Affine embeddings with a finite number of orbits

I. V. Arzhantsev

Abstract

Let G be a simple algebraic group and let H be a reductive subgroup of G. We classify all pairs $(G,\ H)$ such that for any affine G-variety X with a dense G-orbit isomorphic to G/H the number of G-orbits in X is finite.

1 Introduction.

Let G be a connected reductive algebraic group over an algebraically closed field K of characteristic zero and let H be an algebraic subgroup of G. Let us recall that an irreducible algebraic variety X is said to be *an embedding* of the homogeneous space G/H if G acts on X with a dense orbit isomorphic to G/H. We shall denote this by $G/H \hookrightarrow X$.

By definition, *the complexity* of G-variety X is an integer number $c(X)$ equal to the codimension of the generic B-orbit in X for the restricted action $B : X$, where B is a Borel subgroup of G, see [1] and [2]. Normal G-varieties of complexity zero are called *spherical*. A homogeneous space G/H and a subgroup $H \subset G$ are said to be spherical if G/H is a spherical G-variety with respect to the natural G-action.

Theorem 1 (F. J. Servedio [3], D. Luna and Th. Vust [2], D. N. Akhiezer [4]). *The number of G-orbits in X for any embedding $G/H \hookrightarrow X$ of the homogeneous space G/H is finite if and only if G/H is spherical.*

To be more precise, F. J. Servedio proved that any affine spherical variety contains a finite number of G-orbits, D. Luna, Th. Vust and D. N. Akhiezer extended this result to an arbitrary spherical variety and D. N. Akhiezer constructed a projective embedding with infinite number of G-orbits for any homogeneous space of positive complexity.

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Let us say that an embedding $G/H \hookrightarrow X$ is *affine* if the variety X is affine. In many problems of invariant theory, representation theory and other branches of mathematics only affine embeddings of homogeneous spaces are considered. Hence for a homogeneous space G/H it is natural to ask: does there exist an affine embedding $G/H \hookrightarrow X$ with infinite number of G-orbits ?

Note that a given homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety), see [5, Theorem 1.6]. In this situation the subgroup H is said to be *observable* in G. For a description of observable subgroups see [6]. The homogeneous space G/H is affine if and only if H is reductive, see [5, Th. 4.17]. In particular, any reductive subgroup is observable. In the sequel, we suppose that H is an observable subgroup of G. The main problem for this paper is to characterize all quasi-affine homogeneous spaces G/H of a reductive group G with the property:

(AF) *the number of G-orbits in X for any affine embedding $G/H \hookrightarrow X$ is finite.*

Example 1. For any spherical quasi-affine homogeneous space property (AF) holds (Theorem 1).

Example 2. Property (AF) holds for any homogeneous space of the group $SL(2)$. In fact, here $dim\ X \leq 3$ and only a one-parameter family of one-dimensional orbits can appear. But $SL(2)$ contains no two-dimensional observable subgroup.

Example 3. Let T be a maximal torus in G and let V be a finite-dimensional G-module. Suppose that a vector $v \in V$ is T-fixed. Then the orbit Gv is closed in V, see [7]. This shows that property (AF) holds for any subgroup H such that $T \subseteq H$.

Definition 1. An affine homogeneous space G/H is called *affinely closed* if it admits only one affine embedding $X = G/H$.

Homogeneous spaces G/H from Example 3 are affinely closed. Denote by $N_G(H)$ the normalizer of a subgroup H in G. The following theorem is a reformulation of a result due to D. Luna [8]:

Theorem 2. *Let H be a reductive subgroup of a reductive group G. The homogeneous space G/H is affinely closed if and only if the group $N_G(H)/H$ is finite.*

This theorem provides many examples of homogeneous spaces with property (AF). Let us note that the complexity of the space G/T can be arbitrary large and property (AF) cannot be characterized in terms of the complexity only.

In this paper we show that the union of two conditions – the sphericity and the finiteness of the group $N_G(H)/H$ – is very close to characterizing all affine homogeneous spaces of a reductive group G with property (AF). Namely, it follows from Theorem 3 below that if H is reductive then the only case, which is not covered by these two conditions, is the case, where $c(G/H) = 1$, the rank of $N_G(H)/H$ is equal to one, and any extension of H by a one-parameter subgroup of $N_G(H)$ is a spherical subgroup of G.

For a simple group G there is a list of all affine homogeneous spaces of complexity one [9]. In this situation we prove that if $rk\ N_G(H)/H = 1$ and an extension of H by a one-parameter subgroup of $N_G(H)$ is a spherical subgroup in G then property (AF) holds (Proposition 2). This completes the classification of affine homogeneous spaces of simple groups with property (AF) (Theorem 4).

As a corollary, we obtain that for a simple group G there exists only one series of affine homogeneous spaces of complexity one that admit an affine embedding with infinite number of G-orbits. Here $G = SL(n),\ n > 4$ and $H^0 = SL(n-2) \times K^*$, where $SL(n-2)$ is embedded in $SL(n)$ as the stabilizer of the first two basis vectors e_1 and e_2 in the minimal representation of $SL(n)$, and K^* acts on e_1 and e_2 with weights α_1 and α_2 such that $\alpha_1 + \alpha_2 = 2 - n$, $\alpha_1 \neq \alpha_2$, and on $< e_3, \ldots, e_n >$ by scalar multiplication.

Let us fix the following notation:
K^* is the multiplicative group of non-zero elements of the field K;
F^0 is the identity component of an algebraic group F;
$Z(F)$ is the center of a group F;
$T \subset B$ are a maximal torus and a Borel subgroup of the reductive group G;
U is the maximal unipotent subgroup of B;
$N_G(H)$ is the normalizer of a subgroup H in a group G;
$W(H)$ is the quotient group $N_G(H)/H$;
$\gamma : N_G(H) \to W(H)$ is the quotient homomorphism;
V^F is the set of F-fixed points in a F-module V;
F_v is the isotropy subgroup of a vector v in a F-module V;
$V^{\otimes n}$ is the tensor power $V \otimes \ldots \otimes V$ (n times) of a vector space V;
$\Xi(G)_+$ is the semigroup of all dominant weights of G;
V_μ is an irreducible G-module with highest weight μ;
$V_{\mu^*} = V_\mu^*$ is the dual module to V_μ;
QA is the field of quotients of an integral algebra A;
Spec A is the affine variety corresponding to a finitely generated algebra A without nilpotent elements.

2 Embeddings with infinite number of G-orbits.

Theorem 3. *Let H be an observable subgroup in a reductive group G. Suppose that there is a non-trivial one-parameter subgroup $\lambda : K^* \to W(H)$ such that the subgroup $H_1 = \gamma^{-1}(\lambda(K^*))$ is not spherical in G. Then there exists an affine embedding $G/H \hookrightarrow X$ with infinite number of G-orbits.*

We shall prove this theorem in the next section. The idea of the proof is to apply the Akhiezer construction for the non-spherical homogeneous space G/H_1

and to consider the affine cone over a projective embedding of G/H_1 with infinite number of G-orbits.

Corollary 1. *Let G be a reductive group with infinite center $Z(G)$ and let H be an observable subgroup in G that does not contain $Z(G)^0$. Then property (AF) holds for G/H if and only if H is a spherical subgroup of G.*

Proof. If H does not contain $Z(G)^0$ then there exists a non-trivial one-parameter subgroup $\lambda(K^*)$ in $Z(G)$ with finite intersection with H. The corresponding extension H_1 is spherical iff H is spherical in G.

Corollary 2. *Let H be a connected reductive subgroup in a reductive group G. Suppose that there exists a reductive non-spherical subgroup H_1 in G such that $H \subset H_1$ and dim $H_1 =$ dim $H+1$. Then property (AF) does not hold for G/H.*

Proof. Under these assumptions there exists a non-trivial one-parameter subgroup of H_1 with finite intersection with H, which normalizes (and even centralizes) H.

Corollary 3. *Let H be a reductive subgroup in a reductive group G such that the complexity of the homogeneous space G/H is greater than one. Then the number of G-orbits in any affine embedding of G/H is finite if and only if G/H is affinely closed.*

Proof. If the subgroup H is reductive then the group $W(H)$ is reductive too [8]. If $W(H)$ is not finite then it contains a non-trivial one-parameter subgroup $\lambda(K^*)$. For $H_1 = \gamma^{-1}(\lambda(K^*))$ we have $c(G/H_1) \geq 1$.

3 Proof of Theorem 3.

Lemma 1. *If property (AF) holds for a homogeneous space G/H then it holds for any homogeneous space G/H', where H' is an overgroup of H of finite index.*

Proof. Suppose that there exists an affine embedding $G/H' \hookrightarrow X$ with infinite number of G-orbits. Consider the morphism $G/H \to G/H'$. It determines an embedding $K[G/H'] \subset K[G/H]$. Let A be the integral closure of the image of the subalgebra $K[X] \subset K[G/H']$ in the field of rational functions $K(G/H)$.

$$\begin{array}{ccccc} A & \hookrightarrow & K[G/H] & \hookrightarrow & K(G/H) \\ \uparrow & & \uparrow & & \uparrow \\ K[X] & \hookrightarrow & K[G/H'] & \hookrightarrow & K(G/H') \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}\, A & \hookleftarrow & G/H \\ \downarrow & & \downarrow \\ X & \hookleftarrow & G/H' \end{array}$$

We have a natural G-action on the affine variety Spec A and we can consider Spec A as an affine embedding of G/H. The embedding $K[X] \subset A$ defines a finite (surjective) morphism Spec $A \to X$ and therefore, Spec A contains infinitely many G-orbits. This contradiction completes the proof.

Remark 1. The converse statement does not hold. Indeed, set $G = SL(3)$ and $H = (t, t, t^{-2}) \subset T \subset SL(3)$. We can extend H by the one-parameter subgroup $(t, t^{-1}, 1)$. Then $H_1 = T$ is not a spherical subgroup in $SL(3)$ and by Theorem 3 property (AF) does not hold here. On the other hand, there is an overgroup H' of H of finite index such that the group $W(H')$ is finite. By Theorem 2 property (AF) holds for G/H'.

Lemma 2. *1) In the notation of Theorem 3 there exists a finite-dimensional G-module V and a H_1-eigenvector $v \in V$ such that G_v is an overgroup of H of finite index;*

2) If H is reductive then one can suppose that $G_v = H$.

Proof. 1) Since H is observable, there exists a finite-dimensional G-module U and a vector $u \in U$ such that $G_u = H$, see [5, Th. 1.5 and Th 1.6]. We have the representation $W(H) : U^H$ and the vector u has the trivial stabilizer in $W(H)$. Let $u_1, \ldots, u_k$ be an eigenbasis for the action $\lambda(K^*) : U^H$ and let $n_1, \ldots, n_k$ be the weights of $\lambda(K^*)$ on the vectors $u_1, \ldots, u_k$ respectively. Here $n_1, \ldots, n_k$ are integer numbers such that $gcd(n_1, \ldots, n_k) = 1$.

Choose a basis $B_2 = (b_1^2, \ldots, b_k^2), \ldots, B_k = (b_1^k, \ldots, b_k^k)$ of the sublattice $\mathbb{Z}^{k-1} \subset \mathbb{Z}^k$ defined by the equation $n_1x_1 + \ldots + n_kx_k = 0$. Set $b = max \mid b_i^j \mid$. Let $A = (a_1, \ldots, a_k)$ be an integral vector such that $a_i > b$ for any i and $a_1n_1 + \ldots + a_kn_k = N_1 \neq 0$. Set $N = \mid N_1 \mid$. Then the vectors $A_1 = A, A_2 = A + B_2, \ldots, A_k = A + B_k$ generate a sublattice in $\mathbb{Z}^k$ of index N. The coordinates (a_i^j) of the vectors A_i are positive integers. Set $c_j = a_1^j + \ldots + a_k^j$.

Consider the G-module

$$V = U^{\otimes c_1} \oplus \ldots \oplus U^{\otimes c_k}$$

and the vector

$$v = (u_1^{\otimes a_1^1} \otimes \ldots \otimes u_k^{\otimes a_k^1}, \ldots, u_1^{\otimes a_1^k} \otimes \ldots \otimes u_k^{\otimes a_k^k}) \in V.$$

The vector v is a $\lambda(K^*)$-eigenvector of weight N. The stabilizer of this vector (in the group G) is contained in the intersection of stabilizers of the lines $< u_1 >, \ldots, < u_k >$ and is an overgroup of H of index N.

2) If H is reductive then one can suppose that the orbit Gu is closed in U. This implies that the orbits $W(H)u$ and $\lambda(K^*)u$ are closed (in U and in U^H). Consequently the numbers $n_1, \ldots, n_k$ cannot be all positive or all negative, and there exist positive integer numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$ such that $a_1n_1 + \ldots + a_kn_k = 1$, $b_1n_1 + \ldots + b_kn_k = 0$, and $gcd(b_1, \ldots, b_k) = 1$.

Arguing as above, we construct a set of integer vectors $A_1, \ldots, A_k$, a space V and a vector $v \in V$ such that $G_v = H$ (here $N = 1$). This completes the proof.

Remark 2. For an arbitrary observable subgroup statement 2) of the previous lemma does not hold. For example, let G be the group SL_3 and $H = U$ be a maximal unipotent subgroup normalized by T. Consider the subtorus $T' = diag(t^2, t^3, t^{-5})$ in T as a one-parameter subgroup $\lambda(K^*)$. Any H-stable vector in a finite-dimensional G-module is a sum of highest weight vectors. The restriction of any dominant weight to T' has a non-trivial kernel and the stabilizer of such a vector contains H as a proper subgroup.

Proof of Theorem 3. Let V be the G-module from Lemma 2. In the projective space $\mathbb{P}(V)$ the point $< v >$ has the stabilizer H_1. Let Y be the closure of the orbit $G < v >$ in $\mathbb{P}(V)$. Now we shall recall the Akhiezer construction. By assumption, the complexity of the homogeneous space G/H_1 is positive. This implies that there exists a character $\xi : H_1 \to K^*$ such that for the corresponding line bundle L_ξ on G/H_1 the G-module $H^0(G/H_1, L_\xi)$ of regular sections contains two different isomorphic irreducible G-submodules, say W_1 and W_2, see [11, Theorem 1]. Choose two associated bases of T-eigenvectors $\{\phi_1, \ldots, \phi_m\}$ and $\{\psi_1, \ldots, \psi_m\}$ in W_1 and W_2 such that ϕ_1 and ψ_1 are highest weight vectors.

Consider the rational G-equivariant morphism $f : Y - \to \mathbb{P}^{2m-1}$ defined on the open G-orbit by the formula:

$$f(gH_1) = [\phi_1(g) : \ldots : \phi_m(g) : \psi_1(g) : \ldots : \psi_m(g)].$$

Let $\overline{X}$ be the closure of the graph of f in $Y \times \mathbb{P}^{2m-1}$. Set

$$Z_c = \{[a : b] \in \mathbb{P}^{2m-1} \mid a_i = cb_i \text{ for } i = 1, \ldots, m\},$$

where c is a parameter. Then Z_c is a closed G-invariant subvariety in $\mathbb{P}^{2m-1}$ and for $c_1 \neq c$ the intersection of Z_c and Z_{c_1} is empty. It is shown in [4] that the subsets $f^{-1}(Z_c)$ in $\overline{X}$ are non-empty for infinitely many values of c. This proves that $\overline{X}$ contains infinitely many G-orbits.

Let X be the affine cone over $\overline{X}$ with respect to the Segre embedding of $\mathbb{P}(V) \times \mathbb{P}^{2m-1}$. We fix a basis in V such that v is the first basis vector. The point $(< v >: f(< v >))$ on $\overline{X}$ corresponds to the line l on X with coordinates

$$(\alpha\phi_1(eH_1), \ldots, \alpha\phi_m(eH_1), \alpha\psi_1(eH_1), \ldots, \alpha\psi_m(eH_1), 0, \ldots, 0),$$

where α is a coordinate on the line $< v >$. The values $\phi_i(eH_1)$ and $\psi_i(eH_1)$ do not change under the action of $\lambda(K^*)$, the line l is H_1-invariant, and the stabilizer of any non-zero point on l is an overgroup H' of H of finite index. The cone over the dense G-orbit in $\overline{X}$ isomorphic to G/H_1 is a dense G-orbit in X isomorphic to G/H' and X contains infinitely many G-invariant cones corresponding to G-orbits in $\overline{X}$. Lemma 1 completes the proof.

4 Very symmetric embeddings.

The group of G-equivariant automorphisms of a homogeneous space G/H is isomorphic to $W(H)$ (the action $W(H) : G/H$ is induced by the action $N_G(H) : G/H$ by right multiplication). Let $G/H \hookrightarrow X$ be an affine embedding. The group $Aut_G X$ of G-equivariant automorphisms of X is a subgroup of $W(H)$.

Definition 2. An embedding $G/H \hookrightarrow X$ is said to be *very symmetric* if $W(H)^0 \subseteq Aut_G X$.

Any spherical affine variety is very symmetric. In fact, for a spherical homogeneous space G/H any isotypic component $K[G/H]_\mu$ of the G-algebra $K[G/H]$ is irreducible G-module (see [10] or Section 6) and $W(H)$ acts on $K[G/H]_\mu$ by scalar multiplication. This shows that any G-invariant subalgebra in $K[G/H]$ is $W(H)$-invariant too.

In the case of affine $SL(2)$-embeddings only the embedding $X = SL(2)$ is very symmetric, in all other cases the group $Aut_{SL(2)}X$ is isomorphic to a Borel subgroup in $SL(2)$, see [10, III.4.8, Satz 1]. More generally, if X is an affine embedding of the homogeneous space $G/\{e\}$ then X is very symmetric if and only if the action $G : X$ can be extended to an action of the group $G \times G$ with an open orbit isomorphic to $(G \times G)/H$, where H is the diagonal in $G \times G$. Hence X can be considered as an affine $(G \times G)/H$-embedding. Theorem 2 implies that if G is a semisimple group then $X = (G \times G)/H$, for other proofs see [12] and [13, Proposition 1].

If G is a reductive group then the set of all very symmetric embeddings of the homogeneous space $G/\{e\}$ is exactly the set of all affine algebraic monoids with G as group of units [13]. This demonstrates that very symmetric embeddings have a natural characterization in the variety of all affine $G/\{e\}$-embeddings. The classification of reductive algebraic monoids is obtained in [13] and [14].

Now we are interested in the following problem: when does there exist a very symmetric affine embedding of a homogeneous space G/H with infinite number of G-orbits ? The example of $SL(3)/\{e\}$-embeddings shows that the latter property is not equivalent to (AF).

Proposition 1. *Let H be a reductive subgroup in a reductive group G.*

1) If $W(H)^0$ is semisimple then there exists only one very symmetric affine embedding of G/H, namely $X = G/H$.

2) If $W(H)^0$ is not semisimple and $c(G/N_G(H)) \geq 1$ then there exists a very symmetric affine embedding of G/H with infinite number of G-(and even $G \times W(H)^0$-) orbits.

Proof. If X is a very symmetric affine embedding of G/H then one can consider X as a $(G \times W(H)^0)$-variety. The stabilizer of a point in a dense $(G \times W(H)^0)$-orbit in X is conjugated to $\tilde{H} = \{(h, hH) \mid h \in \gamma^{-1}(W(H)^0)\}$.

1) If $W(H)^0$ is semisimple then the Lie subalgebras in $Lie(G \times W(H)^0)$ corresponding to the group $\tilde{H}$ and to the normalizer of $\tilde{H}$ coincide. To check this, one may use the fact that the normalizer of a reductive Lie subalgebra in a reductive Lie algebra is the sum of this subalgebra and its centralizer. Hence Theorem 2 implies statement 1).

2) By assumption, there exists a non-trivial central one-parameter subgroup in $W(H)^0$. Extending $\tilde{H}$ by this subgroup we obtain a subgroup in $N_G(H) \times W(H)^0$,

which is not spherical in $G \times W(H)^0$. Theorem 3 implies that there exists an affine $(G \times W(H)^0)/\tilde{H})$-embedding with infinite number of $(G \times W(H)^0)$-orbits.

5 Classifications.

Let H be a reductive subgroup of G. We are interested in property (AF) for the homogeneous space G/H. Corollary 1 allows us to suppose that G is semisimple. If either H is spherical in G or the group $W(H)$ is finite then property (AF) holds. If $W(H)$ is not finite then there exists a non-trivial one-parameter subgroup $\lambda(K^*) \subset W(H)$ ($W(H)$ is reductive) and we can extend H by this subgroup. If we obtain a non-spherical subgroup in G then property (AF) does not hold for G/H (Theorem 3). So the only unclear case is the following:

H is a non-spherical subgroup in G but any extension of H by a one-parameter subgroup of $W(H)$ is spherical in G.

In this case the homogeneous space G/H has complexity one. Moreover, the rank of the group $W(H)$ is equal to one. Indeed, the field of B-invariant rational functions $K(G/H)^B$ is a field of rational functions in one variable. There is a natural action $W(H) : K(G/H)^B$. If T' is a torus in $W(H)$ and $dim\ T' \geq 2$ then the restricted action $T' : K(G/H)^B$ has a non-trivial kernel. The extension of H by this kernel is a non-spherical subgroup in G.

If the group G is simple then there is a classification of all connected reductive subgroups such that $c(G/H) = 1$, see [9, Table 1]. Below we list all such pairs $(G,\ H)$:

1) $(SL(n),\ SL(n-2) \times (K^*)^2),\ n \geq 3;$

2) $(Sp(2n),\ Sp(2n-4) \times SL(2) \times SL(2)),\ n \geq 3;$

3) $(SL(6),\ Sp(4) \times SL(2) \times K^*), \quad R(\tilde{\phi_1}) \mid_H = R(\phi_1) \otimes 1 \otimes \epsilon + 1 \otimes R(\phi'_1) \otimes \epsilon^{-2};$

4) $(SO(9),\ G_2 \times K^*), \quad R(\tilde{\phi_1}) \mid_H = R(\phi_1) \otimes 1 + 1 \otimes \epsilon + 1 \otimes \epsilon^{-1};$

5) $(SO(11),\ SL(2) \times Spin(7)), \quad R(\tilde{\phi_1}) \mid_H = R(2\phi_1) \otimes 1 + 1 \otimes R(\phi'_3);$

6) $(Sp(4),\ SL(2)),\ R(\tilde{\phi_1}) \mid_H = R(3\phi_1);$

7) $(E_6,\ Spin(9) \times K^*), \quad R(\tilde{\phi_1}) \mid_H = 1 \otimes 1 + R(\phi_1) \otimes 1 + R(\phi_4) \otimes \epsilon + 1 \otimes \epsilon;$

8) $(F_4,\ Spin(8)), \quad R(\tilde{\phi_1}) \mid_H = R(\phi_1) + R(\phi_3) + R(\phi_4) + 2;$

9) $(SL(2n),\ SL(n) \times SL(n));$

10) $(SO(n),\ SO(n-2)),\ n > 4;$

11) $(SO(2n+1),\ SL(n)),\ n>2,\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)+R(\phi_{n-1})+1;$

12) $SO(4n),\ SL(2n)),\ n>1,\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)+R(\phi_{n-1});$

13) $(Sp(2n),\ Sp(2n-2)),\ n>1,\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)+2;$

14) $(Sp(2n),\ SL(n)),\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)+R(\phi_{n-1});$

15) $(E_7,\ E_6),\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)+R(\phi_5)+2;$

16) $(SO(10),\ Spin(7)),\quad R(\tilde{\phi_1})\mid_H = R(\phi_3)+2;$

17) $(SL(n),\ SL(n-2)\times K^*),\ n>4,\quad R(\tilde{\phi_1})\mid_H = R(\phi_1)\otimes\epsilon+1\otimes\epsilon^{\alpha_1}+1\otimes\epsilon^{\alpha_2},$
$\alpha_1+\alpha_2=2-n,\ \alpha_1\neq\alpha_2.$

Comments. In the line $R(\tilde{\phi_1})\mid_H = \ldots$ we indicate the restriction of the simplest representaton of G to H. The fundamental weights of simple components of the semisimple part of H are denoted by $\phi_i, \phi_i', \ldots$, the corresponding irreducible representations are denoted by $R(\phi_i), R(\phi_i'), \ldots$; ϵ is a faithful 1-dimensional representation of the multiplicative group K^*; 1 is the trivial 1-dimensional representation.

Theorem 4. *Let G be a simple group and H be a reductive subgroup in G. Then there exists an affine embedding $G/H \hookrightarrow X$ with infinite number of G-orbits if and only if (G,H) satisfies one of the following conditions:*

1) $c(G/H)\geq 2$ and the group $N_G(H)/H$ is infinite;

2) $G=SL(n),\ n>4$ and $H^0=SL(n-2)\times K^*$, where $SL(n-2)$ is embedded in $SL(n)$ as the stabilizer of the first two basis vectors e_1 and e_2 in the minimal representation of $SL(n)$, and K^* acts on e_1 and e_2 with weights α_1 and α_2 such that $\alpha_1+\alpha_2=2-n$, $\alpha_1\neq\alpha_2$, and on $<e_3,\ldots,e_n>$ by scalar multiplication.

Proof. We have to consider only the case $c(G/H)=1$. Suppose that H is connected. The pairs 1)-8) from the above list are covered by Theorem 2. The pairs 9)-16) are considered in Proposition 2 in the next section. By Lemma 1, property (AF) holds for any finite extension of H in these cases. Finally, the pair 17) (and all finite extension of H here) satisfies the conditions of Theorem 3.

6 Embeddings of complexity one.

Now we need some more specific information about homogeneous spaces of complexity one. Let X be a G-variety. The decomposition of the algebra of regular functions

$$K[X]=\bigoplus_{\mu\in\Xi(X)} K[X]_\mu,$$

where $\mu\in\Xi(G)_+$ and $K[X]_\mu$ is the sum of all irreducible G-submodules in $K[X]$ isomorphic to V_μ, is called *the isotypic decomposition* of the algebra $K[X]$. Here

$\Xi(X)$ is the subset in $\Xi(G)_+$ consisting of all dominant weights μ such that $K[X]_\mu \neq \{0\}$. This subset is a subsemigroup in $\Xi(G)_+$, see [10]. The semigroup $\Xi(X)$ is called *the rank semigroup* of a G-variety X. In particular, if H is an observable subgroup in G then we obtain the rank semigroup $\Xi(G/H)$ for the quasi-affine homogeneous space G/H. It follows from Frobenius reciprocity that

$$\Xi(G/H) = \{\mu \in \Xi(G)_+ \mid V_{\mu^*}^H \neq 0 \}.$$

Denote by m_μ the multiplicity of the irreducible G-submodule V_μ in the G-algebra $K[G/H]$. An observable subgroup $H \subset G$ is spherical iff $m_\mu = 1$ for any $\mu \in \Xi(G/H)$, see [11, Theorem 1].

Theorem 5 ([9, Theorem 2]). *Let G be a simply connected semisimple group, and H be an observable subgroup in G without rational characters. If $c(G/H) = 1$ then there exists a unique weight $\omega \in \Xi(G/H)$ such that*

1) $m_\omega = 2$;

2) if $\mu \in \Xi(G/H)$, $\mu = e\omega + \delta$, $e \in \mathbb{N}$, $\delta \in \Xi(G/H)$, and $\delta - \omega \notin \Xi(G/H)$ then $m_\mu = e + 1$.

Definition 3. The weight $\omega \in \Xi(G/H)$ is called *the remarkable* weight for G/H.

If a subgroup H is reductive then the semigroup $\Xi(G/H)$ is stable under the Weyl involution $\mu \to \mu^*$ and $\omega^* = \omega$.

Without loss of generality it can be assumed that G is simply connected.

Proposition 2. *Let G be a simply connected simple group and H be a connected reductive subgroup in G such that $W(H)^0 \neq \{e\}$ and any extension of H by a one-parameter subgroup $\lambda(K^*)$ of $W(H)$ is spherical in G. Then property (AF) holds for G/H.*

Proof. Applying the normalization, we have to consider only normal embeddings. The center of any spherical reductive subgroup in a simple group is at most one-dimensional. Hence H is semisimple in our case. The rank semigroup $\Xi(G/H) = <\omega, \mu_1, \ldots, \mu_k>$ is free and the remarkable weight is one of the generators of $\Xi(G/H)$, see [9, 3.1]. We have $K[G/H]^U = K[x,y] \otimes K[f_{\mu_1}, \ldots, f_{\mu_k}]$, where x and y are highest weight vectors with weight ω, and $f_{\mu_1}, \ldots, f_{\mu_k}$ are other generators of $K[G/H]^U$, $\mu_i \neq \mu_j$ for $i \neq j$. There are a G-equivariant $\lambda(K^*)$-action on $K[G/H]$ and a T-equivariant $\lambda(K^*)$-action on $K[G/H]^U$.

Fact 1. One can choose x and y in $K[X]_\omega^U$ that are $\lambda(K^*)$-eigenvectors with opposite weights.

Fact 2. For the pairs 9)-15) $f_\mu \in K[G/H]^{\lambda(K^*)}$ for any $\mu \in <\mu_1, \ldots, \mu_k>$.

In order to check these facts one can compare the rank semigroups $\Xi(G/H)$ [9, Table 1] and $\Xi(G/H_1)$ [17, Table 1], where $H_1 = \lambda(K^*)H$.

Let $G/H \hookrightarrow X$ be a normal affine embedding. Then $X \setminus (G/H) = \bigcup_i D_i$, where D_i are irreducible G-stable divisors in X. Denote by ν_i the valuation of the field

$K(X)$ defined by the divisor D_i. A function $f \in K[G/H]$ is regular on X iff $\nu_i(f) \geq 0$ for all i, and the restriction of f to D_i is a non-zero function iff $\nu_i(f) = 0$.

Suppose that for a divisor D_i,

(*) *there exists a linear form $z = \alpha x + \beta y$ such that $\nu_i(x) \neq \nu_i(z)$.*

Let $\mu \in <\mu_1, \ldots, \mu_k>$. The T-isotypic component of $K[G/H]^U$ of weight $\mu + n\omega$ consists of the functions $f_\mu(a_0 x^n + a_1 x^{n-1} z + \ldots + a_n z^n)$. Here $\nu_i(x^{n-j_1} z^{j_1}) \neq \nu_i(x^{n-j_2} z^{j_2})$ for $j_1 \neq j_2$ and there exists at most one j such that $\nu_i(f_\mu) + \nu_i(x^{n-j} z^j) = 0$. Hence the algebra $K[D_i]$ is multiplicity free as a G-module. This proves that the number of G-orbits in D_i is finite. Let D be a union of all divisors D_i with property (*).

Consider $\tilde{X} = X \setminus D$. This is a quasi-affine unirational G-variety of complexity one and by a result of F. Knop [16] the algebra of regular functions $K[\tilde{X}]$ is finitely generated. Here $\nu_i(x) = \nu_i(z)$ for any $D_i \subset \tilde{X}$ and for any $z = \alpha x + \beta y$. We have $\nu_i(f_\mu(x^n + a_1 x^{n-1} y + \ldots + a_n y^n)) = \nu_i(f_\mu(c_1 x + d_1 y) \ldots (c_n x + d_n y)) = \nu_i(f_\mu) + n\nu_i(x)$. Hence either $K[G/H]_{\mu+n\omega} \subset K[\tilde{X}]$ or $K[G/H]_{\mu+n\omega} \cap K[\tilde{X}] = \{0\}$ for any $\mu + n\omega \in \Xi(G/H)$. This implies that the G-equivariant $\lambda(K^*)$-action on G/H can be extended to $\tilde{X}$.

Consider the quotient morphism $\pi : \tilde{X} \to S = \text{Spec } K[\tilde{X}]^{\lambda(K^*)}$. The affine variety S carries a natural G-action. We claim that $K[\tilde{X}]^{K^*} \neq K$. In fact, $K[\tilde{X}]$ is not a multiplicity free G-module and there exist $\mu \in <\mu_1, \ldots, \mu_k>$ and $n > 0$ such that $K[G/H]^U_{\mu+n\omega} \subset K[\tilde{X}]$. In cases 9)-15), we have $f_\mu \in K[G/H]^{K^*}$ for all μ (see Fact 1 and Fact 2) and then

(C) *there exist $\mu \in <\mu_1, \ldots, \mu_k>$ and $n > 0$ such that $f_\mu x^n y^n \in K[\tilde{X}]^{\lambda(K^*)}$*

In the exceptional case 16) ($SO(10)$, $Spin(7)$) we check that $\nu_i(x) \geq 0$ for all i. Indeed, consider the affine G-variety $X' = \text{Spec } K[\tilde{X}]$. There is an embedding $\tilde{X} \subset X'$ and it suffices to prove that the number of G-orbits in X' is finite. If the closed G-orbit in X' is isomorphic to G/L, where L is reductive and $L \neq G$ then $X' \cong G *_L X''$, where X'' is an affine embedding of L/H, see [5, Theorem 6.7]. In our case there are only two candidates for L: $SO(8)$ and $SO(8) \times SO(2)$, the pair (L, H) is spherical and therefore the number of G-orbits in X' is finite. If X' contains G-fixed point then, by the Bogomolov theorem [15], there exists a surjective G-equivariant morphism $X' \to C(\sigma)$ for some $\sigma \in \Xi_+(G)$, $\sigma \neq 0$, where $C(\sigma)$ is the closure of the orbit of a highest weight vector v_σ in V_σ. In this case H is contained in the proper parabolic subgroup $P_\sigma = G_{<v_\sigma>}$. But the subgroup $Spin(7)$ is contained only in one proper parabolic subgroup of $SO(10)$. Denote by $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ the fundamental weights of G. Then $\sigma = k\tilde{\phi}_1 = k\omega$ (here $\omega = \tilde{\phi}_1$, see [9, Table 1]) for some $k > 0$. The morphism $X' \to C(k\omega) \subset V_{k\omega}$ induces a homomorphism $K[V_{k\omega}] \to K[X']$. This implies that there exists a B-semi-invariant function in $K[X']$ of weight $k\omega$. This function can be written as $f = a_0 x^k + a_1 x^{k-1} y + \ldots + a_k y^k$ and $\nu_i(f) = k\nu_i(x) \geq 0$. Hence $\nu_i(x) \geq 0$ for all $D_i \subset \tilde{X}$. Finally we have $xy \in K[X']^{\lambda(K^*)}$ (see Fact 2).

The action $G : S$ is quasihomogeneous. Let G/F be a dense G-orbit in S. Recall that $H_1 = \gamma^{-1}(\lambda(K^*))$. We have the restricted map $G/H_1 \to G/F$ and hence $H_1 \subset F$. The subgroup F is an observable subgroup in G. We claim that F is reductive. In fact, if $rk(H_1) = rk(G)$ then $rk(F) = rk(G)$ and F is reductive [6]. In the exceptional case $(SO(10),\ Spin(7))$ it is also possible to show that $H_1 = Spin(7) \times SO(2)$ is not contained in any proper quasiparabolic subgroup (for the definition see [6]) of $SO(10)$ and thus H_1 is not contained in any observable non-reductive subgroup of $SO(10)$.

Case 1. $F = H_1$.

Then $S = G/H_1$ by Theorem 2. Any fiber of the morphism π is the closure of $K^* \cong H_1/H$ in $\tilde{X}$. Hence any fiber is isomorphic either to K or to K^* and the number of G-orbits in $\tilde{X}$ is at most two.

Case 2. $F \neq H_1$.

There are only three possibilities for F. In all cases $S = G/F$ by Theorem 2.

1) $G = SO(2n+1)$, $F = SO(2n)$, $H = SL(n)$, $n > 3$. Here $\omega = \tilde{\phi}_n$ [9] and $\Xi(G/F) =< \tilde{\phi}_1 >$ [17]. This contradicts condition (C).

2) $G = Sp(2n)$, $F = Sp(2n-2) \times SL(2)$, $H = Sp(2n-2)$, $n > 1$. Here $\omega = \tilde{\phi}_1$ [9] and $\Xi(G/F) =< \tilde{\phi}_2 >$ [17]. This contradicts condition (C).

3) $G = SO(10)$, $F = SO(2) \times SO(8)$, $H = Spin(7)$. Consider the restricted morphism $G/H \to G/F$. The preimage of the point eF is F/H. The closure Z of F/H in $\tilde{X}$ is a spherical F-variety (for the restricted action $F : \tilde{X}$) and it contains finitely many F-orbits. We have $\tilde{X} \cong G *_F Z$ and the number of G-orbits in $\tilde{X}$ is finite. The proof of Proposition 2 is completed.

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References

[1] E. B. Vinberg, Complexity of actions of reductive groups, Funktsional'nyi Analiz i ego Prilog. **20 (1)** (1986), 1-13 (in Russian); English transl.: Func. Anal. and its Appl. **20 (1)** (1986), 1-11.

[2] D. Luna and Th. Vust, Plogements d'Espaces Homogénes, Comment. Math. Helvetici **58** (1983), 186-245.

[3] F. J. Servedio, Prehomogeneous vector spaces and variety, Trans. Amer. Math. Soc. **176** (1973), 421-444.

[4] D. N. Akhiezer, Actions with a finite numbr of orbits, Funktsional'nyi Analiz i ego Prilog. **19 (1)** (1985), 1-5 (in Russian); English transl.: Func. Anal. and its Appl. **19 (1)** (1985), 1-4.

[5] V. L. Popov and E. B. Vinberg, Invariant theory, Itogi Nauki i Tekhniki, Sovremennie Problemy Mat. Fundamentalnie Napravleniia, vol. 55, VINITI, Moscow 1989 (in Russian); English transl., Algebraic Geometry IV, Encyclopaedia of Math. Sciences, vol. 55, Springer-Verlag, Berlin 1994, pp. 123–278.

[6] A. A. Sukhanov, Description of the observable subgroups of linear algebraic groups, Mat. Sbornik **130** (1986), 310-334 (in Russian); English transl.: Math. USSR-Sb. **58** (1987), 311-335.

[7] B. Kostant, Lie group representations on polynomial rings, Amer. J. Math. **85** (1963), 327-404.

[8] D. Luna, Adhérences d'orbite et invariants, Inventiones Math. **29** (1975), 231-238.

[9] D. I. Panyushev, Complexity of quasiaffine homogeneous varieties, t-decompositions, and affine homogeneous spaces of complexity 1, Advaces in Soviet Mathematics, Volume 8, ed. by E. B. Vinberg, 1992, 151-166.

[10] H. Kraft, Geometrische Methoden in der Invariantentheorie, Vieweg Verlag, Braunschweig 1985; Russian transl.: Mir, Moscow 1987.

[11] B. N. Kimel'feld and E. B. Vinberg, Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups, Funktsional'nyi Analiz i ego Prilog. **12 (3)** (1978), 12-19 (in Russian); English transl.: Func. Anal. and its Appl. **12 (3)** (1978), 168-174.

[12] W. C. Waterhouse, The unit group of affine algebraic monoids, Proc. Amer. Math. Soc. **85** (1982), 506-508.

[13] E. B. Vinberg, On reductive algebraic semigroups, In: Lie Groups and Lie Algebras, E. B. Dynkin Seminar, Amer. Math. Soc. Transl (2) **169** (1995), 145-182.

[14] A. Rittatore, Algebraic monoids and group embeddings, Transformation Groups **3 (4)** (1998), 375-396.

[15] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, Izv. Akad. Nauk SSSR, Ser. Mat. **42 (6)** (1979), 1227-1287 (in Russian); English transl.: Math USSR, Izv. **13** (1979), 499-555.

[16] F. Knop, Über Hilberts vierzehntes Problem für Varietäten mit Kompliziertheit eins, Math. Z. **213** (1993), 33-35.

[17] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen, Composit. Math. **38** (1979), 129-153.

Ivan Arzhantsev
Department of Mathematics
Moscow State Pedagogical University
ul. Krasnoprudnaia 14, Moscow, Russia

e-mail: ivan@arjantse.mccme.ru

Current address:

Université de Grenoble I
Institut Fourier
Laboratoire de Mathématiques
UMR 5582 CNRS - UJF
B. P. 74
38402 ST MARTIN D'HÈRES Cedex (France)

e-mail: arjantse@mozart.ujf-grenoble.fr