Eigenvalue and "Twisted" Eigenvalue Problems, Applications to CMC Surfaces

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Abstract. In this paper we investigate an eigenvalue problem which appears naturally when one considers the second variation of a constant mean curvature immersion. In this geometric context, the second variation operator is of the form $\Delta_g + b$, where b is a real valued function, and it is viewed as acting on smooth functions with compact support *and* with mean value zero. The condition on the mean value comes from the fact that the variations under consideration preserve some balance of volume. This kind of eigenvalue problem is interesting in itself. In the case of a compact manifold, possibly with boundary, we compare the eigenvalues of this problem with the eigenvalues of the usual (Dirichlet) problem and we in particular show that the two spectra are interwined (in fact strictly interwined generically). As a by-product of our investigation of the case of a complete manifold with infinite volume we prove, under mild geometric conditions when the dimension is at least 3, that the strong and weak Morse indexes of a constant mean curvature hypersurface coincide.

Résumé. Dans cet article, nous étudions un problème de valeurs propres qui apparaît naturellement quand on considère la variation seconde d'une immersion à courbure moyenne constante. Dans ce contexte géométrique, l'opérateur de variation seconde est de la forme $\Delta_g + b$, où b est une fonction à valeurs réelles et on fait agir l'opérateur sur les fonctions lisses à support compact *et* dont la valeur moyenne est nulle. La condition de valeur moyenne nulle vient du fait que les variations considérées doivent préserver une contrainte de volume. Ce type de problème de valeurs propres est intéressant en lui-même. Dans le cas compact (éventuellement à bord), nous comparons les valeurs propres de ce problème avec celle du problème de Dirichlet (usuel) et nous montrons en particulier que les deux spectres sont entrelacés (en fait strictement entrelacés pour un potentiel générique). Comme sous-produit de l'étude de ce problème spectral dans le cas d'une variété complète de volume infini nous montrons, sous une hypothèse géométrique faible en dimension supérieure ou égale à 3, que les indices de Morse faible et fort d'une hypersurface à courbure moyenne constante coincident.

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1 Introduction

Motivations. Let $i : (M^n, g) \to \overline{M}^{n+1}$ be an isometric immersion with mean curvature $H\nu$, where ν is a unit normal vector field along $i(M)$. Let i_t be a variation of the immersion i, associated to the normal vector field $f\nu$, with $f \in C_0^\infty(M)$ The corresponding variation of the *area functional* $A(t)$ is given by

$$A'(0) = -n \int_M Hf\, dv_g.$$

Minimal immersions are critical points of the area functional. Constant mean curvature immersions can be viewed as critical points of the area functional restricted to variations which preserve the *enclosed volume*, *i.e.* to functions $f \in C_0^\infty(M)$ which satisfy the additional condition $\int_M f\, dv_g = 0$.

For such critical points, the second variation of the area functional is given by $A''(0) = \int_M fLf\, dv_g$, with $L := \Delta_g - |A|^2 + \overline{r}(\nu)$, where Δ_g is the non-negative Laplacian, A the second fundamental form of the immersion and $\overline{r}(\nu)$ the Ricci curvature of $\overline{M}$ evaluated on the unit vector ν. The operator L is called the *stability operator* of the immersion.

In the minimal case, the stability operator is viewed as acting on $C_0^\infty(M)$; in the case of a constant mean curvature immersion, the stability operator is viewed as acting on $C_0^\infty(M) \cap \{\int_M f\, dv_g = 0\}$. This is geometrically meaningful as the following observation shows. The stability operator of a geodesic sphere M in a space form $\overline{M}$ has a negative eigenvalue on $C^\infty(M)$ whereas it is non-negative on $C_0^\infty(M) \cap \{\int_M f\, dv_g = 0\}$ (geodesic spheres are stable as constant mean curvature immersions) [2].

Of particular interest in this context is the index (*i.e.* the number of negative eigenvalues) of the quadratic form $\int_M fLf\, dv_g$ on either $C_0^\infty(M)$, the so-called *strong index*, or on $C_0^\infty(M) \cap \{\int_M f\, dv_g = 0\}$, the so-called *weak index*. As pointed out precedingly, the weak index is the more geometrically natural in the context of constant mean curvature immersions. On the other-hand, it is easier for example to use the strong stability condition than the weak stability condition (stability meaning zero index). Our main motivation for writing this paper was an attempt at understanding the relationship between these two notions. We refer the reader to [2, 4, 8, 9, 12] for more details on index questions.

Contents. Let (M, g) be a Riemannian manifold with boundary and let b be a continuous real valued function on M. We want to investigate the eigenvalue problem, and in particular the index, associated with the quadradic form

$$q(f) := \int_M (|df|_g^2 + bf^2)\, dv_g, \tag{1.1}$$

on smooth functions subject to the conditions

$$f|\partial M = 0 \quad \text{and} \quad \int_M f\, dv_g = 0. \tag{1.2}$$

We call this eigenvalue problem *twisted* because the condition $\int_M f\, dv_g = 0$ is in general not related to the *ordinary* Dirichlet problem associated with the quadratic form (1.1) on smooth functions subject to the sole condition

$$(1.3) \qquad f|\partial M = 0.$$

We shall in fact consider two cases:

(a) M is compact with boundary ∂M (possibly empty);

(b) M is complete, with infinite volume.

In case (a), the condition $\int_M f\, dv_g = 0$ is in general not related to the spectral properties of the operator $\Delta + b$. When M has a boundary, we will only consider Dirichlet boundary conditions (one could also consider Neumann boundary conditions). In case (b), the condition $\int_M f\, dv_g = 0$ does not a priori make sense and we will have to find out what the natural eigenvalue problem is.

We will consider the following questions.

1. Investigate the operator and the eigenvalue problem associated with (1.1) and (1.2). We will call this eigenvalue problem the *twisted eigenvalue problem* (or T-eigenvalue problem).
2. Properties of the T-eigenvalues: comparison with the eigenvalues of the (ordinary) problem (1.1) and (1.3), multiplicities, monotonicity properties, generic properties.
3. Properties of the T-eigenfuctions (unique continuation, Courant property).
4. Explicit examples of T-eigenvalues (intervals, cylinders, Euclidean balls).
5. Applications to geometry (second variation of constant mean curvature immersions).

Section 2 is devoted to the compact case. The basic properties of the twisted Dirichlet eigenvalues and eigenfunctions are stated in Proposition 2.2, in parallel with those of the ordinary Dirichlet problem. The proofs are similar for both cases; we only sketch them. We also give the structure of the eigenspaces when the twisted and the ordinary Dirichlet problems have a common eigenvalue (Proposition 2.4). In Section 2.3, we show that the T-eigenvalues are generically simple (with respect to the potential function b) and that they generically strictly intertwine the Dirichlet eigenvalues. The methods for proving such genericity results may not be so well-known; we have decided to give them with some details. Section 2.4 is devoted to some examples in which one can describe the T-eigenvalues quite explicitly (with $b = 0$). These examples provide negative answers to some natural questions.

In Section 3 we deal with the case of complete manifolds with infinite volume. For such manifolds, the condition $\int_M f\, dv_g = 0$ does not make sense for arbitrary L^2 functions and it is not clear then what the spectral problem actually is.
We consider the operators

$$L := (C_0^\infty(M), \Delta_g + b)$$

and

$$L_T := \left(C_0^\infty(M) \cap \left\{ f \mid \int_M f\, dv_g = 0 \right\}, \Delta_g + b \right).$$

We first show that they are densely defined in $L^2(M)$ (this follows from the infinite volume assumption). Under a mild volume growth assumption on the manifold M, Assumption (3.24), and an assumption on the potential function b, Assumption (3.29), we prove that the Friedrichs extensions of the operators L and L_T coincide (Theorem 3.5).

We finally investigate index questions (Theorems 3.7 and 3.10). In terms of our geometric motivations, we prove that the weak and strong Morse indexes of a constant mean curvature surface into hyperbolic space coincide. We also prove that the same result holds, under mild assumptions on the volume growth of the manifold, when the dimension is at least 3, for constant mean curvature hypersurfaces (see Section 4). As a corollary, we prove da Silveira's result that a weakly stable surface of constant mean curvature 1 in hyperbolic space (with sectional curvature -1) is a horosphere [6, 16].

2 Compact manifolds

In this section we investigate the twisted Dirichlet problem (to be defined in Section 2.2) and we compare the associated eigenvalues with the eigenvalues of the (ordinary) Dirichlet problem on the manifold. This is the purpose of Proposition 2.2.

In Section 2.3, we consider the generic properties of the eigenvalues (with respect to variations of the potential function b).

In Section 2.4, we give examples in which one can explicitly describe the eigenvalues of the twisted Dirichlet problem.

2.1 Notations

• Let (M, g) be a compact Riemannian manifold. We use dv_g to denote the Riemannian measure, $|df|_g$ to denote the norm of the differential of the function f and Δ_g to denote the (non-negative) Laplacian (*i.e.* $\Delta_g = -\frac{d^2}{dx^2}$ on $\mathbb{R}$). In the sequel, b will denote a continuous real valued function on M.

• The spaces $L^p(M)$ will be understood with respect to the Riemannian measure v_g. In particular, the L^2-norm is given by

$$\|f\|^2 := \int_M |f|^2\, dv_g. \tag{2.4}$$

Since M is compact, $L^2(M)$ embeds continuously into $L^1(M)$ and we can define the Hilbert space

$$L^2_T(M) := \{ f \in L^2(M) \mid \int_M f\, dv_g = 0 \}. \tag{2.5}$$

We denote by $\mathcal{D}(M)$, *resp.* by $\mathcal{D}_T(M)$, the space of smooth functions with compact support in the interior of M, *resp.* the subspace of those functions in $\mathcal{D}(M)$ which have mean value zero,

$$\text{(2.6)}\quad \mathcal{D}(M) := C_0^\infty(M) \;\text{ and }\; \mathcal{D}_T(M) := \{ f \in \mathcal{D}(M) \mid \int_M f\, dv_g = 0 \}.$$

• Whenever it is defined, the H^1-norm is given by

$$\text{(2.7)}\qquad \|f\|_1^2 := \int_M \left(|df|_g^2 + f^2 \right) dv_g .$$

We denote by $H_0^1(M)$ the closure of $\mathcal{D}(M)$ with respect to the H^1-norm defined by (2.7). When $\partial M = \emptyset$, $H_0^1(M)$ is just the closure $H^1(M)$ of $C^\infty(M)$ with respect to the H^1-norm. We also define the space

$$\text{(2.8)}\qquad H_{0,T}^1(M) := H_0^1(M) \cap L_T^2(M).$$

• We denote by $H^2(M)$ the closure of $C^\infty(M)$ with respect to the H^2-norm

$$\|f\|_2^2 := \int_M \left(|D^g df|_g^2 + |df|_g^2 + f^2 \right) dv_g$$

where $|D^g df|_g^2$ is the norm of the Hessian of the function f (with respect to the metric g).

• We introduce two linear forms Φ_g and Ψ_g, given by the following formulas whenever they make sense:

$$\text{(2.9)}\qquad \begin{cases} \Phi_g(f) := \int_M f\, dv_g , \\ \Psi_g(f) := \mathrm{Vol}\,(M)^{-1} \int_M (\Delta_g + b) f\, dv_g . \end{cases}$$

• We let q denote the quadratic form

$$\text{(2.10)}\qquad q(f) := \int_M \left(|df|_g^2 + b f^2 \right) dv_g$$

which we will consider on various function spaces.

• Finally, we fix a function ρ such that

$$\text{(2.11)}\qquad \rho \in \mathcal{D}(M), \quad 0 \le \rho \le 1, \quad \int_M \rho\, dv_g = 1.$$

2.2 Preliminary results

The (ordinary) Dirichlet problem for the operator $\Delta_g + b$ on M is classically associated with the quadratic form q^M given by q, with domain $H_0^1(M)$.

We call *twisted Dirichlet problem* the problem associated with the quadratic q_T^M given by q, with domain $H_{0,T}^1(M)$.

The twisted Dirichlet problem arises naturally when one considers the stability operator of a constant mean curvature hypersurface. The twisted condition comes from the fact that one then considers variations which keep some balance of volume fixed (see [2] for more details). Since we will work on the space $L^2_T(M)$, we state the following lemma for future reference.

Lemma 2.1 *The space $\mathcal{D}_T(M)$ is dense in $L^2_T(M)$ and in $H^1_{0,T}(M)$ with respect to the L^2-norm and to the H^1-norm respectively.*

The twisted Dirichlet problem has eigenvalues and eigenfunctions which have properties similar to those of the (ordinary) Dirichlet problem. The eigenvalues of both problems are intertwined. These properties are summarized in the following proposition.

Proposition 2.2 *The Dirichlet and the twisted Dirichlet eigenvalue problems have the following properties.*

1. Quadratic forms and operators:

The quadratic form $q^M = \big(H^1_0(M), q\,\lvert H^1_0(M)\big)$ *is closed and is associated with the self-adjoint operator* $\big(H^2(M) \cap H^1_0(M), \Delta_g + b\big)$ *on $L^2(M)$. The corresponding eigenvalue problem on $L^2(M)$ is given by* (2.12) $\begin{cases} (\Delta_g + b)u = \lambda u \\ u\,\lvert\partial M = 0 \end{cases}$	*The quadratic form* $q^M_T = \big(H^1_{0,T}(M), q\,\lvert H^1_{0,T}(M)\big)$ *is closed and is associated with the self-adjoint operator* $\big(H^2(M) \cap H^1_{0,T}(M), \Delta_g + b - \Psi_g\big)$ *on $L^2_T(M)$. The corresponding eigenvalue problem on $L^2_T(M)$ is given by* (2.13) $\begin{cases} (\Delta_g + b)u - \Psi_g(u) = \lambda u \\ u\,\lvert\partial M = 0 \\ \Phi_g(u) = 0 \end{cases}$

2. The spectra:

The spectrum $\delta(M; g, b)$ of the Dirichlet problem consists of eigenvalues with finite multiplicities $\lambda^D_1(M) < \lambda^D_2(M) \leq \lambda^D_3(M) \leq \ldots$	*The spectrum $\delta^T(M; g, b)$ of the twisted Dirichlet problem consists of eigenvalues with finite multiplicities* $\lambda^T_1(M) \leq \lambda^T_2(M) \leq \lambda^T_3(M) \leq \ldots$

The eigenvalues satisfy the min-max principle and they are deacreasing functions of the domain (with respect to inclusion).

3. Properties of the eigenfunctions: *The eigenfunctions satisfy the unique continuation property.*

An eigenfunction associated with the k-th eigenvalue has at most k nodal domains.	*An eigenfunction associated with the k-th eigenvalue has at most $(k+1)$ nodal domains.*

4. Comparison of the eigenvalues: *The eigenvalues of the Dirichlet and of the twisted Dirichlet problem are intertwined,*

a) $\lambda_1^D(M) < \lambda_1^T(M)$,

b) $\forall k \geq 1, \quad \lambda_k^D \leq \lambda_k^T \leq \lambda_{k+1}^D$.

Proof. We have written the statements in two columns to make the comparison between the ordinary and the twisted problems easier. The assertions concerning the (ordinary) Dirichlet problem are well known. The assertions concerning the twisted problem can be proved using similar methods.

Proof of Assertion 1. The fact that q_T^M is closed follows immediately from the fact that b is bounded and from the definitions (see [13], §VI.1.3, page 313 ff).

In order to identify the self-adjoint operator associated with q_T^M, we use the first representation theorem, Theorem VI.2.1 of [13], and its corollaries. Consider the operator $J_T f := (\Delta_g + b)f - \Psi_g(f)$, with domain $D(J_T) = \mathcal{D}_T(M)$, where Ψ_g is defined by (2.9). Since $\mathcal{D}_T(M)$ is dense in $L^2_T(M)$, J_T is densely defined. Furthermore, J_T is symmetric and its adjoint J_T^* is given by

$$D(J_T^*) = H^2(M) \cap L^2_T(M),$$

$$J_T^* u = (\Delta_g + b)u - \Psi_g(u),$$

where Δ_g acts in the sense of distributions.

Let us denote by S the operator

$$D(S) = H^2 \cap H^1_{0,T}\,,$$

$$S(u) = (\Delta_g + b)u - \Psi_g(u), \forall u \in D(S).$$

Integrating by parts, we have $\langle S(u), v\rangle = q_T^M[u,v]$ for any $u \in D(S), v \in \mathcal{D}_T$ and it follows from Corollary 2.4 in [13] (§VI.2.1, page 323) that $S \subset Q_T^M$, the self-adjoint operator associated with the quadratic form q_T^M. Since $J_T \subset S$, we have $S \subset Q_T^M \subset S^* \subset J_T^*$. For $u \in D(S), v \in D(S^*)$, we have $\langle Su, v\rangle = \langle u, S^* v\rangle$. This implies that $\langle u, (\Delta_g + b)v\rangle = \langle(\Delta_g + b)u, v\rangle$ and hence, integrating by part, that $\int_{\partial M} v \frac{\partial u}{\partial \nu} = 0$ for all $u \in D(S)$. It follows that v must vanish on ∂M *i.e.* that $v \in D(S)$. This implies that S is self-adjoint and hence that $S = Q_T^M$. □

Proof of Assertion 2. The easiest method to prove Assertion 2, at least for our purposes, is the variational method. The proof is exactly the same as for the (ordinary) Dirichlet problem (see [3] for example). This methods provides the existence result and the fact that the eigenvalues satisfy the min-max principle, from which we deduce that they are decreasing functions of the domain with respect to inclusion.

Proof of Assertion 3. Let u be an eigenfunction of the twisted Dirichlet problem in M,

$$\begin{cases} (\Delta_g + b)u = \lambda u + \Psi_g(u), \\ u|\partial M \text{ and } \Phi_g(u) = 0, \end{cases}$$

and assume that u vanishes at infinite order at some point x_0. Evaluating the above equation at this point, we find that $\Psi_g(u) = 0$ and hence that u is an eigenfunction of the (ordinary) Dirichlet problem. The assertion follows from the unique continuation of Dirichlet eigenfunctions.

The proof of Courant's theorem for the eigenfunctions of the twisted Dirichlet problem is the same as in the case of the Dirichlet problem (see [3] for example). Assuming that some eigenfunction u associated with λ_k^T has at least $(k+2)$ nodal domains, one uses the min-max principle to construct another non-trivial eigenfuction v associated with λ_k^T and which vanishes on an open set, a contradiction with the preceding assertion. Note that $(k+1)$ would not suffice because we have to write that v has mean value zero and is orthogonal to the first k eigenfunctions.

Proof of Assertion 4. The proof of the second assertion follows immediately from the min-max principle.

In order to prove the first assertion, we only have to prove that the equality case is not possible. Assume that $\lambda_1^D(M) = \lambda_1^T(M) =: \lambda$. Then there are corresponding eigenfunctions which satisfy $(\Delta_g + b)u^D = \lambda u^D$ and $(\Delta_g + b)u^T = \lambda u^T + \Psi_g(u^T)$, with $u^D > 0$, $\int_M u^T = 0$. Multiplying the first equation by u^T, the second by u^D and integrating by parts, we obtain $\Psi_g(u^T)(\int_M u^D) = 0$. This implies that $\Psi_g(u^T) = 0$ and hence that u^T is an eigenfunction for the Dirichlet problem, associated with the first eigenvalue λ_1^D. This is not possible since $\int_M u^T = 0$.

We have concluded the proof of Proposition 2.2. ∎

The following corollary is a direct consequence of the unique continuation property of the eigenfunctions of the twisted Dirichlet problem.

Corollary 2.3 *Assume that M_1 and M_2 are two relatively compact domains in some Riemannian manifold M, and that $M_1 \subset M_2$. If furthermore $M_2 \setminus M_1$ has non empty interior, then*

$$\forall k, \ \ \lambda_k^T(M_1) > \lambda_k^T(M_2).$$

Remark 1. The *Courant theorem* for the twisted Dirichlet eigenfunctions is best possible because any eigenfunction associated with $\lambda_1^T(M)$ has at least two nodal domains.

Remark 2. The general picture for the eigenvalues is therefore

$$\lambda_1^D < \lambda_1^T \leq \lambda_2^D \leq \lambda_2^T \leq \lambda_3^D \leq \ \cdots \ \leq \lambda_k^D \leq \lambda_k^T \leq \lambda_{k+1}^D \ \leq \cdots$$

The equality $\lambda_k^D = \lambda_k^T$ will in particular occur whenever $\lambda_{k+1}^D = \lambda_k^D$ or $\lambda_{k-1}^T = \lambda_k^T$. The following proposition describes what happens when $\lambda_k^D = \lambda_k^T$.

Proposition 2.4 *Let the compact Riemannian manifold (M, g) and the potential b be given. Assume that there exists some common eigenvalue λ to the Dirichlet and to the twisted Dirichlet spectra $\lambda \in \delta(M; g, b) \cap \delta^T(M; g, b)$. Let E_λ^T denote the eigenspace for the twisted Dirichlet problem associated with $\lambda \in \delta^T(M; g, b)$ and let E_λ^D denote the eigenspace of the Dirichlet problem associated with $\lambda \in \delta(M; g, b)$. Let*

$$E_\lambda^{T,0} := \{u \in E_\lambda^T \mid \Psi_g(u) = 0\},$$

$$E_\lambda^{D,0} := \{u \in E_\lambda^D \mid \Phi_g(u) = 0\}.$$

Then,

1. $E_\lambda^{T,0} = E_\lambda^{D,0}$,
2. *For* E_λ^T *and* E_λ^D, *one has one of the following possibilities*
   (a) $E_\lambda^T = E_\lambda^{T,0} = E_\lambda^{D,0} = E_\lambda^D$,
   (b) $E_\lambda^T = E_\lambda^{T,0}$ *and there exists* $v \in E_\lambda^D$ *such that* $E_\lambda^D = E_\lambda^{D,0} \overset{\perp}{\oplus} \mathbb{R}v$, *with* $\Phi_g(v) \neq 0$,
   (c) $E_\lambda^D = E_\lambda^{D,0}$ *and there exists* $v \in E_\lambda^T$ *such that* $E_\lambda^T = E_\lambda^{T,0} \overset{\perp}{\oplus} \mathbb{R}v$, *with* $\Psi_g(v) \neq 0$.

Proof. The first assertion is clear. Take $u \in E_\lambda^T$ and $v \in E_\lambda^D$ (associated with the same eigenvalue λ). Multiplying the equality $(\Delta_g + b)u = \lambda u + \Psi_g(u)$ by v and integrating, taking into account the properties of u and v, we obtain

$$\Phi_g(v)\,\Psi_g(u) = 0.$$

On the other-hand, it is clear that there is a subspace of codimension at most 1 in E_λ^T (*resp.* in E_λ^D) on which Ψ_g (*resp.* Φ_g) vanishes. The second assertion follows from these observations. ∎

Questions. We may ask two natural questions.

$$\text{(2.14)} \qquad \begin{cases} \text{a)} & \text{May the equality } \lambda_1^T = \lambda_2^D \text{ occur ?} \\ \text{b)} & \text{Does } \lambda_1^T \text{ always have multiplicity one ?} \end{cases}$$

Examples show that the answer to Question (2.14a) is *Yes* and that answer to Question (2.14b) is *No* in general (see Section 2.4). On the other hand, one can show that for generic potentials b, $\lambda_1^T(b) < \lambda_2^D(b)$ and that $\lambda_1^T(b)$ has multiplicity one. We prove that in the next Section.

2.3 Generic properties of eigenvalues and eigenfunctions

To simplify the presentation, we limit ourselves to the case of perturbations by potentials.

Let (M, g) be a compact Riemannian manifold, possibly with boundary. With the notations as in Proposition 2.2, we have the following genericity result.

Proposition 2.5 *Let the compact Riemannian manifold* (M, g) *be given. Then,*

1. *There exists a residual set* U *in* $C^0(M)$ *such that for* $b \in U$, *the spectra* $\delta(M; g, b)$ *and* $\delta^T(M; g, b)$ *are simple;*
2. *There exists a residual set* V *in* $C^0(M)$ *such that for* $b \in V$, *the spectra* $\delta(M; g, b)$ *and* $\delta^T(M; g, b)$ *are simple and disjoint.*

Consider the (Dirichlet) eigenvalue problem

$$(2.15)\qquad \begin{cases} (\Delta_g + b)u = \lambda u \quad \text{in} \quad M, \\ u|\partial M = 0. \end{cases}$$

Let

$$(2.16)\qquad \begin{cases} \lambda_1^D(g,b) < \lambda_2^D(g,b) \leq \ldots \leq \lambda_k^D(g,b) \leq \ldots \nearrow \infty \\ u_1(g,b), u_2(g,b), \ldots, u_k(g,b) \ldots \end{cases}$$

be the set of eigenvalues and an orthonormal basis of associated eigenfunctions.

Let us state a preliminary result.

Proposition 2.6 *The following properties hold.*

1. *For a fixed metric g on M, the set*

$$\mathcal{B}_g := \{b \in C^0(M) \mid \lambda_k^D(g,b) \ \ \textit{is simple for all} \ \ k \geq 1\}$$

is residual in $C^0(M)$.

2. *For a fixed metric g on M, and a fixed continuous linear functional Φ on $L^2(M)$, the set*

$$\mathcal{B}_g := \{b \in C^0(M) \mid \lambda_k^D(g,b) \ \ \textit{simple and} \ \ \Phi(u_k(g,b)) \neq 0 \ \ \textit{for all} \ \ k \geq 1\}$$

is residual in $C^0(M)$.

Proof. Assertion 1 is well known; Assertion 2 does not seem to appear explicitly in the literature. The proofs follow classical lines [17, 1]. ∎

Proof of Proposition 2.5. For $k \geq 0$, define the sets

$$U_k := \{b \in C^0(M) \mid \lambda_1^T, \ldots, \lambda_k^T \ \text{ and } \ \lambda_1^D, \ldots, \lambda_k^D \ \text{ are simple}\}.$$

Then, clearly,

$$C^0(M) = U_0 \supset U_1 \supset \ldots \supset U_k \ldots \supset U_\infty = \bigcap_{k=1}^{\infty} U_k.$$

The sets U_k are open because the eigenvalues of both the Dirichlet and the twisted Dirichlet problem are continuous with respect to variations of the potential function b (this follows from the min-max principle).

Claim: U_{k+1} is dense in U_k.

Take $b \in U_k$. If λ_{k+1}^T and λ_{k+1}^D are both simple, $b \in U_{k+1}$ and we have nothing to prove. We now assume that $\lambda_k^T \neq \lambda^T := \lambda_{k+1}^T = \ldots = \lambda_{k+m}^T \neq \lambda_{k+m+1}^T$ and $\lambda_k^D \neq \lambda := \lambda_{k+1}^D = \ldots = \lambda_{k+p}^D \neq \lambda_{k+p+1}^D$. We consider the perturbation $L_t := \Delta_g + b + t\beta$ of the operator L_0, where β is a real valued continuous function. According to the perturbation theory of self-adjoint operators (see [13], Chap. II and VII), there exist analytic families of eigenvalues and orthonormal eigenfunctions

$$\begin{cases} \lambda^T + t\dot{\mu}_j^T + o(t), & 1 \le j \le m, \\ w_j^T + t\dot{w}_j^T + o(t), & 1 \le j \le m, \end{cases}$$

where $\{w_j^T\}$ is an orthonormal basis of the λ^T-eigenspace, and

$$\begin{cases} \lambda + t\dot{\mu}_j + o(t), & 1 \le j \le p, \\ w_j + t\dot{w}_j + o(t), & 1 \le j \le p, \end{cases}$$

where $\{w_j\}$ is an orthonormal basis of the λ-eigenspace, such that

$$(\Delta_g + b)\dot{w}_j^T + \beta w_j^T = \lambda^T \dot{w}_j^T + \dot{\mu}_j^T w_j^T + \int_M \beta w_j^T + \int_M (\Delta_g + b)\dot{w}_j^T$$

and

$$(\Delta_g + b)\dot{w}_j + \beta w_j = \lambda \dot{w}_j + \dot{\mu}_j w_j$$

respectively. Integrating the first relation against w_q^T and the second against w_q, we obtain the relations

$$\int_M \beta w_j^T w_q^T = \dot{\mu}_j^T \delta_{jq}, \quad 1 \le j, q \le m \tag{2.17}$$

and

$$\int_M \beta w_j w_q = \dot{\mu}_j \delta_{jq}, \quad 1 \le j, q \le p. \tag{2.18}$$

We can now use the relations (2.17) and (2.18) to show that U_{k+1} is dense in U_k (see [17, 1] for more details) and this proves Assertion (i).

Let us now introduce the sets

$$V_k := \{b \in U_k \mid \lambda_1^D < \lambda_1^T < \lambda_2^D < \ldots < \lambda_k^T \le \lambda_{k+1}^D\}.$$

This set is clearly open (by the continuity of the eigenvalues) and contained in U_k. Assume that for some $b \in U_k$, $\lambda = \lambda_j^T = \lambda_j^D$ or $\lambda = \lambda_j^T = \lambda_{j+1}^D$. Denote by u^T, v corresponding normalized eigenfunctions. Since the eigenvalues are simple, the relations (2.17) and (2.18) can be written as

$$\int_M \beta (u^T)^2 = \dot{\mu}^T, \quad \int_M \beta v^2 = \dot{\mu}.$$

If these numbers are different for some choice of β, we can make a small perturbation $b + t\beta$, so that the eigenvalues separate. If not, this means that $(u^T)^2 = v^2$ identically. Looking at the equations, it follows that $\Psi_g(u^T) = 0$ and hence that u^T is also an eigenfunction for the untwisted Dirichlet problem. Since the eigenvalues are simple, we must have $u^T = v$ or $u^T = -v$ identically. In particular, $\Phi_g(v) = 0$. It now suffices to apply the second assertion in Proposition 2.6. ∎

2.4 Examples

2.4.1 Example 1

The simplest example for which one can make explicit computations is the case of an interval with zero (or constant) potential function. Consider $M = [0, a] \subset \mathbb{R}$. The twisted Dirichlet eigenvalue problem on M is given by

$$(2.19) \qquad \begin{cases} \ddot{y} + \lambda y = \Psi(y), & \text{in }]0, a[, \\ y(0) = y(a) = 0, \\ \int_0^a y(t)\, dt = 0, \end{cases}$$

where $\Psi(y) = a^{-1} \int_0^a \ddot{y}(t)\, dt$.

Fact. *The pairs (eigenvalue, eigenfunction) for the twisted Dirichlet problem (2.19) are given by*

$$\left(\frac{4k^2\pi^2}{a^2}, \sin\left(\frac{2k\pi x}{a}\right)\right), k \in \mathbb{N}^\bullet$$

and

$$\left(\frac{4\tau_m^2}{a^2}, \cos\left(\frac{2\tau_m x}{a} - \tau_m\right) - \tau_m\right), m \in \mathbb{N}^\bullet,$$

where the τ_m are the zeros of the equation $x = \tan x$. They satisfy $\tau_m \sim (2m + 1)\pi/2$ when m is large.

In this particular case, the eigenvalues λ_j^D (*resp.* λ_j^T) of the Dirichlet problem (*resp.* of the twisted Dirichlet problem) are arranged as follows,

$$\lambda_1^D < \lambda_1^T = \lambda_2^D < \lambda_2^T < \lambda_3^D < \lambda_3^T = \lambda_4^D < \lambda_4^T < \lambda_5^D < \lambda_5^T = \lambda_6^D < \dots$$

and this shows in particular that the answer to Question (2.14a) is *Yes* in general.

2.4.2 Example 2

A simple example, of geometric interest, for which one can still make explicit computations, is that of a cylinder in Euclidean space. We consider the cylinder with radius r, $F : \mathbb{R} \times S_r^1 \to \mathbb{R}^3$, given by

$$\mathbb{R}\times]0, 2\pi r[\ni (t, \theta) \xrightarrow{F} (t, r\cos(\frac{\theta}{r}), r\sin(\frac{\theta}{r})).$$

In these coordinates, the metric is given by the identity matrix and we consider the operator

$$L := \Delta - r^{-2}$$

in a domain of the form $\Omega_a = [0, a] \times S_r^1$ on the cylinder. Here, r^{-2} can be viewed as the norm square of the second fundamental form of the cylinder, whose mean curvature is $(2r)^{-1}$), and hence L can be viewed as the stability operator of the cylinder.

Since there is an isometric circle action on the cylinder, the eigenspaces of both the Dirichlet and of the twisted Dirichlet problems decompose under this action and we can look at eigenfunctions of the form $u(t)\cos(k\theta/r), k \in \mathbb{N}$ and $u(t)\sin(k\theta/r), k \in \mathbb{N}^{\bullet}$. Since the sine and cosine functions have mean value 0 over S^1_r for $k \geq 1$, it follows that the spectra of the Dirichlet problem and of the twisted Dirichlet problem agree except possibly for the rotation invariant functions and we are therefore reduced to the case we considered in Example 1.

From the geometric point of view, we are interested in the *strong Morse index* of the domain Ω_a (*resp.* in the *weak Morse index*) *i.e.* in the number of negative eigenvalues of the operator $\Delta - r^{-2}$ in Ω_a for the Dirichlet problem (*resp.* for the twisted Dirichlet problem). It is easy to check that, for any a, the only negative eigenvalues come from rotation invariant eigenfunctions. To count them, we have to look at the eigenvalues of the Laplacian in Ω_a for the (twisted) Dirichlet problem which are less that r^{-2}. When a tends to infinity, the number of such eigenvalues, *i.e.* the strong and weak indexes, grow to infinity as well (like a) and, for "most values" of a, they differ by one. This is in contrast with Theorem 3.10.

2.4.3 Example 3

We now consider the case in which M is a ball in Euclidean space. We can take $M = B(0,1) \subset \mathbb{R}^n$. As in Example 2, the Dirichlet and twisted Dirichlet spectra can only differ for eigenvalues corresponding to rotation invariant eigenfunctions.

Lemma 2.7 *Let $M = B(0,1)$ be the unit ball in Euclidean n-space. Write $n =: (2\nu + 2)$. The eigenvalues of the twisted Dirichlet eigenvalue problem in $M = B(0,1)$, corresponding to radial eigenfunctions, are the squares of the positive roots $j_{\nu+2,m}\,, m \geq 1$, of the equation $J_{\nu+2}(x) = 0$ (where J_a is the Bessel function of order a). The corresponding eigenfunctions are given, up to a multiplicative constant, by*

$$u(r) = F_\nu(j_{\nu+2,m}\, r) - F_\nu(j_{\nu+2,m}),$$

where

$$F_\nu(x) = (\frac{x}{2})^\nu\, J_\nu(x).$$

It is a well-known fact that the eigenvalues of the Dirichlet problem in M are the squares of the positive roots $j_{\nu+k,m}\,, m \geq 1, k \in \mathbb{N}$, of the equations $J_{\nu+k}(x) = 0$.

Let us consider the case $n = 2$, *i.e.* $\nu = 0$. Numerical computations easily give the inequalities

$$j_{0,1} < j_{1,1} < j_{2,1} < j_{0,2} < \dots .$$

The results for the 2-dimensional Dirichlet problem can be summarized as follows:

k	eigenfunction(s)	eigenvalue	m	mult.
$k=0$	$J_0(j_{0,m}r)$	$j_{0,m}^2$	$m \geq 1$	1
$k=1$	$J_1(j_{1,m}r)\cos(\theta)$ $J_1(j_{1,m}r)\sin(\theta)$	$j_{1,m}^2$	$m \geq 1$	2
$k=2$	$J_2(j_{2,m}r)\cos(2\theta)$ $J_2(j_{2,m}r)\sin(2\theta)$	$j_{2,m}^2$	$m \geq 1$	2
...	...	...	...	

The results for the 2-dimensional twisted Dirichlet problem can be summarized as follows:

k	eigenfunction(s)	eigenvalue	m	mult.
$k=0$	$J_0(j_{2,m}\,r) - J_0(j_{2,m})$	$j_{2,m}^2$	$m \geq 1$	1
$k=1$	$J_1(j_{1,m}\,r)\cos(\theta)$ $J_1(j_{1,m}\,r)\sin(\theta)$	$j_{1,m}^2$	$m \geq 1$	2
$k=2$	$J_2(j_{2,m}\,r)\cos(2\theta)$ $J_2(j_{2,m}\,r)\sin(2\theta)$	$j_{2,m}^2$	$m \geq 1$	2
...	...	...	...	

The preceding results and numerical computations show that

$$\lambda_1^D < \underbrace{\lambda_1^T = \lambda_2^D = \lambda_2^T = \lambda_3^D}_{\alpha} < \overbrace{\lambda_3^T = \lambda_4^D = \lambda_4^T = \lambda_5^D = \lambda_5^T}^{\beta} < \lambda_6^D$$

which proves that the answer to Question (2.14b) is *No* in general. More precisely, we have $\text{mult}(\lambda_1^T) = 2$ and $\text{mult}(\lambda_3^T) = 3$. We note (compare with Proposition 2.4) that

$$\begin{cases} E_\alpha^T = E_\alpha^D = E_\alpha^{T,0} = E_\alpha^{D,0}, \\ \text{and} \\ E_\beta^{T,0} = E_\beta^{D,0} = E_\beta^D, \\ E_\beta^T = E_\beta^{T,0} \oplus \mathbb{R}\, v. \end{cases}$$

One may also observe that δ^D_{radial} and δ^T_{radial} are simple and that $\delta^D_{\text{radial}} \cap \delta^T_{\text{radial}} = \emptyset$.

3 Complete manifolds, the Morse index revisited

3.1 Notations

Let (M,g) be a complete non-compact Riemannian manifold and let $b : M \to \mathbb{R}$ be a continuous function.

• We let

$$\mathcal{D}(M) := C_0^\infty(M),$$

$$\mathcal{D}_T(M) := \left\{ f \in C_0^\infty(M) \;\middle|\; \int_M f\, dv_g = 0 \right\}.$$

• For any domain $\Omega \subset M$, we let $\mathcal{D}(\Omega)$, *resp.* $\mathcal{D}_T(\Omega)$, denote the set of smooth functions with compact support contained in Ω, *resp.* the set of smooth functions u, with compact support contained in Ω, satisfying $\int_M u\, dv_g = 0$.

• Given any relatively compact domain $\Omega \subset\subset M$, we let $\mathrm{Ind}\,(\Omega)$, *resp.* $\mathrm{Ind}\,_T(\Omega)$, denote the index (*i.e.* the number of negative eigenvalues) of the quadratic form $\int_M (|df|_g^2 + bf^2)\, dv_g$ on $\mathcal{D}(\Omega)$, *resp.* on $\mathcal{D}_T(\Omega)$.

• We denote by $\mathrm{Ind}\,(M)$ and $\mathrm{Ind}\,_T(M)$ the numbers

$$\begin{cases} \mathrm{Ind}\,(M) := \sup \left\{ \mathrm{Ind}\,(\Omega) \mid \Omega \subset\subset M \right\}, \\ \mathrm{Ind}\,_T(M) := \sup \left\{ \mathrm{Ind}\,_T(\Omega) \mid \Omega \subset\subset M \right\}. \end{cases} \tag{3.20}$$

• We consider the operators L and L_T defined by

$$\begin{cases} L := \left(\mathcal{D}(M), \Delta_g + b\right), \\ L_T := \left(\mathcal{D}_T(M), \Delta_g + b\right). \end{cases} \tag{3.21}$$

• Finally, we introduce the spaces

$$\begin{cases} H^1(M) := \text{ the closure of } \mathcal{D}(M) \text{ for the norm } \|\cdot\|_1 , \\ H^1_T(M) := \text{ the closure of } \mathcal{D}_T(M) \text{ for the norm } \|\cdot\|_1. \end{cases} \tag{3.22}$$

3.2 Assumptions and immediate consequences

We shall now make some assumptions on M and on the potential function b and deduce immediate consequences.

Geometric assumptions

We assume that our complete manifold has infinite volume,

$$\mathrm{Vol}\,(M,g)=\infty \tag{3.23}$$

and we point out that this is true whenever M is a constant mean curvature hypersurface in a space form with non-positive curvature [10, 16]. Under this assumption, we have

Lemma 3.1 *For all $\varphi \in L^2(M)$,*

$$\lim_{R\to\infty} V(R)^{-1/2}\int_{B(R)} \varphi\, dv_g = 0.$$

Furthermore, the space $\mathcal{D}_T(M)$ is dense in $L^2(M)$.

As a matter of fact, the condition that the volume of M is infinite is not sufficient for our purposes. We will use the following stronger assumption,

$$\begin{cases} \mathrm{Vol}\,(M,g)=\infty \quad \text{and} \\ \exists C>0, \quad \forall R, \quad V(R+1)\le C\,V(R), \end{cases} \tag{3.24}$$

where $V(R)$ denotes the volume of the geodesic ball $B(x_0,R)\subset M$, for some fixed point x_0 (the assumption does not depend on the choice of x_0).

Remark. If the manifold (M,g) satisfies $\mathrm{Ric}_g \ge -k^2(n-1)g$ then, by the Bishop–Gromov comparison theorem,

$$\frac{V(R+1)}{V(R)}\le \frac{V_k(R+1)}{V_k(R)}$$

where $V_k(R)$ denotes the volume function in the simply-connected model space with constant sectional curvatures $-k^2$, and hence (M,g) satisfies the second line in (3.24).

An analytic consequence of Assumption (3.24) is the following result whose proof is easy.

Proposition 3.2 *Assume that (M,g) is a complete Riemannian manifold satisfying Assumption (3.24). Then, the space $\mathcal{D}_T(M) := \{f\in C_0^\infty(M)\mid \int_M f\,dv_g\}$ is dense in $L^2(M)$ and in $H^1(M)$ for the norms $\|\cdot\|$ and $\|\cdot\|_1$ respectively.*

For future reference, we introduce the following notations. Define ψ_R to be a piecewise C^1 function on $\mathbb{R}$ such that

$$\begin{cases} \psi_R(t)=1 \ \text{ for } \ t\le R \ \text{ and } \ \psi_R(t)=0 \ \text{ for } \ t\ge R+1, \\ 0\le\psi_R\le 1 \ \text{ and } \ |\psi_R'|\le 2, \end{cases} \tag{3.25}$$

and define $\theta_R(x)$ to be the function $\psi_R(d(x_0,x))$, where $d(x_0,x)$ is the Riemannian distance from x to x_0. The function θ_R is in $H^1(M)$ and it can be approximated by smooth functions with compact support in M, *i.e.* by elements of $\mathcal{D}(M)$. Define

$$(3.26) \qquad \begin{cases} A(R) := \int_M \theta_R \, dv_g, \\ \\ \varphi_R := A(R)^{-1/2}\theta_R. \end{cases}$$

Then

$$(3.27) \qquad \begin{cases} \mathrm{Supp}(\varphi_R) \subset B(R+1) \quad \text{and} \quad \int_M \varphi_R \, dv_g = A(R)^{1/2}, \\ \\ \int_M \varphi_R^2 \, dv_g = A(R)^{-1} \int_M \theta_R^2 \, dv_g \leq A(R)^{-1} \int_M \theta_R \, dv_g = 1, \\ \\ V(R) \leq A(R) \leq V(R+1). \end{cases}$$

Assumptions on the potential function

We will now make assumptions on the function b. We point out that they will be satisfied in the geometric context we are interested in (see Section 4).

$$(3.28) \quad \exists D > 0, \quad \forall u \in \mathcal{D}(M), \quad -D \int_M u^2 \, dv_g \leq \int_M \left(|du|_g^2 + bu^2\right) dv_g$$

i.e. the operator L is semi-bounded from below, or

$$(3.29) \begin{cases} b := b_+ - b_- \quad \text{and} \quad \exists B > 0, \quad \forall x \in M, \quad 0 \leq b(_+x) \leq B \\ \\ \exists D > 0, \quad \forall u \in \mathcal{D}(M), \quad -D \int_M u^2 \, dv_g \leq \int_M \left(|du|_g^2 - b_- u^2\right) dv_g \end{cases}$$

where $b_\pm$ denote the positive and negative parts of b.

3.3 Spectral results for L and L_T.

The purpose of this section is to investigate the spectral properties of the operators L and L_T and in particular their relationships with the numbers $\mathrm{Ind}\,(M)$ and $\mathrm{Ind}_T(M)$.

Proposition 3.3 *Let (M,g) be a complete Riemannian manifold. With the above notations, we have*

(1) $\forall \Omega \subset\subset M, \quad Ind_T(\Omega) \leq Ind(\Omega) \leq Ind_T(\Omega) + 1\,;$

(2) $Ind(M) < \infty \Longleftrightarrow Ind_T(M) < \infty\,;$

(3) *If* $Ind(M) < \infty$, *then* L *is semi-bounded from below on* $\mathcal{D}(M)$.

Proof. The first assertion follows easily from the min-max and yields the second one. We restate the third assertion as the following lemma [8], whose proof we will need later on.

Lemma 3.4 *Let M be a complete Riemannian manifold. If Ind$(M) < \infty$, then the operator $\Delta_g + b$ is positive outside a compact set and is semi-bounded from below on $\mathcal{D}(M)$,* i.e. *there exists a positive constant A_M such that*

$$-A_M \int_M \varphi^2 \, dv_g \leq \int_M \varphi(\Delta_g + b)\varphi \, dv_g =: q(\varphi)$$

for any function $\varphi \in \mathcal{D}(M)$.

Proof of Lemma 3.4. It follows from [8], Proposition 1, p. 123, that Ind$(M) < \infty$ implies that the operator $\Delta_g + b$ is positive outside a compact set, *i.e.*

$$(a) \qquad \exists R_0, \ \ \forall\, \varphi \in C_0^\infty(M \setminus B(R_0)), \quad q(\varphi) := \int_M \varphi(\Delta_g + b)\varphi \, dv_g > 0$$

Fix R_0 and take $R \gg R_0$. Given such an R, choose a Lipschitz function η_1 such that $0 \leq \eta_1 \leq 1$, $\eta_1 = 0$ on $B(R)$, $\eta_1 = 1$ on $M \setminus B(2R)$ and $|d\eta_1| \leq 2R^{-1}$. We also define $\eta := 1 - (1 - \eta_1)^2$ for which we have

$$(b) \qquad |d\eta|^2 = 4(1 - \eta^2)|d\eta_1|^2 \leq 16R^{-2}(1 - \eta^2) \leq 16R^{-2}.$$

For $R \gg R_0$ to be chosen later, and for any $\varphi \in C_0^\infty(M)$, we apply inequality (a) to the function $\eta\varphi$ and we obtain the inequality

$$(c) \left\{ \begin{array}{ll} -\int_M b(\eta\varphi)^2 \, dv_g & \leq \int_M |d(\eta\varphi)|_g^2 \, dv_g = \int_M \eta^2 |d\varphi|_g^2 \, dv_g \\ & +2 \int_M \eta\varphi \langle d\eta, d\varphi \rangle \, dv_g + \int_M \varphi^2 |d\eta|_g^2 \, dv_g. \end{array} \right.$$

Adding $q(\varphi) = \int_M (|d\varphi|_g^2 + b\varphi^2) \, dv_g$ to both sides of (c), we obtain

$$(d) \left\{ \begin{array}{ll} \int_M (1 - \eta^2)(|d\varphi|_g^2 + b\varphi^2) \, dv_g & \leq q(\varphi) + 2 \int_M \eta\varphi \langle d\eta, d\varphi \rangle \, dv_g \\ & + \int_M \varphi^2 |d\eta|_g^2 \, dv_M. \end{array} \right.$$

Since $2 \int_M \eta\varphi \langle d\eta, d\varphi \rangle \, dv_g \leq \int_M \varphi^2 \, dv_g + \int_M |d\varphi|_g^2 |d\eta|_g^2 \, dv_g$, using (d), we obtain

$$(e) \left\{ \begin{array}{ll} \int_M (1 - \eta^2)(|d\varphi|_g^2 + b\varphi^2) \, dv_g & \leq q(\varphi) + \int_M \varphi^2 \, dv_g \\ & + \int_M |d\eta|^2 (|d\varphi|_g^2 + \varphi^2) \, dv_g. \end{array} \right.$$

Using (b) and taking into account the fact that $1 - \eta^2$ vanishes outside $B(2R)$, we obtain

$$(f) \ \int_M (1 - \eta^2)(1 - 16R^{-2})|d\varphi|^2 \, dv_g \leq q(\varphi) + c(R, b) \int_M \varphi^2 \, dv_M$$

where $c(R, b) := \sup_{B(2R)} |b| + 2$. We now fix $R \geq R_0 + 4$ and we conclude that

$$-c(R,b)\int_M \varphi^2\,dv_g \le q(\varphi)$$

for any $\varphi \in C_0^\infty(M)$. □

We have finished the proof of Proposition 3.3. ∎

Observations

• Under the geometric assumption (3.23), the operators L and L_T are densely defined (Lemma 3.1) and they admit self-adjoint extensions by von Neumann's theorem (since they commute with complex conjugation, [15] Theorem X.3).

• Under the analytic assumption (3.28), the operator L is essentially self-adjoint (M is complete and there is a real number in the resolvent set of L, see [4]). Under the assumptions (3.23) and (3.28), the operator L is essentially self-adjoint and the operator L_T admits a Friedrichs extension L_T^F in $L^2(M)$.

Theorem 3.5 *Let (M,g) be a complete Riemannian manifold with infinite volume. Assume furthermore that M satisfies*

$$\exists C>0,\quad \forall R,\quad V(R)\le V(R+1)\le C\,V(R)$$

and that the function b satisfies

$$b := b_+ - b_- \quad \textit{and} \quad \exists B>0,\quad \forall x\in M,\quad 0\le b(_+x)\le B,$$

$$\exists D>0,\quad \forall u\in\mathcal{D}(M),\quad -D\int_M u^2\,dv_g \le \int_M \left(|du|_g^2 - b_- u^2\right)dv_g.$$

Then, the operators L and L_T have the same Friedrichs extension, $L_T^F = \overline{L}$.

Proof. With the constant D as in the statement of the theorem or as in (3.29) and with obvious notations, we let

$$\|\varphi\|_{+1}^2 := \int_M |d\varphi|^2 + (D+1-b_-)\varphi^2 \ge \int_M \varphi^2.$$

Since $b_- \ge 0$, we also have $\|\varphi\|_{+1} \le (D+1)^{1/2}\|\varphi\|_1$ and it follows that $H^1(M)$ is contained in the closure $\mathcal{H}_{+1}$ of $\mathcal{D}(M)$ with respect to the norm $\|\cdot\|_{+1}$. Furthermore, since L is essentially self-adjoint, the closure $\overline{L}$ of the operator L is associated with the quadratic form q, see (2.10), on $\mathcal{H}_{+1}$.

Let $\mathcal{H}_{T,+1}$ denote the closure of $\mathcal{D}_T(M)$ for the norm $\|\cdot\|_{+1}$. The Friedrichs extension L_T^F of the operator L_T is precisely the operator associated with the quadratic form q on $\mathcal{H}_{T,+1}$.

It is clear that $\mathcal{H}_{T,+1}\subset\mathcal{H}_{+1}$. In order to prove the theorem, we have to show the reverse inclusion.

Take some $f\in\mathcal{H}_{+1}$. Then, there exists a sequence $\{f_n\}\subset\mathcal{D}(M)$ such that $\|f-f_n\|_{+1}$ tends to 0. Choose an increasing sequence $R_n \nearrow \infty$ such that $\mathrm{Supp}(f_n)\subset B(R_n+1)$ and let $\varphi_n := \varphi_{R_n}$, where φ_R is the family defined in (3.27). Finally, let

$$h_n := f_n - A(R_n)^{-1/2}\left(\int_{B(R_n+1)} f_n\right)\varphi_n.$$

Then, $h_n \in \mathcal{D}_T(M)$ and

$$\|f-h_n\|_{+1} \le \|f-f_n\|_{+1} + A(R_n)^{-1/2}\left|\int_{B(R_n+1)} f\right| \, \|\varphi_n\|_{+1}.$$

Notice that $\|\varphi_n\|_{+1} \le C_1\|\varphi_n\|_1 \le C_2$ for some constants C_1, C_2. On the other-hand, in view of (3.27) and (3.24), one has

$$A(R_n)^{-1/2}\left|\int_{B(R_n+1)} f\right| \le C_3\left\{\|f-f_n\|_{+1} + V(R_n+1)^{-1/2}\left|\int_{B(R_n+1)} f\right|\right\}$$

whose right-hand side tends to zero. It follows that $\|f-h_n\|_{+1}$ tends to zero and hence that $f \in \mathcal{H}_{T,+1}$. This finishes the proof of Theorem 3.5. ∎

It is not clear whether L_T is actually essentially self-adjoint. We have the following result in this direction (compare with a similar result in [7]).

Proposition 3.6 *Let M be a complete Riemannian manifold with infinite volume such that*

$$\exists C>0, \quad \forall R, \quad V(R) \le V(R+1) \le C\,V(R).$$

Assume the function b is bounded on M. Assume furthermore that there exists a family ρ_R of functions, $\rho_R : M \to I\!R_+$ such that

$$\text{(3.30)} \qquad \begin{cases} 0 \le \rho_R \le 1 \quad \textit{and} \quad \textit{Supp}(\rho_R) \subset B(R+1) \\ \rho_R\,|B(R) \equiv 1 \\ \exists A, \quad \forall R, \quad |d\rho_R|_g\,, \ |\Delta_g \rho_R| \ \le A. \end{cases}$$

Then, the operators L and L_T have the same closure, in particular the operator L_T is essentially self-adjoint.

Remark. The functions θ_R defined in (3.25) satisfy (3.30), except possibly the last inequality $|\Delta_g \rho_R| \le A$.

Proof. By analogy with (3.26), we define

$$A(R) := \int_M \rho_R\, dv_g \quad \text{and} \quad \varphi_R := A(R)^{-1/2}\rho_R$$

and we have, under Assumption (3.30),

$$\begin{cases} V(R) \le A(R) \le V(R+1) \le C\,V(R) \\ \int_M \varphi_R\, dv_g = A(R)^{1/2} \quad \text{and} \quad \int_M \varphi_R^2\, dv_g = 1. \end{cases}$$

Let $u \in L^2$ and let $\{u_n\} \subset \mathcal{D}$ be a sequence such that $u_n \to u$ in L^2. Choose an increasing sequence R_n such that $\text{Supp}(u_n) \subset B(R_n)$.

Fact 1. There exists a constant C_1 such that for any $R \geq 0$, $\|d\varphi_R\|, \|\Delta_g \varphi_R\| \leq C_1$ (L^2 norms).

This follows immediately from the from Assumption (3.30).

Fact 2. For the sequence defined above,

$$\lim_{n \infty} A(R_n)^{-1/2} \int_M u_n \, dv_g = 0.$$

This follows immediately from the inequalities

$$A(R_n)^{-1/2} \left| \int_{B(R_n+1)} (u - u_n) \, dv_g \right| \leq \left(\frac{V(R_n+1)}{A(R_n)}\right)^{1/2} \|u - u_n\|,$$

and

$$A(R_n)^{-1/2} \left| \int_{B(R_n+1)} u \, dv_g \right| \leq \left(\frac{V(R_n+1)}{A(R_n)}\right)^{1/2} \big(V(R_n+1)\big)^{-1/2} \left| \int_{B(R_n+1)} u \, dv_g \right|$$

whose right hand sides tend to zero by Assumption (3.30) and Lemma 3.1.

Fact 3. If $u, \Delta_g u \in L^2$ and if there exists a sequence u_n such that $u_n \to u$ and $Lu_n \to Lu$ in L^2 (*i.e.* if $u \in D(\overline{L})$) then, under the assumption of the proposition, $u \in D(\overline{L_T})$.

Indeed, let $f_n := u_n - A(R_n)^{-1/2} (\int_M u_n \, dv_g) \varphi_{R_n}$, where R_n is an increasing sequence such that $\mathrm{Supp}(u_n) \subset B(R_n)$. We can then write

$$\|u - f_n\| \leq \|u - u_n\| + A(R_n)^{-1/2} \left| \int_M u_n \, dv_g \right| \, \|\varphi_{R_n}\| \to 0$$

since both terms on the right hand side go to zero because $\|\varphi_{R_n}\| = 1$. Similarly,

$$\|du - df_n\| \leq \|du - du_n\| + A(R_n)^{-1/2} \left| \int_M u_n \, dv_g \right| \, \|d\varphi_{R_n}\| \to 0$$

and

$$\|\Delta_g (u - f_n)\| \leq \|\Delta_g (u - u_n)\| + A(R_n)^{-1/2} \left| \int_M u_n \, dv_g \right| \, \|\Delta_g \varphi_{R_n}\| \to 0$$

using Fact 1. This proves the proposition. ∎

Theorem 3.7 *Let (M, g) be a complete Riemannian manifold with infinite volume.*

1. *If $Ind(M)$ and $Ind_T(M)$ are finite, the operators L and L_T admit Friedrichs extensions, denoted by L^F and L_T^F respectively, and $Ind(M) = Ind(L^F)$, $Ind_T(M) = Ind(L_T^F)$.*

2. *If* $Ind(M)$ *and* $Ind_T(M)$ *are infinite, the quadratic form* $\int_M \left(|du|_g^2 + bu^2\right) dv_g$ *has infinite index on both* $\mathcal{D}(M)$ *and* $\mathcal{D}_T(M)$.

Proof. The first part of Assertion (1), $\mathrm{Ind}\,(M) = \mathrm{Ind}\,(L^F)$, is an amplification of [8], Proposition 2, page 124 (see also [4], Proposition 3). The second part of Assertion (1), $\mathrm{Ind}_T(M) = \mathrm{Ind}\,(L_T^F)$, is an analogous statement, for the operator L_T. We give the proof of this second part for completeness.

Let us assume that $\mathrm{Ind}_T(M) < \infty$. It follows from Proposition 3.3 that L_T has a Friedrichs extension. In the sequel, we denote by $\mathrm{Ind}_T(B(\rho))$ the index of the quadratic form $q_T^{B(\rho)}$ associated with the twisted Dirichlet problem in the ball $B(\rho)$.

Let $N := \mathrm{Ind}_T(M)$. By the monotonicity of the eigenvalues of the twisted Dirichlet problem (Proposition 2.2, Assertion (2)), there exists a constant R_1 such that the index $\mathrm{Ind}_T(B(\rho))$ is equal to N for any $\rho \geq R_1$. Denote the negative eigenvalues of the quadratic form $q_T^{B(\rho)}$ by $\lambda_{1,\rho} \leq \ldots \leq \lambda_{N,\rho}$ and denote an orthonormal basis of associated eigenfunctions by $f_{1,\rho}, \ldots, f_{N,\rho}$, with the functions being extended by zero outside the ball $B(\rho)$.

Lemma 3.8 *With the above notations, there exist positive constants* a_M, A_M, C_M *and* R_M *such that for all* $\rho \geq R_M$, *for all* $R \geq R_M$ *and for all* $j \in \{1, \ldots, N\}$,

(i) $\lambda_{j,\rho} \in [-A_M, -a_M]$,

(ii) $\int_{M \setminus B(2R)} f_{j,\rho}^2 \, dv_g \leq C_M R^{-2}$,

(iii) $1 \leq \|f_{j,\rho}\|_{+1}^2 := \int_M \left(|df_{j,\rho}|_g^2 + (A_M + 1 + b) f_{j,\rho}^2\right) dv_g \leq C_M$,

(iv) there exists a constant $C(R)$ *such that* $\int_{B(R)} \left(|df_{j,\rho}|_g^2 + f_{j,\rho}^2\right) dv_g \leq C(R)$.

With the notations as in the preceding Lemma, let $\mathcal{H}_{T,+1}$ denote the domain of the quadratic form q_T^M, *i.e.* the closure of $\mathcal{D}_T(M)$ with respect to the norm $\|\cdot\|_{+1}$ (see the proof of Theorem 3.5).

Lemma 3.9 *With the above notations, one can find functions* $f_1, \ldots, f_N \in L^2(M) \cap \mathcal{H}_{T,+1}$, *forming an* L^2*-orthonormal basis, numbers* $\lambda_1, \ldots, \lambda_N$ *in the interval* $[-A_M, -a_M]$ *and a sequence* $\rho_k \nearrow \infty$ *such that*

(i) f_{j,ρ_k} *converges to* f_j, *strongly in* L^2 *and weakly in* $\mathcal{H}_{T,+1}$,

(ii) $\lambda_{j,\rho_k} \to \lambda_j$,

(iii) $(\Delta_g + b) f_j = \lambda_j f_j$ *in the sense of distributions.*

The second part of Assertion (1) in Theorem 3.7 follows from Claim 1 below and the second part of Assertion (2) follows from Claim 2.

Claim 1. If $N := \mathrm{Ind}_T(M)$ is finite, then $\mathrm{Ind}\,(q_T^M)$ is finite and these numbers are equal.

Under the assumption of the claim, it follows from Lemma 3.9 that for all $j \in \{1, \ldots, N\}$, the function f_j is in the domain of q_T^M, and satisfies $q_T^M(f_j, \varphi) = \langle (\Delta_g + b)\varphi, f_j \rangle = \lambda_j \langle \varphi, f_j \rangle$ for all $\varphi \in \mathcal{D}$. It follows from [13], Theorem 2.1, Chapter VI.2.1, page 322, that $f_j \in D(\overline{L_T})$, the domain of the Friedrichs extension of L_T. It also follows that $\mathrm{Ind}\,(q_T^M) \geq N$.

Let us now prove that $\overline{L_T}$ is non-negative on $\{f_1, \dots, f_N\}^\perp$. Take an element $\varphi \in D(\overline{L_T}) \cap \{f_1, \dots, f_N\}^\perp$. There exists a sequence $\{\varphi_n\} \in \mathcal{D}_T(M)$ which converges to φ in L^2 and such that $q_T^M(\varphi_n)$ converges to $q_T^M(\varphi)$. Choose a sequence ρ_n such that for all n, $\mathrm{Supp}(\varphi_n) \subset B(\rho_n)$. Write

$$\varphi_n = \sum_{j=1}^{N} a_{j,n} f_{j,n} + \psi_n$$

where we have set $f_{j,n} := f_{j,\rho_n}$ for simplicity. Let us also write $\lambda_{j,n} := \lambda_{j,\rho_n}$.

Since φ_n is in the domain of the operator $\Delta_g + b$ with Dirichlet boundary conditions on $B(\rho_n)$, we can easily compute

$$q_T^M(\varphi_n) = \sum_{j=1}^{N} \lambda_{j,n} a_{j,n}^2 + q_T^{B(\rho_n)}(\psi_n).$$

Since the $\lambda_{j,n}$ are the negative eigenvalues of the quadratic form $q_T^{B(\rho_n)}$, it follows that $q_T^{B(\rho_n)}(\psi_n) \geq 0$. On the other-hand, it follows from the definitions of $\varphi_n, a_{j,n}$ and from Lemma 3.9 that $a_{j,n}$ tends to zero for all j when n goes to infinity and hence we conclude that $q_T^M(\varphi) = \lim q_T^M(\varphi_n) \geq 0$. This shows that $\mathrm{Ind}\,(q_T^M) \leq N$ and the claim is proved.

Claim 2. If $\mathrm{Ind}\,(q_T^M)$ is finite then so is $\mathrm{Ind}_T(M)$.

Let W be the direct sum of eigenspaces of q_T^M corresponding to negative eigenvalues. Its dimension $\mathrm{Ind}\,(q_T^M)$ is finite by assumption. Assume that $\infty \geq \mathrm{Ind}_T(M) \geq \dim W + 1$. This means that there exists some relatively compact open subset $\Omega \subset M$ such that $\mathrm{Ind}\,(q_T^\Omega) \geq \dim W + 1$ and hence that there exists some subspace $\mathcal{L} \subset \mathcal{D}_T(\Omega)$ whose dimension is at least $\dim W + 1$, on which the quadratic form q_T^Ω is negative. One can in particular find some non-zero $\varphi \in \mathcal{L}$ which is also orthogonal to W and for which $q_T^M(\varphi) < 0$, a contradiction. This proof in particular shows that $\mathrm{Ind}_T(M) \leq \mathrm{Ind}\,(q_T^M)$. ∎

Proof of Lemma 3.8. Take R_0 as in the proof of Lemma 3.4 and choose $R_1 \geq R_0 + 8$ such that $\rho \geq R_1$ implies that $\mathrm{Ind}_T(L; B(\rho)) = N$ and we use the above notations $\lambda_{j,\rho}, f_{j,\rho}$. Because the eigenvalues are non increasing functions of ρ, there is some positive constant a_M such that $\lambda_{N,\rho} \leq -a_M$ for any $\rho \geq R_1$. We can now apply Lemma 3.4 with $\varphi = f_{j,\rho}$, $j \in \{1, \dots, N\}$ and $\rho \geq R_1$ and conclude that $\lambda_{j,\rho} \in [-A_M, -a_M]$.

Since $\int_M \eta^2 |d\varphi|_g^2 \, dv_g = \int_M \langle d\varphi, d(\eta^2\varphi) - 2\eta\varphi d\eta \rangle \, dv_g$, we have that

$$\int_M \eta^2 |d\varphi|_g^2 \, dv_g + 2\int_M \eta\varphi \langle d\eta, d\varphi \rangle \, dv_g = \int_M \langle d\varphi, d(\eta^2\varphi) \rangle \, dv_g = \int_M \eta^2 \varphi \Delta_g \varphi \, dv_g$$

and relation (c) applied to $\varphi = f_{j,\rho}$ gives

$$-\lambda_{j,\rho} \int_M \eta^2 f_{j,\rho}^2 \, dv_g \leq \int_M f_{j,\rho}^2 |d\eta|_g^2 \, dv_g \leq 16 R^{-2}$$

since the functions $f_{j,\rho}$ form an orthonormal basis. It follows that

$$\exists C_1,\ \ \forall R \geq R_1,\ \ \forall \rho \geq R_1,\quad \int_{M\setminus B(2R)} f_{i,\rho}^2\, dv_g \leq C_1 R^{-2}.$$

Assertion (iii) is clear.

Inequality (f) also gives, with $\varphi = f_{j,\rho}$,

$$\forall R \geq R_1,\ \ \exists C_2(R),\ \ \forall \rho \geq R_1, \int_{B(R)} (|df_{i,\rho}|_g^2 + f_{i,\rho}^2)\, dv_g \leq C_2(R).$$

This proves Lemma 3.8 □

Proof of Lemma 3.9. Let $g_j := f_{1,\rho_j}$. The sequence $\{g_j\}$ being bounded in $\mathcal{H}_{T,+1} \subset L^2$, there exists a function $g \in \mathcal{H}_{T,+1}$ and a subsequence, which we still denote $\{g_j\}$, which converges weakly to g. Taking another subsequence if necessary, we can assume that the sequence λ_{1,ρ_j} converges to some $\lambda_1 \in [-A_M, -a_M]$.

We claim that the sequence $\{g_j\}$ is relatively compact in L^2. To show this, it suffices to show that is is totally bounded. Take $\varepsilon > 0$ and, using Lemma 3.8, choose R big enough such that $\int_{M\setminus B(2R)} g_j^2\, dv_g \leq \varepsilon$. Since the inclusion $H^1(B(3R)) \to L^2(B(3R))$ is compact and since $\{g_j\}$ is bounded in $H^1(B(3R))$ by Lemma 3.8, we can find $k_1, \ldots, k_{N(\varepsilon)}$ such that for any j, there exists some $k(j) \in \{1, \ldots, N(\varepsilon)\}$ such that $\|g_j - g_{k(j)}\|_{L^2(B(3R))} \leq \varepsilon$. It follows easily that $\|g_j - g_{k(j)}\|_{L^2(M)} \leq 3\varepsilon$, which proves our claim.

From our subsequence $\{g_j\}$, we can therefore extract a subsequence which converges strongly in L^2 to some function $\tilde{g}$ and weakly in $\mathcal{H}_{T,+1}$ to the function g. It follows that $\tilde{g} = g$.

We can now repeat this argument with $j = 2, \ldots, N$ and this proves Lemma 3.9. □

Theorem 3.10 *Let (M, g) be a complete Riemannian manifold with infinite volume. Assume furthermore that M satisfies*

$$\exists C > 0,\ \ \forall R,\ \ V(R) \leq V(R+1) \leq C\, V(R)$$

and that b satisfies

$$b := b_+ - b_- \ \ \text{and}\ \ \exists B > 0,\ \ \forall x \in M,\ \ 0 \leq b(_+x) \leq B,$$

$$\exists D > 0,\ \ \forall u \in \mathcal{D}(M),\quad -D \int_M u^2\, dv_g \leq \int_M \left(|du|_g^2 - b_- u^2\right) dv_g .$$

Then, $Ind(M) = Ind_T(M)$.

Proof. This is a direct consequence of Theorems 3.7 and 3.5. ∎

4 Applications

Let $i : M^n \to \mathbb{H}^{n+1}$ be an isometric immersion with constant mean curvature H into the hyperbolic space with curvature -1. The stability operator of the immersion is the operator $\Delta - n(H^2-1) - |A^0|^2$, where A^0 is the traceless second fundamental form (see [6] Section 6.2.2, with a different sign convention for the Laplacian).

For such an immersion, there are two different notions of *Morse index*. We call *weak Morse index* of the immersion i the index of the stability operator acting on all smooth functions with compact support *and* with mean value equal to zero; we call *strong Morse index* of the immersion i the index of the stability operator acting on all smooth functions with compact support (our terminology differs from that of [12]).

Comparing with Section 3, we see that the strong Morse index corresponds to $\mathrm{Ind}\,(M)$, while the weak Morse index corresponds to $\mathrm{Ind}_T(M)$, when the operator under consideration is $\Delta - n(H^2-1) - |A^0|^2$ in place of $\Delta_g + b$. We say that the immersion is *weakly stable* (*resp. strongly stable*) if it has weak index (*resp.* strong index) equal to zero. We can reformulate Theorems 3.7 and 3.10 in this geometric context as:

Theorem 4.1 *Let $i : M^n \to \mathbb{H}^{n+1}$ be a complete non-compact isometric immersion with constant mean curvature H into the hyperbolic space with curvature -1.*
(1) The strong and weak Morse indexes of M are simultaneously finite or infinite.
(2) Assume furthermore that M satisfies the assumption

$$\exists\, C \quad \text{such that} \quad \forall R, \quad V(R) \leq V(R+1) \leq C\, V(R),$$

where $V(R)$ is the volume of the geodesic ball of radius R centered at some fixed point x_0. Then, the weak and strong Morse indexes of M are equal. In particular, the notions of weak stability *and of* strong stability *coincide.*

Theorem 4.2 *Let $i : M^2 \to \mathbb{H}^3$ be a complete non-compact isometric immersion with constant mean curvature H into the hyperbolic space with curvature -1. Then, the weak and strong Morse indexes of M are equal. In partiuclar, the notions of* weak stability *and of* strong stability *coincide.*

Proof. Assertion (1) follows from Theorem 3.7 and Assertion (2) from Theorem 3.10. This proves Theorem 4.1.

In dimension 2, we can apply [6], Theorem 4.2 and conclude that $|A^0|$ and the Gaussian curvature are uniformly bounded on $i(M)$. In particular, Assumptions (3.24) and (3.29) are satisfied and it follows that the quadratic forms q_T^M and q^M coincide. We can then apply Theorem 4.1. ∎

Using the above result, one can give a new proof of the following result of A. da Silveira [16].

Corollary 4.3 *A weakly stable complete surface with constant mean curvature* 1 *in hyperbolic space* $\mathbb{H}^3$ *with curvature* -1 *is a horosphere.*

Proof. By the above theorem, weakly stable implies strongly stable and according to [6], the traceless second fundamental form of our surface satisfies the inequality

$$|A^0(x_0)| \leq C\, R^{-1}$$

for any ball $B(x_0, R)$, with a constant C which does not depend on R. Letting R tend to infinity, we conclude that $A^0 = 0$. ∎

Remark. In the geometric context of constant mean curvature immersions, Theorem 3.10 shows that when M satisfies Assumption (3.24) and has finite Morse index, then it is equivalent to consider the stability operator $\Delta - n(H^2 - 1) - |A^0|^2$ on $C_0^\infty(M)$ or on $C_0^\infty(M) \cap \{f \mid \int_M f\, dv_g = 0$. Ph. Castillon ([7], Proposition 3.1) proved a similar result when the mean curvature H satifies $|H| < 1$ and when M^n has finite total curvature, $\int_M |A^0|^n\, dv_g < \infty$. As shown in [5], a constant mean curvature immersion $M \to \mathbb{H}^{n+1}$ with finite total curvature actually satisfies Assumption (3.24). Indeed, $|A^0|$ tends to zero at infinity and hence the Ricci curvature of M is bounded from below.

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