M)$ relative to subgroups of the modular group $\Gamma^n$.

Important part in the theory of symplectic $HS$-rings is played by the duality defined by means of the anti-automorphism of second degree

$$M \mapsto M^* = \mu(M)M^{-1}$$

of the semigroup $\Sigma^n$. If $\Sigma_1$ is a subsemigroup invariant with respect to the anti-automorphism, then the linear map of a $HS$-ring of the semigroup given on double classes by

$$(M) \mapsto (M)^* = (M^*) \qquad (M \in \Sigma_1) \tag{0-10}$$

is an anti-isomorphism of the second degree of the ring, called *the star map.* Elements $T$ and $T^*$ will be referred as *dual (with respect to the star map).*

Multiplicative properties of double classes $(M)_G$ for a congruence subgroup $G$ of level $q$ of $\Gamma^n$, such as $\Gamma^n_0(q)$ or $\Gamma^n(q,q)$, essentially depend upon the cases whether the multipliers $\mu(M)$ of the matrices $M$ are coprime with the level $q$, or divide a power of $q$ (we write then that $\mu(M)|q^\infty$). We shall call such matrices along with their left and double classes modulo $G$ *regular* (or *q-regular*) and *singular* (or *q-singular*), respectively and denote by

$$\Sigma^n_{(q)} = \{M \in \Sigma^n;\quad \gcd(\mu(M), q) = 1\} \text{ and } \Sigma^n_q = \{M \in \Sigma^n;\quad \mu(M)|q^\infty\} \tag{0-11}$$

the subsemigroups of $q$-regular and $q$-singular matrices of $\Sigma^n$. The semigroups can be used to build corresponding $HS$-rings of the group $G$, consisting, respectively, of linear combinations of regular and the singular double classes modulo $G$. The star map (0-10) transforms the subrings into themselves.

**0.3. Petersson, Hecke, and Frobenius operators.** Hecke–Shimura rings act on modular forms by means of linear representation given by Hecke operators. First of all, matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\Sigma = \Sigma^n$ act on functions $F$ on $\mathbb{H} = \mathbb{H}_n$ by *(normalized) Petersson operators of weight* $k$ given by

$$F \mapsto F|_kM = \mu(M)^{nk - \frac{n(n+1)}{2}} \det(CZ + D)^{-k} F(M\langle Z\rangle), \tag{0-12}$$

where $M\langle Z\rangle = (AZ + B)(CZ + D)^{-1}$. The operators map holomorphic functions into holomorphic functions and satisfy the relations

$$|_kMM_1 = |_kM|_kM_1 \quad (M, M_1 \in \Sigma). \tag{0-13}$$

Using Petersson operators, the functional equations (0-6) can be rewritten in the form

$$F|_k M = F \qquad (F \in \mathfrak{M}_k^n(q,\, q), \quad M \in \Gamma^n(q,\, q)). \tag{0-14}$$

As a rule, Petersson operators do not map modular forms to modular forms for the same group, but certain linear combinations of the operators coming from corresponding Hecke–Shimura rings do. For example, if $G = \Gamma^n(q,\, q)$,

$$F \in \mathfrak{M}(G) = \mathfrak{M}_k^n(q,\, q), \quad \text{and} \quad T = \sum_i a_i(GM_i) \in \mathcal{H}(G,\, \Sigma),$$

then it follows from (0-13), (0-14), and the definition of $HS$-rings that the function

$$F|T = F|_k T = \sum_i a_i F|_k M_i \tag{0-15}$$

does not depend on the choice of representatives $M_i \in GM_i$ and belongs to the space $\mathfrak{M}(G)$. The operators (0-15) are called *Hecke operators* (*of weight $k$ for the group $G$*). It follows from the definition of multiplication in $HS$-rings that the map $T \to |T$ is a linear representation of the ring $\mathcal{H}(G,\, \Sigma)$ on the space $\mathfrak{M}(G)$. The subspace of cusp forms $\mathfrak{N}(G) = \mathfrak{N}_k^n(q,\, q)$ is invariant with respect to all of the Hecke operators.

The special interest for us will be presented by linear combinations of Petersson operators of the form

$$F \mapsto F|\Pi(m) = F|_k\Pi(m) = \sum_{B \in \mathbb{S}_n/m\mathbb{S}_n} F|_k \begin{pmatrix} E & B \\ 0 & mE \end{pmatrix}, \tag{0-16}$$

where $E = E_n$ and $\mathbb{S}_n$ is the set of all integral symmetric matrices of order $n$, operating on Fourier series of the form (0-2). The reason is that the operators have very simple action on Fourier coefficients of the series:

(0-17)

$$\begin{aligned} F|\Pi(m) &= \sum_{A \in \mathbb{E}_n} f(A) m^{nk - \frac{n(n+1)}{2} - nk} \exp(\pi i Tr(A(Z+B)/m)) \\ &= \sum_{A \in \mathbb{E}_n} f(A) \exp(\pi i Tr(AZ/m)) m^{-\frac{n(n+1)}{2}} \sum_{B \in \mathbb{S}_n/m\mathbb{S}_n} \exp(\pi i Tr(AB/m)) \\ &= \sum_{A \in \mathbb{E}_n} f(mA) \exp(\pi i Tr(AZ)). \end{aligned}$$

The operators (0-16) will be called *Frobenius operators*. We shall see in §3 that the Frobenius operators with numbers $m$ dividing some powers of the level $q$ are actually Hecke operators of weight $k$ for the groups $\Gamma_0^n(q)$ and $\Gamma^n(q,\, q)$, and so map the corresponding spaces of modular forms into themselves.

**0.4. Eigenfunctions of Hecke operators and zeta functions.** It is well known that the spaces $\mathfrak{N}_k^n(q,\, q)$ are spanned by common eigenfunctions of sufficiently big rings of regular Hecke operators. The questions of simultaneous diagonalization of singular Hecke operators for the groups $\Gamma_0^2(q)$ were recently considered in [An(98)]. Here in §4 we show that singular Frobenius operators for the groups $\Gamma^2(q,\, q)$ of prime levels $q$ can be simultaneously diagonalized together with regular Hecke operators on certain subspaces of the space $\mathfrak{N}_k^2(q,\, q)$. The restriction to prime levels considerably simplifies consideration, although scarcely is crucial.

Explicit relations between Fourier coefficients of eigenfunctions of Hecke operators and corresponding eigenvalues reveal certain multiplicative properties of the coefficients. For example, the Fourier coefficients $f(A)$ of an eigenfunction of the Frobenius operator $|\Pi(m)$ with the eigenvalue $\lambda(m)$ satisfy, by (0-17), the relations

$$f(mA) = \lambda(m) f(A) \qquad (\forall \quad A \in \mathbb{E}_n). \tag{0-18}$$

In general, the relations between Fourier coefficients and eigenvalues are naturally formulated in the form of identities among Dirichlet series constructed by Fourier coefficients of eigenfunctions, from one side, and Euler products (zeta-functions) formed by the corresponding eigenvalues, from another. In §5 we shall deduce relations between Dirichlet series of the form

$$R_F(s,\, A) = \sum_{m=1}^{\infty} \frac{f(mA)}{m^s} \qquad (A \in \mathbb{E}_2,\, A > 0)$$

for eigenfunctions $F \in \mathfrak{M}_k^2(q,\, q)$ with Fourier coefficients $f(A)$ and spinor Euler products. The relations open a way to approach analytic and functional properties of the Euler products, but it lies beyond the scope of this work.

The main result of the paper could be interpreted as a theorem of semisimplicity of certain properly defined Hecke–Shimura algebras (compare with [Hi(93)], p.218). We hope also to prove a $\Lambda$-adic version of the result which is important for better understanding of $\Lambda$-adic spinor $L$-functions for $GSp_4$ (see [Pa-PSh(98)]). Note also that a definition of local factors and analytic properties of the $L$-functions were studied from a representation-theoretic point of view in [PSh(98)].

**0.5. Why with characters ?** What are the reasons of considering modular forms for the groups $\Gamma_0^n(q)$ with Dirichlet characters, whereas the case of the unit character has been already treated? The first reason is a trivial one: modular forms with characters appear quite naturally in many of arithmetical questions, for example, as theta series of integral quadratic forms. The second reason is rather a mystical one and based on the belief that natural mathematical questions must have nice answers, at least in some cases. Studies of zeta functions of modular forms for congruence subgroups indicate that individual functional equations can only be expected for zeta functions of modular forms with certain Dirichlet characters (see [Li(75)]), whereas, in general, zeta functions satisfy only matrix functional equations. The future research will show whether the belief is justified.

**Sources.** All omitted details and proofs on Siegel modular forms, regular Hecke–Shimura rings, and Hecke operators can be found in [An(87)] or [An-Z(90)]; the singular rings and operators for the groups $\Gamma_0^n(q)$ were considered in [An(98)].

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**Notation.** We fix the letters $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ for the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively.

$\mathbb{A}_n^m$ is the set of all $m \times n$-matrices with entries in the set $\mathbb{A}$. We use the notation $\mathbb{E}_n$ and $\mathbb{S}_n$ introduced in (0-2) and (0-16) for the sets of all even matrices and integral symmetric matrices of order $n$, respectively.

If $M$ is a matrix, ${}^tM$ always denotes the transposed matrix. $E_n$ is the unit matrix of order $n$.

## §1. Neutral rings and operators

If $G \subset K$ are two congruence subgroups of level $q$ of the modular group, then *the neutral HS–ring*

$$\mathcal{N}(G\backslash K) = \mathcal{H}(G,\, K)$$

is contained both in regular and singular $HS$-rings of $G$, since $\Sigma_{(q)} \cap \Sigma_q = \Gamma$.

We shall start with the groups

$$G = \Gamma^n(q,\, q), \quad \text{and} \quad K = \Gamma_0^n(q). \tag{1-1}$$

The group $G$ is clearly a normal subgroup of $K$, and the map

$$K \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \det A \bmod q$$

defines an isomorphism of the factor group $G\backslash K$ with the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$ of invertible elements of the residue class ring modulo $q$. For an integer $r$ coprime with $q$, we denote by

$$P(r) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K, \quad \det A \equiv r \pmod{q} \tag{1-2}$$

a representative of the inverse image of $r \bmod q$ under the map. All such matrices with a given $r \bmod q$ form the single left and double coset $GP(r) = GP(r)G$ modulo the group $G$. The following proposition is an easy consequence of definitions.

**Proposition 1.** (1) *The double classes modulo the group $G$ of the matrices $P(r)$,*

$$\rho(r) = (P(r))_G = (GP(r)), \tag{1-3}$$

*depend only on $r \bmod q$ and span the ring $\mathcal{N}(G\backslash K)$ over $\mathbb{Q}$;*

(2) *The classes* (1-3) *satisfy the relations*

$$\rho(r)\rho(r') = \rho(rr'), \quad \rho(r)^* = \rho(r^{-1}) \qquad (r, r' \in (\mathbb{Z}/q\mathbb{Z})^*),$$

*where the first star stands for the map* (0-10)*;*

(3) *The subspaces $\mathfrak{M}_k^n(q, \chi) \subset \mathfrak{M}_k^n(q, q)$ and $\mathfrak{N}_k^n(q, \chi) \subset \mathfrak{N}_k^n(q, q)$ of modular forms and cusp forms of weight $k$ and character $\chi$ for the group $K$ can be characterized as subsets of all functions $F$ of the corresponding spaces satisfying the relations*

$$F|_k\rho(r) = \chi(r)^{-1}F \qquad (r \in (\mathbb{Z}/q\mathbb{Z})^*).$$

In [An(98)], §3.2 the neutral ring $\mathcal{N}(\Gamma_0^n(q)\backslash\Gamma_0^n(q/p))$ was studied for each prime divisor $p$ of $q$ not dividing the quotient $q/p$. Here we shall only quote a simplified version of the result in the particular case, when $n = 2$ and $q = p$.

**Proposition 2.** *Let $q$ be a prime number, and let $n = 2$. Then the following assertions hold:*

(1) *The neutral ring $\mathcal{N}(K\backslash\Gamma)$ of the group $\Gamma = \Gamma^2$ and subgroup $K = \Gamma_0^2(q)$ consists of linear combinations of the three different double classes*

$$\xi_0 = (E)_K, \quad \xi_1 = (I)_K, \quad \text{and} \quad \xi_2 = (J)_K \tag{1-4}$$

*of the matrices*

$$E = E_4, \quad I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad J = J_2 = \begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix};$$

(2) *The double classes have the following decompositions into left classes modulo $K$:*

$$\xi_0 = (KE), \quad \xi_1 = \sum_{U \in \mathbf{U}_1} (KIU), \quad \xi_2 = \sum_{U \in \mathbf{U}_2} (KJU), \tag{1-5}$$

*where $\mathbf{U}_1 = \mathbf{U}_{11} \cup \mathbf{U}_{12}$ with*

$$\mathbf{U}_{11} = \left\{ \begin{pmatrix} 1 & v & w & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -v & 1 \end{pmatrix}; v, w \bmod q \right\},$$

$$\mathbf{U}_{12} = \left\{ \begin{pmatrix} 0 & -1 & 0 & w \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; w \bmod q \right\},$$

*and where*

$$\mathbf{U}_2 = \left\{ \begin{pmatrix} E_2 & B \\ 0 & E_2 \end{pmatrix}; \quad B \in \mathbb{S}_2/q\mathbb{S}_2 \right\};$$

(3) *The double classes are invariant under the star map; in particular, the ring* $\mathcal{N}(K\backslash\Gamma)$ *is commutative;*

(4) *The following formula is valid:*

$$\xi_2^2 = q^3\xi_0 + q^2(q-1)\xi_1 + q^2(q-1)\xi_2.$$

For further applications we shall need analogous facts for the group $\Gamma^2(q, q)$ in place of $\Gamma_0^2(q)$, which turn out to be rather more complicated even for primes $q$.

**Proposition 3.** *Let $q$ be an odd prime number, and let $n = 2$. Then the following assertions hold:*

(1) *The neutral ring $\mathcal{N}(G\backslash\Gamma)$ of the group $\Gamma = \Gamma^2$ and subgroup $G = \Gamma^2(q, q)$ consists of linear combinations of $2q$ different double classes*

$$\text{(1-6)} \qquad \begin{cases} \rho(r) = (P(r))_G \quad (r \in (\mathbb{Z}/q\mathbb{Z})^*), \\ \xi_1^+ = (I)_G, \\ \xi_1^- = (P(d)I)_G, \\ \rho(r)\widetilde{\xi}_2 = (P(r)J)_G \quad (r \in (\mathbb{Z}/q\mathbb{Z})^*) \end{cases}$$

*with $\widetilde{\xi}_2 = (J)_G$, where $P(r)$ are matrices* (1-2), *matrices $I$ and $J$ were defined in* (1-4), *and where $d$ is a quadratic non-residue modulo $q$;*

(2) *The double classes* (1-6) *have the following decompositions into left classes modulo $G$:*

$$\text{(1-7)} \qquad \begin{cases} \rho(r) = (GP(r)) \quad (r \in (\mathbb{Z}/q\mathbb{Z})^*), \\ \xi_1^+ = \sum_{t^2 \in (\mathbb{Z}/q\mathbb{Z})^*, U \in \mathbf{U}_1} (GP(t^2)U), \\ \xi_1^- = \sum_{t^2 \in (\mathbb{Z}/q\mathbb{Z})^*, U \in \mathbf{U}_1} (GP(dt^2)U), \\ \rho(r)\widetilde{\xi}_2 = \sum_{U \in \mathbf{U}_2} (GP(r)JU) \quad (r \in (\mathbb{Z}/q\mathbb{Z})^*), \end{cases}$$

*where the sets $\mathbf{U}_1$ and $\mathbf{U}_2$ were defined in* (1-5)*;*

(3) *The class* $\widetilde{\xi}_2$ *satisfies the following relations*

$$(\rho(r)\widetilde{\xi}_2)^* = \widetilde{\xi}_2\rho(r^{-1}) = \rho(r)\widetilde{\xi}_2 \qquad (r \in (\mathbb{Z}/q\mathbb{Z})^*), \tag{1-8}$$

*where the star stands for the map* (0-10), *and the relation*

$$(\widetilde{\xi}_2)^2 = q^3\rho(1) + q^2(\xi_1^+ + \xi_1^-) + \sum_{r=1}^{q-1}(q^2 + \left(\frac{-r}{q}\right) q)\rho(r)\widetilde{\xi}_2, \tag{1-9}$$

*where* $\left(\frac{-r}{q}\right)$ *is the Legendre symbol;*

(4) *The restriction of the Hecke operator of weight* $k$ *corresponding to the class* $\widetilde{\xi}_2$ *on the subspace* $\mathfrak{M}_k^2(q,\chi_1) \subset \mathfrak{M}_k^2(q,q)$, *where* $\chi_1$ *is the trivial character, coincides with the operator corresponding to* $\xi_2$.

*Proof.* First of all, we shall choose a convenient system of representatives of the left cosets $G\backslash\Gamma$. Since one can, clearly, take

$$G\backslash\Gamma = \{G\backslash K\} \times \{K\backslash\Gamma\},$$

where $K = \Gamma_0^2(q)$, and according to Propositions 1 and 2, respectively, one can take

$$G\backslash K = \{P(1),\dots,P(q-1)\} \quad \text{and} \quad K\backslash\Gamma = \{E \cup I\mathbf{U}_1 \cup J\mathbf{U}_2\},$$

it follows that one can take

$$\begin{aligned} K\backslash\Gamma = \{P(1),\dots,P(q-1)\} &\bigcup \{P(1)I\mathbf{U}_1 \cup \dots \cup P(q-1)I\mathbf{U}_1\} \\ &\bigcup \{P(1)J\mathbf{U}_2 \cup \dots \cup P(q-1)J\mathbf{U}_2\}. \end{aligned} \tag{1-10}$$

In order to proceed we shall prove the following three lemmas.

**Lemma 4.** *Two matrices of the set* (1-10) *can belong to the same double coset modulo* $G$ *only when they both belong to the same of the three subsets in curly brackets.*

*Proof.* All matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of a fixed double coset modulo the group $G$ (or the group $K$) have, clearly, the same value of the rank $r_q(C)$ of the block $C$ over the field of $q$ elements. Since this value for the matrices of the sets in curly brackets is, respectively, 0, 1, and 2, the lemma follows.

□

**Lemma 5.** *Two matrices $P(r)IU$ and and $P(r')IU'$ with $1 \le r, r' \le q-1$, and $U, U' \in \mathbf{U}_1$ belong to the same double coset modulo $G$, if and only if the Legendre symbols of $r$ and $r'$ modulo $q$ equal:*

$$\left(\frac{r}{q}\right) = \left(\frac{r'}{q}\right). \tag{1-11}$$

*Proof.* If the matrices $P(r)IU$ and $P(r')IU'$ belong to the same double coset modulo $G$, then it is also true for the matrices $P(r)I$ and $P(r')I$, since the set $\mathbf{U}_1$ is contained in $G$. It means that there are matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ of $G$ such that

$$P(r)I\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} P(r')I,$$

whence

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = P(r)I\begin{pmatrix} A & B \\ C & D \end{pmatrix} I^{-1}P(r')^{-1}.$$

Let us set $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, use similar notation for the matrices $B$, $C$, and $D$, set $P(r) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, and $P(r') = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$. Then we can rewrite the last relation in the form

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d_1 & -c_2 & -c_1 & d_2 \\ -b_3 & a_4 & a_3 & b_4 \\ -b_1 & a_2 & a_1 & b_2 \\ -d_3 & c_4 & c_3 & d_4 \end{pmatrix} \begin{pmatrix} {}^t\delta' & -{}^t\beta' \\ -{}^t\gamma' & {}^t\alpha' \end{pmatrix}.$$

Since $C' \equiv \gamma \equiv \gamma' \equiv 0 \pmod q$, it follows that $a_2 \equiv b_1 \equiv d_3 \equiv 0 \pmod q$. Since $\det A' \equiv 1 \pmod q$, we conclude that

$$\det\left(\alpha \begin{pmatrix} d_1 & -c_2 \\ b_3 & a_4 \end{pmatrix} {}^t\delta'\right) \equiv (r/r')d_1a_4 \equiv 1 \pmod q. \tag{1-12}$$

Further, by the congruence $A^tD \equiv E_2 \pmod q$, we get the congruence $a_1d_1 \equiv 1 \pmod q$, and by $\det A \equiv 1 \pmod q$, the congruence $a_1a_4 \equiv 1 \pmod q$. With the congruences we obtain from (1-12) the congruence

$$r'/r \equiv d_1a_4 \equiv a_4/a_1 \equiv a_4^2/a_1a_4 \equiv a_4^2 \pmod q,$$

which proves necessity of the condition (1-11). Conversely, if the condition (1-11) is satisfied, then $r' \equiv rt^2 \pmod q$, and we can take a matrix $P(r')$ in the form

$$P(r') = P(rt^2) = P(r)P(t^2) \quad \text{with} \quad P(t^2) \equiv diag(t, t, t^{-1}, t^{-1}) \pmod q.$$

Then it follows that

$$GP(rt^2)I = GP(r)II^{-1}P(t^2)I \in GP(r)IG,$$

since

$$I^{-1}P(t^2)I \equiv diag(t^{-1}, t, t, t^{-1}) \pmod q.$$

□

**Lemma 6.** *Two matrices $P(r)JU$ and $P(r')JU'$ with $1 \le r, r' \le q-1$ and $U$, $U' \in \mathbf{U}_2$ belong to the same double coset modulo $G$, if and only if $r = r'$.*

*Proof.* Since the set $\mathbf{U}_2$ is contained in $G$, it is sufficient to consider the matrices $P(r)J$ and $P(r')J$. Similarly to the proof of previous lemma, if the matrices belong to the same double coset modulo $G$, then there are matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ in $G$ such that

$$P(r)J\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}P(r')J,$$

whence

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = P(r)J\begin{pmatrix} A & B \\ C & D \end{pmatrix}J^{-1}P(r')^{-1} = P(r)\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}P(r')^{-1}.$$

If $P(r) = \begin{pmatrix} \alpha & * \\ * & * \end{pmatrix}$ and $P(r') = \begin{pmatrix} * & * \\ * & \delta' \end{pmatrix}$, then it follows that

$$1 \equiv \det A' \equiv \det(\alpha D^t \delta') \equiv \det\alpha \det\delta' \equiv r/r' \pmod q.$$

□

We can now continue the proof of Proposition 3. By using the system of representatives (1-10), one can easily see that the assertions (1) and (2) of the proposition directly follow from Propositions 1 and 2 and Lemmas 4, 5, and 6. The part (4) follows from decompositions (1-7) and (1-5).

Let us turn now to the part (3). Since $J^{-1} = -J$, and the coset $GP(r)^{-1}$ contains a representative of the form $P(r^{-1}) = JP(r)J^{-1}$, we get

$$(\rho(r)\widetilde{\xi}_2)^* = (P(r)J)^*_G = (J^{-1}P(r)^{-1})_G = (P(r)J^{-1})_G,$$

which proves the formulas (1-8).

By (1-7) and definition of multiplication in $HS$-rings, we can write

$$(\widetilde{\xi}_2)^2 = \sum_{U,U' \in bU_2} (GJUJU') = \sum_{H \in G\backslash\Gamma/G} c(H)(H)_G,$$

where $H$ runs a system of representatives of double cosets of $\Gamma$ modulo $G$, and the coefficients $c(H)$ can be found by the formula

$$\begin{aligned} c(H) &= \nu(J)\nu(H)^{-1}\#\{U \in \mathbf{U}_2; \quad JUJ \in GHG\} \\ &= q^3\nu(H)^{-1}\#\left\{B \in \mathbb{S}_2/q\mathbb{S}_2; \quad \begin{pmatrix} -E_2 & 0 \\ B & -E_2 \end{pmatrix} \in GHG\right\}, \end{aligned} \tag{1-13}$$

where $\nu(J) = q^3$ and $\nu(H)$ are the numbers of left cosets modulo $G$ contained in the double cosets $GJG$ and $GHG$, respectively (see, for example, [An(87)],Lemma 3.1.5). According to the first part of the proposition, one can take $H = P(r)$, $I$, $P(d)I$, or $P(r)J$, with $r = 1, \dots, q-1$ and a quadratic non-residue $d$ modulo $q$. If $H = P(r)$, the formula (1-13) gives

$$c(P(r)) = q^3\#\left\{B \in \mathbb{S}_2/q\mathbb{S}_2; \quad \begin{pmatrix} -E_2 & 0 \\ B & -E_2 \end{pmatrix} \in GP(r)G = GP(r)\right\}$$

which is $q^3$, if $r = 1$, and 0, if $r \neq 1$, since the matrix $B$ must satisfy the congruence $B \equiv 0 \pmod{q}$. If $H = I$ or $H = P(d)I$, then the corresponding matrices $B$ in (1-13) must have rank 1 over the field of $q$ elements, i.e. be of the form

$$B = \begin{pmatrix} c & cb \\ cb & cb^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \tag{1-14}$$

with, say, $c = 1, \dots, q-1$ and $b = 0, 1, \dots, q-1$. Using the decompositions (1-7) for the classes $\xi_1^{\pm}$, it is not hard to check that the number of matrices $B$ of the form (1-14) with a given $c$ satisfying

$$\begin{pmatrix} -E_2 & 0 \\ B & -E_2 \end{pmatrix} \in GP(r)I\mathbf{U}_1$$

is $q+1$, if $r \equiv c^{-1} \pmod{q}$, and is zero, otherwise. Summing up over quadratic residues (resp., non-residues) modulo $q$, we conclude that the number of matrices $B$ in formula (1-13) for $c(I)$ (resp., $c(P(d)I)$) is $(q+1)(q-1)/2$. Since the number of left classes in the double classes $\xi_1^+$ and $\xi_1^-$ is clearly $q(q+1)(q-1)/2$, it follows that corresponding coefficients are equal to $q^2$.

Finally, if $H = P(r)J$, then the corresponding matrices $B$ in (1-13) must have rank 2 over the field of $q$ elements. It is easy to see that the number of matrices $B \in \mathbb{S}_2/q\mathbb{S}_2$ with a fixed modulo $q$ the value of $\det B \equiv b \not\equiv 0 \pmod{q}$ and satisfying

$$\begin{pmatrix} -E_2 & 0 \\ B & -E_2 \end{pmatrix} \in GP(r)J\mathbf{U}_2$$

is

$$\#\{B \in \mathbb{S}_2/q\mathbb{S}_2; \quad \det B \equiv b \pmod{q}\}, \tag{1-15}$$

if $r \equiv b^{-1} \pmod q$, and is zero, otherwise. It is not difficult to verify that the number (1-15), i.e. the number of solutions of the congruence $xy - z^2 \equiv b \pmod q$ depends only on whether $-b$ is a quadratic residue or non-residue module $q$ and is equal to $q^2 + \left(\frac{-b}{q}\right) q$. Then, by (1-13), we get

$$c(P(r)J) = q^2 + \left(\frac{-r^{-1}}{q}\right) q = q^2 + \left(\frac{-r}{q}\right) q.$$

□

## §2. Regular rings and operators

We shall define *the regular HS–rings of the groups* $G$ and $K$ as Hecke–Shimura rings

$$\text{(2-1)} \qquad \mathcal{H}_r(G) = \mathcal{H}(G,\, R(G)) \quad \text{and} \quad \mathcal{H}_r(K) = \mathcal{H}(K,\, R(K))$$

of the semigroups

$$R(G) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma^n_{(q)}; \quad C \equiv 0,\, \det A \equiv 1 \pmod q \right\}$$

and

$$R(K) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Sigma^n_{(q)}; \quad C \equiv 0 \pmod q \right\}$$

and the corresponding groups (see (0-11)). The first ring acts on the spaces

$$\text{(2-2)} \qquad \mathfrak{M} = \mathfrak{M}^n_k(q,\, q) \quad \text{and} \quad \mathfrak{N} = \mathfrak{N}^n_k(q,\, q)$$

by the Hecke operators (0-15), which will be referred as *regular.*

**Lemma 7.** *Each element of the ring* $\mathcal{H}_r(G)$ *commutes with each element* $\rho(r)$ *of the form* (1-3)*; in particular, the regular Hecke operators map into itself each of the subspaces*

$$\text{(2-3)} \qquad \mathfrak{M}(\chi) = \mathfrak{M}^n_k(q,\, \chi) \subset \mathfrak{M} \quad \textit{and} \quad \mathfrak{N}(\chi) = \mathfrak{N}^n_k(q,\, \chi) \subset \mathfrak{N}.$$

*Proof.* It follows from [An(87)], Theorem 3.3.3(3) that

$$(P(r)^{-1}MP(r))_G = (M)_G, \quad \text{if} \quad M \in R(G) \quad \text{and} \quad r \in (\mathbb{Z}/q\mathbb{Z})^*,$$

whence

$$(M)_G\rho(r) = (P(r)P(r)^{-1}MP(r))_G = \rho(r)(P(r)^{-1}MP(r))_G = \rho(r)(M)_G;$$

□

The action of the ring $\mathcal{H}_r(G)$ on each of the subspaces $\mathfrak{M}(\chi)$ or $\mathfrak{N}(\chi)$ can be interpreted as the action of the regular ring $\mathcal{H}_r(K)$ of the group $K$ given by *Hecke operators of weight $k$ and character $\chi$*:

$$\text{(2-4)} \qquad F|_{k,\chi}T = \sum_i a_i F|_{k,\chi}M_i \qquad (F \in \mathfrak{M}(\chi);\ T = \sum_i a_i(KM_i) \in \mathcal{H}_r(K)),$$

where

$$F \mapsto F|_{k,\chi}M = \chi(\det A)F|_k M \qquad (M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R(K))$$

is *the Petersson operator of weight $k$ and character $\chi$*, and $|_k M$ is the operator (0-12). More generally, we have the following theorem.

**Theorem 8.** *In the above notation, the following assertion hold:*

(1) *If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in R(K)$, then $P(M)^{-1}M \in R(G)$, where $P(M) = P(\det A)$, and $P(r)$ are the matrices* (1-2)*; a decomposition*

$$(M)_K = \sum_i (KM_i)$$

*implies the decomposition*

$$(P(M)^{-1}M)_G = \sum_i (GP(M_i)^{-1}M_i)$$

*and vice versa;*

(2) *The linear mappings*

$$\text{(2-5)} \qquad \mathbf{i}: \quad \mathcal{H}_r(K) \to \mathcal{H}_r(G), \qquad \mathbf{j}: \quad \mathcal{H}_r(G) \to \mathcal{H}_r(K)$$

*defined on double classes $(M)_K$ and $(M')_G$ of matrices $M \in R(K)$ and $M' \in R(G)$ by*

$$\mathbf{i}(M)_K = (P(M)^{-1}M)_G, \quad \mathbf{j}(M')_G = (M')_K$$

*are mutually inverse isomorphisms of the $\mathbb{Q}$-algebras;*

(3) *The maps* (2-5) *are compatible with the action of corresponding Hecke operators on each of the spaces $\mathfrak{M}(\chi)$, where the ring $\mathcal{H}_r(K)$ acts by the operators* (2-4) *of weight $k$ and character $\chi$.*

*Proof.* The assertions (1) and (2) follows from definitions and [An(87)], Theorem 3.3.3. As to assertion (3), by Proposition 1(3), we have

$$\begin{aligned} F|_k \mathbf{i}(M)_K &= \sum_{M_i \in K\backslash KMK} F|_k P(M_i)^{-1}M_i \\ &= \sum_{M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in K\backslash KMK} \chi(\det A_i)F|_k M_i = F|_{k,\chi}(M)_K, \end{aligned}$$

if $F \in \mathfrak{M}(\chi)$ and $M \in R(K)$.

□

The regular rings (2-1) are commutative integral domains generated by *local subrings* $\mathcal{H}_p(G)$ and $\mathcal{H}_p(K)$ consisting of linear combinations of double classes of matrices contained, respectively, in $R(G)$ and $R(K)$ whose multipliers are powers of a prime number $p$, where $p$ runs over all primes not dividing the level $q$. Each of the local rings is a polynomial ring over $\mathbb{Q}$ in $n+1$ algebraically independent generators. For example, the ring $\mathcal{H}_p(\Gamma_0^2(q))$ is generated by the double classes

$$(2\text{-}6) \qquad \begin{cases} T(p) = T^2(p) = (diag(1,1,p,p))_K, \\ T_1(p^2) = T_1^2(p^2) = (diag(1,p,p^2,p))_K, \\ [p] = T_2^2(p^2) = (pE_4)_K, \end{cases}$$

where $K = \Gamma_0^2(q)$.

## §3. Singular Frobenius elements

We start with general definitions for the groups $G$ and $K$ of the form (1-1) with arbitrary $n$ and $q$, but later on we shall switch again to the particular case $n=2$ and a prime $q$. Elements

(3-1)
$$\Pi(m) = \left( \begin{pmatrix} E_n & 0 \\ 0 & mE_n \end{pmatrix} \right)_K \quad \text{and} \quad \widetilde{\Pi}(m) = \left( \begin{pmatrix} E_n & 0 \\ 0 & mE_n \end{pmatrix} \right)_G \quad \text{with} \quad m | q^\infty,$$

of the subrings of singular elements of the corresponding $HS$-rings are called (*singular*) *Frobenius elements for the groups $K$ and $G$*, respectively.

**Proposition 9.** *The following assertions hold:*

(1) *The double classes* (3-1) *have the decompositions into left classes as follows:*

$$(3\text{-}2) \quad \Pi(m) = \sum_{B \in \mathbb{S}/m\mathbb{S}} \left( K \begin{pmatrix} E & B \\ 0 & mE \end{pmatrix} \right), \quad \widetilde{\Pi}(m) = \sum_{B \in \mathbb{S}/m\mathbb{S}} \left( G \begin{pmatrix} E & B \\ 0 & mE \end{pmatrix} \right),$$

*where $E = E_n$ and $\mathbb{S} = \mathbb{S}_n$;*

(2) *The Frobenius elements satisfy relations*

$$(3\text{-}3) \qquad \Pi(mm') = \Pi(m)\Pi(m'), \quad \widetilde{\Pi}(mm') = \widetilde{\Pi}(m)\widetilde{\Pi}(m') \qquad (m,\, m' | q^\infty),$$

*and relations*

$$(3\text{-}4) \qquad \rho(r)\widetilde{\Pi}(m) = \widetilde{\Pi}(m)\rho(r) \qquad (m|q^\infty,\, r \in (\mathbb{Z}/q\mathbb{Z})^*),$$

*where $\rho(r)$ are the classes* (1-3)*;*

(3) *The Frobenius elements commute with each element of the corresponding regular $HS$-ring* (2-1)*;*

(4) *Each of the subspaces $\mathfrak{M}(\chi)$ and $\mathfrak{N}(\chi)$ defined by* (2-3) *is invariant under all of the operators $|_k\widetilde{\Pi}(m)$. The restriction of $|_k\widetilde{\Pi}(m)$ to the subspace $\mathfrak{M}(\chi_1)$ with the trivial character $\chi_1$ coincides with the operator $|_k\Pi(m)$. The operators $|_k\Pi(m)$ and $|_k\widetilde{\Pi}(m)$ on the spaces $\mathfrak{M}(\chi_1)$ and $\mathfrak{M} = \mathfrak{M}_k^n(q, q)$, respectively, are Frobenius operators in the sence of Introduction.*

*Proof.* (1) Let us prove, for example, the second decomposition (3-2). One can easily see that all of the left cosets in the decomposition are different and belong to the double coset $G\begin{pmatrix} E & 0 \\ 0 & mE \end{pmatrix}G$. A representative of every left coset contained in the double coset can be obviously taken in the form $\begin{pmatrix} E & 0 \\ 0 & mE \end{pmatrix}M$ with $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$. Then, if $B'$ is a matrix of $\mathbb{S}/m\mathbb{S}$ satisfying the congruence $B' \equiv \alpha^{-1}\beta \pmod{m}$, we have the relation

$$\begin{pmatrix} E & 0 \\ 0 & mE \end{pmatrix} M = M' \begin{pmatrix} E & B' \\ 0 & mE \end{pmatrix} \quad \text{with} \quad M' \in G.$$

(2) Multiplicative relations (3-3) directly follow from decompositions (3-2). Let $P(r) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then, if $B$, $B'$ are matrices of $\mathbb{S}$ satisfying the congruence $B' \equiv (\alpha + B\gamma)^{-1}(\beta + B\delta) \pmod{m}$, the matrix

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} E & B \\ 0 & mE \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E & B' \\ 0 & mE \end{pmatrix}^{-1}$$

belongs to $K$ and satisfies $\det \alpha' \equiv \det \alpha \equiv r \pmod{q}$. It follows that the left coset of the matrix modulo $G$ is $GP(r)$, and so

$$\begin{aligned} \widetilde{\Pi}(m)\rho(r) &= \sum_{B \in \mathbb{S}/m\mathbb{S}} \left( G \begin{pmatrix} E & B \\ 0 & mE \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) \\ &= \sum_{B' \in \mathbb{S}/m\mathbb{S}} \left( G \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} E & B' \\ O & mE \end{pmatrix} \right) \\ &= \sum_{B' \in \mathbb{S}/m\mathbb{S}} \left( GP(r) \begin{pmatrix} E & B' \\ 0 & mE \end{pmatrix} \right) = \rho(r)\widetilde{\Pi}(m). \end{aligned}$$

(3) By (3-3), we can assume that $m = p$ is a prime divisor of $q$. The assertion relating to the element $\Pi(p)$ and elements of the ring $\mathcal{H}_r(K)$ follows from [An(87)],Proposition 3.4.11(1) and Theorem 3.3.12. Let now $M \in R(G)$ and

$$(M)_G = \sum_i (GM_i).$$

Then, by Theorem 8(1), we have the decomposition

$$(M)_K = \sum_i (KM_i).$$

We already know that the last element commutes with $\Pi(p)$, i.e.

$$\sum_{i,B} \left( KM_i \begin{pmatrix} E & B \\ 0 & pE \end{pmatrix} \right) = \sum_{j,B'} \left( K \begin{pmatrix} E & B' \\ 0 & pE \end{pmatrix} M_j \right).$$

This means that, for each $i$ and $B$, there are $B'$ and $j$ such that

$$M_i \begin{pmatrix} E & B \\ 0 & pE \end{pmatrix} = L_i \begin{pmatrix} E & B' \\ 0 & pE \end{pmatrix} M_j \quad \text{with} \quad L_i \in K.$$

If $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ and $L_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$, then the last relation implies the congruence

$$\alpha_i \equiv a_i {}^t d_j \mu(M)^{-1} \equiv a_i a_j^{-1} \pmod{q},$$

whence $\det \alpha_i \equiv 1 \pmod{q}$, and so $L_i \in G$.

The part (4) follows from the part (1) and relations (0-17).

□

Let us consider now the double classes

(3-5) $$\Omega = (\omega)_K \quad \text{and} \quad \widetilde{\Omega} = (\omega)_G$$

of the matrix $\omega = \begin{pmatrix} 0 & E_n \\ -qE_n & 0 \end{pmatrix}$ modulo the groups (1-1).

**Proposition 10.** (1). *The classes* (3-5) *have the following properties:*

(3-6) $$\Omega = (K\omega), \quad \widetilde{\Omega} = (G\omega),$$

(3-7) $$\Omega^2 = [q]_K = (qE_{2n})_K, \quad \widetilde{\Omega}^2 = [-q]_G = (-qE_{2n})_G,$$

(3-8) $$\Omega^* = \Omega, \quad \widetilde{\Omega}^* = (-\omega)_G = [-1]_G \widetilde{\Omega},$$

*where the star stands for the map* (0-10), *and*

(3-9) $$\rho(r)\widetilde{\Omega} = \widetilde{\Omega}\rho(r^{-1}), \qquad (r \in (\mathbb{Z}/q\mathbb{Z})^*);$$

(2). *The restriction of the Hecke operator* $|_k\widetilde{\Omega}$ *on the subspace* $\mathfrak{M}(\chi_1)$ *with the trivial character* $\chi_1$ *coincides with the operator* $|_k\Omega$.

*Proof.* From obvious relations

$$\omega^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\omega = \begin{pmatrix} D & -C/q \\ -qB & A \end{pmatrix} \tag{3-10}$$

follow relations

$$\omega^{-1}K\omega = K, \quad \omega^{-1}G\omega = G, \tag{3-11}$$

which imply the decompositions (3-6). The relations (3-7) follow from (3-6), since $\omega^2 = -qE_{2n}$. The last relation shows that $q\omega^{-1} = -\omega$, which implies (3-8). By (1-3), (3-6), and (3-10), we get

$$\rho(r)\widetilde{\Omega} = (GP(r)\omega)_G = (G\omega\omega^{-1}P(r)\omega)_G = \widetilde{\Omega}\rho(r^{-1}).$$

The assertion (2) follows from (3-6).

□

**Proposition 11.** *The Frobenius elements* (3-1) *with* $m = q$ *have the following factorizations:*

$$\Pi(q) = \Omega(J)_K, \quad \widetilde{\Pi}(q) = \widetilde{\Omega}(J)_G, \tag{3-12}$$

*where* $J = J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$.

*Proof.* It is sufficient to prove the decompositions

$$(J)_K = \sum_{B\in\mathbb{S}/q\mathbb{S}} (KJ\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}), \quad (J)_G = \sum_{B\in\mathbb{S}/q\mathbb{S}} (GJ\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}), \tag{3-13}$$

where $\mathbb{S} = \mathbb{S}_n$ and $E = E_n$. In order to prove, for example, the second one, we note that, clearly,

$$(J)_G = \sum_{M\in G\cap J^{-1}GJ\backslash G} (GJM).$$

Since

$$G\cap J^{-1}GJ = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G; \quad B \equiv 0 \pmod q\right\},$$

the decomposition follows.

□

In order to conclude with singular Frobenius elements, we note that their duals under the star map (0-10),

$$\Pi^*(m) = \left(\begin{pmatrix} mE & 0 \\ 0 & E \end{pmatrix}\right)_K \quad \widetilde{\Pi}^*(m) = \left(\begin{pmatrix} mE & 0 \\ 0 & E \end{pmatrix}\right)_G \quad \text{with} \quad m|q^\infty, \tag{3-14}$$

where $E = E_n$, have, mainly, similar properties:

**Proposition 12.** *The following assertions hold:*

(1) *The double classes* (3-14) *have decompositions into left classes as follows:*
(3-15)
$$\Pi^*(m) = \sum_{B\in\mathbb{S}/m\mathbb{S}} \left(K\begin{pmatrix} mE & 0 \\ qB & E\end{pmatrix}\right), \quad \widetilde{\Pi}^*(m) = \sum_{B\in\mathbb{S}/m\mathbb{S}} \left(G\begin{pmatrix} mE & 0 \\ qB & E\end{pmatrix}\right),$$

*where* $\mathbb{S} = \mathbb{S}_n$*;*

(2) *The elements* (3-14) *satisfy relations*

$$\text{(3-16)} \qquad \Pi^*(mm') = \Pi^*(m)\Pi^*(m'), \quad \widetilde{\Pi}^*(mm') = \widetilde{\Pi}^*(m)\widetilde{\Pi}^*(m') \quad (m,\, m'|q^\infty),$$

*and relations*

$$\text{(3-17)} \qquad \rho(r)\widetilde{\Pi}^*(m) = \widetilde{\Pi}^*(m)\rho(r) \qquad (m|q^\infty,\, r\in(\mathbb{Z}/q\mathbb{Z})^*),$$

*where* $\rho(r)$ *are the classes* (1-3)*;*

(3) *The elements* $\Pi^*(m)$ *commute with each element of the regular ring* $\mathcal{H}_r(K)$ *defined by* (2-1)*;*

(4) *Each of the subspaces* $\mathfrak{M}(\chi)$ *and* $\mathfrak{N}(\chi)$ *defined by* (2-3) *is invariant under all of the operators* $|_k\widetilde{\Pi}^*(m)$*. Restrictions of operators* $|_k\widetilde{\Pi}^*(m)$ *to the subspace* $\mathfrak{M}(\chi_1)$ *coincides with operators* $|_k\Pi^*(m)$*;*

(5) *The following factorizations hold:*

$$\text{(3-18)} \qquad \Pi^*(q) = (J)_K\Omega, \quad \widetilde{\Pi}^*(q) = (J)_G\widetilde{\Omega},$$

*where* $J = J_n$*.*

*Proof.* (1) Let us prove, for example, the second of decompositions (3-15). By the second of decompositions (3-2), we obtain

$$\omega^{-1}G\omega\omega^{-1}\begin{pmatrix} E & 0 \\ 0 & mE\end{pmatrix}\omega\omega^{-1}G\omega = \bigcup_{B\in\mathbb{S}/m\mathbb{S}} \omega^{-1}G\omega\omega^{-1}\begin{pmatrix} E & B \\ 0 & mE\end{pmatrix}\omega,$$

where $\omega$ is defined in (3-5). By (3-10) and (3-11), the desired decomposition follows.

(2) Since the map (0-10) is anti-isomorphism, relations (3-16) and (3-17) follow from (3-3), (3-4), and Proposition 1(2).

By the same reason, the part (3) follows from Proposition 9(3), since each element of the ring $\mathcal{H}_r(K)$ is invariant under the star map (see, for example, [An(87)],formula (3.3.14)).

The part (4), clearly, follows from the part (1) and relations (3-17).

Finally, by applying the star map to both sides each of the factorizations (3-12) and using (3-8), we obtain the factorizations (3-18), since $(J)^*_K = (-J)_K = (J)_K$ and $(J)^*_G = [-1]_G(J)_G$.

□

## §4. New forms and eigenforms

In the course of the section we assume that the genus $n = 2$ and the level $q$ is an odd prime number. (The case of $q = 2$ was, in fact, considered in [An(98)], since $\Gamma^n(2, 2) = \Gamma_0^n(2)$.)

We keep the notation

$$\Gamma = \Gamma^2 = \Gamma_0^2(1), \quad K = \Gamma_0^2(q), \quad \text{and} \quad G = \Gamma^2(q, q), \tag{4-1}$$

denote by

$$\mathfrak{N}(\Gamma) = \mathfrak{N}_k(\Gamma) = \mathfrak{N}_k^2(1, 1) \quad \text{and} \quad \mathfrak{N}(K) = \mathfrak{N}_k(K) = \mathfrak{N}_k^2(q, \chi_1) \tag{4-2}$$

the spaces of all cusp forms of weight $k$ for the groups $\Gamma$ and $K$, respectively, where $\chi_1$ is the trivial character, and set

$$\mathfrak{N}(\chi) = \mathfrak{N}_k^2(q, \chi), \quad \mathfrak{N} = \mathfrak{N}_k^2(q, q) = \sum_{\chi \bmod q} \mathfrak{N}(\chi), \tag{4-3}$$

where $\chi$ runs over all Dirichlet characters modulo $q$. The purpose of this section is to define subspaces of the space $\mathfrak{N}$, where all regular Hecke operators and the Hecke operators corresponding to singular Frobenius elements $\widetilde{\Pi}(q^\delta)$ can be simultaneously diagonalized. We follow the pattern of [An(98)], §4.2, §4.3, using corresponding results and technique.

We shall define the subspace $\mathbf{O}$ of *old forms* of $\mathfrak{N}$ by

$$\mathbf{O} = O_k^2(q) = \sum_{i=0}^{3} \mathfrak{N}(\Gamma) \mid_k \widetilde{\Pi}^*(q^i). \tag{4-4}$$

By Theorem 4.4 of [An(98)], the subspace can be written in the form

$$\mathbf{O} = \sum_{i=0}^{3} \mathfrak{N} \mid_k \tau(q)\widetilde{\Pi}^*(q^i), \tag{4-5}$$

where

$$\tau(q) = [\Gamma : G]^{-1} \sum_{M \in G\backslash\Gamma} (GM) \in \mathcal{N}(G\backslash\Gamma) \tag{4-6}$$

is *the trace idempotent of the pair* $(G, \Gamma)$. Note that, by Proposition 12(4), the operators $|\widetilde{\Pi}^*(q^i)$ in (4-4) can be replaced by the operators $|\Pi^*(q^i)$, and so the space $\mathbf{O}$ of old forms is contained in the space $\mathfrak{N}(K)$ of cusp forms of weight $k$ for the group $K$:

$$\mathbf{O} = \sum_{i=0}^{3} \mathfrak{N}(\Gamma)|\Pi^*(q^i) \subset \mathfrak{N}(K). \tag{4-7}$$

We remind that the space $\mathfrak{N}$ is a finite-dimensional Hilbert space with respect to Petersson scalar product

$$\mathfrak{N} \ni F,\, F' \to (F,\, F')$$

defined by an invariant integration on a fundamental domain of the group $G$ (see [An(87)],§2.5 or [An(98)],§1.4). We shall note here only one of the properties of the scalar product, namely that Hecke operators on $\mathfrak{N}$ corresponding to elements of the $HS$-ring $\mathcal{H}(G,\, \Sigma^2)$, which are dual relatively to the star map (0-10), are conjugated with respect to the scalar product:

$$(F|T,\, F') = (F,\, F'|T^*) \qquad (F,\, F' \in \mathfrak{N}, \quad T \in \mathcal{H}(G,\, \Sigma^2)) \tag{4-8}$$

(see [An(98)], Theorem 1.9).

Now we shall define the subspace $\mathbf{N}$ of *new forms* of $\mathfrak{N}$ as the orthogonal complement with respect to the Petersson scalar product of the subspace of old forms:

$$\mathbf{N} = N_k^2(q,\, q) = \{F \in \mathfrak{N}; \quad (F,\, \mathbf{O}) = 0\}. \tag{4-9}$$

By Theorem 4.4 of [An(98)] and (4-8), the subspace $\mathbf{N}$ can be presented in the form

$$\mathbf{N} = \{F \in \mathfrak{N}; \quad F|\widetilde{\Pi}(q^i)\tau(q) = 0 \quad (i = 0, 1, 2, 3)\}. \tag{4-10}$$

It follows from Proposition 1(2) and (4-8) that the decomposition

$$\mathfrak{N} = \sum_{\chi \bmod q} \mathfrak{N}(\chi) = \mathfrak{N}(K) + \sum_{\chi \neq \chi_1} \mathfrak{N}(\chi)$$

is orthogonal with respect to the scalar product, whence, by (4-7), we obtain an orthogonal decomposition

$$\mathbf{N} = \mathbf{N}(K) + \sum_{\chi \neq \chi_1} \mathfrak{N}(\chi), \tag{4-11}$$

where

$$\mathbf{N}(K) = \{F \in \mathfrak{N}(K); \quad (F,\, \mathbf{O}) = 0\}$$

is the subspace of new forms of $\mathfrak{N}(K)$; in particular, every cusp form of weight $k$ and a non-trivial character $\chi$ for the group $G$ is a new form.

Let us consider now the action of Hecke operators on the space of new forms.

**Theorem 13.** *Let $q$ be an odd prime number, $k$ an integer, and $n=2$. Then the following assertions hold:*

(1) *The subspace $\mathbf{N}=N_k^2(q,q)$ of new forms of $\mathfrak{N}$ together with each of the subspaces $\mathbf{N}(K)$ and $\mathfrak{N}(\chi)$ with $\chi\neq\chi_1$ entering into the decomposition* (4-11) *are invariant with respect to all Hecke operators of the regular ring $\mathcal{H}_r(G)$;*

(2) *The regular operators commute with each other and are normal on the subspaces.*

*Proof.* By Theorem 8(3), each of the subspaces $\mathfrak{N}(\chi)\subset\mathfrak{N}$ is invariant under all of the Hecke operators $|\,T=|_kT$ with $T\in\mathcal{H}_r(G)$, and the operators can be interpreted on each subspace $\mathfrak{N}(\chi)$ as the operators $|_{k,\chi}\mathbf{j}(T)$ with $\mathbf{j}(T)\in\mathcal{H}_r(K)$, where $\mathbf{j}$ is the isomorphism defined in Theorem 8(2). It remains only to add that the subspace $\mathbf{N}(K)\subset\mathfrak{N}(K)$ is invariant under all Hecke operators of $\mathcal{H}_r(K)$, by Theorem 4.6(1) of [An(98)].

The assertion (2) follows, since the ring $\mathcal{H}_r(G)$ is commutative and corresponding Hecke operators on $\mathfrak{N}$ are normal, by Proposition 4.1.7 of [An(87)].

□

**Theorem 14.** *Let $q$ be an odd prime number, $k$ an integer, and $n=2$. Then the following assertions hold:*

(1) *The subspace $\mathbf{N}=N_k^2(q,q)$ of new forms of $\mathfrak{N}$ together with each of the subspaces $\mathbf{N}(K)$ and $\mathfrak{N}(\chi)$ with $\chi\neq\chi_1$ entering into the decomposition* (4-11) *are invariant with respect to Frobenius operators $|\widetilde{\Pi}(q^\delta)=|_k\widetilde{\Pi}(q^\delta)$ and its conjugated $|\widetilde{\Pi}^*(q^\delta)=|_k\widetilde{\Pi}^*(q^\delta)$ with $\delta=0,1,2,\ldots$;*

(2) *The Frobenius operators are normal on the subspace $\mathbf{N}(K)$ and on each of the subspaces $\mathfrak{N}(\chi)$ with a non-trivial character $\chi$ different from the quadratic character $\chi_2(r)=\left(\frac{r}{q}\right)$. More precisely, the operators satisfy relations*

$$F|\widetilde{\Pi}(q^\delta)\widetilde{\Pi}^*(q^\delta)=F|\widetilde{\Pi}^*(q^\delta)\widetilde{\Pi}(q^\delta)=q^{(2k-4)\delta}F, \tag{4-12}$$

*if $F\in\mathbf{N}(K)$, and relations*

$$F|\widetilde{\Pi}(q^\delta)\widetilde{\Pi}^*(q^\delta)=F|\widetilde{\Pi}^*(q^\delta)\widetilde{\Pi}(q^\delta)=q^{(2k-3)\delta}F, \tag{4-13}$$

*if $F\in\mathfrak{N}(\chi)$ with $\chi\neq\chi_1,\chi_2$.*

We shall prove first three lemmas.

**Lemma 15.** *The subspace $\mathbf{N}(K)$ is invariant under the Hecke operators $|\widetilde{\xi}_2$ and $|\widetilde{\Omega}$ of weight $k$ corresponding, respectively, to double classes defined in Proposition 3 and in* (3-5).

*Proof.* By Proposition 3(4) and Proposition 10(2), restrictions of the operators $|\widetilde{\xi}_2$ and $|\widetilde{\Omega}$ on the space $\mathfrak{N}(\chi_1)=\mathfrak{N}(K)$ coincide, respectively, with the operators $|\xi_2$ and $|\Omega$, and so the lemma is true, by Lemma 4.10 of [An(98)].

□

**Lemma 16.** *Each of the subspaces* $\mathfrak{N}(\chi)$ *is mapped by the operators* $|\widetilde{\xi}_2$ *and* $|\widetilde{\Omega}$ *into the subspace* $\mathfrak{N}(\chi^{-1})$.

*Proof.* The lemma follows from relations (1-8) and (3-9).

□

**Lemma 17.** *The operators* $|\widetilde{\Omega}^2$ *and* $|\widetilde{\xi}_2^2$ *satisfy relations*

$$F|\widetilde{\Omega}^2 = q^{2k-6}F \qquad (F \in \mathfrak{M} = \mathfrak{M}_k^2(q,\, q)), \tag{4-14}$$

*and*

$$F|\widetilde{\xi}_2^2 = \begin{cases} q^2F, & \text{if} \quad F \in \mathbf{N}(K), \\ q^3F, & \text{if} \quad F \in \mathfrak{N}(\chi) \quad (\chi \neq \chi_1, \chi_2), \\ q^2F, & \text{if} \quad F \in \mathfrak{N}'(\chi_2), \\ q^4F, & \text{if} \quad F \in \mathfrak{N}''(\chi_2), \end{cases} \tag{4-15}$$

*where*

$$\mathfrak{N}'(\chi_2) = \{F = (-1)^{\frac{q-1}{2}} q^2 F' - F'|\widetilde{\xi}_2; \quad F' \in \mathfrak{N}(\chi_2)\},$$

*and*

$$\mathfrak{N}''(\chi_2) = \{F = (-1)^{\frac{q-1}{2}} qF' + F'|\widetilde{\xi}_2; \quad F' \in \mathfrak{N}(\chi_2)\}.$$

*Proof.* The relations (4-14) follow from (3-7) and (0-12). In order to prove relations (4-15), we note first that, by Proposition 3(1),(2), the trace idempotent (4-6) can be written in the form

$$\tau(q) = [\Gamma : G]^{-1} \left( \sum_{r=1}^{q-1} \rho(r) + \xi_1^+ + \xi_1^- + \sum_{r=1}^{q-1} \rho(r)\widetilde{\xi}_2 \right).$$

It follows from the formula (1-9) that, for every modular form $F \in \mathfrak{M}_k^2(q,\, q)$, we have the relation

$$\begin{aligned} F|\widetilde{\xi}_2^2 &= F| \left\{ q^3\rho(1) + q^2(\xi_1^+ + \xi_1^-) + \sum_{r=1}^{q-1} (q^2 + \left(\frac{-r}{q}\right) q)\rho(r)\widetilde{\xi}_2 \right\} \\ &= q^3F|\rho(1) + q^2[\Gamma : G]F|\tau(q) - q^2F|\sum_{r=1}^{q-1}\rho(r) + qF|\sum_{r=1}^{q-1}\left(\frac{-r}{q}\right)\rho(r)\widetilde{\xi}_2. \end{aligned}$$

By definition, we have $F|\rho(1) = F$. By (4-10), if $F$ is a new form, we conclude that $F|\tau(q) = 0$. By Proposition 1(3), we obtain for $F \in \mathfrak{M}(\chi)$ relations

$$F|\sum_{r=1}^{q-1}\rho(r) = \left(\sum_{r=1}^{q-1}\chi(r)^{-1}\right) F = \begin{cases} (q-1)F, & \text{if} \quad \chi = \chi_1, \\ 0, \quad \text{if} & \chi \neq \chi_1, \end{cases}$$

and

$$F|\sum_{r=1}^{q-1}\left(\frac{-r}{q}\right)\rho(r)\widetilde{\xi}_2 = \left(\frac{-1}{q}\right)\left\{\sum_{r=1}^{q-1}\chi(r)^{-1}\left(\frac{r}{q}\right)\right\}F|\widetilde{\xi}_2$$

$$= \begin{cases} (-1)^{\frac{q-1}{2}}(q-1)F|\widetilde{\xi}_2, & \text{if} \quad \chi = \chi_2, \\ 0, \quad \text{if} \quad \chi \neq \chi_2. \end{cases}$$

By using the above formulas, we conclude: if $F \in \mathbf{N}(K)$, then

$$F|\widetilde{\xi}_2^2 = q^3F - q^2(q-1)F = q^2F;$$

if $F \in \mathfrak{N}(\chi)$ with $\chi \neq \chi_1, \chi_2$, then $F|\widetilde{\xi}_2^2 = q^3F$; but if $F \in \mathfrak{N}(\chi_2)$, then

$$F|\widetilde{\xi}_2^2 = q^3F + (-1)^{\frac{q-1}{2}}q(q-1)F|\widetilde{\xi}_2.$$

Using the last relation, we get: if $F \in \mathfrak{N}'(\chi_2)$, then

$$F|\widetilde{\xi}_2 = \left((-1)^{\frac{q-1}{2}}q^2F' - F'|\widetilde{\xi}_2\right)|\widetilde{\xi}_2 = -q^3F' + (-1)^{\frac{q-1}{q}}qF'|\widetilde{\xi}_2$$
$$= -(-1)^{\frac{q-1}{2}}q\left((-1)^{\frac{q-1}{2}}q^2F' - F'|\widetilde{\xi}_2\right) = (-1)^{\frac{q+1}{2}}qF;$$

but if $F \in \mathfrak{N}''(\chi_2)$, then, similarly, we obtain

$$F|\widetilde{\xi}_2 = \left((-1)^{\frac{q-1}{2}}F' + F'|\widetilde{\xi}_2\right)|\widetilde{\xi}_2 = (-1)^{\frac{q-1}{2}}q^2F.$$

□

*Proof of Theorem 14.* By Proposition 9(4) and Proposition 12(4), the operators $|\,\widetilde{\Pi}(q^\delta)$ and $|\,\widetilde{\Pi}^*(q^\delta)$ coincide on the subspace $\mathfrak{N}(K)$, respectively, with the operators $|\,\Pi(q^\delta)$ and $|\,\Pi^*(q^\delta)$, and so, by Lemma 4.10 [An(98)], they map the subspace $\mathbf{N}(K)$ into itself. On the other hand, by Proposition 1(3), formulas (3-4) and (3-17), the operators map each of the subspaces $\mathfrak{N}(\chi)$ into themselves. The part (1) is proved. By using the multiplicative relations (3-3) and (3-16), and decompositions (3-12) and (3-18), we obtain the relation

$$F \mid \widetilde{\Pi}(q^\delta)\widetilde{\Pi}^*(q^\delta) = F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Pi}(q)\widetilde{\Pi}^*(q)\widetilde{\Pi}^*(q^{\delta-1})$$
$$= F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Omega}\widetilde{\xi}_2^2\widetilde{\Omega}\widetilde{\Pi}^*(q^{\delta-1}).$$

If $F \in \mathbf{N}(K)$ (resp., $F \in \mathfrak{N}(\chi)$ with $\chi \neq \chi_1, \chi_2$), then the function $F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Omega}$ belongs to $\mathbf{N}(K)$ (resp., to $\mathfrak{N}(\chi^{-1})$), by the first part, Lemma 15, and Lemma 16, and so, by Lemma 17, the last expression is equal to

$$q^2F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Omega}^2\widetilde{\Pi}^*(q^{\delta 1}) = q^2q^{2k-6}F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Pi}^*(q^{\delta-1}),$$

if $F \in \mathbf{N}(K)$, and is $q^3 q^{2k-6} F \mid \widetilde{\Pi}(q^{\delta-1})\widetilde{\Pi}^*(q^{\delta-1})$, if $F \in \mathfrak{N}(\chi)$ with $\chi \neq \chi_1, \chi_2$. Similarly, we get

$$\begin{aligned} F \mid \widetilde{\Pi}^*(q^\delta)\widetilde{\Pi}(q^\delta) &= F \mid \widetilde{\Pi}^*(q^{\delta-1})\widetilde{\xi}_2\widetilde{\Omega}^2\widetilde{\xi}_2\widetilde{\Pi}(q^{\delta-1}) \\ &= q^{2k-6} F \mid \widetilde{\Pi}^*(q^{\delta-1})\widetilde{\xi}_2^2\widetilde{\Pi}(q^{\delta-1}), \end{aligned}$$

which is $q^{2k-4} F \mid \widetilde{\Pi}^*(q^{\delta-1})\widetilde{\Pi}(q^{\delta-1})$, if $F \in \mathbf{N}(K)$, and is $q^{2k-3} F \mid \widetilde{\Pi}^*(q^{\delta-1})\widetilde{\Pi}(q^{\delta-1})$, if $F \in (\chi)$, $\chi \neq \chi_1, \chi_2$. The relations (4-12), (4-13) follow then, by induction.

□

**Theorem 18.** *Let $q$ be an odd prime number. Then the subspace $\mathbf{N}(K)$ of new forms of an integral weight $k$ for the group $K = \Gamma_0^2(q)$ as well as each of the subspaces $\mathfrak{N}(\chi) = \mathfrak{N}_k^2(q, \chi)$ of cusp form of weight $k$ and character $\chi$ for the group $K$, where $\chi$ is a Dirichlet character modulo $q$ different from the trivial character $\chi_1$ and the quadratic character $\chi_2(r) = \left(\frac{r}{q}\right)$, has an orthonormal basis consisting of common eigenfunctions of all Hecke operators corresponding to elements of the regular $HS$-ring $\mathcal{H}_r(G)$ of the group $G = \Gamma^2(q, q)$ and to all of the singular Frobenius elements $\widetilde{\Pi}(q^\delta)$ with $\delta = 0, 1, 2, \ldots$ for the group $G$. Eigenvalues $\lambda(q^\delta)$ of the operators $\mid \widetilde{\Pi}(q^\delta)$ satisfy*

$$\mid \lambda(q^\delta) \mid = \begin{cases} q^{(k-2)\delta} & \text{on} \quad \mathbf{N}(K), \\ q^{(k-3/2)\delta} & \text{on} \quad \mathfrak{N}(\chi) \quad (\chi \neq \chi_1, \chi_2). \end{cases} \tag{4-16}$$

*Proof.* By Theorem 13 and Theorem 14, each of the subspaces is invariant under all of the indicated operators, and the operators are normal on the subspaces. By Theorem 13(2) and Proposition 9(2),(3), all of the operators commute with each other. Then the first statement follows, by a well known theorem of linear algebra. By (4-8), the operators $|\widetilde{\Pi}(q^\delta)$ and $|\widetilde{\Pi}^*(q^\delta)$ are conjugated, and so they have complex conjugate eigenvalues. The relations (4-16) follow then from relations (4-12) and (4-13).

□

We note that the assumption $\chi \neq \chi_2$ was rather unexpected for us. It has appeared because we could not prove that at least one of the subspaces $\mathfrak{N}'(\chi_2)$ or $\mathfrak{N}''(\chi_2)$ introduced in Lemma 17 is invariant under the operator $\widetilde{\Omega}$. May be it means that the Frobenius operators $|\widetilde{\Pi}(q^\delta)$ are not always diagonalizable on the space $\mathfrak{N}(\chi_2)$, which, therefore, should be replaced by a subspace of "very new forms".

## §5. Euler factorization of radial Dirichlet series

E. Hecke has discovered in [He(37)] that Fourier coefficients of eigenfunctions of certain operators on spaces of modular forms in one variable, called now Hecke operators, have definite multiplicative properties. After his works it was just natural to expect that Fourier coefficients of Siegel modular forms also have some multiplicative properties. But even formulation of the question in Siegel case is far from being definite. Let us take, for example, a modular form $F$ for the group $\Gamma^n(q,\, q)$. According to (0-7), the Fourier coefficients $f(A)$ of $F$ are values of a function on classes of even matrices of order $n$ modulo *the proper integral equivalence*

$$A \mapsto {}^tUAU \qquad (A \in \mathbb{E}_n,\, U \in SL_n(\mathbb{Z})), \tag{5-1}$$

or, in other words, on classes of integral quadratic forms $\frac{1}{2}{}^tXAX$ with respect to the equivalence. A priori it is not clear in what sence such a function can be multiplicative, because, in general, there is no natural multiplication of the classes. Instead of the general question, one can ask whether one or another function of one integral variable associated to $f$ is multiplicative. The simplest of possible functions are *the radial functions*

$$m \mapsto f(mA)$$

with fixed $A \in \mathbb{E}_n$, and the simplest question on multiplicativity of such functions is whether the corresponding *radial Dirichlet series*

$$R_F(s,\, \eta,\, A) = \sum_{m=1}^{\infty} \frac{\eta(m) f(mA)}{m^s}, \tag{5-2}$$

where $\eta(m)$ is a completely multiplicative function, have, at least formally, an Euler type factorization over all prime numbers. In the case of modular forms in one variable Euler factorizations of the radial Dirichlet series associated to eigenfunctions of "all" Hecke operators are well known (see, for example, [Li(75)]). In Siegel cases little is known, if the genus $n > 2$. But if $n = 2$, Euler factorizations of radial Dirichlet series were obtained in [An(74)] for the case of eigenfunctions $F$ of all Hecke operators for the full modular group $\Gamma^2$ and $\eta(m) \equiv 1$. The result was generalized in [An(87)], §4.3.2 to *regular radial series*

$$R_F^*(s,\, \eta,\, A) = \sum_{m \geq 1,\, \gcd(m,\, q)=1} \frac{\eta(m) f(mA)}{m^s} \tag{5-3}$$

corresponding to eigenfunctions $F$ of all regular Hecke operators for the groups $\Gamma^2(q,\, q)$. Here we shall extend the factorization to full radial series of the form (5-2) under assumption that $F$ is also an eigenfunction of all singular Frobenius operators for the groups. The extension makes sence in view of the results on existence of eigenfunctions obtained above and in [An(98)], Theorem 4.15.

**Theorem 19.** *Let $k$, $q$ be positive rational integers, $\chi$ a Dirichlet character modulo $q$, and let*

$$F(Z) = \sum_{A\in\mathbb{E}_2\,,\ A\geq 0} f(A)\exp(\pi i\,Tr(AZ)) \in \mathfrak{M}_k^2(q,\,\chi)$$

*be an eigenfunction of all regular Hecke operators $|_{k,\,\chi}$ for the group $\Gamma_0^2(q)$,*

$$F|_{k,\,\chi}T = \lambda_F(T)F \qquad (T\in\mathcal{H}_r(\Gamma_0^2(q))),$$

*and all of the singular Frobenius operators $|_k\widetilde{\Pi}(q')$ for the group $\Gamma^2(q,\,q)$,*

$$F|_k\widetilde{\Pi}(q') = \lambda_F(q')F \qquad (q'|q^\infty). \tag{5-4}$$

*For a negative rational integer $\Delta$, let $d$ denotes the discriminant of the imaginary quadratic field $\mathcal{F} = \mathbb{Q}(\sqrt{\Delta})$, $l = \sqrt{\Delta/d}$, $\mathcal{O}_l$ the subring of index $l$ in the ring $\mathcal{O}$ of all integers of the field, let $H(\Delta)$ be the group of classes of equivalent complete modules in $\mathcal{F}$ with the ring of multipliers $\mathcal{O}_l$, realized as the group with respect to the Gauss composition of classes $\{A\}$ modulo the proper equivalence* (5-1) *of matrices $\begin{pmatrix} 2a & b\\ b & 2c\end{pmatrix}\in\mathbb{E}_2$ (or corresponding quadratic forms) with $b^2-4ac=\Delta$ and $\gcd(a,\,b,\,c)=1$, and let $\psi$ be a character of the group $H(\Delta)$. Then, for every completely multiplicative function $\eta:\{1,2,\dots\}\to\mathbb{C}$, the following assertions hold:*

(1) *Radial series* (5-2) *satisfy the formal identity*

$$\sum_{\{A_i\}\in H(\Delta)} \psi(\{A_i\}) \sum_{m=1}^{\infty} \frac{\eta(m)f(mA_i)}{m^s} \tag{5-5}$$

$$= \rho(s)\left\{\prod_{\mathfrak{p},\,N(\mathfrak{p})\nmid(ql)^2}\left(1-\frac{\eta(N(\mathfrak{p}))\chi(N(\mathfrak{p}))\psi(\mathfrak{p})}{N(\mathfrak{p})^{s-k+2}}\right)\right\}\left\{\prod_{p\nmid q} Q_{p,\,F}^{-1}\left(\frac{\eta(p)}{p^s}\right)\right\}$$
$$\times\left\{\prod_{p|q}\left(1-\frac{\eta(p)\lambda_F(p)}{p^s}\right)^{-1}\right\},$$

*where $A_i$ on the left runs a system of representatives of proper classes $\{A_i\}$ in $H(\Delta)$, $\rho(s)$ on the right is the finite sum given by*

(5-6)

$$\rho(s) = \sum_{\{A_i\}\in H(\Delta)} \psi(\{A_i\}) \sum_{\substack{t,t_1\geq 1,\,t|t_1|l,\\ \gcd(tt_1,\,q)=1}} \frac{\eta(tt_1)\chi(t^2t_1)\mu(t)\mu(t_1)}{t^{s-2k+3}t_1^{s-k+2}} f\left(\frac{t_1}{t}(A_i\times\mathcal{O}_{l/t_1})\right)$$

*with the Möbius function $\mu$, subrings $\mathcal{O}_\nu$ of index $\nu$ of the ring $\mathcal{O}$, and the natural epimorphism $\{A\}\to\{A\times\mathcal{O}_\nu\}$ of the group $H(\Delta)=H(dl^2)$ on $H(d\nu^2)$, and where*

$\mathfrak{p}$ *runs over all regular prime ideals of the ring* $\mathcal{O}_l$ *whose norm* $N(\mathfrak{p})$ *is coprime with* $ql$*,* $p$ *runs over all prime numbers satisfying corresponding conditions, and, for each of prime numbers* $p$ *not dividing* $q$*,* $Q_{p,F}(x)$ *is the polynomial of degree four of the form*

$$Q_{p,F}(x) = 1 - \lambda_F(T(p))x + \left\{p\lambda_F(T_1(p^2)) + \chi(p^2)p^{2k-5}(p^2+1)\right\}x^2 - \chi(p^2)p^{2k-3}\lambda_F(T(p))x^3 + \chi(p^4)p^{4k-6}x^4, \tag{5-7}$$

*with the elements* $T(p)$*,* $T_1(p^2)$ *of the regular ring* $\mathcal{H}_p(\Gamma_0^2(q))$ *defined by* (2-6)*;*

(2) *Suppose that radial series* $R_F(s, \eta, A')$ *are formally equal to zero for every matrix* $A' = \begin{pmatrix} 2a' & b' \\ b' & 2c' \end{pmatrix}$ *with* $\gcd(a', b', c') = 1$ *and* $(b')^2 - 4a'c' = d\nu^2$*, where* $\nu | l$ *and* $\nu < l$*, then the factor* $\rho(s)$ *is a constant:*

$$\rho(s) = \sum_{\{A_i\}\in H(\Delta)} \psi(\{A_i\})f(A_i); \tag{5-8}$$

(3) *If* $\eta$ *satisfies* $|\eta(m)| \le cm^\sigma$ *with some constants* $c$ *and* $\sigma$*, then the Dirichlet series on the left in* (5-5) *and infinite products on the right converge absolutely and uniformly in each of the right half-plane of the form* $Re\, s > 2ke + \sigma + 1 + \varepsilon$ *with arbitrary* $\varepsilon > 0$*, where* $e = 1$ *in the general case, and* $e = 1/2$*, if* $F$ *is a cusp form.*

*Proof.* By [An(87)], Theorem 4.3.16, we have the formal identity

$$\sum_{\{A_i\}\in H(\Delta)} \psi(\{A_i\})R_F^*(s, \eta, q'A_i) = \tag{5-9}$$

$$\eta(q')\rho(s, q')\left\{\prod_{\mathfrak{p},\, N(\mathfrak{p})\nmid(ql)^2}\left(1 - \frac{\eta(N(\mathfrak{p}))\chi(N(\mathfrak{p}))\psi(\mathfrak{p})}{N(\mathfrak{p})^{s-k+2}}\right)\right\}\prod_{p\nmid q} Q_{p,F}^{-1}\left(\frac{\eta(p)}{p^s}\right),$$

for every $q'$ dividing $q^\infty$, where the factor $\rho(s, q')$ is equal to

$$\sum_{\{A_i\}\in H(\Delta)} \psi(\{A_i\}) \sum_{\substack{t,t_1\ge 1,\, t|t_1|l,\\ \gcd(tt_1, q)=1}} \frac{\eta(tt_1)\chi(t^2t_1)\mu(t)\mu(t_1)}{t^{s-2k+3}t_1^{s-k+2}} f\left(\frac{q't_1}{t}(A_i \times \mathcal{O}_{l/t_1})\right),$$

and where formulas for coefficients of polynomials $Q_{p,F}(x)$ follow from formulas obtained in [An(87)] (see Proposition 3.3.35 and Exercise 3.3.38), since

$$F|_{k,\chi}[p] = F|_{k,\chi}(\Gamma_0^2(q)(pE_4)) = \chi(p^2)p^{2k-6}F.$$

It follows from (5-4) and (0-18) that

$$f(q'A') = \lambda_F(q')f(A'), \qquad \text{if} \quad q'|q^\infty \quad \text{and} \quad A' \in \mathbb{E}_2,$$

whence

$$\rho(s,\,q') = \lambda_F(q')\rho(s) \tag{5-10}$$

with the sum $\rho(s)$ given by (5-6). It follows from (5-9) and (5-10) that the series in the left side of (5-5) is formally equal to

$$\sum_{q'|q^\infty} \frac{\eta(q')\lambda_F(q')\rho(s)}{(q')^s} \left\{ \prod_{\mathfrak{p},\, N(\mathfrak{p})\nmid (ql)^2} \left(1 - \frac{\eta(N(\mathfrak{p}))\chi(N(\mathfrak{p}))\psi(\mathfrak{p})}{N(\mathfrak{p})^{s-k+2}}\right)\right\} \prod_{p\nmid q} Q_{p,\,F}^{-1}\left(\frac{\eta(p)}{p^s}\right),$$

which proves the identity (5-5), since

$$\sum_{q'|q^\infty} \frac{\eta(q')\lambda_F(q')}{(q')^s} = \prod_{p|q}\sum_{\delta=0}^{\infty} \frac{\eta(p^\delta)\lambda_F(p^\delta)}{p^{\delta s}} = \prod_{p|q}\left(1 - \frac{\eta(p)\lambda_F(p)}{p^s}\right)^{-1}.$$

The formula (5-8) follows from (5-6), since, by the assumption of part (2),

$$f\left(\frac{t_1}{t}(A_i \times \mathcal{O}_{l/t_1})\right) = 0,$$

unless $t = t_1 = 1$.

The part (3) follows from corresponding assertions on series and products in (5-9), proved in [An(87)], Theorem 4.3.16.

□

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