INDEX OF TRANSVERSALLY ELLIPTIC $\mathcal{D}$-MODULES

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1. Introduction

We consider the situation of the Lefschetz-Atiyah-Bott formula of [2], i.e. M is a compact manifold, $\varphi : M \to M$ a smooth map, $\mathcal{E}^\cdot$ an "elliptic complex" on M and $u^\cdot : \varphi^*\mathcal{E}^\cdot \to \mathcal{E}^\cdot$ a "lifting" of φ to $\mathcal{E}^\cdot$; the cohomology groups of $\mathcal{E}^\cdot$ are finite-dimensional and the trace of the morphism $\Gamma(u)$ induced by u on the cohomology is given by a fixed point formula. We are interested in deformations of φ and u, $\phi : T \times M \to M$, $u' : \phi^*\mathcal{E}^\cdot \to p^*\mathcal{E}^\cdot$, where $p : T \times M \to M$ is the projection. For each $t \in T$ they restrict to a map $\phi_t : M \to M$ and a lifting of ϕ_t, $u'_t : \phi_t^*\mathcal{E}^\cdot \to \mathcal{E}^\cdot$ and $t \mapsto \operatorname{tr}\Gamma(u'_t)$ is a function on T. In fact, following Atiyah's idea about transversally elliptic operators (see [1]), it is possible to weaken the hypothesis of ellipticity on $\mathcal{E}^\cdot$ (the cohomology is no longer finite-dimensional) and still get a hyperfunction on T, which corresponds to a trace in a generalized sense. We do this in the more general framework of $\mathcal{D}$-modules and constructible sheaves using the character cycle construction of Kashiwara in [13] and the microlocal Euler class of Schapira and Schneiders in [20]. Our construction is local on the space of parameters, which is not supposed to be a Lie group.

More precisely, let Z, X be complex analytic manifolds, $Z_\mathbb{R}$ a real submanifold of Z whose Z is a complexification, $\phi : Z \times X \to X$ a map such that for each $z \in Z$ the map $\phi_z : X \to X$, $x \mapsto (z, x)$ is smooth and proper. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$; we consider "liftings" of ϕ for $\mathcal{M}$ and F, i.e. $u : \underline{\phi}^{-1}\mathcal{M} \to \underline{p}^{-1}\mathcal{M}$ a $\mathcal{O}_Z \underline{\boxtimes} \mathcal{D}_X$-linear morphism and $v : \phi^{-1}F \to \mathbb{C}_{Z_\mathbb{R}} \boxtimes F$ (in the above setting X should be a complexification of M, $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}^\cdot$ and $F = \mathbb{C}_M$). The motivation for these definitions is the example of quasi-equivariant $\mathcal{D}$-modules and equivariant sheaves, in which case we assume moreover that Z is a group and u and v are compatible with the law of the group. However for the main results of this paper we will not need that Z be a group.

For each $z \in Z_\mathbb{R}$, the liftings restrict to $u_z : \underline{\phi_z}^{-1}\mathcal{M} \to \mathcal{M}$, $v_z : \phi^{-1}F \to F$ and induce a morphism on the global solutions of $\mathcal{M}$ and F:

$$S(u_z, v_z) : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).$$

Hence we obtain $\pi_i : Z_\mathbb{R} \times \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$. We want to compute the "generalized trace" of π_i as a hyperfunction on $Z_\mathbb{R}$. This generalized trace should be understood as in representation theory. Let $\pi : G \to \mathrm{End}(E)$ be a continuous representation of a Lie group; we assume that for each infinitely differentiable form ω with compact support on G the endomorphism of E, $\pi_\omega : x \mapsto \int_G \pi(g)(x) \cdot \omega$ is trace class and that $\chi : \omega \mapsto \operatorname{tr} \pi_\omega$ is a distribution. Then χ is

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called the character of E. We note that this definition makes sense also if G is not a group and π is just a family of endomorphisms of E.

In our case the vector spaces $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ have in general no natural separated topology (we will consider also the particular case when Z is a semi-simple Lie group and X its flag manifold; for this case Kashiwara and Schmid have proved in [16] that the $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ are continuous representations of $Z_{\mathbb{R}}$). But $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ is well-defined in the derived category of Fréchet nuclear spaces and continuous linear maps. We can build directly π_i in this category if we send $Z_{\mathbb{R}}$ into $\Gamma_c(Z_{\mathbb{R}}; \mathcal{B}^{(dz)}_{Z_{\mathbb{R}}})$ by the map $z \mapsto \delta_z$ (δ_z being the Dirac function at z). Indeed in section 4 we will see that u and v define a morphism

$$S(u,v) : \Gamma_c(Z_{\mathbb{R}}; \mathcal{B}^{(dz)}_{Z_{\mathbb{R}}}) \otimes R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X),$$

such that for $z \in Z_{\mathbb{R}}$, $S(u_z, v_z) = S(u,v)(\delta_z \otimes \cdot)$ and, more generally, π_ω above corresponds to $S(u,v)(\omega \otimes \cdot)$.

In section 2 we show that the notions of nuclear map and trace of a nuclear map extend well to the derived category (the important point here is the fact that nuclear maps from nuclear spaces are well-behaved with respect to quotient and inclusion).

In sections 5 and 6 we attach to u and v a hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$ on $Z_{\mathbb{R}}$ by a cohomological trace formula and a microlocal product. More precisely, to u we associate its "kernel" $k(\phi, \mathcal{M}, u)$ (see definition 5.2) with value in a $\mathcal{D}$-module supported by the graph, Γ, of ϕ in $Z \times X \times X$, and we take its image by a diagonal trace map. We microlocalize along Γ in order to keep the information carried by the characteristic variety of $\mathcal{M}$; we obtain a cohomology class:

$$c(\phi, \mathcal{M}, u) \in H^0_{\Lambda_1}(T^*(Z \times X \times X); \mu_\Gamma(\mathcal{O}_Z \boxtimes \delta_! \omega_X)),$$

where Λ_1 is a subset of $T^*_\Gamma(Z \times X \times X)$ depending on $\mathrm{char}\,\mathcal{M}$. For F and v we obtain also a kernel $k(\phi, F, v)$ and a similar class:

$$c(\phi, F, v) \in H^{dz}_{\Lambda_2}(T^*(Z \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_Z \boxtimes \delta_! \omega_X)),$$

where Λ_2 depends on $SS(F)$. If $\Lambda_1 \cap \Lambda_2^a$ is contained in the zero-section we can make the "microlocal product" of the two classes and take the direct image to Z. We obtain a microfunction

$$\chi(\phi, \mathcal{M}, F, u, v) \in H^{dz}_\Lambda(T^*Z; \mu_{Z_{\mathbb{R}}}(\mathcal{O}_Z)),$$

where Λ is a bound expressed in terms of the fixed points of ϕ in $\mathrm{char}\,\mathcal{M} \cap SS(F)$ (in the case of a group action it coincides with the bound given by Berline and Vergne in [4]). The condition on Λ_1 and Λ_2 has a nice expression on X. Let us introduce the following subset of T^*X associated to ϕ:

$$\Lambda_\phi = p_3(T^*_\Gamma(Z \times X \times X) \cap T^*_{Z \times \Delta}(Z \times X \times X)),$$

where $p_3 : T^*Z \times T^*X \times T^*X \to T^*X$ is the projection to the third factor, and Δ the diagonal of $X \times X$. When ϕ is a group action, Λ_ϕ is the conormal to the orbits. We see that $\Lambda_1 \cap \Lambda_2^a$ is in the zero-section if the pair $(\mathcal{M}, F)$ is "transversally elliptic", i.e.

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \cap \Lambda_\phi \subset T^*_X X.$$

In particular if $\phi_{\mathbb{R}} : G_{\mathbb{R}} \times M \to M$ is a real group action which can be complexified into $\phi : G \times X \to X$, then, for $F = \mathbb{C}_M$, $SS(F) \cap \Lambda_\phi = T^*_M X \cap \Lambda_\phi$ can be identified with $T^*_{G_{\mathbb{R}}} M$. Hence if $\mathcal{M}$ is associated to an equivariant differential operator P, the above condition is satisfied if and only if P is transversally elliptic in the sense of Atiyah.

To make the link between $\chi(\phi,\mathcal{M},F,u,v)$ and the trace of $S(u,v)$ we will use that $\chi(\phi,\mathcal{M},F,u,v)$ is the trace of the microlocal product of the kernels $k(\phi,\mathcal{M},u)$ and $k(\phi,F,v)$. Unfortunately we need a stronger hypothesis on $(\mathcal{M},F)$ to make the product of these kernels because their supports are bigger than Λ_1 and Λ_2. We set:

$$\Lambda'_\phi = p_3(T^*_\Gamma(Z\times X\times X)\cap(T^*_Z Z\times T^*(X\times X))).$$

In general Λ_ϕ is strictly included in Λ'_ϕ but for a group action they are equal. We say that $(\mathcal{M},F)$ is strongly transversally elliptic if

$$\operatorname{char}(\mathcal{M})\cap SS(F)\cap\Lambda'_\phi\subset T^*_X X.$$

The main result of the paper is theorem 8.2 which says that if $Z_\mathbb{R}$ is compact, $\mathcal{M}$ is good and $(\mathcal{M},F)$ is strongly transversally elliptic then, for an analytic form ω on $Z_\mathbb{R}$, $S(u,v)(\omega\otimes\cdot)$ is nuclear with trace $\int_{Z_\mathbb{R}}\omega\cdot\chi(\phi,\mathcal{M},F,u,v)$. (In particular, for actions of compact Lie groups, we obtain in section 10 that our cohomological index coincides with Atiyah's index of transversally elliptic operators.)

The idea of the proof is roughly that, if $(\mathcal{M},F)$ is strongly transversally elliptic, then $S(u,v)(\omega\otimes\cdot)$ can be defined with a "smoothing operator". We first prove that, for $\mathcal{M}$ good, a morphism from $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)$ to itself induced by a kernel with value in

$$\Omega_{X\times X}\otimes^L_{\mathcal{D}_{X\times X}}((\mathcal{M}\otimes F)\boxtimes(\underline{\mathrm{D}}\,\mathcal{M}\otimes\mathrm{D}'\,F))$$

(this could be compared to a smoothing operator) is nuclear with trace the cohomological trace of the kernel. For this we use the realification of a $\mathcal{D}$-module introduced by Schapira and Schneiders. Now let k be the microlocal product of the kernels $k(\phi,\mathcal{M},u)$ and $k(\phi,F,v)$. For an analytic form ω on $Z_\mathbb{R}$, let k_ω be the direct image on $X\times X$ of $k\cdot\omega$. This is a kernel on $X\times X$ of the kind above; hence it has a well-defined trace. The morphism associated to k_ω is nothing but $S(u,v)(\omega\otimes\cdot)$ (see proposition 6.5) and the theorem follows.

In section 9 we assume that the graph of ϕ, Γ, is transversal to the diagonal, $Z\times\Delta$, of $Z\times X\times X$ (for a group action this means that X is homogeneous). In this case Λ_ϕ is included in the zero-section so that any pair $(F,\mathcal{M})$ is transversally elliptic and the microlocal product reduces to a usual cup-product on the zero-section. If we assume moreover that $\mathcal{M}$ arises from a complex of vector bundles and u from a morphism of complexes, we can show that $c(\phi,\mathcal{M},u)$ is the image of a holomorphic form on the fixed points manifold $\tilde{Z}=\Gamma\cap(Z\times\Delta)$. (If Z is a point this means that $\phi: X\to X$ is a map transversal to id and we obtain the Atiyah-Bott formula of [2] for a "linear" lifting.)

In section 10 we consider in particular the action of a complex semi-simple Lie group, G, on its flag manifold, X. For $\mathcal{M}=\mathcal{D}_X$ and F a $G_\mathbb{R}$-equivariant sheaf on X ($G_\mathbb{R}$ being a real form of G), the pair $(\mathcal{M},F)$ is strongly transversally elliptic and our formula for $\chi(\phi,\mathcal{M},F,u,v)$ is the formula given by Kashiwara in [14]. We prove that $\chi(\phi,\mathcal{M},F,u,v)$ has a well-defined restriction to any translate, $g\cdot K$, of a maximal compact subgroup, K, of $G_\mathbb{R}$. Hence we obtain (theorem 10.4) that $\chi(\phi,\mathcal{M},F,u,v)$ is the character of $G_\mathbb{R}$ in $R\mathrm{Hom}(F,\mathcal{O}_X)$ (which is a continuous representation of $G_\mathbb{R}$ by Kashiwara-Schmid results), as conjectured in [14] (see [21] for another proof).

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Notations. We will mainly follow the definitions and notations of [15]. For a manifold X, $\pi_X : T^*X \to X$ (or π is there is no risk of confusion) is the projection from the cotangent bundle to X. For a morphism of manifolds $f : X \to Y$, we have the induced maps on the cotangent bundles:

$$T^*X \xleftarrow{{}^tf'} X \times_Y T^*Y \xrightarrow{f_\pi} T^*Y.$$

If Λ is a closed conic subset of T^*Y, we say that f is non-characteristic for Λ if ${}^tf'$ is proper on $f_\pi^{-1}(\Lambda)$. We denote by Λ^a the image of Λ by the antipodal map of T^*Y, $(y,\xi) \mapsto (y,-\xi)$. We denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$) the bounded derived category of sheaves (resp. $\mathbb{R}$-constructible sheaves) on X.

The topological dualizing sheaf is $\omega_X = a^!\mathbb{C}$, for a the projection from X to a point. More generally, for $f : X \to Y$, we set $\omega_{X|Y} = f^!\mathbb{C}_Y$. For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$, its dual and its naive dual are:

$$\mathrm{D}\, F = R\mathcal{H}om\,(F, \omega_X), \quad \mathrm{D}'\, F = R\mathcal{H}om\,(F, \mathbb{C}_X).$$

If M is a submanifold of X, the conormal to M is denoted by $T^*_M X$ and Sato's microlocalization functor along M is denoted by μ_M. The diagonal of $X \times X$ is usually denoted by Δ_X or Δ. The functor μhom is defined by:

$$\mu hom(F,G) = \mu_\Delta R\mathcal{H}om\,(q_2^{-1}F, q_1^!G),$$

where q_i is the projection from $X \times X$ to the i^{th} factor. The micro-support of $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ is denoted by $SS(F)$.

For a complex analytic manifold X, we denote by d_X its complex dimension, by Ω_X or $\mathcal{O}_X^{(d_X)}$ the sheaf of holomorphic maximal degree forms on X. For a product of complex manifolds we denote by $\mathcal{O}_{X\times Y}^{(a,b)}$ the holomorphic forms of degree a on X and b on Y. A $\mathcal{D}_X$-module is "good" if, in a neighborhood of any compact subset of X, it admits a finite filtration by coherent $\mathcal{D}_X$-submodules, such that each quotient of this filtration can be endowed with a good filtration. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$) the bounded derived category of complexes of $\mathcal{D}_X$-modules with coherent (resp. good) cohomology. If $f : X \to Y$ is a morphism of complex analytic manifolds, the inverse and direct images for $\mathcal{D}$-modules are denoted by $\underline{f}^{-1}$ and $\underline{f}_*$. The dualizing sheaf for left $\mathcal{D}_X$-modules is

$$\mathcal{K}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{D}_X)[d_X].$$

It has two left $\mathcal{D}_X$-module structures. The dual of a left $\mathcal{D}_X$-module $\mathcal{M}$ is the left $\mathcal{D}_X$-module:

$$\underline{\mathrm{D}}\,\mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X).$$

The characteristic variety of $\mathcal{M}$ is denoted by $\mathrm{char}\,\mathcal{M}$. We say that a map is non-characteristic for a $\mathcal{D}$-module $\mathcal{M}$ or a sheaf F if it is non-characteristic for $\mathrm{char}\,\mathcal{M}$ or $SS(F)$. We denote by $\boxtimes$ and $\underline{\boxtimes}$ the external tensor products for sheaves and $\mathcal{D}$-modules.

2. Nuclear maps in the derived category

We will need a notion of trace for a morphism in the derived category of Fréchet nuclear spaces (FN-spaces), or DFN-spaces. We prove that the notions of nuclear map and trace of a nuclear map extend well to the derived category.

The derived category of FN-spaces and linear continuous maps is constructed as follows (see [3] and also [22]). Let $\mathbf{C}^{\mathrm{b}}(FN)$ be the category of bounded complexes of FN-spaces. The category $\mathbf{K}^{\mathrm{b}}(FN)$ is obtained from $\mathbf{C}^{\mathrm{b}}(FN)$ by identifying to 0 a morphism homotopic to 0. The complexes which are algebraic exact form a null system in $\mathbf{K}^{\mathrm{b}}(FN)$. The derived category $\mathbf{D}^{\mathrm{b}}(FN)$ is defined as the localization of $\mathbf{K}^{\mathrm{b}}(FN)$ by this null system. Since the topological tensor product $\bar{\otimes}$ is exact on the category of FN-spaces, it extends to the derived category. The category $\mathbf{D}^{\mathrm{b}}(DFN)$ is defined similarly.

In [9] Grothendieck develops a theory of "p-summable" Fredholm kernels and introduces the nuclear spaces. We give a brief summary of the results we will need. In the following we write LCTVS for locally convex topological vector space. For G, F two LCTVS and p a real number such that $0 < p \leq 1$ let $G \overset{(p)}{\otimes} F$ be the set of elements of $G\bar{\otimes}F$ which can be written $\sum_i \lambda_i\, x_i \otimes y_i$ with $\sum_i |\lambda_i|^p < \infty$ and (x_i) (resp. (y_i)) in a bounded convex circled subset A (resp. B) of G (resp. F) such that the associated normed space G_A (resp. F_B) be complete.

By [9], chapter II, corollary 4 of theorem 4, we know that for a LCTVS E the natural map $E' \overset{(2/3)}{\otimes} E \to L(E,E)$ is injective (here E' is the strong dual of E and $L(E,E)$ is the set of continuous linear maps from E to itself). Hence a map $u \in L(E,E)$ which belongs to the image of $E' \overset{(2/3)}{\otimes} E$ has well-defined determinant and trace, namely the determinant and trace of its unique kernel in $E' \overset{(2/3)}{\otimes} E$.

Let $u \in E' \overset{(1)}{\otimes} E$ and let $\tilde{u}$ be its image in $L(E,E)$. The link between the determinant of u and the eigenvalues of $\tilde{u}$ is explained in [8], chapter II, theorem 4. For $\lambda \in \mathbb{C} \smallsetminus \{0\}$ set $E_{1/\lambda} = \bigcup_{p\in\mathbb{N}} \ker(1-\lambda\tilde{u})^p$. Then $n = \dim E_{1/\lambda}$ is finite and λ is a zero of order n of $\det(1-zu)$. Moreover, if $F_{1/\lambda} = \operatorname{im}(1-\lambda\tilde{u})^n$ then E is the topological direct sum of $E_{1/\lambda}$ and $F_{1/\lambda}$, and $(1-\lambda\tilde{u}) : F_{1/\lambda} \to F_{1/\lambda}$ is an isomorphism.

If $u \in E' \overset{(2/3)}{\otimes} E$ and $(\lambda_i)_{i\in\mathbb{N}}$ is the sequence of eigenvalues of $\tilde{u}$, with multiplicities, we have also (see [9] chapter II, corollary 4 of theorem 4):

$$\det(1-zu) = \prod_{i\in\mathbb{N}}(1-z\lambda_i), \qquad \sum_{i\in\mathbb{N}} |\lambda_i| < \infty, \qquad \operatorname{tr} u = \sum_{i\in\mathbb{N}} \lambda_i.$$

These results apply in particular to nuclear maps from nuclear spaces because, by [9] chapter II, corollary 3 of theorem 11, any bounded map from a nuclear quasi-complete space E to itself is in the image of $E' \overset{(p)}{\otimes} E$ for any $p > 0$.

If $u : E \to E$ is a nuclear map between two LCTVS and F is a closed subspace of E such that $u(F) = \{0\}$, the induced map $E/F \to G$ is in general not nuclear (see [9] chapter I, remark 9 after proposition 16). However this is true if E is a nuclear space.

Lemma 2.1. *Let $u : E \to G$ be a nuclear map between two LCTVS and assume that E is a nuclear space.*

i) *Assume F is a closed subspace of E such that $u(F) = 0$. Then the induced map $u' : E/F \to G$ is nuclear.*

ii) *Assume F is a closed subspace of G such that $u(E) \subset F$. Then the induced map $u'' : E \to F$ is nuclear.*

Proof. *i)* Since u is nuclear it decomposes as $E \xrightarrow{a} B_1 \xrightarrow{b} B_2 \xrightarrow{c} G$ where B_1, B_2 are Banach spaces and b is nuclear. We can factor c through the quotient of B_2 by $\ker(c)$ and hence assume that c is injective. Then $\ker(u) = \ker(b \circ a)$ and u' decomposes as $E/F \xrightarrow{u_1} B_2 \xrightarrow{c} G$. Now E/F is nuclear too and any continuous linear map from a nuclear space to a Banach space is nuclear. Hence u_1, and u', are nuclear.

ii) We write $u = c \circ b \circ a$ as above. Since $\operatorname{im}(a) \subset (c \circ b)^{-1}(F)$ we may replace B_1 by $(c \circ b)^{-1}(F)$ and hence assume that $\operatorname{im}(c \circ b) \subset F$. Then u'' decomposes as $E \xrightarrow{u_2} B_1 \xrightarrow{c \circ b} F$ and it is nuclear because u_2 is. ☐

Definition 2.2. Let $E^\cdot$, $F^\cdot$ be objects of $\mathbf{D}^{\mathrm{b}}(FN)$. A morphism $u : E^\cdot \to F^\cdot$ in $\mathbf{D}^{\mathrm{b}}(FN)$ is called nuclear if there exists a morphism of complexes $v^\cdot : E^\cdot \to F^\cdot$ in $\mathbf{C}^{\mathrm{b}}(FN)$ such that all maps $v^i : E^i \to F^i$ are nuclear and $u = v^\cdot$ in $\mathbf{D}^{\mathrm{b}}(FN)$. A nuclear morphism in $\mathbf{D}^{\mathrm{b}}(DFN)$ is defined in the same way.

The following lemma implies that a nuclear morphism in $\mathbf{D}^{\mathrm{b}}(FN)$ or $\mathbf{D}^{\mathrm{b}}(DFN)$ has a well-defined trace which depends only on the (purely algebraic) morphism induced on the cohomology. It is convenient to introduce the following notations and terminology. For an endomorphism $w : G \to G$ of a $\mathbb{C}$-vector space and $\lambda \in \mathbb{C}$, we write:

$$G_\lambda = \bigcup_{n \in \mathbb{N}} \ker(w - \lambda)^n, \qquad {}_\lambda G = \bigcap_{n \in \mathbb{N}} \operatorname{im}(w - \lambda)^n.$$

We say that w has a "naive trace" if, setting $m_\lambda = \dim G_\lambda$, we have:

$$\forall \lambda \in \mathbb{C}^*,\ m_\lambda < \infty \qquad \text{and} \qquad \sum_{\lambda \in \mathbb{C}^*} m_\lambda \cdot |\lambda| < \infty.$$

If this is the case we set $\operatorname{tr}_n w = \sum_{\lambda \in \mathbb{C}^*} m_\lambda \cdot \lambda$.

Lemma 2.3. *Let $E^\cdot$ be a complex of nuclear spaces and $u^\cdot : E^\cdot \to E^\cdot$ a morphism of complexes such that each u^i is nuclear. Then, for each i, $H^i(u^\cdot) : H^i(E^\cdot) \to H^i(E^\cdot)$ has a "naive trace" and:*

$$\sum_i (-1)^i \operatorname{tr}_n H^i(u^\cdot) = \sum_i (-1)^i \operatorname{tr} u^i.$$

Proof. We prove the lemma by induction on the length of the complex $E^\cdot$. If it is of length 1 this is a restatement of the properties of nuclear maps in nuclear spaces recalled above, namely that they have a "naive trace" equal to the trace of their kernel.

Let us assume $E^\cdot$ is of length n (with $E^i = 0$ for $i < 1$ and $i > n$) and the result is true for complexes of length less than $n-1$. Let us consider the truncated complex $F^\cdot = \tau_{<n} E^\cdot$ and the endomorphism $v^\cdot$ of $F^\cdot$ induced by $u^\cdot$. By definition $F^i = E^i$ for $i \leq n-2$, $F^{n-1} = \ker d_E^{n-1}$, $F^i = 0$ for $i \geq n$. The F^i are also FN-spaces and, by lemma 2.1, v^{n-1} is nuclear so that the induction hypothesis applies to $F^\cdot$ and $v^\cdot$. It just remains to prove that $H^n(u^\cdot) : E^n / \operatorname{im} d_E^{n-1} \to E^n / \operatorname{im} d_E^{n-1}$ has a naive trace and

$$\operatorname{tr}_n H^n(u^\cdot) = \operatorname{tr} u^n - \operatorname{tr} u^{n-1} + \operatorname{tr} v^{n-1}.$$

Since u^n, u^{n-1}, v^{n-1} are nuclear maps in nuclear spaces they have a naive trace equal to their trace as nuclear maps. Hence our lemma will follow from the exactness of the sequence:

$$0 \to F^{n-1}_\lambda \to E^{n-1}_\lambda \to E^n_\lambda \to (E^n/\operatorname{im} d_E^{n-1})_\lambda \to 0,$$

for all $\lambda \in \mathbb{C}^*$.

The exactness at the first two terms is obvious. Recall that $E^n = E^n_\lambda \oplus {}_\lambda E^n$ and $E^{n-1} = E^{n-1}_\lambda \oplus {}_\lambda E^{n-1}$ and since d_E^{n-1} commutes with $u^\cdot$, d_E^{n-1} respects this decomposition. Hence an element of E^n_λ which is in $\operatorname{im} d_E^{n-1}$ is in fact the image of an element of E^{n-1}_λ. This prove the exactness at the third term. Let us prove the surjectivity at the last term. Let $x \in E^n$ be such that $(u^n - \lambda)^k(x) \in \operatorname{im} d_E^{n-1}$. We have to find an element of E^n_λ in the class of x modulo $\operatorname{im} d_E^{n-1}$. We may as well assume that k is great enough so that $\operatorname{im}(u^n-\lambda)^k = {}_\lambda E^n$. Since $(u^n-\lambda)^k(x)$ belongs to ${}_\lambda E^n \cap \operatorname{im} d_E^{n-1}$ there exists $y \in {}_\lambda E^{n-1}$ such that $d_E^{n-1} y = (u^n - \lambda)^k(x)$ (again because d_E^{n-1} respects the decomposition of E^n and E^{n-1}). We know that $u^{n-1}-\lambda : {}_\lambda E^{n-1} \to {}_\lambda E^{n-1}$ is an isomorphism, so that we may write $y = (u^{n-1} - \lambda)^k(y')$ with $y' \in {}_\lambda E^{n-1}$ and we have $(u^n - \lambda)^k(x - d_E^{n-1} y') = 0$. Hence $x' = x - d_E^{n-1} y'$ belongs to $E^n_\lambda \cap (x + \operatorname{im} d_E^{n-1})$ and this prove the surjectivity. □

Definition 2.4. Let $u : E^\cdot \to E^\cdot$ be a nuclear morphism in $\mathbf{D}^{\mathrm{b}}(FN)$ or $\mathbf{D}^{\mathrm{b}}(DFN)$ and let $v^\cdot : E^\cdot \to E^\cdot$ be a morphism of complexes representing u with the v^i nuclear. We call trace of u the number $\operatorname{tr} u = \sum_i (-1)^i \operatorname{tr} v^i$ which only depends on u by the preceding lemma.

Remarks 2.5. 1) Since the trace of nuclear maps between topological vector spaces is additive, the trace we have defined is also additive.

2) From the algebraic description of the trace it is easy to see that if E, F are objects of $\mathbf{D}^{\mathrm{b}}(FN)$ or $\mathbf{D}^{\mathrm{b}}(DFN)$ and $u : E \to F$ and $v : F \to E$ are two morphisms such that $u \circ v$ and $v \circ u$ are nuclear then $\operatorname{tr} u \circ v = \operatorname{tr} v \circ u$.

3) The lemma implies in particular that if id_E is nuclear for an object E of $\mathbf{D}^{\mathrm{b}}(FN)$ then the cohomology groups of E are of finite dimension.

4) If $u : E \to E$ is a nuclear morphism in $\mathbf{D}^{\mathrm{b}}(FN)$ and $\operatorname{im} d_E^i$ is closed for a given i, then $H^i(E)$ is an FN-space and $H^i(u) : H^i(E) \to H^i(E)$ a nuclear map. In particular $H^i(u)$ has a trace as a nuclear map and $\operatorname{tr} H^i(u) = \operatorname{tr}_n H^i(u)$.

5)For three LCTVS E, F, G and two continuous linear maps $u : E \to F$, $v : F \to G$, the composition $v \circ u$ is nuclear as soon as u or v is nuclear. The same is true in the category $\mathbf{D}^{\mathrm{b}}(FN)$ for our notion of nuclear morphism as will follow easily from the next lemma.

Lemma 2.6. *Let $E^\cdot$, $F^\cdot$, $G^\cdot$ be objects of $\mathbf{C}^{\mathrm{b}}(FN)$.*

- *i) Let $u^\cdot : E^\cdot \to F^\cdot$, $\varphi^\cdot : E^\cdot \to G^\cdot$ be morphisms in $\mathbf{C}^{\mathrm{b}}(FN)$ such that each $u^i : E^i \to F^i$ is nuclear and $\varphi^\cdot$ is a quasi-isomorphism. Then there exists a morphism of complexes $v^\cdot : G^\cdot \to F^\cdot$ such that each $v^i : G^i \to F^i$ is nuclear and $v^\cdot \circ \varphi^\cdot$ is homotopic to $u^\cdot$.*
- *ii) The same with reversed arrows.*

Proof. $i)$ Let us denote by $d_E^\cdot$, $d_F^\cdot$, $d_G^\cdot$ the differentials of $E^\cdot$, $F^\cdot$, $G^\cdot$. We consider the mapping cone $M^\cdot$ of $\varphi^\cdot$, i.e. $M^i = E^{i+1} \bigoplus G^i$ with differential $d_M^i = \begin{pmatrix} -d_E^{i+1} & 0 \\ \varphi^{i+1} & d_G^i \end{pmatrix}$. Since $\varphi^\cdot$ is a quasi-isomorphism, $M^\cdot$ is (algebraically) exact. The morphism $u^\cdot$

induces a morphism $u'^{\cdot} = (u^{\cdot}, 0)$ from $M^{\cdot}$ to $F^{\cdot}[1]$, where $F^{\cdot}[1]$ is the complex with components $F^i[1] = F^{i+1}$ and differential $-d^{\cdot}_F$. Each u'^i is of course nuclear. We claim that there exists a homotopy $s^j : M^j \to F^j$ such that each s^j is nuclear and $u'^j = -d^j_F \circ s^j + s^{j+1} \circ d^j_M$. Since the complexes are bounded we may prove this by increasing induction. We assume that s^j has been built for $j \leq i$ and we construct s^{i+1}. We consider $a = u'^i + d^i_F \circ s^i$. Since $a \circ d^{i-1}_M = 0$ and $\operatorname{im} d^{i-1}_M = \ker d^i_M$, a factors through a map $b : M^i / \ker d^i_M \to F^{i+1}$, which is nuclear by lemma 2.1. By the open mapping theorem (M^i and $\operatorname{im} d^i_M = \ker d^{i+1}_M$ are Fréchet spaces) the injection $i : M^i / \ker d^i_M \to M^{i+1}$ is an isomorphism of $M^i / \ker d^i_M$ onto its image. In this situation b factors through a nuclear map $s^{i+1} : M^{i+1} \to F^{i+1}$ (see [9] chapter I, proposition 16) so that $u'^i = -d^i_F \circ s^i + s^{i+1} \circ d^i_M$. Hence we have proved the existence of the homotopy.

Now we decompose $s^i = (s^{i+1}_E, s^i_G)$ and write the preceding equality in terms of this decomposition. We obtain:

$$u^i = -d^i_F \circ s^{i+1}_E - s^{i+2}_E \circ d^{i+1}_E + s^{i+1}_G \circ \varphi^{i+1} \tag{1}$$

$$0 = -d^i_F \circ s^i_G + s^{i+1}_G \circ d^i_G. \tag{2}$$

Let us set $v^i = s^i_G$; this is a nuclear map since s^i is. Formula (2) shows that $v^{\cdot} : G^{\cdot} \to F^{\cdot}$ is a morphism of complexes and formula (1) shows that $u^{\cdot}$ and $v^{\cdot} \circ \varphi^{\cdot}$ are homotopic.

ii) The proof is similar. We consider the morphism from $F^{\cdot}$ to the mapping cone of $\varphi^{\cdot}$ induced by $u^{\cdot}$ and we show that this morphism is homotopic to 0 by a nuclear homotopy. This is done by decreasing induction, using a property of lifting of nuclear maps (see [9] chapter I, proposition 16 or also [20], proposition 2.3 of the third part):

> Let $u : A \to B$, $v : C \to B$ be two morphisms of Fréchet spaces with u surjective and v nuclear. Then there exists a nuclear map $w : C \to A$ such that $v = u \circ w$.

□

Now let $u : E^{\cdot} \to F^{\cdot}$, $v : F^{\cdot} \to G^{\cdot}$ be two morphisms in $\mathbf{D}^{\mathrm{b}}(FN)$ and assume u is nuclear. Then u is equal in $\mathbf{D}^{\mathrm{b}}(FN)$ to a morphism of complexes $u'^{\cdot} : E^{\cdot} \to F^{\cdot}$ such that each u'^i is nuclear and there exist a complex $H^{\cdot}$ and morphisms of complexes $\varphi^{\cdot} : H^{\cdot} \to F^{\cdot}$, $v'^{\cdot} : H^{\cdot} \to G^{\cdot}$ such that φ is a quasi-isomorphism and $v = v' \circ \varphi^{-1}$. Hence $v \circ u = v' \circ \varphi^{-1} \circ u'$ in $\mathbf{D}^{\mathrm{b}}(FN)$. By the previous lemma there exists a morphism of complexes $u_1^{\cdot} : E^{\cdot} \to H^{\cdot}$ such that the u_1^i are nuclear and u' is homotopic to $\varphi \circ u_1$. Hence $v \circ u = v' \circ u_1$ in $\mathbf{D}^{\mathrm{b}}(FN)$. Since the composition of a nuclear map with a continuous linear map is nuclear this show that $v \circ u$ is nuclear. In the same way we can prove that $v \circ u$ is nuclear if v is.

3. Review on the microlocalization functor

Since we will make constant use of Sato's microlocalization functor we recall here some of its main properties as stated in [15].

Let X be a real manifold, M a closed submanifold of X. Let $i : M \to X$, $j : T^*_M X \to T^*X$ be the inclusions and $\pi_X : T^*X \to X$ the projection. The microlocalization along M, μ_M is a functor from $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ to $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*_M X})$. We will often write μ_M for $j_*\mu_M$. The functor μ_M has the following properties (see paragraph 4.3 of [15]).

Proposition 3.1. *Let $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. We have:*

$$\mu_M(F)|_M \simeq R\pi_{X*}\mu_M(F) \simeq i^! F,$$
$$R\pi_{X!}\mu_M(F) \simeq R\Gamma_M(\mu_M(F))|_M \simeq i^{-1}F \otimes \omega_{M|X},$$
$$\operatorname{supp} \mu_M(F) \subset SS(F) \cap T^*_M X.$$

By the first isomorphism, for any closed conic subset Λ of $T^*_M X$, we have a morphism:

$$H^0_\Lambda(T^*X; \mu_M(F)) \to H^0_S(X; F),$$

where $S = \pi_X(\Lambda)$. We will call this morphism "projection to the zero-section". More generally, for a submanifold M' of X such that $M \subset M' \subset X$, we have a morphism, setting $T = T^*_M X \cap T^*_{M'} X$:

$$\mu_M(F)|_T \to R\Gamma_T \mu_{M'}(F).$$

If $L \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ is locally constant then $\mu_M(F \otimes L) \simeq \mu_M(F) \otimes \pi_X^{-1} L$. Microlocalization behaves well with respect to non-characteristic inverse image as shown in the next proposition. Let $f : Y \to X$ be a morphism of manifolds, N a closed submanifold of Y such that $f(N) \subset M$. Let us denote by ${}^t f'_N$ and $f_{N\pi}$ the restrictions of ${}^t f' : Y \times_X T^*X \to T^*Y$ and $f_\pi : Y \times_X T^*X \to T^*X$ to $N \times_M T^*_M X$. The following result is contained in proposition 4.3.5 and corollary 6.7.3 of [15].

Proposition 3.2. *Let $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. We have a commutative diagram:*

$$\begin{array}{ccc}
R\,{}^t f'_{N!} f^{-1}_{N\pi} \mu_M(F) & \xrightarrow{\ r\ } & \mu_N(f^{-1}F) \otimes \pi_Y^{-1}(\omega_{Y|X} \otimes \omega^{-1}_{N|M}) \\
\downarrow & & \downarrow{\scriptstyle s} \\
R\,{}^t f'_{N*} f^!_{N\pi} \mu_M(F) \otimes \pi_Y^{-1} \omega^{-1}_{N|M} & \longleftarrow & \mu_N(f^! F) \otimes \pi_Y^{-1} \omega^{-1}_{N|M}
\end{array}$$

compatible with the projection to the zero-section. If f is non-characteristic for F and $f|_N : N \to M$ is smooth then r is an isomorphism.

Remark 3.3. The compatibility with the projection to the zero-section means the following. Let $\pi_X : T^*X \to X$, $\pi_Y : T^*Y \to Y$, $\tau : Y \times_X T^*X \to Y$ be the projections. We set for short:

$$F_1 = f^{-1}R\Gamma_M(F),\ F_2 = R\Gamma_N(f^{-1}F) \otimes \omega_{Y|X} \otimes \omega^{-1}_{N|M},\ F_3 = R\Gamma_N(f^! F) \otimes \omega^{-1}_{N|M}.$$

The first isomorphism of proposition 3.1 induces:

$$a : R\pi_{Y*}(R\,{}^t f'_{N!} f^{-1}_{N\pi} \mu_M(F)) \to F_1,$$
$$b : R\pi_{Y*}(\mu_N(f^{-1}F) \otimes \pi_Y^{-1}(\omega_{Y|X} \otimes \omega^{-1}_{N|M})) \xrightarrow{\sim} F_2,$$
$$c : R\pi_{Y*}(\mu_N(f^! F) \otimes \pi_Y^{-1} \omega^{-1}_{N|M}) \xrightarrow{\sim} F_3.$$

Morphism a is obtained from the morphism of functors $R\,{}^t f'_{N!}(\cdot) \to R\,{}^t f'_{N*}(\cdot)$ and the isomorphism $R\tau_* f^{-1}_{N\pi}(G) \simeq f^{-1}R\pi_{X*}(G)$ for any conic object G of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*X})$. There is in general no morphism from F_1 to F_2 (this is in fact the reason why we need to microlocalize). However we have two natural morphisms $F_1 \to F_3$ and $F_2 \to F_3$ described by the following compositions:

$$t_0 : F_1 \to (R\Gamma_M(F))_N \to R\Gamma_N R\Gamma_M(F) \otimes \omega^{-1}_{N|M} \simeq F_3,$$
$$s_0 : F_2 \to R\Gamma_N(f^{-1}F \otimes \omega_{Y|X}) \otimes \omega^{-1}_{N|M} \to F_3.$$

The diagram of the proposition is compatible with the projection to the zero-section in the sense that $R\pi_{Y*}(s) = s_0$ and $R\pi_{Y*}(s \circ r) = t_0 \circ a$. Roughly speaking, t_0 can be factorized through s_0 for sections of F_1 arising from "microlocal sections of F" whose support is non-characteristic for f (see also proposition 3.8 below).

The inverse image morphism r induces a morphism on the global sections whose support is in good position. Let Λ be a closed conic subset of T^*X. Assume that f is non-characteristic for Λ. Then ${}^tf'$ is proper on $f_\pi^{-1}(\Lambda)$ and, setting $\Lambda' = {}^tf'(f_\pi^{-1}(\Lambda))$, morphism r gives us a morphism:

$$H^0_\Lambda(T^*X; \mu_M(F)) \to H^0_{\Lambda'}(T^*Y; \mu_N(f^{-1}F) \otimes \pi_Y^{-1}(\omega_{Y|X} \otimes \omega^{-1}_{N|M})). \tag{3}$$

When f is a diagonal embedding this morphism yields a product between microlocal classes. This is the way the product of Euler classes is defined in [20]. We will need such a product in the following situation.

Lemma 3.4. *Let X be a real manifold, M and N submanifolds of X, with $N \subset M$. Let $F, G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ and let Λ_1 and Λ_2 be closed conic subsets of T^*_MX. We assume that $\Lambda_1 \cap \Lambda_2^a \subset T^*_XX$. Then morphism (3) induces a "microlocal product":*

$$\begin{aligned} &H^0_{\Lambda_1}(T^*X; \mu_M(F)) \times H^0_{\Lambda_2}(T^*X; \mu_N(G)) \\ &\qquad\qquad \to H^0_{\Lambda_1+\Lambda_2}(T^*X; \mu_N(F \otimes G) \otimes \pi_X^{-1}\omega_{M|X}) \end{aligned}$$

compatible with the projection to the zero-section in the sense of remark 3.3.

Proof. The external product defines a morphism from the left hand side to

$$H^0_{\Lambda_1\times\Lambda_2}(T^*(X \times X); \mu_{M\times N}(F \boxtimes G)).$$

Let $\delta : X \to X \times X$ be the diagonal embedding. The assumption $\Lambda_1 \cap \Lambda_2^a \subset T^*_XX$ is equivalent to the fact that δ is non-characteristic for $\Lambda_1 \times \Lambda_2$. Hence we may compose the external product with the morphism (3) where Y, X, N, M, f are replaced by X, $X \times X$, N, $N \times M$, δ. This gives the desired morphism if we identify $\omega^{-1}_{N|N\times M} \otimes \omega_{X|X\times X}$ and $(\omega_{M|X})_N$. ☐

Remark 3.5. This microlocal product is compatible with the inverse image in the following situation. We keep the notations of lemma 3.4 and consider moreover a morphism of manifolds $f : X' \to X$, M', N' submanifolds of X' such that $N' \subset M'$, $f(M') \subset M$ and $f(N') \subset N$. We assume that f is non-characteristic for $\Lambda_1 + \Lambda_2$ (hence also for Λ_1 and Λ_2) and we set $\Lambda_1' = {}^tf'(f_\pi^{-1}(\Lambda_1))$, $\Lambda_2' = {}^tf'(f_\pi^{-1}(\Lambda_2))$. Let us set for short:

$$\begin{gathered} A_1 = H^0_{\Lambda_1}(T^*X; \mu_M(F)), \qquad A_2 = H^0_{\Lambda_2}(T^*X; \mu_N(G)), \\ A = H^0_{\Lambda_1+\Lambda_2}(T^*X; \mu_N(F \otimes G) \otimes \pi_X^{-1}\omega_{M|X}), \\ A_1' = H^0_{\Lambda_1'}(T^*X'; \mu_{M'}(f^{-1}F) \otimes \pi_{X'}^{-1}(\omega_{X'|X} \otimes \omega^{-1}_{M'|M})), \\ A_2' = H^0_{\Lambda_2'}(T^*X'; \mu_{N'}(f^{-1}F) \otimes \pi_{X'}^{-1}(\omega_{X'|X} \otimes \omega^{-1}_{N'|N})), \\ A' = H^0_{\Lambda_1'+\Lambda_2'}(T^*X'; \mu_{N'}(f^{-1}(F \otimes G)) \otimes \pi_{X'}^{-1}\omega), \end{gathered}$$

where $\omega = f^{-1}(\omega_{M|X}) \otimes \omega_{X'|X} \otimes \omega^{-1}_{N'|N}$. Since f is non-characteristic for Λ_1, Λ_2, $\Lambda_1 + \Lambda_2$, we have inverse image morphisms as (3): $r_1 : A_1 \to A_1'$, $r_2 : A_2 \to A_2'$, $r : A \to A'$. Since f is non-characteristic for $\Lambda_1 + \Lambda_2$ and $\Lambda_1 \cap \Lambda_2^a \subset T^*_XX$, we have also $\Lambda_1' \cap {\Lambda_2'}^a \subset T^*_{X'}X'$, so that there exists a microlocal product from $A_1' \times A_2'$ to A'.

The microlocal products, starting from $A_1 \times A_2$ and $A'_1 \times A'_2$, commute with the inverse images $r_1 \times r_2$ and r. This is a consequence of the fact that the inverse image as (3) is compatible with the composition of morphisms of manifolds.

Remark 3.6. We keep the notations of the preceding remark. We set moreover $\Lambda''_2 = \Lambda_2 \cap T^*_M X$ and $A''_2 = H^0_{\Lambda''_2}(T^*X; \mu_M(G))$. Since $N \subset M$ we have a morphism $A_2 \to A''_2$. Setting:

$$A'' = H^0_{\Lambda_1+\Lambda''_2}(T^*X; \mu_M(F \otimes G) \otimes \pi_X^{-1}\omega_{M|X}),$$

we have also a morphism $A \to A''$ and, since $\Lambda_1 \cap \Lambda''^a_2 \subset T^*_X X$, the microlocal product from $A_1 \times A''_2$ to A'' is well-defined. We obtain a commutative diagram:

$$\begin{array}{ccc} A_1 \times A_2 & \longrightarrow & A \\ \downarrow & & \downarrow \\ A_1 \times A''_2 & \longrightarrow & A'' \end{array}$$

where the horizontal arrows are microlocal products and the vertical arrows are projections to $T^*_M X$. In view of this diagram it is useless to consider two submanifolds of X to define the microlocal product of lemma 3.4 if we are only interested in the projection of the result to the zero-section, because N does not appear in the second line. However the bound for the support of the product obtained in the first line is more precise than the bound obtained in the second line.

Remark 3.7. The microlocal product is related to the cup-product when M is non-characteristic for F, i.e. $T^*_M X \cap SS(F) \subset T^*_X X$. In this case we have $F \otimes \omega_{M|X} \xrightarrow{\sim} R\Gamma_M(F)$ and $\operatorname{supp} \mu_M(F) \subset T^*_X X$ so that π_X is proper on $\operatorname{supp} \mu_M(F)$ and the projection to the zero-section gives isomorphisms:

$$H^0_{\Lambda_1}(T^*X; \mu_M(F)) \simeq H^0_{S_1}(X; R\Gamma_M(F)) \simeq H^0_{S_1}(X; F \otimes \omega_{M|X}),$$

where $S_i = \pi_X(\Lambda_i)$. Let us also identify the sheaves corresponding to F_1, F_2, F_3 of remark 3.3 in our case. We have:

$$F_1 = R\Gamma_M(F) \otimes R\Gamma_N(G),\ F_2 = R\Gamma_N(F \otimes G) \otimes \omega_{M|X},\ F_3 = R\Gamma_N(\delta^!(F \boxtimes G)) \otimes \omega_M.$$

Since we have also $F_1 \simeq \omega_{M|X} \otimes F \otimes R\Gamma_N(G)$, there exists a morphism $F_1 \to F_2$ which factorizes the morphism t_0 of remark 3.3. Hence the compatibility of the microlocal product and the projection to the zero-section gives the commutative diagram:

$$\begin{array}{ccc} A_1 \times A_2 & \longrightarrow & A \\ \downarrow & & \downarrow \\ H^0_{S_1}(X; F \otimes \omega_{M|X}) \times H^0_{S_2}(X; G) & \longrightarrow & H^0_{S_1 \cap S_2}(X; F \otimes G \otimes \omega_{M|X}), \end{array}$$

where the bottom arrow is the usual cup-product.

We will also need a slightly different version of the microlocal product. Let F, F', G, G' be objects of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$. Let $\delta : X \to X \times X$ be the diagonal embedding. We consider $\mu hom(F, G)$ and $\mu hom(G', F')$; they are objects of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*X})$ and satisfy

$$R\pi_* \mu hom(F, G) \simeq R\mathcal{H}om\,(F, G) \quad R\pi_* \mu hom(G', F') \simeq R\mathcal{H}om\,(G', F').$$

We have the canonical morphisms:

$$\text{Hom}\,(F,G)\otimes\text{Hom}\,(G',F')\to\text{Hom}\,(R\mathcal{H}om\,(F',F),R\mathcal{H}om\,(G',G)) \tag{4}$$

$$\text{Hom}\,(R\mathcal{H}om\,(F',F),G\otimes\mathrm{D}'\,G')\to\text{Hom}\,(R\mathcal{H}om\,(F',F),R\mathcal{H}om\,(G',G)). \tag{5}$$

We are looking for conditions which imply that morphism (4) can be factorized through (5). This is the case if the morphisms in $\text{Hom}\,(F,G)$ and $\text{Hom}\,(G',F')$ arise from sections of $\mu hom(F,G)$ and $\mu hom(G',F')$ with suitable supports. The following result is contained in the proof of proposition 4.4.8 of [15].

Proposition 3.8. *Let Λ, Λ' be closed conic subsets of T^*X satisfying $\Lambda^a\cap\Lambda'\subset T^*_XX$. There exists a natural morphism:*

$$\begin{aligned}H^0_\Lambda(T^*X;\mu hom(F,G))\otimes H^0_{\Lambda'}(T^*X;\mu hom(G',F'))&\\ \to H^0_{\Lambda+\Lambda'}(T^*X;\mu hom(R\mathcal{H}om\,(F',F),G\otimes\mathrm{D}'\,G'))&\\ \to\text{Hom}\,(R\mathcal{H}om\,(F',F),G\otimes\mathrm{D}'\,G'),&\end{aligned}$$

whose composition with morphism (5) *coincides with the composition of the projection to the zero-section and morphism* (4).

4. Liftings and action on global sections

In this section we introduce liftings of an application for a $\mathcal{D}$-module and a constructible sheaf and define their action on the solutions. In the rest of the paper we will be interested in the trace of this action.

The general situation will be the following. Let X and Z be complex analytic manifolds. We consider a family of maps from X to itself parameterized by Z. By this we mean a morphism of manifolds $\phi : Z\times X\to X$; for $z\in Z$ we denote by $i_z : X\to Z\times X$ the embedding $x\mapsto(z,x)$ and we set $\phi_z=\phi\circ i_z : X\to X$. We will always make the following hypothesis on ϕ:

$$\forall z\in Z,\ \phi_z : X\to X \text{ is smooth and proper.} \tag{6}$$

We consider also a real analytic submanifold $Z_\mathbb{R}$ of Z such that Z is a complexification of $Z_\mathbb{R}$. To simplify the exposition we will always assume that $Z_\mathbb{R}$ is oriented. We denote by $\phi_\mathbb{R} : Z_\mathbb{R}\times X\to X$ the restriction of ϕ, by $p : Z\times X\to X$ and $p_\mathbb{R} : Z_\mathbb{R}\times X\to X$ the projections, by $\Gamma\subset Z\times X\times X$ and $\Gamma_\mathbb{R}\subset Z_\mathbb{R}\times X\times X$ the graphs of ϕ and $\phi_\mathbb{R}$.

Definition 4.1. Let $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. A lifting of ϕ for $\mathcal{M}$ is a morphism:

$$u\in\text{Hom}_{\mathcal{O}_Z\boxtimes\mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}),\underline{p}^{-1}(\mathcal{M})).$$

A lifting of $\phi_\mathbb{R}$ for F is a morphism:

$$v\in\text{Hom}\,({\phi_\mathbb{R}}^{-1}(F),{p_\mathbb{R}}^{-1}(F)).$$

These definitions are motivated by the example of group actions and quasi-equivariant $\mathcal{D}$-modules. Indeed if a complex Lie group G acts on a complex manifold X with action $\phi : G\times X\to X$ then a $\mathcal{D}_X$-module $\mathcal{M}$ is quasi-equivariant if there is an isomorphism $\underline{\phi}^{-1}\mathcal{M}\simeq\underline{p}^{-1}\mathcal{M}$ which is $\mathcal{O}_G\boxtimes\mathcal{D}_X$-linear (but in general not $\mathcal{D}_{G\times X}$-linear) and compatible with the law of the group. In our definition we just forget the fact that Z is a group (and of course the compatibility with a group law).

We will often consider v as a morphism from $\phi^{-1}(F)$ to $\mathbb{C}_{Z_\mathbb{R}} \boxtimes F$ through the isomorphism:

$$\mathrm{Hom}\,(\phi_\mathbb{R}{}^{-1}(F), p_\mathbb{R}{}^{-1}(F)) \simeq \mathrm{Hom}\,(\phi^{-1}(F), \mathbb{C}_{Z_\mathbb{R}} \boxtimes F). \tag{7}$$

In the same way it will be convenient to change our $\mathcal{O}_Z \boxtimes \mathcal{D}_X$-linear lifting into a $\mathcal{D}_{Z\times X}$-linear one through the isomorphism:

$$R\mathcal{H}om_{\mathcal{O}_Z\boxtimes\mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M})) \simeq R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}(\mathcal{M}), \mathcal{K}_Z \boxtimes \mathcal{M}). \tag{8}$$

This isomorphism is a particular case of the following one. Let $\mathcal{N}$ be a $\mathcal{D}_{Z\times X}$-module which is a coherent $\mathcal{O}_Z \boxtimes \mathcal{D}_X$-module and $\mathcal{P}$ a coherent $\mathcal{D}_X$-module; then

$$R\mathcal{H}om_{\mathcal{O}_Z\boxtimes\mathcal{D}_X}(\mathcal{N}, \mathcal{O}_Z \boxtimes \mathcal{P}) \simeq R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\mathcal{N}, \mathcal{K}_Z \boxtimes \mathcal{P}).$$

It is enough to check this for $\mathcal{P} = \mathcal{D}_X$. Since it is local on $Z \times X$ we may take a resolution of $\mathcal{N}$ by finite free $\mathcal{O}_Z \boxtimes \mathcal{D}_X$-modules and we are reduced to $\mathcal{O}_Z \boxtimes \mathcal{D}_X \simeq R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\mathcal{O}_Z \boxtimes \mathcal{D}_X, \mathcal{K}_Z \boxtimes \mathcal{D}_X)$, which is a consequence of $R\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{O}_Z, \mathcal{K}_Z) \simeq \mathcal{O}_Z$.

For $z \in Z_\mathbb{R}$ the base change by the embedding $\{z\} \to Z$ transforms $\underline{\phi}^{-1}\mathcal{M}$ into $\underline{\phi_z^{-1}}\mathcal{M}$ and the lifting u into a lifting of ϕ_z for $\mathcal{M}$, $u_z \in \mathrm{Hom}_{\mathcal{D}_X}(\underline{\phi_z^{-1}}\mathcal{M}, \mathcal{M})$. The inverse image of v by i_z gives also a lifting of ϕ_z for F, $v_z \in \mathrm{Hom}\,(\phi_z^{-1}F, F)$. From u_z and v_z we obtain a morphism from $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ to itself as follows. We first remark that we have a natural morphism:

$$R\mathrm{Hom}_{\mathcal{D}_X}(\underline{\phi_z^{-1}}\mathcal{M} \otimes \phi_z^{-1}F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X). \tag{9}$$

Indeed the Cauchy-Kowalevski-Kashiwara theorem (which we may apply since ϕ_z is smooth) and standard adjunction formulas for sheaves give:

$$\begin{aligned} R\mathrm{Hom}_{\mathcal{D}_X}(\underline{\phi_z^{-1}}\mathcal{M} \otimes \phi_z^{-1}F, \mathcal{O}_X) &\simeq R\mathrm{Hom}\,(\phi_z^{-1}F, R\mathcal{H}om_{\mathcal{D}_X}(\underline{\phi_z^{-1}}\mathcal{M}, \mathcal{O}_X)) \\ &\simeq R\mathrm{Hom}\,(\phi_z^{-1}F, \phi_z^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \\ &\simeq R\mathrm{Hom}\,(R\phi_{z!}\phi_z^{-1}F, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)), \end{aligned} \tag{10}$$

where in the third isomorphism we used $\phi_z^{-1} \simeq \phi_z^!$. Since ϕ_z is proper we have an adjunction morphism $F \to R\phi_{z!}\phi_z^{-1}F$. Composing it with (10) and using the adjunction between $R\mathcal{H}om\,(\cdot,\cdot)$ and $\cdot \otimes \cdot$ we obtain (9). The tensor product of u_z and v_z gives a morphism:

$$R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\underline{\phi_z^{-1}}\mathcal{M} \otimes \phi_z^{-1}F, \mathcal{O}_X),$$

whose composition with (9) gives:

$$S(u_z, v_z) : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).$$

Taking cohomology we obtain morphisms $\pi_{i,z} : \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M}\otimes F, \mathcal{O}_X) \to \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M}\otimes F, \mathcal{O}_X)$ and letting z run over $Z_\mathbb{R}$ we obtain maps $\pi_i : Z_\mathbb{R} \times \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$. We want to say that these maps are continuous in some sense but the topology of the Ext groups is in general not separated; hence we have to stay in the derived category. Indeed $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ is well-defined as an object of $\mathbf{D}^\mathrm{b}(FN)$ and any relatively compact subset U of $Z_\mathbb{R}$, contained in a compact subset K of $Z_\mathbb{R}$ has a natural embedding $j : U \to \Gamma_K(Z_\mathbb{R}; \mathcal{B}^{(dz)}_{Z_\mathbb{R}})$, where

$j(z) = \delta_z$ is the Dirac function at z. Now $\Gamma_K(Z_\mathbb{R}; \mathcal{B}^{(d_Z)}_{Z_\mathbb{R}})$ is also a FN-space and we define a morphism:

$$S_K(u,v) : \Gamma_K(Z_\mathbb{R}; \mathcal{B}^{(d_Z)}_{Z_\mathbb{R}}) \mathbin{\bar{\otimes}} R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X),$$

such that the π_i are obtained by taking the cohomology of $S_K(u,v)$ and composing with j. We note that $\Gamma_K(Z_\mathbb{R}; \mathcal{B}^{(d_Z)}_{Z_\mathbb{R}}) \simeq R\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{K}_Z \otimes \mathbb{C}_K, \mathcal{O}_Z)[2d_Z]$, and we obtain $S_K(u,v)$ similarly as $S(u_z, v_z)$ by composing a morphism deduced from the tensor product of u and v:

$$\begin{aligned}(11) \quad & R\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{K}_Z \otimes \mathbb{C}_K, \mathcal{O}_Z) \mathbin{\bar{\otimes}} R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)[2d_Z] \\ & \longrightarrow R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}((\mathcal{K}_Z \boxtimes \mathcal{M}) \otimes (\mathbb{C}_K \boxtimes F), \mathcal{O}_{Z\times X})[2d_Z] \\ & \xrightarrow{S'_K(u,v)} R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M} \otimes \phi^{-1}F \otimes \mathbb{C}_{K\times X}, \mathcal{O}_{Z\times X})[2d_Z],\end{aligned}$$

and a natural morphism:

$$(12) \quad \begin{aligned} & R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M} \otimes \phi^{-1}F \otimes \mathbb{C}_{K\times X}, \mathcal{O}_{Z\times X})[2d_Z] \\ & \qquad \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).\end{aligned}$$

The last one is defined as (9) by the following sequence of morphisms, where we remark that $\phi^{-1} \simeq \phi^!\,[2d_Z]$ since ϕ is smooth, and $R\phi_!\mathbb{C}_{K\times X} \simeq R\phi_*\mathbb{C}_{K\times X}$ since ϕ is proper on $K \times X$:

$$\begin{aligned} & R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M} \otimes \phi^{-1}F \otimes \mathbb{C}_{K\times X}, \mathcal{O}_{Z\times X})[2d_Z] \\ & \quad \simeq R\mathrm{Hom}\,(\mathbb{C}_{K\times X}, R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M} \otimes \phi^{-1}F, \mathcal{O}_{Z\times X}))[2d_Z] \\ & \quad \simeq R\mathrm{Hom}\,(\mathbb{C}_{K\times X}, \phi^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X))[2d_Z] \\ & \quad \simeq R\mathrm{Hom}\,(R\phi_!\mathbb{C}_{K\times X}, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)) \\ & \quad \to R\mathrm{Hom}\,(\mathbb{C}_X, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)) \\ & \quad \simeq R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).\end{aligned}$$

For $\omega \in \Gamma_K(Z_\mathbb{R}; \mathcal{B}^{(d_Z)}_{Z_\mathbb{R}})$ we denote by

$$(13) \qquad S(u,v)(\omega) : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$$

the morphism induced by $S_K(u,v)$ (it does not depend on K). For $z \in Z_\mathbb{R}$ we have $S(u_z, v_z) = S(u,v)(\delta_z)$. The purpose of the paper is to show that, when $Z_\mathbb{R}$ is compact and ω is an analytic form on $Z_\mathbb{R}$, $S(u,v)(\omega)$ is nuclear with a trace given by a cohomological formula.

If the topological vector space $E = \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ is separated the fact that $S_K(u,v)$ is well-defined in the derived category of Fréchet nuclear spaces implies the continuity of π_i. Indeed $\pi_i : K \times E \to E$ is the composition of the continuous maps $j \times id_E$ and $H^i(S_K(u,v))$ and is itself continuous.

Remark 4.2. When Z is a group and the data are equivariant and $\mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ is separated, π_i is a representation of $Z_\mathbb{R}$. Under suitable hypothesis this representation is admissible. The following construction is used to define the character of an admissible representation of a Lie group. Let ω be a maximal degree $\mathcal{C}^\infty$-form with compact support in $Z_\mathbb{R}$. We set for $x \in E$, $\pi_{i,\omega}(x) = \int_{Z_\mathbb{R}} \pi_i(z,x)\omega(z)$. (When $Z_\mathbb{R}$ is a semi-simple Lie group and E is admissible, $\pi_{i,\omega}$ is trace-class and

$\omega \mapsto \operatorname{tr} \pi_{i,\omega}$ is a distribution on $Z_\mathbb{R}$, the character of π_i.) The definition of $\pi_{i,\omega}$ makes sense without assuming that $Z_\mathbb{R}$ be a group and we have:

$$\pi_{i,\omega} = S(u,v)(\omega).$$

Remark 4.3. In the above computations we can replace $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$ by $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F)$ and obtain a morphism similar to $S(u,v)(\omega)$:

$$S_1(u,v)(\omega) : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F),$$

which commutes with $S(u,v)(\omega)$ and the contraction morphism:

$$R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).$$

5. Cohomology classes associated to a lifting

In this section we will build microlocal cohomology classes from u and v. In the next section we will make the product of these classes, under the assumption that the pair $(\mathcal{M}, F)$ is "transversally elliptic", and obtain a hyperfunction on $Z_\mathbb{R}$.

The method for defining these cohomology classes is taken from [13] and from [20] for the microlocal aspect. We identify our lifting with the section of a "kernel" and apply a trace morphism to it. The following general result appears in slightly different form in [20] (see also [6] for integral transforms in the framework of $\mathcal{D}$-modules).

Lemma 5.1. *Let $f : Y \to Y'$ be a morphism of complex analytic manifolds and $i_f : Y \to Y \times Y'$, $y \mapsto (y, f(y))$ the graph embedding of f. Let $\mathcal{P} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$, $\mathcal{P}' \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{Y'})$. We set for short $\mathcal{S} = R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{P}, \mathcal{O}_Y)$ and $\mathcal{S}' = R\mathcal{H}om_{\mathcal{D}_{Y'}}(\mathcal{P}', \mathcal{O}_{Y'})$. If f is non-characteristic for $\mathcal{P}'$ we have two natural morphisms:*

$$R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}(\mathcal{P}'), \mathcal{P}) \to i_f^!(\Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{P} \boxtimes \underline{\mathrm{D}}\, \mathcal{P}'))[d_{Y'} - d_Y],$$

$$i_f^!(\Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{P} \boxtimes \underline{\mathrm{D}}\, \mathcal{P}'))[d_{Y'} - d_Y] \to R\mathcal{H}om(\mathcal{S}, f^{-1}\mathcal{S}'),$$

whose composition coincides with the image by the functor $R\mathcal{H}om_{\mathcal{D}_Y}(\cdot, \mathcal{O}_Y)$.

We note that $f^{-1}\mathcal{S}' \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}(\mathcal{P}'), \mathcal{O}_Y)$ because f is non-characteristic for $\mathcal{P}'$. When $Y = Y'$, $f = id$ and $\mathcal{P} = \mathcal{P}' = \mathcal{D}_Y$ the morphisms of the lemma are Sato's morphism $\mathcal{D}_Y \to H^{d_Y}_{\Delta_Y}(\mathcal{O}^{(0,d_Y)}_{Y\times Y})$ and $H^{d_Y}_{\Delta_Y}(\mathcal{O}^{(0,d_Y)}_{Y\times Y}) \to R\mathcal{H}om(\mathcal{O}_Y, \mathcal{O}_Y)$.

Proof. Since f is non-characteristic for $\mathcal{P}'$ we have, by the duality isomorphism of [19], theorem 3.5.6, $\underline{\mathrm{D}}\, \underline{f}^{-1}\mathcal{P}' \simeq \underline{f}^{-1}\, \underline{\mathrm{D}}\, \mathcal{P}'$. Let $\Gamma_f \subset Y \times Y'$ be the graph of f and p_2 the projection from $Y \times Y'$ to Y'. Let us set $\mathcal{B}^{(a,b)}_{\Gamma_f|Y\times Y'} = \mathcal{O}^{(a,b)}_{Y\times Y'} \otimes_{\mathcal{O}_{Y\times Y'}} \mathcal{B}_{\Gamma_f|Y\times Y'}$. Since $\mathcal{D}_{Y \xrightarrow{f} Y'} \simeq i_f^{-1}(\mathcal{B}^{(0,d_{Y'})}_{\Gamma_f|Y\times Y'})$, we have:

$$\begin{aligned}\underline{f}^{-1}\, \underline{\mathrm{D}}\, \mathcal{P}' &\simeq i_f^{-1}(\mathcal{B}^{(0,d_{Y'})}_{\Gamma_f|Y\times Y'} \otimes^L_{p_2^{-1}\mathcal{D}_{Y'}} p_2^{-1}\, \underline{\mathrm{D}}\, \mathcal{P}') \\ &\simeq i_f^{-1}(\mathcal{B}^{(d_Y,d_{Y'})}_{\Gamma_f|Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{K}_Y \boxtimes \underline{\mathrm{D}}\, \mathcal{P}'))[-d_Y].\end{aligned}$$

Composing this isomorphism with $\mathcal{B}^{(d_Y,d_{Y'})}_{\Gamma_f|Y\times Y'} \to R\Gamma_{\Gamma_f}(\Omega_{Y\times Y'})[d_{Y'}]$ we obtain finally:

$$R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{P}', \mathcal{K}_Y) \to i_f^!(\Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{K}_Y \boxtimes \underline{\mathrm{D}}\, \mathcal{P}'))[d_{Y'} - d_Y].$$

The tensor product with $\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{P}$ gives the first morphism of the lemma.

For the second one we will in fact build a morphism:

$$\Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{P} \underline{\boxtimes}\, \underline{\mathrm{D}}\, \mathcal{P}') \to R\mathcal{H}om\,(p_1^{-1}\mathcal{S}, p_2^{-1}\mathcal{S}')[d_Y + d_{Y'}]. \tag{14}$$

Since f is non-characteristic for $\mathcal{S}'$ again, we have $f^{-1}\mathcal{S}' \simeq f^!\mathcal{S}'[2d_{Y'} - 2d_Y]$ and we deduce the isomorphism

$$i_f^! R\mathcal{H}om\,(p_1^{-1}\mathcal{S}, p_2^{-1}\mathcal{S}')[2d_{Y'}] \simeq R\mathcal{H}om\,(\mathcal{S}, f^{-1}\mathcal{S}').$$

Together with (14) this gives the second morphism of the lemma. Let K be the left hand side of (14). By adjunction between $R\mathcal{H}om$ and $\otimes$, morphism (14) corresponds to a morphism from $K \otimes p_1^{-1}\mathcal{S}$ to $p_2^{-1}\mathcal{S}'[d_Y + d_{Y'}]$. A contraction gives

$$K \otimes p_1^{-1}\mathcal{S} \to \Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{O}_Y \underline{\boxtimes}\, \underline{\mathrm{D}}\, \mathcal{P}').$$

By the Cauchy-Kowalevski-Kashiwara theorem we have the isomorphisms:

$$\begin{aligned}\Omega_{Y\times Y'} \otimes^L_{\mathcal{D}_{Y\times Y'}} (\mathcal{O}_Y \underline{\boxtimes}\, \underline{\mathrm{D}}\, \mathcal{P}') &\simeq R\mathcal{H}om_{\mathcal{D}_{Y\times Y'}}(\underline{p_2^{-1}}\mathcal{P}', \mathcal{O}_{Y\times Y'})[d_Y + d_{Y'}] \\ &\simeq p_2^{-1}\mathcal{S}'[d_Y + d_{Y'}]\end{aligned}$$

and thus we obtain (14). □

We apply this lemma to the situation of section 4; we keep the notations introduced there. Because of formula (8) we may consider the lifting u for $\mathcal{M}$ as an element of $\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}(\mathcal{M}), \mathcal{K}_Z \underline{\boxtimes}\, \mathcal{M})$. Let us set:

$$L_{\mathcal{M}} = \Omega_{Z\times X\times X} \otimes^L_{\mathcal{D}_{Z\times X\times X}} (\mathcal{K}_Z \underline{\boxtimes}\, \mathcal{M} \underline{\boxtimes}\, \underline{\mathrm{D}}\, \mathcal{M})[-d_Z]. \tag{15}$$

Lemma 5.1 above applied to $Y = Z \times X$, $Y' = X$, $f = \phi$, $\mathcal{P} = \mathcal{K}_Z \underline{\boxtimes}\, \mathcal{M}$ and $\mathcal{P}' = \mathcal{M}$ yields a morphism

$$R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}(\mathcal{M}), \mathcal{K}_Z \underline{\boxtimes}\, \mathcal{M}) \to R\Gamma_\Gamma(L_{\mathcal{M}}). \tag{16}$$

The duality contraction between $\mathcal{M}$ and $\underline{\mathrm{D}}\,\mathcal{M}$ defines the trace morphism of [20]:

$$\mathcal{M} \underline{\boxtimes}\, \underline{\mathrm{D}}\, \mathcal{M} \to \delta_!\mathcal{K}_X \simeq \mathcal{B}_{\Delta|X\times X}[d_X].$$

Since $\Omega_{X\times X} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{B}_{\Delta|X\times X}[d_X] \simeq \delta_!\omega_X$ and $\mathcal{B}_{\Delta|X\times X}$ is holonomic this gives a morphism

$$L_{\mathcal{M}} \to \mathcal{O}_Z \boxtimes \delta_!\omega_X. \tag{17}$$

Composing this with (16) and taking global sections we obtain from the lifting u a cohomology class in $H^0_\Gamma(Z \times X \times X; \mathcal{O}_Z \boxtimes \delta_!\omega_X)$. However we will need the microlocal information carried by the characteristic variety of $\mathcal{M}$. Hence before we apply the trace morphism, we microlocalize along the graph of ϕ; we have:

$$R\Gamma_\Gamma(L_{\mathcal{M}}) \simeq R\pi_*\mu_\Gamma(L_{\mathcal{M}}) \simeq R\pi_* R\Gamma_{A_1}\mu_\Gamma(L_{\mathcal{M}}),$$

where

$$\begin{aligned}A_1 &= \mathrm{supp}\,\mu_\Gamma(L_{\mathcal{M}}) = SS(L_{\mathcal{M}}) \cap T^*_\Gamma(Z \times X \times X) \\ &= (T^*Z \times \mathrm{char}\,\mathcal{M} \times \mathrm{char}\,\mathcal{M}) \cap T^*_\Gamma(Z \times X \times X).\end{aligned}$$

Taking global sections we obtain finally the following two morphisms:

$$\mathrm{Hom}_{\mathcal{O}_Z \underline{\boxtimes} \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M})) \to H^0_{A_1}(T^*(Z \times X \times X); \mu_\Gamma(L_{\mathcal{M}})) \tag{18}$$

$$H^0_{A_1}(T^*(Z \times X \times X); \mu_\Gamma(L_{\mathcal{M}})) \to H^0_{A_1}(T^*(Z \times X \times X); \mu_\Gamma(\mathcal{O}_Z \boxtimes \delta_!\omega_X)), \tag{19}$$

where

$$\Lambda_1 = A_1 \cap (T^*Z \times T^*_\Delta(X \times X)).$$

In the "transversal case" (see section 9) we will not need to know the support Λ_1. Forgetting the support is equivalent to the projection to the zero-section:

$$H^0_{\Lambda_1}(T^*(Z \times X \times X); \mu_\Gamma(\mathcal{O}_Z \boxtimes \delta_! \omega_X)) \to H^0_\Gamma(Z \times X \times X; \mathcal{O}_Z \boxtimes \delta_! \omega_X). \tag{20}$$

This last group is isomorphic to $H^0_{\tilde{Z}}(Z \times X; \mathcal{O}_Z \boxtimes \omega_X)$, where $\tilde{Z}$ is the fixed points set of ϕ, $\tilde{Z} = \Gamma \cap Z \times \Delta_X$.

Definition 5.2. Let Z and X be complex manifolds, $\phi : Z \times X \to X$ a morphism of manifolds, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Let $u \in \mathrm{Hom}_{\mathcal{O}_Z \underline{\boxtimes} \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M}))$ be a lifting of ϕ for $\mathcal{M}$. The image of u by the morphism (18) above will be denoted by $k(\phi, \mathcal{M}, u)$. The image of $k(\phi, \mathcal{M}, u)$ by the morphism (19) will be denoted by $c(\phi, \mathcal{M}, u)$ and its projection to the zero-section by (20) will be denoted by $c_0(\phi, \mathcal{M}, u)$.

Remark 5.3. Through formula (8) we deduce a structure of $\mathcal{D}_Z \boxtimes \mathbb{C}_X$-module on $R\mathcal{H}om_{\mathcal{O}_Z \underline{\boxtimes} \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M}))$ since $\mathcal{K}_Z$ has two structures of left $\mathcal{D}_Z$-module. In the same way $L_\mathcal{M}$ and $\mathcal{O}_Z \boxtimes \delta_! \omega_X$ also have a $\mathcal{D}_Z$-linear structure. It is immediate from the construction that the maps $u \mapsto k(\phi, \mathcal{M}, u)$ and $u \mapsto c(\phi, \mathcal{M}, u)$ are $\mathcal{D}_Z$-linear.

Example 5.4. The $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X$ has a natural lifting whose class is related to the fundamental class of the graph, Γ, of ϕ in $Z \times X \times X$. We have the following isomorphism:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{O}_Z \underline{\boxtimes} \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{D}_X), \underline{p}^{-1}(\mathcal{D}_X)) &\simeq R\mathcal{H}om_{\mathcal{D}_{Z \times X}}(\mathcal{D}_{Z \times X \xrightarrow{\phi} X}, \mathcal{K}_Z \boxtimes \mathcal{D}_X) \\ &\simeq \mathcal{D}_{X \xleftarrow{\phi} Z \times X} \otimes_{\mathcal{O}_{Z \times X}} (\mathcal{O}^{(d_Z, 0)}_{Z \times X})^* \\ &\simeq H^{d_X}_{[\Gamma]}(\mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}), \end{aligned}$$

where $H^{d_X}_{[\Gamma]}(\cdot)$ is the algebraic local cohomology with support in Γ. The fundamental class of Γ, $\delta_\Gamma \in H^{d_X}_\Gamma(Z \times X \times X; \mathcal{O}^{(d_X)}_{Z \times X \times X})$, gives by the projection $\mathcal{O}^{(d_X)}_{Z \times X \times X} \to \mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}$ a canonical lifting, l_ϕ, of ϕ for $\mathcal{D}_X$. We have also $L_\mathcal{M} \simeq \mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}[d_X]$ and morphism (16) is just the natural inclusion:

$$H^{d_X}_{[\Gamma]}(\mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}) \to H^{d_X}_\Gamma(\mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}).$$

The trace map (17), $\mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}[d_X] \to \mathcal{O}_Z \boxtimes \delta_! \omega_X$, is decomposed through the restriction to the diagonal and the map $\Omega_X[d_X] \to \omega_X$:

$$\mathcal{O}^{(0, d_X, 0)}_{Z \times X \times X}[d_X] \to i_{\Delta *} \mathcal{O}^{(0, d_X)}_{Z \times X}[d_X] \to \mathcal{O}_Z \boxtimes \delta_! \omega_X, \tag{21}$$

where i_Δ is the diagonal embedding of $Z \times X$ in $Z \times X \times X$. Setting $\tilde{Z} = \Gamma \cap Z \times \Delta_X$, we have an isomorphism between $H^{d_X}_\Gamma(i_{\Delta *} \mathcal{O}^{(0, d_X)}_{Z \times X})$ and $H^{d_X}_{\tilde{Z}}(\mathcal{O}^{(0, d_X)}_{Z \times X})$. In particular $c_0(\phi, \mathcal{M}, l_\phi)$ is the image of a class in $H^{d_X}_{\tilde{Z}}(\mathcal{O}^{(0, d_X)}_{Z \times X})$, but, even if Γ and $Z \times \Delta_X$ are transversal, so that $\tilde{Z}$ is a submanifold of $Z \times X$, this class is not the projection of the fundamental class of $\tilde{Z}$ in $Z \times X$ (see section 9).

Example 5.5. The preceding example generalizes to the case of a $\mathcal{D}_X$-module induced by a fiber bundle, $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}$, where $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module. We have:

$$\underline{\phi}^{-1}(\mathcal{M}) \simeq \mathcal{D}_{Z\times X \xrightarrow{\phi} X} \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{E}, \quad \underline{p}^{-1}(\mathcal{M}) \simeq \mathcal{D}_{Z\times X \xrightarrow{p} X} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{E},$$

$$R\mathcal{H}om_{\mathcal{O}_Z\boxtimes\mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M})) \simeq \mathcal{H}om_{\mathcal{O}_{Z\times X}}(\phi^*\mathcal{E}, p^*\mathcal{E}) \otimes_{\mathcal{O}_{Z\times X}} H^{d_X}_{[\Gamma]}(\mathcal{O}^{(0,d_X,0)}_{Z\times X\times X}).$$

Let us call a lifting of ϕ for $\mathcal{E}$ an $\mathcal{O}_{Z\times X}$-linear morphism $u' : \phi^*\mathcal{E} \to p^*\mathcal{E}$. The last isomorphism says that u' determines a lifting $u = u' \otimes l_\phi$ of ϕ for the associated $\mathcal{D}_X$-module $\mathcal{M}$. We set $\mathcal{F} = \mathcal{O}_Z \boxtimes \mathcal{E} \boxtimes \mathcal{E}^*$ so that

$$i_\Gamma^*\mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_{Z\times X}}(\phi^*\mathcal{E}, p^*\mathcal{E}) \quad \text{and} \quad L_{\mathcal{M}} \simeq \mathcal{O}^{(0,d_X,0)}_{Z\times X\times X} \otimes_{\mathcal{O}_{Z\times X\times X}} \mathcal{F}[d_X]$$

and the morphism (16) corresponds to the tensor product of Sato's morphism with $\mathcal{F}$. The trace morphism is also just the tensor product of (21) and the contraction $\mathcal{F} \to i_{\Delta *}\mathcal{O}_{Z\times X}$.

Remark 5.6. When $\mathcal{M}$ is represented by a complex $\mathcal{N}^\cdot$, $\underline{\phi}^{-1}\mathcal{M}$ and $\underline{p}^{-1}\mathcal{M}$ are represented by the complexes $\underline{\phi}^{-1}\mathcal{N}^\cdot$ and $\underline{p}^{-1}\mathcal{N}^\cdot$ since ϕ and p are smooth. If we assume that u is given by a morphism of complexes $u^\cdot : \underline{\phi}^{-1}\mathcal{N}^\cdot \to \underline{p}^{-1}\mathcal{N}^\cdot$ we have

$$c(\phi, \mathcal{M}, u) = \sum (-1)^i c(\phi, \mathcal{N}^i, u^i).$$

A similar construction holds for sheaves. We just state the result.

Lemma 5.7. *Let $f : Y \to Y'$ be a morphism of real manifolds. Let $\Gamma_f \subset Y \times Y'$ be the graph of f, which we identify with Y. Let $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_Y)$, $G' \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_{Y'})$ We have a natural isomorphism:*

$$R\mathcal{H}om(f^{-1}G', G) \xrightarrow{\sim} R\Gamma_{\Gamma_f}(G \boxtimes \mathrm{D}\, G').$$

We apply this to the situation of section 4, with $Y = Z_{\mathbb{R}} \times X$, $Y' = X$, $f = \phi_{\mathbb{R}}$, $G = \mathbb{C}_{Z_{\mathbb{R}}} \boxtimes F$, $G' = F$. We microlocalize and take global sections so that we obtain an isomorphism:

$$\mathrm{Hom}\,({\phi_{\mathbb{R}}}^{-1}(F), {p_{\mathbb{R}}}^{-1}(F)) \xrightarrow{\sim} H^0(T^*(Z_{\mathbb{R}} \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_{Z_{\mathbb{R}}} \boxtimes F \boxtimes \mathrm{D}\, F)).$$

The class associated to v will be defined as its image through this isomorphism. But we need to consider it on $T^*(Z \times X \times X)$, like the class of u, rather than on $T^*(Z_{\mathbb{R}} \times X \times X)$. For this we use the following identification. Let i be the inclusion of $Z_{\mathbb{R}} \times X \times X$ in $Z \times X \times X$; we identify $\Gamma_{\mathbb{R}} \subset Z_{\mathbb{R}} \times X \times X$ with its image by i. The inverse image morphism r of proposition 3.2 applied to i gives the following isomorphism, for $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{X\times X})$, Λ a closed conic subset of $T^*(X \times X)$, $\Lambda' = T^*_Z Z \times \Lambda$, $\Lambda'' = T^*_{Z_{\mathbb{R}}} Z_{\mathbb{R}} \times \Lambda$:

$$H^0_{\Lambda'}(T^*(Z \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_Z \boxtimes G)) \xrightarrow{\sim} H^0_{\Lambda''}(T^*(Z_{\mathbb{R}} \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\omega_{Z_{\mathbb{R}}|Z} \boxtimes G)).$$

Since $Z_{\mathbb{R}}$ is oriented we have in fact $\omega_{Z_{\mathbb{R}}|Z} \simeq \mathbb{C}_{Z_{\mathbb{R}}}[-d_Z]$. There is also a trace morphism for F:

$$F \boxtimes \mathrm{D}\, F \to \delta_! \omega_X.$$

Setting $L_F = \mathbb{C}_Z \boxtimes F \boxtimes \mathrm{D}\, F$, we obtain finally two morphisms:

$$\mathrm{Hom}\,({\phi_{\mathbb{R}}}^{-1}(F), {p_{\mathbb{R}}}^{-1}(F)) \to H^{d_Z}_{A_2}(T^*(Z \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(L_F)) \tag{22}$$

$$\to H^{d_Z}_{\Lambda_2}(T^*(Z \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_Z \boxtimes \delta_! \omega_X)), \tag{23}$$

where

$$A_2 = (T^*_Z Z \times SS(F) \times SS(F)^a) \cap T^*_{\Gamma_{\mathbb{R}}}(Z \times X \times X)$$
$$\Lambda_2 = A_2 \cap (T^*_Z Z \times T^*_\Delta(X \times X)).$$

We have also the projection to the zero-section:

$$H^{d_Z}_{\Lambda_2}(T^*(Z \times X \times X); \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_Z \boxtimes \delta_! \omega_X)) \to H^{d_Z}_{\Gamma_{\mathbb{R}}}(Z \times X \times X; \mathbb{C}_Z \boxtimes \delta_! \omega_X) \tag{24}$$

Definition 5.8. With the notations of section 4, the image of a lifting v of ϕ for F by the morphism (22) will be denoted by $k(\phi, F, v)$. The image of $k(\phi, F, v)$ by (23) will be denoted by $c(\phi, F, v)$ and its projection to the zero-section will be denoted by $c_0(\phi, F, v)$.

Remark 5.9. The action $S_K(u, v)$ defined in section 4 can be recovered from the "kernels" $k(\phi, \mathcal{M}, u)$ and $k(\phi, F, v)$ associated to u and v in the definitions above. Indeed let $k_0(u)$ and $k_0(v)$ be the projections of $k(\phi, \mathcal{M}, u)$ and $k(\phi, F, v)$ to the zero-section. Then $k_0(u) \in H^0_\Gamma(Z \times X \times X; L_\mathcal{M})$ and $k_0(v) \in H^{d_Z}_{\Gamma_{\mathbb{R}}}(Z \times X \times X; L_F)$. By lemma 5.7

$$\begin{aligned} H^{d_Z}_{\Gamma_{\mathbb{R}}}(Z \times X \times X; L_F) &\simeq H^0_{\Gamma_{\mathbb{R}}}(Z_{\mathbb{R}} \times X \times X; \mathbb{C}_{Z_{\mathbb{R}}} \boxtimes F \boxtimes \mathrm{D}\, F) \\ &\simeq \mathrm{Hom}\,({\phi_{\mathbb{R}}}^{-1}(F), {p_{\mathbb{R}}}^{-1}(F)) \end{aligned}$$

and the image of $k_0(v)$ by this isomorphism is of course v; hence we can even recover v from $k(\phi, F, v)$. We cannot recover u from $k(\phi, \mathcal{M}, u)$ but only its action on the solutions:

$$u' : R\mathcal{H}om_{\mathcal{D}_{Z \times X}}(\mathcal{K}_Z \underline{\boxtimes} \mathcal{M}, \mathcal{O}_{Z \times X}) \to R\mathcal{H}om_{\mathcal{D}_{Z \times X}}(\underline{\phi}^{-1}(\mathcal{M}), \mathcal{O}_{Z \times X}).$$

Indeed $k_0(u)$ is nothing else than the image of u by (16), hence its image by the second morphism of lemma 5.1 is u'. Now the data of u' and v are sufficient to recover the morphism (11) and hence $S_K(u, v)$.

Remark 5.10. From $k(\phi, \mathcal{M}, u)$ and $k(\phi, F, v)$ we can also obtain microlocal analogues of the morphisms v and u' of the preceding remark. Let us set:

$$G = R\mathcal{H}om_{\mathcal{D}_{Z \times X}}(\mathcal{K}_Z \underline{\boxtimes} \mathcal{M}, \mathcal{O}_{Z \times X}), \quad G' = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Let $p_{12} : Z \times X \times X \to Z \times X$ and $p_3 : Z \times X \times X \to X$ be the projections on the first two and on the third factors. Morphism (14) in the proof of lemma 5.1 yields in fact a morphism:

$$L_\mathcal{M} \to R\mathcal{H}om\,(p_{12}^{-1} G, p_3^{-1} G')[2d_X].$$

Let i_Γ be the embedding of the graph of ϕ in $Z \times X \times X$. By proposition 4.4.5 of [15] (which is a consequence of proposition 3.2), we have, since ϕ is smooth:

$$\mu hom(G, \phi^{-1} G') \simeq R\, {}^t i'_{\Gamma !} {i_\Gamma}_\pi^{-1}(\mu_\Gamma R\mathcal{H}om\,(p_{12}^{-1} G, p_3^{-1} G'))[2d_X].$$

Let us set $A'_1 = {}^t i'_\Gamma({i_\Gamma}_\pi^{-1}(A_1))$. Hence $k(\phi, \mathcal{M}, u)$ gives a section of $\mu hom(G, \phi^{-1} G')$,

$$u'_\mu \in H^0_{A'_1}(T^*(Z \times X); \mu hom(G, \phi^{-1} G')),$$

whose projection to the zero-section coincides with u'.

In the same way we have an isomorphism:

$$\mu hom(\phi^{-1} F, p^{-1} F)^a \simeq R\, {}^t i'_{\Gamma !} {i_\Gamma}_\pi^{-1}(\mu_\Gamma(\mathbb{C}_Z \boxtimes F \boxtimes \mathrm{D}\, F)),$$

where we denote by $(\cdot)^a$ the inverse image by the antipodal map of $T^*(Z \times X)$. Since $\Gamma_{\mathbb{R}}$ is a closed subset of Γ we have a morphism of functors $\mu_{\Gamma_{\mathbb{R}}}(\cdot) \to R\Gamma_T \mu_\Gamma(\cdot)$, where $T = T^*_{\Gamma_{\mathbb{R}}}(Z \times X \times X) \cap T^*_\Gamma(Z \times X \times X)$. Let us set $A'_2 = {}^t i'_\Gamma(i_{\Gamma\pi}^{-1}(A_2))$. Hence $k(\phi, F, v)$ gives

$$v'_\mu \in H^{dz}_{A'_2}(T^*(Z \times X); \mu hom(\phi^{-1}F, p^{-1}F)^a),$$

whose projection to the zero-section is v. (Note that since $\pi_* A'_2 \subset Z_{\mathbb{R}} \times X$ the projection of v'_μ to the zero-section belongs to $H^{dz}_{Z_{\mathbb{R}} \times X}(Z \times X; R\mathcal{H}om\,(\phi^{-1}F, p^{-1}F))$ which is isomorphic to $\mathrm{Hom}\,(\phi_{\mathbb{R}}{}^{-1}F, p_{\mathbb{R}}{}^{-1}F)$.) Under geometric hypothesis on $\mathrm{char}\,\mathcal{M}$ and $SS(F)$ we will be able to make the microlocal product of u'_μ and v'_μ and this will give another construction of $S(u,v)$ showing that it is trace-class in some sense.

6. Micro-product of the characteristic classes

In this section we will make the microlocal product (defined in lemma 3.4) of the cohomology classes $c(\phi, \mathcal{M}, u)$, $c(\phi, F, v)$ and $k(\phi, \mathcal{M}, u)$, $k(\phi, F, v)$ obtained in section 5, under geometric assumptions on the characteristic varieties of $\mathcal{M}$ and F. The product of $c(\phi, \mathcal{M}, u)$ and $c(\phi, F, v)$ will give a hyperfunction on $Z_{\mathbb{R}}$. The product of $k(\phi, \mathcal{M}, u)$ and $k(\phi, F, v)$ will give a "kernel" from which we can recover the morphism $S(u,v)(\omega)$ of formula (13). It will be used to show that $S(u,v)(\omega)$ has a trace.

Before we state the result we introduce a subset of T^*X associated to ϕ, which corresponds to the conormal to the orbits when ϕ is a group action.

Definition 6.1. Let Z, X be complex analytic manifolds and $\phi : Z \times X \to X$ a morphism of manifolds. We set

$$\Lambda_\phi = p_3(T^*_\Gamma(Z \times X \times X) \cap T^*_{Z \times \Delta}(Z \times X \times X)),$$

where $p_3 : T^*Z \times T^*X \times T^*X \to T^*X$ is the projection to the third factor, and Δ the diagonal of $X \times X$. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. We say that the pair $(\mathcal{M}, F)$ is transversally elliptic for ϕ if

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \cap \Lambda_\phi \subset T^*_X X.$$

In local coordinates we have:

$$\Lambda_\phi = \{(x,\xi) \in T^*X; \quad \exists z \in Z \;\; \phi(z,x) = x,\; {}^t\phi'_{(z,x)}(\xi) = (0,\xi)\}.$$

If Z is a point and ϕ is the identity map, then $\Lambda_\phi = T^*X$ and $(\mathcal{M}, F)$ is transversally elliptic if it is elliptic in the sense of [20]. The following proposition associates a hyperfunction on Z to liftings of ϕ for a transversally elliptic pair.

Proposition-Definition 6.2. *We consider complex analytic manifolds, Z, X and $\phi : Z \times X \to X$ a map satisfying* (6). *Let $Z_{\mathbb{R}}$ be a real, oriented submanifold of Z such that Z is a complexification of $Z_{\mathbb{R}}$. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ such that $(\mathcal{M}, F)$ is transversally elliptic for ϕ and* $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is compact. Then the construction of microlocal classes and the microlocal product define a natural product:*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_Z \boxtimes \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M})) \times \mathrm{Hom}\,(\phi_{\mathbb{R}}{}^{-1}(F), p_{\mathbb{R}}{}^{-1}(F)) \\ \to H^{dz}_\Lambda(T^*Z; \mu_{Z_{\mathbb{R}}}(\mathcal{O}_Z)), \end{aligned} \tag{25}$$

with the following bound for the wave front set of the hyperfunction so obtained:

$$\Lambda = \{(z,\eta) \in T^*_{Z_\mathbb{R}}Z;\ \exists (x,\xi) \in \mathrm{char}(\mathcal{M}) \cap SS(F)^a,\ \phi(z,x) = x \text{ and } {}^t\phi'_{(z,x)}(\xi) = (\eta,\xi)\}.$$

Let u and v be liftings of ϕ for $\mathcal{M}$ and F (see definition 4.1). The hyperfunction image of (u,v) by the previous morphism will be denoted by $\chi(\phi,\mathcal{M},F,u,v)$.

Proof. Let us set for short $T' = T^*(Z \times X \times X)$. The morphisms (19) and (23) sending a lifting to a cohomology class already give us a morphism from the left hand side of (25) to

$$\mathbf{A} = H^0_{\Lambda_1}(T';\mu_\Gamma(\mathcal{O}_Z \boxtimes \delta_!\omega_X)) \times H^{d_Z}_{\Lambda_2}(T';\mu_{\Gamma_\mathbb{R}}(\mathbb{C}_Z \boxtimes \delta_!\omega_X)).$$

Hence we just need to define a morphism from $\mathbf{A}$ to $H^{d_Z}_\Lambda(T^*Z;\mu_{Z_\mathbb{R}}(\mathcal{O}_Z))$. For this we will apply the microlocal product of lemma 3.4 and integrate the result.

The microlocal product is defined if the sets Λ_1 and Λ_2^a have no intersection outside the zero-section of $T^*(Z \times X \times X)$. Recall that

$$\Lambda_1 = (T^*Z \times \mathrm{char}\,\mathcal{M} \times \mathrm{char}\,\mathcal{M}) \cap T^*_\Gamma(Z \times X \times X) \cap (T^*Z \times T^*_\Delta(X \times X)),$$
$$\Lambda_2 = (T^*_ZZ \times SS(F) \times SS(F)^a) \cap T^*_{\Gamma_\mathbb{R}}(Z \times X \times X) \cap (T^*_ZZ \times T^*_\Delta(X \times X)).$$

Let us set $L = \mathrm{char}(\mathcal{M}) \cap SS(F)$. We see that

$$\Lambda_1 \cap \Lambda_2^a \subset (T^*Z \times L^a \times L) \cap T^*_\Gamma(Z \times X \times X) \cap (T^*_ZZ \times T^*_\Delta(X \times X)).$$

This last set is included in the zero-section if and only if $L \cap \Lambda_\phi \subset T^*_XX$; but this is precisely the hypothesis of transversal ellipticity. Hence lemma 3.4 gives us a morphism from $\mathbf{A}$ to

$$H^{d_Z}_{\Lambda_1+\Lambda_2}(T';\mu_{\Gamma_\mathbb{R}}(\mathcal{O}_Z \boxtimes (\delta_!\omega_X \otimes \delta_!\omega_X)) \otimes \pi^{-1}\omega_{\Gamma|Z\times X\times X}) \simeq H^{d_Z}_{\Lambda_1+\Lambda_2}(T';\mu_{\Gamma_\mathbb{R}}(\mathcal{O}_Z \boxtimes \delta_!\omega_X)).$$

Let $p_1 : Z \times X \times X \to Z$ be the first projection. We have a topological integration morphism $Rp_{1!}(\mathcal{O}_Z \boxtimes \delta_!\omega_X) \to \mathcal{O}_Z$ and the compatibility of microlocalization and direct image gives a map from $H^{d_Z}_{\Lambda_1+\Lambda_2}(T';\mu_{\Gamma_\mathbb{R}}(\mathcal{O}_Z \boxtimes \delta_!\omega_X))$ to $H^{d_Z}_\Lambda(T^*Z;\mu_{Z_\mathbb{R}}(\mathcal{O}_Z))$, where $\Lambda = {p_1}_\pi({}^tp_1')^{-1}(\Lambda_1+\Lambda_2)$. This gives the construction of morphism (25). In order to obtain a more explicit description of Λ we notice that:

$$\Lambda_1 = \{(z,x,x,\eta,\xi,-\xi) \in T^*(Z \times X \times X);\ (x,\xi) \in \mathrm{char}(\mathcal{M}),\ \phi(z,x) = x,\ {}^t\phi'_{(z,x)}(\xi) = (\eta,\xi)\},$$
$$\Lambda_2 = \{(z,x,x,0,\xi,-\xi) \in T^*(Z \times X \times X);\ z \in Z_\mathbb{R},\ (x,\xi) \in SS(F),\ \phi(z,x) = x,\ {}^t\phi'_{(z,x)}(\xi) = (\eta,\xi) \text{ where } \eta \in T^*_{Z_\mathbb{R}}Z\}$$

and the expression for Λ is easily deduced. □

In order to understand $\chi(\phi,\mathcal{M},F,u,v)$ as the trace of a nuclear map we will need also the microlocal product of $k(\phi,\mathcal{M},u)$ and $k(\phi,F,v)$. However this last product is defined only under a condition stronger than transversal ellipticity (see definition below). When defined it will yield a "kernel" with value in

$$(26)\qquad L_{\mathcal{M},F} = \Omega_{Z\times X\times X} \otimes^L_{\mathcal{D}_{Z\times X\times X}} (\mathcal{K}_Z \underline{\boxtimes} (\mathcal{M} \otimes F) \underline{\boxtimes} (\underline{\mathrm{D}}\,\mathcal{M} \otimes \mathrm{D}'F))[-d_Z].$$

If $Z_\mathbb{R}$ is compact this kernel will define a morphism

$$H^0(Z_\mathbb{R};\Omega_Z) \otimes R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}'F),$$

which coincides with the morphism $S(u,v)$ of section 4 on the analytic forms. We will show in section 8 that it has a well-defined trace given by $\chi(\phi,\mathcal{M},F,u,v)$.

Definition 6.3. In the situation of definition 6.1 we set

$$\Lambda'_\phi = p_3(T^*_\Gamma(Z\times X\times X)\cap(T^*_Z Z\times T^*(X\times X))),$$

and we say that the pair $(\mathcal{M},F)$ is strongly transversally elliptic for ϕ if

$$\mathrm{char}(\mathcal{M})\cap SS(F)\cap\Lambda'_\phi\subset T^*_X X.$$

For a given point $x\in X$ let us denote by ${}_x\phi : Z\to X$ the function $z\mapsto\phi(z,x)$. In local coordinates we have:

$$\Lambda'_\phi=\{(x,\xi)\in T^*X;\ \exists(z,y)\in Z\times X\ \phi(z,y)=x,\ {}^t({}_y\phi)'_z(\xi)=0\}.$$

We can see from the definitions that $\Lambda_\phi\subset\Lambda'_\phi$ and in general this inclusion is strict. For example if Z is a point, so that ϕ is just a morphism from X to X, and if we assume that ϕ is transversal to id with a (discrete) set of fixed points S, then $\Lambda_\phi=S\times_X T^*_X X$ but $\Lambda'_\phi=T^*_S X$. However, if ϕ is a group action then $\Lambda_\phi=\Lambda'_\phi$ is the conormal to the orbits.

Proposition-Definition 6.4. *In the situation of proposition 6.2 we set moreover* $S=\mathrm{supp}(\mathcal{M})\cap\mathrm{supp}(F)$. *We assume that* $(\mathcal{M},F)$ *is strongly transversally elliptic for* ϕ. *The construction of microlocal kernels and the microlocal product define a morphism:*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_Z\boxtimes\mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}),\underline{p}^{-1}(\mathcal{M}))\times\mathrm{Hom}\,({\phi_{\mathbb{R}}}^{-1}(F),{p_{\mathbb{R}}}^{-1}(F))&\\ \to H^{d_Z}_T(Z\times X\times X;L_{\mathcal{M},F}),&\end{aligned}\tag{27}$$

where $T=\Gamma_{\mathbb{R}}\cap(Z\times S\times S)$. *For* u *and* v, *liftings of* ϕ *for* $\mathcal{M}$ *and* F, *we denote by* $K(\phi,\mathcal{M},F,u,v)$ *the image of* (u,v) *by this morphism.*

Proof. The proof is the same as that of proposition 6.2. We can make the product of $k(\phi,\mathcal{M},u)$ and $k(\phi,F,v)$ if $A_1\cap A_2^a$ is included in the zero-section (A_1 and A_2 are the supports of $k(\phi,\mathcal{M},u)$ and $k(\phi,F,v)$ introduced in (18) and (22)). It is easy to see that this condition is implied by the strong transversal ellipticity. The result of the product belongs to

$$H^{d_Z}_{\Lambda'}(T^*(Z\times X\times X);\mu_{\Gamma_{\mathbb{R}}}(L_{\mathcal{M}}\otimes L_F)\otimes\pi^{-1}\omega_{\Gamma|Z\times X\times X}),$$

where the set Λ' has its projection included in T. We take the image by the projection to the zero-section and we remark that

$$R\Gamma_{\Gamma_{\mathbb{R}}}(L_{\mathcal{M}}\otimes L_F)\otimes\omega_{\Gamma|Z\times X\times X}\simeq R\Gamma_T(L_{\mathcal{M},F}).$$

This gives the product of the proposition. ☐

The tensor product of the duality contractions for $\mathcal{M}$, $\mathcal{M}\underline{\boxtimes}\,\mathrm{D}\,\mathcal{M}\to\delta_!\mathcal{K}_X$, and for F, $F\boxtimes\mathrm{D}'F\to\delta_!\mathbb{C}_X$, define a trace morphism for $L_{\mathcal{M},F}$:

$$\mathrm{tr}: L_{\mathcal{M},F}\to\mathcal{O}_Z\boxtimes\delta_!\omega_X.\tag{28}$$

By functoriality of the microlocal product, $\mathrm{tr}(K(\phi,\mathcal{M},F,u,v))$ coincides with the product of $c(\phi,\mathcal{M},u)$ and $c(\phi,F,v)$.

We want also to recover the action of u and v on the global sections, from $K(\phi,\mathcal{M},F,u,v)$. Let us set:

$$H=R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\mathcal{K}_Z\underline{\boxtimes}(\mathcal{M}\otimes F),\mathcal{O}_{Z\times X}),\quad H'=R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X\otimes\mathrm{D}'F).$$

Let $p_{12} : Z \times X \times X \to Z \times X$ and $p_3 : Z \times X \times X \to X$ be the projections on the first two and on the third factors. We have a morphism

$$L_{\mathcal{M},F} \to R\mathcal{H}om\,(p_{12}^{-1}H, p_3^{-1}H')[2d_X], \tag{29}$$

defined as morphism (14) in the proof of lemma 5.1, by a contraction and an isomorphism:

$$\begin{aligned} L_{\mathcal{M},F} \otimes p_{12}^{-1}H &\to \Omega_{Z\times X\times X} \otimes^L_{\mathcal{D}_{Z\times X\times X}} (\mathcal{O}_{Z\times X} \boxtimes (\underline{\mathrm{D}}\,\mathcal{M} \otimes \mathrm{D}'\,F))[-d_Z] \\ &\simeq \mathbb{C}_{Z\times X} \boxtimes R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}'\,F)[2d_X]. \end{aligned}$$

We have also:

$$\begin{aligned} R\Gamma_{\Gamma_{\mathbb{R}}} R\mathcal{H}om\,(p_{12}^{-1}H, p_3^{-1}H')[2d_X] &\simeq R\Gamma_{Z_{\mathbb{R}}\times X\times X} R\Gamma_{\Gamma} R\mathcal{H}om\,(p_{12}^{-1}H, p_3^{-1}H')[2d_X] \\ &\simeq R\Gamma_{Z_{\mathbb{R}}\times X} R\mathcal{H}om\,(H, \phi^{-1}H'), \end{aligned}$$

so that, taking global sections, we get:

$$H^{d_Z}_T(Z\times X\times X; L_{\mathcal{M},F}) \to R\mathrm{Hom}\,(H \otimes \mathbb{C}_{Z_{\mathbb{R}}\times X}, \phi^{-1}H')[d_Z].$$

Through this morphism, a “kernel” $k \in H^{d_Z}_T(Z\times X\times X; L_{\mathcal{M},F})$ yields a morphism from $H \otimes \mathrm{D}'\,\mathbb{C}_{Z_{\mathbb{R}}\times X}$ to $\phi^{-1}H'$. On the global sections we obtain

$$\begin{aligned} S'(k) : R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\mathcal{K}_Z \,\underline{\boxtimes}\, (\mathcal{M} \otimes F), \mathcal{O}_{Z\times X} \otimes \mathrm{D}'\,\mathbb{C}_{Z_{\mathbb{R}}\times X})[2d_Z] \\ \to R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M}, \mathcal{O}_{Z\times X} \otimes \phi^{-1}(\mathrm{D}'\,F))[2d_Z]. \end{aligned} \tag{30}$$

For $Z_{\mathbb{R}}$ compact and ω an analytic form of degree d_Z on $Z_{\mathbb{R}}$, we will show that $S_{Z_{\mathbb{R}}}(u,v)(\omega)$, defined in section 4, is nuclear. For this we will compare $S'_{Z_{\mathbb{R}}}(u,v)$ (defined in formula (11)) and $S'(K(\phi,\mathcal{M},F,u,v))$; in fact they form a commutative diagram with the natural morphisms c_1 and c_2 described as follows. The inclusion $H^0(Z_{\mathbb{R}};\Omega_Z) \to H^0(Z_{\mathbb{R}};\mathcal{B}^{(d_Z)}_{Z_{\mathbb{R}}})$ corresponds to a morphism of contraction of duality:

$$c_1 : R\Gamma(Z\times X; H \otimes \mathrm{D}'\,\mathbb{C}_{Z_{\mathbb{R}}\times X})[2d_Z] \to R\mathrm{Hom}\,(\mathbb{C}_{Z_{\mathbb{R}}\times X}, H)[2d_Z].$$

We have a similar morphism:

$$c_2 : R\Gamma(Z\times X; \phi^{-1}H')[2d_Z] \to R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}(\mathcal{M}\otimes F), \mathcal{O}_{Z\times X})[2d_Z].$$

Proposition 6.5. *We keep the notations and hypothesis of proposition 6.4. We set for short $k = K(\phi,\mathcal{M},F,u,v)$. We have with the notations above:*

i) $\mathrm{tr}\,k$ *is the microlocal product of* $c(\phi,\mathcal{M},u)$ *and* $c(\phi,F,v)$.

ii) $c_2 \circ S'(k) = S'_{Z_{\mathbb{R}}}(u,v) \circ c_1$.

Proof. The first assertion is a simple consequence of the functoriality of the microlocal product, applied to the morphisms $L_{\mathcal{M}} \to \mathcal{O}_Z \boxtimes \delta_!\omega_X$ and $L_F \to \mathbb{C}_Z \boxtimes \delta_!\omega_X$.

For the second assertion we keep the notations H, H' above and we set:

$$G = R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\mathcal{K}_Z \,\underline{\boxtimes}\, \mathcal{M}, \mathcal{O}_{Z\times X}), \quad G' = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

In remark 5.10 and formula (29) we have already built three similar morphisms, from which we deduced $S'_{Z_{\mathbb{R}}}(u,v)$ and $S'(k)$:

$$\begin{aligned} L_{\mathcal{M}} &\to R\mathcal{H}om\,(p_{12}^{-1}G, p_3^{-1}G')[2d_X], \\ L_F &\xrightarrow{\sim} R\mathcal{H}om\,(p_3^{-1}F, p_{12}^{-1}F)[2d_X], \\ L_{\mathcal{M},F} &\to R\mathcal{H}om\,(p_{12}^{-1}H, p_3^{-1}H')[2d_X]. \end{aligned}$$

We set for short $T' = T^*(Z \times X \times X)$ and $T'' = T^*(Z \times X)$. When we microlocalize along Γ, we get:

$$\begin{aligned} H^0_{A_1}(T'; \mu_\Gamma(L_{\mathcal{M}})) &\to H^0_{A'_1}(T''; \mu hom(G, \phi^{-1}G')), \\ H^{dz}_{A_2}(T'; \mu_\Gamma(L_F)) &\to H^{dz}_{A'_2}(T''; \mu hom(\phi^{-1}F, \mathbb{C}_Z \boxtimes F)), \\ H^{dz}_{A}(T'; \mu_\Gamma(L_{\mathcal{M},F})) &\to H^{dz}_{A'}(T''; \mu hom(H, \phi^{-1}H')), \end{aligned}$$

where $A = A_1 + A_2$ and $A' = A'_1 + A'_2$ (note that A'_2 and A' have a projection included into $Z_\mathbb{R} \times X$). Let u'_μ, v'_μ, w'_μ be the images of $k(\phi, \mathcal{M}, u)$, $k(\phi, F, v)$, k by these last three morphisms. The projection of w'_μ to the zero-section is $S'(k)$. By remarks 5.9 and 5.10 the projections of u'_μ and v'_μ to the zero-section are

$$\begin{aligned} u' &\in H^0(Z \times X; R\mathcal{H}om\,(G, \phi^{-1}G')) \simeq \mathrm{Hom}\,(G, \phi^{-1}G'), \\ v &\in H^{dz}_{Z_\mathbb{R} \times X}(Z \times X; R\mathcal{H}om\,(\phi^{-1}F, \mathbb{C}_Z \boxtimes F)) \simeq \mathrm{Hom}\,(\phi^{-1}F, \mathbb{C}_{Z_\mathbb{R}} \boxtimes F), \end{aligned}$$

where u' is the morphism induced by u on the solutions. One has to be careful that there are two ways of making the product of u' and v:

$$\tag{31} \begin{aligned} &\mathrm{Hom}\,(G, \phi^{-1}G') \times \mathrm{Hom}\,(\phi^{-1}F, \mathbb{C}_{Z_\mathbb{R}} \boxtimes F) \\ &\qquad\qquad \to \mathrm{Hom}\,(R\mathcal{H}om\,(\mathbb{C}_{Z_\mathbb{R}} \boxtimes F, G), R\mathcal{H}om\,(\phi^{-1}F, \phi^{-1}G')), \end{aligned}$$

$$\tag{32} \begin{aligned} &H^0(Z \times X; R\mathcal{H}om\,(G, \phi^{-1}G')) \times H^{dz}_{Z_\mathbb{R} \times X}(Z \times X; R\mathcal{H}om\,(\phi^{-1}F, \mathbb{C}_Z \boxtimes F)) \\ &\quad \to H^{dz}_{Z_\mathbb{R} \times X}(Z \times X; R\mathcal{H}om\,(G, \phi^{-1}G') \otimes R\mathcal{H}om\,(\phi^{-1}F, \mathbb{C}_Z \boxtimes F)) \\ &\quad \to H^{dz}_{Z_\mathbb{R} \times X}(Z \times X; R\mathcal{H}om\,(R\mathcal{H}om\,(\mathbb{C}_Z \boxtimes F, G), R\mathcal{H}om\,(\phi^{-1}F, \phi^{-1}G'))) \\ &\quad \to \mathrm{Hom}\,(R\mathcal{H}om\,(\mathbb{C}_Z \boxtimes F, \mathrm{D}'\,\mathbb{C}_{Z_\mathbb{R}} \otimes G), R\mathcal{H}om\,(\phi^{-1}F, \phi^{-1}G')). \end{aligned}$$

The image of (u', v) by (31) is of course $S'_{Z_\mathbb{R}}(u, v)$ but its image by (32) is $S'_{Z_\mathbb{R}}(u, v) \circ c_1$.

By functoriality of the microlocal product, w'_μ is the product of u'_μ and v'_μ. The product on the zero-section corresponding to this microlocal product is (32). Hence it follows from proposition 3.8 that $S'_{Z_\mathbb{R}}(u, v) \circ c_1$ is equal to $c_2 \circ S'(k)$. □

Remark 6.6. It should be noted that all the constructions in sections 5 and 6, in particular the definitions of $c(\phi, \mathcal{M}, u)$, $c(\phi, F, v)$ and their product $\chi(\phi, \mathcal{M}, F, u, v)$, are "local on Z" in the following sense. Let U be an open subset of Z and ϕ' the restriction of ϕ to $U \times X$. The liftings u and v restrict to liftings u' and v' of ϕ'; the pair $(\mathcal{M}, F)$ is (strongly) transversally elliptic for ϕ' if it is for ϕ and we have for example $\chi(\phi', \mathcal{M}, F, u', v') = \chi(\phi, \mathcal{M}, F, u, v)|_U$.

7. Restriction to a non-characteristic submanifold

We keep the notations of sections 4 and 6. We consider moreover a submanifold ${Z_\mathbb{R}}'$ of $Z_\mathbb{R}$ with a complexification Z' in Z. For suitable ${Z_\mathbb{R}}'$ the pair $(\mathcal{M}, F)$ is still transversally elliptic with respect to Z'. The following proposition asserts that, in this case, the hyperfunction χ' on ${Z_\mathbb{R}}'$ associated to the restriction of the data to Z' is the inverse image of χ.

More precisely, let $\phi' : Z' \times X \to X$ and $p' : Z' \times X \to X$ be the restrictions of ϕ and p. The lifting $u \in \mathrm{Hom}_{\mathcal{O}_Z \boxtimes \mathcal{D}_X}(\underline{\phi}^{-1}(\mathcal{M}), \underline{p}^{-1}(\mathcal{M}))$ of ϕ for $\mathcal{M}$ restricts to a lifting $u' \in \mathrm{Hom}_{\mathcal{O}_{Z'} \boxtimes \mathcal{D}_X}(\underline{\phi'}^{-1}(\mathcal{M}), \underline{p'}^{-1}(\mathcal{M}))$. The restriction of v will be denoted similarly by v'.

Proposition 7.1. *Assume the pair $(\mathcal{M}, F)$ is transversally elliptic with respect to ϕ and to ϕ'. Then Z' is non-characteristic for the wave-front set of the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$ and the restriction of $\chi(\phi, \mathcal{M}, F, u, v)$ to $Z_{\mathbb{R}}{}'$ is $\chi(\phi', \mathcal{M}, F, u', v')$.*

Proof. Let us denote by $i : Z' \to Z$ and $j : Z' \times X \times X \to Z \times X \times X$ the inclusions, by Γ and Γ' the graphs of ϕ and ϕ'. In (19), we obtained the following bound for the support of $c(\phi, \mathcal{M}, u)$:

$$\Lambda_1 = (T^*Z \times \operatorname{char}\mathcal{M} \times \operatorname{char}\mathcal{M}) \cap T^*_\Gamma(Z \times X \times X) \cap (T^*Z \times T^*_\Delta(X \times X)).$$

Since Λ_1 is included in the conormal bundle to the graph of a map from $Z \times X$ to X (in our case ϕ) it is non-characteristic for j. Moreover, if we set $\Lambda_1' = {}^t j'(j_\pi^{-1}(\Lambda_1))$, we have:

$$\Lambda_1' = (T^*Z' \times \operatorname{char}\mathcal{M} \times \operatorname{char}\mathcal{M}) \cap T^*_{\Gamma'}(Z' \times X \times X) \cap (T^*Z' \times T^*_\Delta(X \times X)),$$

and Λ_1' is the bound of the support of $c(\phi', \mathcal{M}, u')$. Let us denote by T and T' the cotangent bundles of $Z \times X \times X$ and $Z' \times X \times X$. We have the inverse image morphism of proposition 3.2 (see (3)):

$$H^0_{\Lambda_1}(T; \mu_\Gamma(\mathcal{O}_Z \boxtimes \delta_! \omega_X)) \to H^0_{\Lambda_1'}(T'; \mu_{\Gamma'}(i^{-1}(\mathcal{O}_Z) \boxtimes \delta_! \omega_X)).$$

Let us denote by r_1 the composition of this morphism with the map $i^{-1}(\mathcal{O}_Z) \to \mathcal{O}_{Z'}$. The same reasoning for F yields a similar morphism, r_2 (Λ_2 is also non-characteristic for j because it is contained in $T^*_Z Z \times T^*(X \times X)$). In view of remark 3.5 on the compatibility of the microlocal product with the inverse image, the proposition will be proved if we show that j is non-characteristic for $\Lambda_1 + \Lambda_2$ and

$$r_1(c(\phi, \mathcal{M}, u)) = c(\phi', \mathcal{M}, u'), \quad r_2(c(\phi, F, v)) = c(\phi', F, v').$$

Let us show that j is non-characteristic for $\Lambda_1 + \Lambda_2$. Let $p \in Z' \times X \times X$ and $\xi_1 \in \pi^{-1}(p) \cap \Lambda_1$, $\xi_2 \in \pi^{-1}(p) \cap \Lambda_2$ be such that ${}^t j'(\xi_1 + \xi_2) = 0$. Then ${}^t j'(\xi_1) = -{}^t j'(\xi_2)$ belongs to $\Lambda_1' \cap \Lambda_2'^a$. But this last set is contained in the zero-section because $(\mathcal{M}, F)$ is transversally elliptic for ϕ' and we have ${}^t j'(\xi_1) = {}^t j'(\xi_2) = 0$. Since j is non-characteristic for Λ_1 and Λ_2 this implies $\xi_1 = \xi_2 = 0$. This proves that j is non-characteristic for $\Lambda_1 + \Lambda_2$.

Now we show that $r_1(c(\phi, \mathcal{M}, u)) = c(\phi', \mathcal{M}, u')$ (the proof for r_2 is similar). We set as in formula (15):

$$L_\mathcal{M} = \Omega_{Z \times X \times X} \otimes^L_{\mathcal{D}_{Z \times X \times X}} (\mathcal{K}_Z \,\underline{\boxtimes}\, \mathcal{M} \,\underline{\boxtimes}\, \underline{\mathrm{D}}\mathcal{M})[-d_Z],$$
$$L'_\mathcal{M} = \Omega_{Z' \times X \times X} \otimes^L_{\mathcal{D}_{Z' \times X \times X}} (\mathcal{K}_{Z'} \,\underline{\boxtimes}\, \mathcal{M} \,\underline{\boxtimes}\, \underline{\mathrm{D}}\mathcal{M})[-d_{Z'}].$$

Since $\mathcal{K}_Z$ has two structures of left $\mathcal{D}_Z$-module, $L_\mathcal{M}$ is a left $\mathcal{D}_Z$-module and we have:

$$\mathcal{D}_{Z' \xrightarrow{i} Z} \otimes^L_{i^{-1}\mathcal{D}_Z} j^{-1} L_\mathcal{M} \simeq L'_\mathcal{M}.$$

In particular the tensor product with the canonical section $\mathbf{1}_i$ of $\mathcal{D}_{Z' \xrightarrow{i} Z}$ gives a morphism $j^{-1}L_\mathcal{M} \to L'_\mathcal{M}$. The construction of $c(\phi, \mathcal{M}, u)$ is in three steps. First we use morphism (16), then we microlocalize with the isomorphism

$$R\Gamma_\Gamma(L_\mathcal{M}) \simeq R\pi_* R\Gamma_{A_1} \mu_\Gamma(L_\mathcal{M}),$$

and finally we apply the trace morphism $L_\mathcal{M} \to \mathcal{O}_Z \boxtimes \delta_! \omega_X$. The trace morphism commutes obviously with the inverse image by j, the microlocalization also in view of proposition 3.2. Hence it remains to prove that we have a commutative diagram:

$$\begin{array}{ccc} j^{-1}R\mathcal{H}om_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}(\mathcal{M}),\mathcal{K}_Z\boxtimes\mathcal{M}) & \longrightarrow & j^{-1}R\Gamma_\Gamma(L_\mathcal{M}) \\ \downarrow & & \downarrow \\ R\mathcal{H}om_{\mathcal{D}_{Z'\times X}}(\underline{\phi'}^{-1}(\mathcal{M}),\mathcal{K}_{Z'}\boxtimes\mathcal{M}) & \longrightarrow & R\Gamma_\Gamma(L'_\mathcal{M}). \end{array}$$

But in this diagram the vertical arrows are just "taking tensor product with $\mathbf{1}_i \in \mathcal{D}_{Z'\xrightarrow{i}Z}$" and they commute with the functorial morphisms described in the proof of lemma 5.1 to obtain morphism (16). □

8. The index as a trace

In this section we will interpret the index built in section 6 as a generalized trace on $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)$, when $(\mathcal{M},F)$ is strongly transversally elliptic, $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ and $Z_\mathbb{R}$ is compact. More precisely, we will show that, for an analytic form ω on $Z_\mathbb{R}$, the morphism $S_{Z_\mathbb{R}}(u,v)(\omega)$ (defined in section 4), from $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)$ to itself, is nuclear, with trace $\int\omega\cdot\chi(\phi,\mathcal{M},F,u,v)$.

8.1. **Trace of kernels.** We consider the action of a "kernel" in the solution space of a $\mathcal{D}$-module and show that it is nuclear, with trace, the trace of the kernel. Let X be a complex analytic space, $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. We set

$$K_{\mathcal{M},F}=\Omega_{X\times X}\otimes^L_{\mathcal{D}_{X\times X}}((\mathcal{M}\otimes F)\boxtimes(\underline{\mathrm{D}}\,\mathcal{M}\otimes\mathrm{D}'F))$$

(this corresponds to the notation $L_{\mathcal{M},F}$ of formula (26) with $Z=\{pt\}$). We have the trace morphism (28), $\mathrm{tr}:K_{\mathcal{M},F}\to\delta_!\omega_X$. We denote also by tr the morphism induced on the global sections, from $H^0(X\times X;K_{\mathcal{M},F})$ to $H^0(X;\omega_X)$. Setting

$$\mathcal{S}=R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)\quad\text{and}\quad\mathcal{S}'=R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X\otimes\mathrm{D}'F),$$

we have a morphism induced by (29) (for $Z=\{pt\}$) on the global sections:

$$R\Gamma_c(X\times X;K_{\mathcal{M},F})\to R\mathrm{Hom}\,(R\Gamma(X;\mathcal{S}),R\Gamma(X;\mathcal{S}')).$$

In particular a "kernel" $k\in H^0_c(X\times X;K_{\mathcal{M},F})$ defines

$$T(k):R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)\to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X\otimes\mathrm{D}'F).$$

We have also the contraction morphism:

$$c:R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X\otimes\mathrm{D}'F)\to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X).$$

The following proposition identifies the trace of $T(k)\circ c$ and the trace of k.

Proposition 8.1. *Let X be a complex analytic space, $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $F\in\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. We assume that $\mathrm{supp}\,\mathcal{M}\cap\mathrm{supp}\,F$ is compact. With the notations above, for $k\in H^0(X\times X;K_{\mathcal{M},F})$, the morphisms $T(k)\circ c$ and $c\circ T(k)$ are nuclear morphisms (in the sense of definition 2.2) respectively in $\mathbf{D}^{\mathrm{b}}(DFN)$ and $\mathbf{D}^{\mathrm{b}}(FN)$. They have the same trace (in the sense of definition 2.4) and:*

$$\mathrm{tr}\,T(k)\circ c=\mathrm{tr}\,c\circ T(k)=\int_X\mathrm{tr}(k).$$

Proof. In the proof $\mathcal{C}_X$ (resp. $\mathcal{A}_X$) is the sheaf of infinitely differentiable (resp. real analytic) functions on X, $\mathcal{C}_X^{(i)}$ (resp. $\mathcal{A}_X^{(i)}$) is the sheaf of forms of degree i on X with coefficients in $\mathcal{C}_X$ (resp. $\mathcal{A}_X$) and, for a product of manifolds, $\mathcal{C}_{X\times X}^{(i,j)}$ is the sheaf of forms of degree i in the first factor and j in the second factor.

In order to represent the kernel k and its action we need resolutions of $K_{\mathcal{M},F}$ by soft sheaves. For this we will use the "realification" of a $\mathcal{D}$-module introduced in [20]. Let us recall some of their definitions and results. We denote by $\mathcal{D}_{X^{\mathbb{R}}}$ the sheaf of real analytic differential operators, i.e. $\mathcal{D}_{X^{\mathbb{R}}} = (\mathcal{D}_{X\times\bar{X}})|_{X^{\mathbb{R}}}$, where $\bar{X}$ is the conjugate manifold of X and $X^{\mathbb{R}}$ is the real analytic manifold underlying X, identified with the diagonal of $X\times\bar{X}$. The realification of a $\mathcal{D}_X$-module $\mathcal{M}$ is the sheaf $\mathcal{M}_{\mathbb{R}} = \mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ with a structure of $\mathcal{D}_{X^{\mathbb{R}}}$-module defined as follows. For $a, f \in \mathcal{A}_X$, $m \in \mathcal{M}$ we set:

$$\frac{\partial}{\partial z_i}(a\otimes m) = \frac{\partial a}{\partial z_i}\otimes m + a\otimes\frac{\partial}{\partial z_i}m \quad \frac{\partial}{\partial \bar{z}_i}(a\otimes m) = \frac{\partial a}{\partial \bar{z}_i}\otimes m \quad f\cdot(a\otimes m) = fa\otimes m.$$

The reason for introducing $\mathcal{M}_{\mathbb{R}}$ is that, for $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{M}_{\mathbb{R}}$ has a finite global resolution by finite free $\mathcal{D}_{X^{\mathbb{R}}}$-modules, in a neighborhood of any compact subset of X (see [20] proposition 3.1).

The links between the de Rham complex, the sheaf of solutions, the dual of $\mathcal{M}$ and $\mathcal{M}_{\mathbb{R}}$ are explained as follows. We have (see lemmas 3.3–3.4 of [20]):

$$\Omega_X \otimes^L_{\mathcal{D}_X} \mathcal{M} \simeq \mathcal{C}_X^{(2d_X)} \otimes^L_{\mathcal{D}_{X^{\mathbb{R}}}} \mathcal{M}_{\mathbb{R}}[-d_X] \tag{33}$$

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X) \simeq R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{M}_{\mathbb{R}},\mathcal{C}_X) \tag{34}$$

We set $\mathcal{K}_{X^{\mathbb{R}}} = \mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X^{(2d_X)},\mathcal{D}_{X^{\mathbb{R}}})$ where the structure of $\mathcal{A}_X$-module of $\mathcal{D}_{X^{\mathbb{R}}}$ is defined by multiplication on the right. Then $\mathcal{K}_{X^{\mathbb{R}}}$ has two compatible structures of left $\mathcal{D}_{X^{\mathbb{R}}}$-module. We have:

$$(\underline{\mathrm{D}}\,\mathcal{M})_{\mathbb{R}} \simeq R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{M}_{\mathbb{R}},\mathcal{K}_{X^{\mathbb{R}}})[2d_X]. \tag{35}$$

In view of these formulas we have:

$$\mathcal{S} \simeq R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{M}_{\mathbb{R}}\otimes F,\mathcal{C}_X), \qquad \mathcal{S}' \simeq R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{M}_{\mathbb{R}},\mathcal{C}_X\otimes \mathrm{D}'F),$$
$$K_{\mathcal{M},F} \simeq \mathcal{C}_{X\times X}^{(2d_X,2d_X)} \otimes^L_{\mathcal{D}_{(X\times X)^{\mathbb{R}}}} ((\mathcal{M}_{\mathbb{R}}\otimes F)\boxtimes R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{M}_{\mathbb{R}},\mathcal{K}_{X^{\mathbb{R}}}\otimes \mathrm{D}'F)).$$

The morphism from $R\Gamma(X\times X;K_{\mathcal{M},F})$ to $R\mathrm{Hom}\,(R\Gamma(X;\mathcal{S}),R\Gamma(X;\mathcal{S}'))$ generalizes immediately to $\mathcal{D}_{X^{\mathbb{R}}}$-modules as follows. Let $\mathcal{N}_1$, $\mathcal{N}_2$ be in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{X^{\mathbb{R}}})$, F_1, F_2 in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ and let us set

$$K = \mathcal{C}_{X\times X}^{(2d_X,2d_X)} \otimes^L_{\mathcal{D}_{(X\times X)^{\mathbb{R}}}} ((\mathcal{N}_1\otimes F_1)\boxtimes R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_2,\mathcal{K}_{X^{\mathbb{R}}}\otimes \mathrm{D}'F_2)).$$

If $\operatorname{supp}\mathcal{N}_1 \cap \operatorname{supp} F_1$ is compact, our morphism is given by the composition of a contraction and a relative integration:

$$R\Gamma(X\times X;K)\otimes R\mathrm{Hom}_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_1\otimes F_1,\mathcal{C}_X)$$

$$\to R\Gamma(X\times X;\mathcal{C}_{X\times X}^{(2d_X,2d_X)} \otimes^L_{\mathcal{D}_{(X\times X)^{\mathbb{R}}}} (\mathcal{C}_X\boxtimes R\mathcal{H}om_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_2,\mathcal{K}_{X^{\mathbb{R}}}\otimes \mathrm{D}'F_2))) \tag{36}$$

$$\to R\mathrm{Hom}_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_2,\mathcal{C}_X\otimes \mathrm{D}'F_2). \tag{37}$$

Note that the integration morphism

$$H^0_c(X;\mathcal{C}_X^{(2d_X)} \otimes^L_{\mathcal{D}_{(X\times X)^{\mathbb{R}}}} \mathcal{C}_X) \simeq H^{2d_X}_c(X;\mathbb{C}_X) \to \mathbb{C}$$

is nothing but $\omega \otimes \varphi \mapsto \int \omega \cdot \varphi$ for a form ω and a function φ.

Now, up to shrinking X to a suitable neighborhood of $\operatorname{supp} \mathcal{M} \cap \operatorname{supp} F$ we may take a global resolution $\mathcal{N}^{\cdot}$ of $\mathcal{M}_{\mathbb{R}}$ of the form $\mathcal{N}^i = \mathcal{D}_{X^{\mathbb{R}}}^{N_i}$. By (35) we have $(\underline{\mathrm{D}}\,\mathcal{M})_{\mathbb{R}} \simeq \mathcal{N}'^{\cdot}$, where $\mathcal{N}'^j = \mathcal{K}_{X^{\mathbb{R}}}^{N_{2d_X - j}}$. Hence, by (33), $\Omega_{X\times X} \otimes^L_{\mathcal{D}_{X\times X}} \mathcal{M} \boxtimes \underline{\mathrm{D}}\,\mathcal{M}$ is quasi-isomorphic to a complex $\mathcal{L}^{\cdot}$ of sheaves of matrices with entries in $\mathcal{C}_{X\times X}^{(2d_X,0)}$, $\mathcal{L}^i = \bigoplus_{p+q=i} \mathrm{Mat}_{N_p \times N_{-q}} \mathcal{C}_{X\times X}^{(2d_X,0)}$.

Now we need a resolution of $F \boxtimes \mathrm{D}' F$. By proposition 3.10 of [20], any $\mathbb{R}$-constructible sheaf has a bounded resolution $G^{\cdot}$ with $G^i = \bigoplus_{U \in I_i} \mathbb{C}_U$, where I_i is a locally finite family of relatively compact open subsets U of X such that $R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathbb{C}_X) \simeq \mathbb{C}_{\overline{U}}$. Up to shrinking X once more we may assume that F has such a resolution, for which the families I_i are finite. Hence $F \boxtimes \mathrm{D}' F$ is quasi-isomorphic to a complex $H^{\cdot}$, where $H^j = \bigoplus_{r+s=j} \bigoplus_{U \in I_r, V \in I_{-s}} \mathbb{C}_{U \times \overline{V}}$.

With the resolutions $\mathcal{L}^{\cdot}$ and $H^{\cdot}$ we can represent the global sections of $K_{\mathcal{M},F} \simeq \mathcal{L}^{\cdot} \otimes H^{\cdot}$. Indeed soft sheaves are acyclic for the functor $\Gamma(X \times X; (\cdot)_{U\times \overline{V}})$, where U, V are open subsets of X. Hence we obtain a representative $K^{\cdot}$ of $R\Gamma(X \times X; K_{\mathcal{M},F})$ with $K^l = \bigoplus_{i+j=l} \Gamma(X \times X; \mathcal{L}^i \otimes H^j)$. We have:

$$\Gamma(X \times X; \mathcal{L}^i \otimes H^j) = \bigoplus \mathrm{Mat}_{N_p \times N_{-q}}(\Gamma(X \times X; (\mathcal{C}_{X\times X}^{(2d_X,0)})_{U\times\overline{V}})),$$

where the sum runs over the couples of integers (p,q) such that $p+q=i$, and the couples of opens sets $(U,V) \in \bigcup_{r+s=j} I_r \times I_{-s}$. In particular the kernel k admits a representative k_0 in K^0, which we assume fixed in what follows. Note that a section of $\Gamma(X \times X; (\mathcal{C}_{X\times X})_{U\times\overline{V}})$ is represented by a function defined on $U \times W$ for W a neighborhood of $\overline{V}$, with support in $C \times W$ for C a compact subset of U.

Using the same resolutions for $\mathcal{M}_{\mathbb{R}}$ and F and isomorphism (34) we obtain resolutions $S^{\cdot}$ of $R\Gamma(X; \mathcal{S})$ and $S'^{\cdot}$ of $R\Gamma(X; \mathcal{S}')$ of the form:

$$S^i = \bigoplus_{p+r=i} \bigoplus_{U \in I_{-r}} \Gamma(U; \mathcal{C}_X)^{N_{-p}} \tag{38}$$

$$S'^i = \bigoplus_{p+r=i} \bigoplus_{U \in I_{-r}} \Gamma(\overline{U}; \mathcal{C}_X)^{N_{-p}}. \tag{39}$$

In view of these resolutions, it just remains to describe morphisms (36) and (37) when $\mathcal{N}_1 = \mathcal{D}_{X^{\mathbb{R}}}^{N_1}[i_1]$, $\mathcal{N}_2 = \mathcal{D}_{X^{\mathbb{R}}}^{N_2}[i_2]$, $F_1 = \mathbb{C}_{U_1}[j_1]$, $F_2 = \mathbb{C}_{U_2}[j_2]$ with $i_1 - i_2 + j_1 - j_2 = 0$. In this case we have:

$$\begin{aligned} R\Gamma(X \times X; K) &\simeq \mathrm{Mat}_{N_1 \times N_2}(\Gamma(X \times X; (\mathcal{C}_{X\times X}^{(2d_X,0)})_{U_1 \times \overline{U}_2})) \\ R\mathrm{Hom}_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_1 \otimes F_1, \mathcal{C}_X) &\simeq \Gamma(U_1; \mathcal{C}_X)^{N_1}[-i_1 - j_1] \\ R\mathrm{Hom}_{\mathcal{D}_{X^{\mathbb{R}}}}(\mathcal{N}_2, \mathcal{C}_X \otimes \mathrm{D}' F_2)) &\simeq \Gamma(\overline{U}_2; \mathcal{C}_X)^{N_2}[-i_2 - j_2]. \end{aligned}$$

Let A be a matrix in $H^0(X \times X; K)$, $\varphi \in H^0(U_1; \mathcal{C}_X)^{N_1}$. Then morphism (36) sends $A \otimes \varphi$ to $\varphi \cdot A$ which is an N_2-vector with entries in $\Gamma(X \times X; (\mathcal{C}_{X\times X}^{(2d_X,0)})_{U_1 \times \overline{U}_2})$ and morphism (37) integrates $\varphi \cdot A$ with respect to the first variable (recall that A has support in $C \times \overline{U}_2$ for C a compact subset of U_1). This gives a N_2-vector, φ', with entries in $\Gamma(U_2; \mathcal{C}_X)$. The map $\varphi \mapsto \varphi'$ is nuclear. Its compositions with the restriction maps, which sends functions defined in a neighborhood of $\overline{U}_1$ or $\overline{U}_2$ to their restrictions to U_1 or U_2, are also nuclear. If $\mathcal{N}_1 = \mathcal{N}_2$ and $F_1 = F_2$ they have the same trace:

$$\sum_i \int_X A_{ii}|_{\Delta_X},$$

which is the image of A by the morphism tr.

Summing over the components of $\mathcal{N}^\cdot$ and $G^\cdot$ we obtain the proposition. □

8.2. **The index as a generalized trace.** In this paragraph we consider the situation described in section 4; $\phi : Z \times X \to X$ is a morphism of complex manifolds satisfying condition (6), $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$. We assume that $\operatorname{supp}\mathcal{M} \cap \operatorname{supp} F$ is compact and moreover that $Z_\mathbb{R}$ is compact. We still denote by u a lifting of ϕ for $\mathcal{M}$ and v a lifting of $\phi_\mathbb{R}$ for F.

Since $Z_\mathbb{R}$ is compact we may consider the morphism

$$S_{Z_\mathbb{R}}(u,v) : \Gamma(Z_\mathbb{R}; \mathcal{B}^{(dz)}_{Z_\mathbb{R}}) \bar{\otimes} R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X),$$

defined in section 4. We show that, for $\omega \in \Gamma(Z_\mathbb{R}; \Omega_Z)$, $S(u,v)(\omega)$ (defined in (13)) is nuclear with trace the evaluation of the index on ω.

Let us set $k = K(\phi, \mathcal{M}, F, u, v)$ for short. Composing the morphism $S'(k)$ defined in formula (30) and the natural morphism

$$R\mathrm{Hom}_{\mathcal{D}_{Z\times X}}(\underline{\phi}^{-1}\mathcal{M}, \mathcal{O}_{Z\times X} \otimes \phi^{-1}\mathrm{D}' F)[2d_Z] \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F)$$

defined similarly as (12), we obtain

$$S(k) : \Gamma(Z_\mathbb{R}; \Omega_Z) \otimes R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F).$$

For $\omega \in \Gamma(Z_\mathbb{R}; \Omega_Z)$ we denote by $S(k)(\omega)$ the morphism from $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F, \mathcal{O}_X)$ to $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F)$ induced by $S(k)$. Let i be the inclusion of $\Gamma(Z_\mathbb{R}; \Omega_Z)$ in $\Gamma(Z_\mathbb{R}; \mathcal{B}^{(dz)}_{Z_\mathbb{R}})$ and c the contraction morphism:

$$c : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X \otimes \mathrm{D}' F) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X).$$

As an immediate consequence of proposition 6.5 we have:

$$c \circ S(k) = S_{Z_\mathbb{R}}(u,v) \circ (i \otimes id).$$

Hence we only need to show that $S(k)(\omega)$ is nuclear; in fact we will see that it is defined by a kernel as in the preceding paragraph. Recall that k is a section of $L_{\mathcal{M},F}$, where $L_{\mathcal{M},F}$ is defined by formula (26). We may multiply k by $\omega \in \Gamma(Z_\mathbb{R}; \Omega_Z)$ and integrate along the projection to $X \times X$. We obtain a kernel $k_\omega \in H^0(X \times X; K_{\mathcal{M},F})$, where $K_{\mathcal{M},F}$ is defined in the preceding paragraph. From the definitions of $S(k)$ and $T(k_\omega)$ we see that $S(k)(\omega) = T(k_\omega)$. By proposition 8.1 this implies that $c \circ S(k)$ is nuclear with trace $\int_X \mathrm{tr}(k_\omega)$. But the trace morphisms $\mathrm{tr} : L_{\mathcal{M},F} \to \mathcal{O}_Z \boxtimes \delta_! \omega_X$ and $\mathrm{tr} : K_{\mathcal{M},F} \to \delta_! \omega_X$ commute with the integration along the projection $q : Z \times X \times X \to X \times X$; hence $\mathrm{tr}(k_\omega) = \int_q \mathrm{tr}(k) \cdot \omega$. Now, denoting by $*_\mu$ the microlocal product, we have by proposition 6.5:

$$\begin{aligned}\int_{Z\times X\times X} \mathrm{tr}(k) \cdot \omega &= \int_{Z\times X\times X} (c(\phi, \mathcal{M}, u) *_\mu c(\phi, F, v)) \cdot \omega \\ &= \int_Z \chi(\phi, \mathcal{M}, F, u, v) \cdot \omega.\end{aligned}$$

Finally we have obtained the desired result:

Theorem 8.2. *We consider complex analytic manifolds,* Z, X *and* $\phi : Z \times X \to X$ *a map satisfying* (6)*. Let* $Z_{\mathbb{R}}$ *be a real, oriented submanifold of* Z *such that* Z *is a complexification of* $Z_{\mathbb{R}}$*. Let* $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$*,* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$*; let* u *be a lifting of* ϕ *for* $\mathcal{M}$ *and* v *a lifting of* $\phi_{\mathbb{R}}$ *for* F*.*

We assume that $(\mathcal{M}, F)$ *is strongly transversally elliptic in the sense of definition 6.3, that* $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ *is compact and that* $Z_{\mathbb{R}}$ *is compact.*

Then for any form $\omega \in \Gamma(Z_{\mathbb{R}}; \Omega_Z)$ *the morphism* (13)*:*

$$S(u,v)(\omega) : R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X) \to R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M} \otimes F, \mathcal{O}_X)$$

in $\mathbf{D}^{\mathrm{b}}(FN)$ *is nuclear and its trace in the sense of definition 2.4 is:*

$$\mathrm{tr}(S(u,v)(\omega)) = \int_{Z_{\mathbb{R}}} \omega \cdot \chi(\phi, \mathcal{M}, F, u, v).$$

9. Transversal case

In this section we will make additional hypothesis on the map $\phi : Z \times X \to X$ and then on the lifting u of ϕ for $\mathcal{M}$, in order to compute more easily the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$. We denote as before the diagonal of $X \times X$ by Δ_X.

9.1. **Transversal case.** Until the end of the section we assume that the graph Γ of ϕ and the graph $Z \times \Delta_X$ of the projection $p : Z \times X \to X$ are transversal in $Z \times X \times X$ (if ϕ is a group action this is the case if and only if the action is homogeneous). This is equivalent to

$$\Lambda_\phi \subset T^*_X X,$$

where Λ_ϕ is the subset of T^*X introduced in definition 6.1. Hence any pair $(\mathcal{M}, F)$ with $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ is transversally elliptic, so that no microlocal information on F and $\mathcal{M}$ is needed to build the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$. In this paragraph we will give a construction without using the microlocalization functor.

In definitions 5.2 and 5.8 we introduced the following cohomology classes associated to liftings of ϕ; they are the projections of $c(\phi, \mathcal{M}, u)$ and $c(\phi, F, v)$ to the zero-section:

$$c_0(\phi, \mathcal{M}, u) \in H^0_\Gamma(Z \times X \times X; \mathcal{O}_Z \boxtimes \delta_! \omega_X),$$
$$c_0(\phi, F, v) \in H^{d_Z}_S(Z \times X \times X; \mathbb{C}_Z \boxtimes \delta_! \omega_X),$$

where $S = \Gamma_{\mathbb{R}} \cap (Z \times \mathrm{supp}\, F \times \mathrm{supp}\, F) \cap (Z \times \Delta_X)$. Let

$$\tilde{Z} = \Gamma \cap (Z \times \Delta_X) = \{(z,x) \in Z \times X;\ \phi(z,x) = x\}$$

be the "fixed points set" of ϕ; by the transversality hypothesis this is a submanifold of $Z \times X \times X$ of dimension $d_{\tilde{Z}} = d_Z$. Let $q : \tilde{Z} \to Z$ be the projection. We have canonical isomorphisms (note that $S \subset \tilde{Z}$):

(40) $$H^0_\Gamma(Z \times X \times X; \mathcal{O}_Z \boxtimes \delta_! \omega_X) \simeq H^0_{\tilde{Z}}(Z \times X; \mathcal{O}_Z \boxtimes \omega_X) \simeq H^0(\tilde{Z}; q^! \mathcal{O}_Z),$$

(41) $$H^{d_Z}_S(Z \times X \times X; \mathbb{C}_Z \boxtimes \delta_! \omega_X) \simeq H^{d_Z}_S(Z \times X; \mathbb{C}_Z \boxtimes \omega_X) \simeq H^{d_Z}_S(\tilde{Z}; \mathbb{C}_{\tilde{Z}}).$$

Let $c'(u) \in H^0(\tilde{Z}; q^! \mathcal{O}_Z)$ and $c'(v) \in H^{d_Z}_T(\tilde{Z}; \mathbb{C}_{\tilde{Z}})$ be the images of $c_0(\phi, \mathcal{M}, u)$ and $c_0(\phi, F, v)$ by these isomorphisms. The cup-product of $c'(u)$ and $c'(v)$ belongs to $H^{d_Z}_S(\tilde{Z}; q^! \mathcal{O}_Z)$. We can integrate it along the projection q using the morphism $Rq_! q^! \to id$.

Lemma 9.1. *If the graph,* Γ, *of* ϕ *is transversal to* $Z \times \Delta_X$ *in* $Z \times X \times X$, *we have, with the notations above:*

$$\chi(\phi, \mathcal{M}, F, u, v) = \int_q c'(u) \cup c'(v).$$

Proof. The lemma is in fact just a consequence of the commutativity of the diagram in remark 3.7. We set for short $T' = T^*(Z \times X \times X)$. In view of remark 3.6 the micro-product of $c(\phi, \mathcal{M}, u)$ and $c(\phi, F, v)$ is also obtained by first sending

$$c(\phi, F, v) \in H^{d_Z}_{\Lambda_2}(T'; \mu_{\Gamma_{\mathbb{R}}}(\mathbb{C}_Z \boxtimes \delta_! \omega_X))$$

to $H^{d_Z}_S(T'; \mu_\Gamma(\mathbb{C}_Z \boxtimes \delta_! \omega_X))$ by the natural morphism associated to the inclusion $\Gamma_{\mathbb{R}} \subset \Gamma$. By the transversality hypothesis Γ is non-characteristic for $\mathbb{C}_Z \boxtimes \delta_! \omega_X$ so that remark 3.7 applies and tells us that the projection of the micro-product to the zero-section is equal to the cup-product of $c'(u)$ and $c'(v)$, after the identification

$$\begin{aligned} H^{d_Z}_S(T'; \mu_\Gamma(\mathbb{C}_Z \boxtimes \delta_! \omega_X)) &\simeq H^{d_Z}_S(Z \times X \times X; (\mathbb{C}_Z \boxtimes \delta_! \omega_X) \otimes \omega_{\Gamma | Z \times X \times X}) \\ &\simeq H^{d_Z}_S(Z \times X \times X; \mathbb{C}_{\tilde{Z}}) \\ &\simeq H^{d_Z}_S(\tilde{Z}; \mathbb{C}_{\tilde{Z}}). \end{aligned}$$

This identification is the same as (41) and integration along q yields the lemma. □

9.2. **Lifting induced by a fiber bundle morphism.** In this paragraph we still make the hypothesis of transversality. We want to describe the class $c'(u) \in H^0(\tilde{Z}; q^! \mathcal{O}_Z)$. In particular we will show that it is related to the fundamental class of $\tilde{Z}$ in $Z \times X$. For a morphism of complex manifolds $f : Y \to Y'$ we have the integration morphism $\sigma : Rf_! \Omega_Y[d_Y] \to \Omega_{Y'}[d_{Y'}]$ which gives $\tau : f_! \Omega_Y \to H^{d_{Y'} - d_Y}_{f(Y')}(\Omega_{Y'})$ (this morphism is used to define the fundamental class, for example in [15], definition 11.1.5).

We would like to substitute the complex $q^! \mathcal{O}_Z$ for something easier to describe. We remark that if q is a local diffeomorphism then $q^! \mathcal{O}_Z \simeq \mathcal{O}_{\tilde{Z}}$, but in general there is no natural map between $q^! \mathcal{O}_Z$ and $\mathcal{O}_{\tilde{Z}}$. However the choice of a volume form ω on Z gives an identification $\mathcal{O}_Z \simeq \Omega_Z$ and the integration morphism $\sigma_Z : Rq_! \Omega_{\tilde{Z}} \to \Omega_Z$ gives by adjunction a natural morphism $\sigma'_Z : \Omega_{\tilde{Z}} \to q^! \Omega_Z$. It is natural to ask whether $c'(u)$ arises from a section $c'_\omega(u) \in H^0(\tilde{Z}; \Omega_{\tilde{Z}})$ by the composition:

$$H^0(\tilde{Z}; \Omega_{\tilde{Z}}) \xrightarrow{\sigma'_Z} H^0(\tilde{Z}; q^! \Omega_Z) \xrightarrow{\otimes \omega^{-1}} H^0(\tilde{Z}; q^! \mathcal{O}_Z),$$

where we write, by abuse of notations, σ'_Z for $H^0(\sigma'_Z)$ or $H^0(\tilde{Z}; \sigma'_Z)$. For this we assume that the $\mathcal{D}$-module $\mathcal{M}$ arises from a "differential complex of fiber bundles", i.e.

$$\mathcal{M} \quad \simeq \quad \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}^i \xrightarrow{d_i} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}^{i+1} \to \cdots,$$

where the $\mathcal{E}^i$ are locally free $\mathcal{O}_X$-modules and the differentials d_i are $\mathcal{D}_X$-linear. We assume moreover that the lifting u of ϕ for $\mathcal{M}$ is induced by $\mathcal{O}_{Z \times X}$-linear morphisms $u'^i : \phi^* \mathcal{E}^i \to p^* \mathcal{E}^i$ as explained in example 5.5. In this case we know from remark 5.6 that

$$c_0(\phi, \mathcal{M}, u) = \sum (-1)^i c_0(\phi, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}^i, u'^i \otimes l_\phi),$$

where l_ϕ is the natural lifting of $\mathcal{D}_X$. Hence we are reduced to $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}$, where $\mathcal{E}$ is a fiber bundle and $u = u' \otimes l_\phi$ for a lifting u' of ϕ for $\mathcal{E}$.

We set $\mathcal{F} = \mathcal{O}_Z \boxtimes \mathcal{E} \boxtimes \mathcal{E}^*$; we have $i_\Gamma^* \mathcal{F} \simeq \mathcal{H}om_{\mathcal{O}_{Z\times X}}(\phi^*\mathcal{E}, p^*\mathcal{E})$. Let i_Γ, i_Δ, i be the embeddings of Γ, $Z \times \Delta_X$, $\tilde{Z}$ in $Z \times X \times X$. By examples 5.5 and 5.4 we know that $c'(u)$ is the image of u by $e \circ d$, where d and e are the compositions (tensor products between $\mathcal{O}$-modules are taken over $\mathcal{O}$):

$$d : H_\Gamma^{d_X}(\mathcal{O}_{Z\times X\times X}^{(0,d_X,0)} \otimes \mathcal{F}) \to H_\Gamma^{d_X}(i_{\Delta *}\mathcal{O}_{Z\times X}^{(0,d_X)}) \xrightarrow{\sim} H_{\tilde{Z}}^{d_X}(\mathcal{O}_{Z\times X}^{(0,d_X)}),$$
$$e : H_{\tilde{Z}}^{d_X}(\mathcal{O}_{Z\times X}^{(0,d_X)}) \to H_{\tilde{Z}}^{0}(\mathcal{O}_Z \boxtimes \omega_X) \xrightarrow{\sim} H^0(q^!\mathcal{O}_Z),$$

where the first morphism in d is induced by the contraction $\mathcal{F} \to i_{\Delta *}\mathcal{O}_{Z\times X}$. We are interested in the image of u by d; in particular we ask if it is related to the fundamental class of $\tilde{Z}$ in $Z \times X$. But we have to be careful that the morphisms induced by the fundamental classes of Γ and $\tilde{Z}$

$$i_{\Gamma *}\mathcal{O}_{Z\times X} \to H_\Gamma^{d_X}(\mathcal{O}_{Z\times X\times X}^{(0,d_X,0)}) \quad \text{and} \quad i_*\mathcal{O}_{\tilde{Z}} \to H_{\tilde{Z}}^{d_X}(\mathcal{O}_{Z\times X}^{(0,d_X)})$$

do not commute with d and the restriction $i_{\Gamma *}\mathcal{O}_{Z\times X} \to i_*\mathcal{O}_{\tilde{Z}}$ (indeed the second morphism could be zero). However the next lemma says that the corresponding morphisms with maximal degree forms on Z fit into a commutative diagram. We consider the morphisms defining the fundamental classes of Γ and $\tilde{Z}$:

$$\tau_\Gamma : i_{\Gamma *}(\mathcal{O}_{Z\times X}^{(d_Z,0)}) \otimes \mathcal{F} \to H_\Gamma^{d_X}(\mathcal{O}_{Z\times X\times X}^{(d_Z,d_X,0)} \otimes \mathcal{F}),$$
$$\tau_{\tilde{Z}} : i_*\Omega_{\tilde{Z}} \to i_{\Delta *}H_{\tilde{Z}}^{d_X}(\Omega_{Z\times X}),$$

where τ_Γ is the integration morphism associated with i_Γ, tensored by $(\mathcal{O}_{Z\times X\times X}^{(0,0,d_X)})^* \otimes \mathcal{F}$. We recall that the canonical lifting, l_ϕ, of $\mathcal{D}_X$ is defined (example 5.4) as the projection of the fundamental class of Γ. In view of the definition of the fundamental class we have also, for a d_Z-form ω on Z and for $\mathcal{F} = \mathcal{O}_{Z\times X\times X}$:

$$\omega \otimes l_\phi = \tau_\Gamma(\omega).$$

Let d' be "$d \otimes \mathcal{O}_{Z\times X\times X}^{(d_Z,0,0)}$" and e' be "$e \otimes \mathcal{O}_{Z\times X\times X}^{(d_Z,0,0)}$":

$$d' : H_\Gamma^{d_X}(\mathcal{O}_{Z\times X\times X}^{(d_Z,d_X,0)} \otimes \mathcal{F}) \to i_{\Delta *}H_{\tilde{Z}}^{d_X}(\Omega_{Z\times X}),$$
$$e' : i_{\Delta *}H_{\tilde{Z}}^{d_X}(\Omega_{Z\times X}) \to H^0(q^!\Omega_Z).$$

By definition of $\tau_{\tilde{Z}}$ we have $\sigma'_Z = e' \circ \tau_{\tilde{Z}}$. Writing $\tilde{Z}$ as the transversal intersection of Γ and $Z \times \Delta_X$, we obtain also a map from $i_{\Gamma *}(\mathcal{O}_{Z\times X}^{(d_Z,0)})$ to $i_*\Omega_{\tilde{Z}}$. Indeed we have the composition of isomorphisms:

$$\begin{aligned} i_*\Omega_{\tilde{Z}} &\simeq i_{\Gamma *}\Omega_{Z\times X} \otimes i_{\Delta *}\Omega_{Z\times X} \otimes \Omega^*_{Z\times X\times X} \\ &\simeq i_{\Gamma *}\mathcal{O}_{Z\times X}^{(d_Z,0)} \otimes i_{\Delta *}(\mathcal{O}_{Z\times X}^{(d_Z,d_X)} \otimes i_\Delta^*(\mathcal{O}_{Z\times X\times X}^{(d_Z,d_X,0)})^*) \\ &\simeq i_{\Gamma *}\mathcal{O}_{Z\times X}^{(d_Z,0)} \otimes i_{\Delta *}\mathcal{O}_{Z\times X}. \end{aligned} \tag{42}$$

The contraction $\mathcal{F} \to i_{\Delta *}\mathcal{O}_{Z\times X}$ composed with (42) yields

$$\alpha_\mathcal{E} : i_{\Gamma *}\mathcal{O}_{Z\times X}^{(d_Z,0)} \otimes \mathcal{F} \to i_*\Omega_{\tilde{Z}}. \tag{43}$$

For the link between $\alpha_\mathcal{E}$ and the inverse image of forms see remark 9.5.

Lemma 9.2. *We assume that Γ and $Z \times \Delta_X$ are transversal in $Z \times X \times X$ and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module. We set $\mathcal{F} = \mathcal{O}_Z \boxtimes \mathcal{E} \boxtimes \mathcal{E}^*$. With the notations above* *we have a commutative diagram:*

$$\begin{array}{ccc} i_{\Gamma *}(\mathcal{O}^{(d_Z,0)}_{Z\times X}) \otimes_{\mathcal{O}_{Z\times X\times X}} \mathcal{F} & \xrightarrow{\alpha_{\mathcal{E}}} & i_*\Omega_{\tilde{Z}} \\ \downarrow \tau_\Gamma & & \downarrow \tau_{\tilde{Z}} \\ H^{d_X}_\Gamma(\mathcal{O}^{(d_Z,d_X,0)}_{Z\times X\times X} \otimes_{\mathcal{O}_{Z\times X\times X}} \mathcal{F}) & \xrightarrow{d'} & i_{\Delta *}H^{d_X}_{\tilde{Z}}(\Omega_{Z\times X}). \end{array}$$

Proof. By the definitions of $\alpha_{\mathcal{E}}$ and d' it is enough to show that the same diagram, with $i_{\Delta *}(\mathcal{O}_{Z\times X})$ in place of $\mathcal{F}$, is commutative. We introduce the following sheaf on $Z \times \Delta_X$, $\Omega_{rel} = \Omega_{Z\times \Delta_X} \otimes i^*_\Delta \Omega^*_{Z\times X\times X}$. Since $\tilde{Z}$ is the transversal intersection of Γ and $Z \times \Delta_X$, the integration morphisms associated to i and i_Γ are related by the commutative diagram:

$$\begin{array}{ccc} i_{\Gamma *}\Omega_{Z\times X} \otimes i_{\Delta *}\Omega_{rel} & \xrightarrow[a]{\sim} & i_*\Omega_{\tilde{Z}} \\ \downarrow & & \downarrow \\ \Omega_{Z\times X\times X}[d_X] \otimes i_{\Delta *}\Omega_{rel} & \xrightarrow{\sim} & i_{\Delta *}\Omega_{Z\times X}[d_X]. \end{array}$$

Since Ω_{rel} is canonically isomorphic to $i^*_\Delta(\mathcal{O}^{(0,0,d_X)}_{Z\times X\times X})^*$, we have also the isomorphisms:

$$i_{\Gamma *}\Omega_{Z\times X} \otimes i_{\Delta *}\Omega_{rel} \xrightarrow[b]{\sim} i_{\Gamma *}\mathcal{O}^{(d_Z,0)}_{Z\times X} \otimes i_{\Delta *}(\mathcal{O}_{Z\times X}),$$

$$\Omega_{Z\times X\times X} \otimes i_{\Delta *}\Omega_{rel} \simeq \mathcal{O}^{(d_Z,d_X,0)}_{Z\times X\times X} \otimes i_{\Delta *}(\mathcal{O}_{Z\times X}).$$

We conclude with the remark that the composition $a \circ b^{-1}$ coincides with $\alpha_{\mathcal{E}}$. $\square$

Now we can write $c'(u)$ as the image of a form on $\tilde{Z}$. Let ω be a volume form on Z. We have

$$c'(u) \cdot \omega \in H^0(\tilde{Z}; q^!\Omega_Z).$$

For $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}$ and $u = u' \otimes l_\phi$ as above, $\omega \otimes u$ is the image of

$$\omega \otimes u' \in i_{\Gamma *}\mathcal{O}^{(d_Z,0)}_{Z\times X} \otimes \mathcal{F} \simeq i_{\Gamma *}(\mathcal{O}^{(d_Z,0)}_{Z\times X} \otimes \mathcal{H}om\,(\phi^*\mathcal{E}, p^*\mathcal{E}))$$

by τ_Γ and we have, by lemma 9.2:

$$\begin{aligned} c'(u) \cdot \omega &= (e' \circ d' \circ \tau_\Gamma)(\omega \otimes u') \\ &= (e' \circ \tau_{\tilde{Z}} \circ \alpha_{\mathcal{E}})(\omega \otimes u') \\ &= (\sigma'_Z \circ \alpha_{\mathcal{E}})(\omega \otimes u'). \end{aligned}$$

If $\mathcal{M}$ is given by a complex $\mathcal{E}^\cdot$ we sum over the components.

Proposition 9.3. *With hypothesis and notations of lemma 9.1 we assume that $\mathcal{M}$ is given by a complex of fiber bundles, $\mathcal{M} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}^\cdot$, and u by morphisms $u'^i \in \mathrm{Hom}_{\mathcal{O}_{Z\times X}}(\phi^*\mathcal{E}^i, p^*\mathcal{E}^i)$. Then, for a volume form ω on $\tilde{Z}$ we have:*

$$c'(u) \cdot \omega = \sigma'_Z\Big(\sum_i (-1)^i \alpha_{\mathcal{E}^i}(\omega \otimes u_i)\Big).$$

Let us set $c'_\omega(u) = \sum (-1)^i \alpha_{\mathcal{E}^i}(\omega \otimes u_i)$; this is a section of $\Omega_{\tilde{Z}}$. For $S \subset \tilde{Z}$ such that $q|_S$ is proper, we denote by $\int_q$ the integration morphism from $H^{d_Z}_S(\tilde{Z}; \Omega_{\tilde{Z}})$ to $H^{d_Z}_{q(S)}(Z; \Omega_Z)$. Finally we obtain the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$ as the direct image of a form on $\tilde{Z}$:

Corollary 9.4. *With hypothesis and notations of lemma 9.1 and proposition 9.3 we have:*

$$\chi(\phi, \mathcal{M}, F, u, v) \cdot \omega = \int_q c'_\omega(u) \cup c'(v).$$

Remark 9.5. Here is the link between the morphism $\alpha_{\mathcal{O}_X}$ of (43) and the inverse image by the projection $q : \tilde{Z} \to Z$ (recall that $\alpha_{\mathcal{E}}$ is just the product of $\alpha_{\mathcal{O}_X}$ and the contraction $\mathcal{F} \to i_{\Delta *}\mathcal{O}_{Z\times X}$). Let $p : Z \times X \to Z$ be the projection and ω a maximal degree form on Z. We set $\tilde{\omega} = \alpha_{\mathcal{O}_X}(p^*\omega)$; the forms $\tilde{\omega}$ and $q^*\omega$ are related as follows. For $z \in Z$ we denote by $\phi_z : X \to X$ the map $x \mapsto \phi(z, x)$. If $(z, x) \in \tilde{Z}$, then $\phi_z(x) = x$ and $\phi'_z(x)$ is an endomorphism of T_xX so that it makes sense to consider the function $D(z, x) = \det(id - \phi'_z(x))$ on $\tilde{Z}$. A local computation gives:

$$q^*\omega = D(z, x) \cdot \tilde{\omega}. \tag{44}$$

In particular, if Z is a point and $\phi : X \to X$ is "transversal to id", the class

$$c'_\omega(u) \in H^{d_X}_{\tilde{Z}}(\Omega_X) \simeq \bigoplus_{x\in\tilde{Z}} \mathbb{C}$$

is given by a complex number at each fixed point of ϕ. With the notations above for a lifting induced by fiber bundles morphisms, this number is at a fixed point x:

$$c'_\omega(u)_x = \sum_i (-1)^i \frac{\operatorname{tr} {u'}^i}{\det(id - \phi'(x))}.$$

This is the Atiyah-Bott formula of [2] for a "linear" lifting (see also [10] for an expression in the framework of "elliptic pairs").

10. Group action case

In this section we consider the previous results in the case of a group action. Our manifold Z is assumed to be a complex Lie group and we denote it by G; $\phi : G \times X \to X$ is a group action. The condition (6) is clearly satisfied. We denote by e the neutral element of G, by $G_\mathbb{R}$ a real form of G, by $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{R}$ the Lie algebras of G and $G_\mathbb{R}$. For $g \in G$, $x \in X$ we denote by $\phi_g : X \to X$ the map $y \mapsto g \cdot y$, by ${}_x\phi : G \to X$ the map $h \mapsto h \cdot x$ and by $m_g : G \to G$ the multiplication on the right $h \mapsto h \cdot g$.

We consider a G-quasi-equivariant good $\mathcal{D}_X$-module, $\mathcal{M}$. This means that there exists an $\mathcal{O}_G \boxtimes \mathcal{D}_X$-linear isomorphism u from $\underline{\phi}^{-1}(\mathcal{M})$ to $\underline{p}^{-1}(\mathcal{M})$ compatible with the multiplication of G (this compatibility with the product is in fact not used for the definition of the index).

In the same way we consider a $G_\mathbb{R}$-equivariant $\mathbb{R}$-constructible sheaf F on X, i.e. we have an isomorphism v from ${\phi_\mathbb{R}}^{-1}(F)$ to ${p_\mathbb{R}}^{-1}(F)$ compatible with the product of $G_\mathbb{R}$. Hence we are in the situation of section 4. Recall the subsets of T^*X associated to ϕ, Λ_ϕ and Λ'_ϕ, introduced in definitions 6.1 and 6.3:

$$\Lambda_\phi = \{(x, \xi) \in T^*X;\ \exists g \in G\ g \cdot x = x,\ {}^t\phi'_{(g,x)}(\xi) = (0, \xi)\}$$
$$\Lambda'_\phi = \{(x, \xi) \in T^*X;\ \exists (g, y) \in G \times X\ g \cdot y = x,\ {}^t({}_y\phi)'_g(\xi) = 0\}.$$

We have already noticed that $\Lambda_\phi \subset \Lambda'_\phi$. For a group there is also the conormal to the orbits defined as follows. Let $\mu : T^*X \to \mathfrak{g}^*$ be the moment map of T^*X. By definition, for $(x,\xi) \in T^*X$, $\mu(x,\xi) = {}^t({}_x\phi)'_e(\xi)$. The conormal to the orbits is

$$T^*_GX = \mu^{-1}(0) = \{(x,\xi) \in T^*X; {}^t({}_x\phi)'_e(\xi) = 0\}.$$

We see on this formula that $T^*_GX \subset \Lambda'_\phi$.

Lemma 10.1. *If $\phi : G \times X \to X$ is a group action we have:*

$$T^*_GX = \Lambda'_\phi = \Lambda_\phi.$$

Proof. *i*) We first show that $\Lambda'_\phi \subset T^*_GX$. Let $(x,\xi) \in \Lambda'_\phi$. By definition there exists $g \in G$ such that, setting $y = g^{-1} \cdot x$, we have ${}^t({}_y\phi)'_g(\xi) = 0$. Then ${}_x\phi = {}_y\phi \circ m_g$ and ${}^t({}_x\phi)'_e = {}^t(m_g)'_e \circ {}^t({}_y\phi)'_g$. Hence we also have ${}^t({}_x\phi)'_e(\xi) = 0$, so that $(x,\xi) \in T^*_GX$.

ii) We show that $T^*_GX \subset \Lambda_\phi$. Let $(x,\xi) \in T^*_GX$, so that ${}^t({}_x\phi)'_e(\xi) = 0$. It is sufficient to show that ${}^t\phi'_{(e,x)}(\xi) = (0,\xi)$. But, in general, we have

$${}^t\phi'_{(g,x)}(\xi) = ({}^t({}_x\phi)'_g(\xi), {}^t(\phi_g)'_x(\xi)).$$

Since ϕ_e is the identity morphism of X the result follows. □

From this lemma the pair $(\mathcal{M}, F)$ is transversally elliptic if and only if

$$\mathrm{char}(\mathcal{M}) \cap SS(F) \cap T^*_GX \subset T^*_XX$$

and this is equivalent to the strong transversal ellipticity. Until the end of the section we assume that $(\mathcal{M}, F)$ is transversally elliptic and that $\mathrm{supp}(\mathcal{M}) \cap \mathrm{supp}(F)$ is compact. Hence the hypothesis of proposition 6.2 are satisfied and we can consider the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$. It is invariant by conjugation because of the equivariance of the data. Indeed let $h \in G_\mathbb{R}$; the conjugation by h, $c_h : G \to G$, $g \mapsto h \cdot g \cdot h^{-1}$ and the action ϕ_h of h on X make the following diagram commute:

$$\begin{array}{ccc} G \times X & \xrightarrow{\phi} & X \\ {\scriptstyle c_h \times \phi_h}\downarrow & & \downarrow{\scriptstyle \phi_h} \\ G \times X & \xrightarrow{\phi} & X. \end{array}$$

We set $\mathcal{M}' = \underline{\phi_h}^{-1}\mathcal{M}$, $F' = \phi_h^{-1}F$ and we let u', v' be the inverse images of u, v. We have $\chi(\phi, \mathcal{M}, F, u, v) = c_h^*(\chi(\phi, \mathcal{M}', F', u', v'))$, because c_h and ϕ_h are diffeomorphisms. But, by equivariance, $\mathcal{M}' \simeq \mathcal{M}$, $F' \simeq F$ and u', v' coincide with u, v, so that $\chi(\phi, \mathcal{M}, F, u, v) = c_h^*(\chi(\phi, \mathcal{M}, F, u, v))$.

We have also a better expression for the bound Λ of the wave-front set of $\chi(\phi, \mathcal{M}, F, u, v)$ given in proposition 6.2. For $g \in G$ let us identify $T^*_eG = \mathfrak{g}^*$ and T^*_gG through m_g; this gives an isomorphism $T^*G \simeq G \times \mathfrak{g}^*$. Let x, $y \in X$ and $g \in G$ be such that $x = g \cdot y$. For $\xi \in T^*_xX$ we have:

$${}^tm'_g({}^t({}_y\phi)'_g(\xi)) = {}^t({}_x\phi)'_e(\xi) = \mu(x,\xi).$$

Hence with the isomorphism $T^*G \simeq G \times \mathfrak{g}^*$ we obtain ${}^t({}_y\phi)'_g(\xi) = (g, \mu(x,\xi))$. We say that $(x,\xi) \in T^*X$ is fixed by $g \in G$ if $g \cdot x = x$ and ${}^t(\phi_g)'_x(\xi) = \xi$ (the second equality makes sense because x is fixed by g). We denote by $\mathfrak{g}_\mathbb{R}^\perp \subset \mathfrak{g}^*$ the orthogonal of $\mathfrak{g}_\mathbb{R}$ in $\mathfrak{g}^*$. We have the following expression for the bound Λ of the wave-front set of $\chi(\phi, \mathcal{M}, F, u, v)$:

$$\begin{aligned} \Lambda = \{(g,\eta) \in G_\mathbb{R} \times \mathfrak{g}_\mathbb{R}^\perp;\ & \exists (x,\xi) \in \mathrm{char}\,\mathcal{M} \cap SS(F) \\ & (x,\xi) \text{ is fixed by } g \text{ and } \eta = \mu(x,\xi)\ \}\ . \end{aligned} \tag{45}$$

This bound coincides with the one given in [4] in the case of a compact group.

Example 10.2. Let G be a complex Lie group with a compact real form $G_{\mathbb{R}}$. We let G operate on $X = G$ by multiplication on the left. We consider the action of $G_{\mathbb{R}}$ on $\Gamma(G_{\mathbb{R}}; \mathcal{A}_{G_{\mathbb{R}}})$, i.e. we consider $\mathcal{M} = \mathcal{D}_X$ which is naturally G-quasi-equivariant, with lifting $u = l_\phi$, and $F = \mathbb{C}_{G_{\mathbb{R}}}$ the constant sheaf on $G_{\mathbb{R}} \subset X$ which is also naturally $G_{\mathbb{R}}$-equivariant, with lifting $v = id_F$. Since X is homogeneous we are in the setting of paragraph 9.2 ($\mathcal{M}$ being associated to the trivial bundle on X). We need to determine the form $c'_\omega(l_\phi)$ of corollary 9.4, for a volume form ω on G. It is the image of ω by the morphism $\alpha_{\mathcal{O}_X}$ defined by formula (43). The fixed points set of the action of G on X is $\tilde{G} = \{e\} \times X$ viewed as a subset of $G \times X$, and we identify $\tilde{G}$ with $G = X$ by the projection $G \times X \to X$. If we choose ω to be invariant we can see, with this identification, that $c'_\omega(l_\phi) = \omega$ as a form on $\tilde{G}$. We have to determine also the class $c'(v)$; it belongs to $H^{d_G}_S(\tilde{G}; \mathbb{C}_{\tilde{G}})$, where $S = (G_{\mathbb{R}} \times \operatorname{supp} F) \cap \tilde{G}$. With our identification $\tilde{G} = X = G$ we have $S = G_{\mathbb{R}}$, $H^{d_G}_S(\tilde{G}; \mathbb{C}_{\tilde{G}}) \simeq H^0(G_{\mathbb{R}}; \mathbb{C}_{G_{\mathbb{R}}}) \simeq \mathbb{C}$ and $c'(v) = 1$. Finally $\chi(\phi, \mathcal{M}, F, u, v) \cdot \omega$ is the direct image of $\omega|_S$ through the projection $q : \tilde{G} \to G$ sending $\tilde{G}$ to $\{e\}$. Hence $\chi(\phi, \mathcal{M}, F, u, v)$ is the Dirac function on $\{e\}$.

10.1. **Real compact Lie group.** In this paragraph we show that the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$ coincides with the character of transversally elliptic operators given by Atiyah in [1]. Let $G_{\mathbb{R}}$ be a real compact Lie group, acting on a real analytic manifold M, $\mathcal{F}_1$, $\mathcal{F}_2$ be equivariant fiber bundles on M and Q be an equivariant differential operator from the sections of $\mathcal{F}_1$ to the sections of $\mathcal{F}_2$. We assume this situation can be complexified, i.e. we assume that there exist a complex Lie group G, with $G_{\mathbb{R}}$ as a real form, acting on a complexification, X, of M, and G-equivariant fiber bundles $\mathcal{E}_1$, $\mathcal{E}_2$ on X endowed with a G-equivariant differential operator P, such that $\mathcal{E}_1$, $\mathcal{E}_2$, P restrict to $\mathcal{F}_1$, $\mathcal{F}_2$, Q on M. We set $F = \omega_{M|X}$ and

$$\mathcal{M} \quad = \quad 0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}_2^* \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E}_1^* \to 0.$$

If we choose an identification between $SS(F) = T^*_M X$ and T^*M we have

$$T^*_{G_{\mathbb{R}}} M \simeq T^*_G X \cap T^*_M X.$$

Let $\sigma_Q : \pi^* \mathcal{F}_1 \to \pi^* \mathcal{F}_2$ be the principal symbol of Q (here π is the projection $T^*M \to M$). We have also with the identification $T^*_M X \simeq T^*M$:

$$\{(x, \xi);\ \sigma_Q(x, \xi) \text{ is not an isomorphism}\} = \operatorname{char} \mathcal{M} \cap T^*_M X.$$

Recall that Q is transversally elliptic in the sense of Atiyah if σ_Q is an isomorphism on $T^*_{G_{\mathbb{R}}} M \setminus T^*_M M$. Hence Q is transversally elliptic if and only if $(\mathcal{M}, F)$ is transversally elliptic in the sense of definition 6.1.

We want to show that the hyperfunction $\chi(\phi, \mathcal{M}, F, u, v)$ agrees with Atiyah's index, which is defined as the trace of the group $G_{\mathbb{R}}$ on the virtual representation $\ker Q - \operatorname{coker} Q$, where Q acts on the infinitely differentiable sections of $\mathcal{F}_1$ and $\mathcal{F}_2$. The equality of this index with $\chi(\phi, \mathcal{M}, F, u, v)$ is nearly an immediate consequence of theorem 8.2 except that we deal with analytic or hyperfunction sections. We have $R\mathcal{H}om\,(F, \mathcal{O}_X) \simeq \mathcal{B}_M$ and $\mathrm{D}'F \otimes \mathcal{O}_X \simeq \mathcal{A}_M$. Let $\mathcal{C}^\infty_M$ be the sheaf of infinitely differentiable functions on M. Let us set for short:

$$A = R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M), \quad C = R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^\infty_M), \quad B = R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M).$$

We have natural morphisms $A \xrightarrow{f} C \xrightarrow{g} B$, and for an analytic form ω on $G_{\mathbb{R}}$ we have, by section 4, morphisms commuting with f and g, say $S_A(\omega) : A \to A$, $S_C(\omega) : C \to C$, $S_B(\omega) : B \to B$. But in fact we know by proposition 6.5 that they are compositions of f, g and a morphism $S(\omega) : B \to A$. This implies that $S_A(\omega)$, $S_B(\omega)$, $S_C(\omega)$ have the same "naive trace" (in the sense of section 2) and hence the same trace.

Proposition 10.3. *Let $G_{\mathbb{R}}$ be a real compact Lie group acting on a real compact manifold M and let Q be a transversally elliptic operator on M. Assume that $G_{\mathbb{R}}$, M, Q can be complexified in G, X, P and let $\mathcal{M}$ be the $\mathcal{D}_X$-module associated to P as above. Then $\chi(\phi, \mathcal{M}, \omega_{M|X}, u, v)$ is equal to the analytic index of Q defined in* [1].

10.2. **Semi-simple Lie group.** In this paragraph G is a connected, semi-simple, complex Lie group, $G_{\mathbb{R}}$ a real form of G, X the flag manifold of G. We consider $\mathcal{M} = \mathcal{D}_X$ which is canonically G-quasi-equivariant, with lifting l_ϕ, and is in fact associated to the trivial line bundle on X; we consider also a $G_{\mathbb{R}}$-equivariant $\mathbb{R}$-constructible sheaf F on X with lifting denoted by v. The action of G on X is homogeneous; hence we are in the setting of paragraph 9.2 and we can apply the results of corollary 9.4. In this case the fixed points set is the following subset of $G \times X$, where X is identified with the set of Borel subgroups of G:

$$\tilde{G} = \{(g, B) \in G \times X;\ g \in B\}.$$

Let $\omega \in \Gamma(G; \Omega_G)$. The form $\tilde{\omega} = c'_\omega(l_\phi) \in \Gamma(\tilde{G}; \Omega_{\tilde{G}})$ of corollary 9.4 is given by formula (44):

$$\tilde{\omega} = \frac{1}{\det(id - \phi'_g(x))} \cdot q^* \omega,$$

where $q : \tilde{G} \to G$ is the projection and $\phi'_g(x)$ is the derivative of ϕ_g at the fixed point $x \in X$. Note that, if g is in a maximal torus $H \subset G$ and x corresponds to a Borel $B \supset H$ determined by a set of positive roots Δ_+, we have:

$$\det(id - \phi'_g(x)) = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})(g).$$

On the other hand, with the notations of paragraph 9.1, the class $c'(v)$ is the characteristic cycle introduced by Kashiwara in [13]. It belongs to $H^{d_G}_S(\tilde{G}; \mathbb{C}_{\tilde{G}})$, where $S = q^{-1}(\operatorname{supp} F) \subset q^{-1}(G_{\mathbb{R}})$. The subset $q^{-1}(G_{\mathbb{R}})$ of $\tilde{G}$ is the fixed points set of $G_{\mathbb{R}}$ in X. But $G_{\mathbb{R}}$ has finitely many orbits in X, say $O_1, \dots, O_n$. Let us denote by $\tilde{G}_i$ the fixed points set of $G_{\mathbb{R}}$ in O_i, i.e. $\tilde{G}_i = (G_{\mathbb{R}} \times O_i) \cap \tilde{G}$. Then $\tilde{G}_i$ is a real smooth submanifold of $\tilde{G}$ of real dimension d_G. Hence $q^{-1}(G_{\mathbb{R}}) = \bigsqcup \tilde{G}_i$ is the union of finitely many submanifolds of $\tilde{G}$ of real dimension d_G. By corollary 9.4 the index is

$$\chi(\phi, \mathcal{D}_X, F, l_\phi, v) = \int_q c'(v) \cup \tilde{\omega} \tag{46}$$

and is in fact the sum of direct images of a multiple of $\tilde{\omega}$ on each $\tilde{G}_i$. This formula coincides with the one given in [13] (see also [21]).

From the results of [16] we know that the complex $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X \otimes F, \mathcal{O}_X) \simeq R\mathrm{Hom}\,(F, \mathcal{O}_X)$ is strict and that the resulting FN-spaces representations of $G_{\mathbb{R}}$,

$$\pi_i : G_{\mathbb{R}} \to \mathrm{End}\,(\mathrm{Ext}^i(F, \mathcal{O}_X)),$$

are admissible. This implies in particular that they have generalized characters. Let us denote by χ_i the character of π_i; for a maximal degree $\mathcal{C}^\infty$-form ω with compact support on $G_{\mathbb{R}}$, the morphism $\pi_{i,\omega} : \mathrm{Ext}^i(F, \mathcal{O}_X) \to \mathrm{Ext}^i(F, \mathcal{O}_X)$, $x \mapsto$

$\int_{G_{\mathbb{R}}} \pi_i(g)(x) \cdot \omega$ is trace-class and, by definition, $< \chi_i, \omega >= \operatorname{tr} \pi_{i,\omega}$. Now we can prove that the character $\sum(-1)^i \chi_i$ is given by formula (46). This was conjectured in [14], paragraph 6.3, and proved in [21]. (Up to the Matsuki correspondence (see [17] and [16]) a similar character formula was also given in [13] and proved in [18], by a comparison method between the character and Kashiwara's formula, using a decomposition on Schubert cells and the Osborne conjecture.)

Theorem 10.4. *Let G be a connected, semi-simple, complex Lie group, $G_{\mathbb{R}}$ a real form of G, F an $\mathbb{R}$-constructible $G_{\mathbb{R}}$-equivariant sheaf on the flag manifold X of G. Let χ_i be the character of the representation $\operatorname{Ext}^i(F, \mathcal{O}_X)$ and c' the characteristic cycle of F. With the notations above, for a volume form ω on G and the associated volume form $\tilde{\omega}$ on $\tilde{G}$, we have:*

$$\sum(-1)^i \chi_i \cdot \omega = \int_q c' \cup \tilde{\omega}.$$

Proof. We have to show that $\chi(\phi, \mathcal{D}_X, F, u, v) = \sum(-1)^i \chi_i$. This would be a particular case of theorem 8.2 if $G_{\mathbb{R}}$ were compact. In fact we prove the result on all translates of a maximal compact subgroup of $G_{\mathbb{R}}$.

Let us set for short $\chi = \chi(\phi, \mathcal{D}_X, F, u, v)$ and $\chi' = \sum(-1)^i \chi_i$. We know that χ' is a central eigendistribution. For χ we remark that l_ϕ itself is annihilated by the image in $\mathcal{D}_G$ of the augmentation ideal $Z_+(\mathfrak{g})$ of $Z(\mathfrak{g})$. Indeed, for $P \in U(\mathfrak{g})$ let P_G and P_X be its images in $\mathcal{D}_G$ and $\mathcal{D}_X$; it is well-known (see for example [5]) that if $P \in Z_+(\mathfrak{g})$ then $P_X = 0$. Hence the claim follows from $P_G \cdot l_\phi = l_\phi \cdot P_X$. This implies that $P_G(\chi) = 0$ for $P \in Z_+(\mathfrak{g})$ because the construction of $\chi(\phi, \mathcal{M}, F, u, v)$ is $\mathcal{D}_G$-linear; hence χ is also a central eigendistribution.

Since both χ and χ' are central eigendistributions on $G_{\mathbb{R}}$, by the results of Harish-Chandra [11], they are determined by their restrictions to the open subset of regular semi-simple elements of $G_{\mathbb{R}}$, say $G_{\mathbb{R}reg}$. Moreover these restrictions to $G_{\mathbb{R}reg}$ are analytic functions which are locally L^1 in $G_{\mathbb{R}}$ (notice that for χ this is also a consequence of formula (46)). Hence we only need to show that $\chi = \chi'$ on $G_{\mathbb{R}reg}$. Let K be a maximal compact subgroup of $G_{\mathbb{R}}$. Let us denote by $K_{\mathbb{C}}$ the complexification of K, by $\mathfrak{k}$ and $\mathfrak{k}_{\mathbb{C}}$ the Lie algebras of K and $K_{\mathbb{C}}$. In fact we will show that χ and χ' have well-defined restrictions to any translate $g \cdot K$ of K and that these restrictions coincide. This implies clearly that $\chi = \chi'$ on $G_{\mathbb{R}reg}$ and then that $\chi = \chi'$.

Let us begin with the existence of the restrictions. Formula (45) gives the following bound for the wave-front set of χ:

$$\Lambda = \{(g, \eta) \in G_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}}^{\perp};\ \exists (x, \xi) \in T^*X,\ (x, \xi) \text{ is fixed by } g \text{ and } \eta = \mu(x, \xi)\ \}\ .$$

In fact, if $q : \tilde{G} \to G$ is the projection we have $\Lambda = q_\pi({}^t q'^{-1}(T^*_{\tilde{G}} \tilde{G}))$. In [12] it is shown that a central eigendistribution with trivial central character is solution of the $\mathcal{D}_G$-module $\underline{q}_* \mathcal{O}_{\tilde{G}}$. Since $\operatorname{char} \underline{q}_* \mathcal{O}_{\tilde{G}}$ is contained in Λ, it is a bound for the wave-front set of χ' too and the existence of the restrictions of χ and χ' to $g \cdot K$ is a consequence of lemma 10.5 below.

Now we need a description of the restrictions of χ and χ'. For χ we will apply proposition 7.1 and theorem 8.2. We fix $g \in G_{\mathbb{R}}$ and we consider the restriction of ϕ to $g \cdot K_{\mathbb{C}}$, say $\psi : g \cdot K_{\mathbb{C}} \times X \to X$. By lemma 10.6 below $\Lambda'_\psi \cap SS(F)$ is contained in the zero-section. Since $\Lambda_\psi \subset \Lambda'_\psi$ the pair $(\mathcal{D}_X, F)$ is transversally elliptic with respect to ψ and it follows from proposition 7.1 that $\chi|_{g \cdot K} = \chi(\psi, \mathcal{D}_X, F, l_\psi, v')$, where v' is the restriction of v. Now theorem 8.2, together with the fact that the complex $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}\otimes F,\mathcal{O}_X)$ is strict, says that for $\omega\in\Gamma(g\cdot K;\Omega_{g\cdot K_{\mathbb{C}}})$ we have

$$<\chi|_{g\cdot K},\omega>=\sum(-1)^i\,\mathrm{tr}\,\pi'_{i,\omega},$$

where π'_i is the restriction of π_i to $g\cdot K$ and $\pi'_{i,\omega}$ is the endomorphism of $\mathrm{Ext}^i(F,\mathcal{O}_X)$ defined by $x\mapsto\int_{g\cdot K}\pi_i(k)(x)\cdot\omega(k)$.

Hence we will have $\chi|_{g\cdot K}=\chi'|_{g\cdot K}$ if we show that

$$<\chi_i|_{g\cdot K},\omega>=\mathrm{tr}\,\pi'_{i,\omega}.$$

This is proved in [7] (for K, not $g\cdot K$, but the proof adapts immediately). For the reader convenience we restate their result in lemma 10.7. (Note that $\pi'_{i,\omega}(x)=\pi_i(g)(\int_K\pi_i(k')(x)\cdot\omega(g\cdot k'))$ to agree with the notations of the lemma). □

Lemma 10.5. *With the notations of the proof of theorem 10.4 we have for any $g\in G_{\mathbb{R}}$:*

$$\Lambda\cap T^*_{g\cdot K_{\mathbb{C}}}G\subset T^*_GG.$$

Proof. The bound Λ is contained in

$$\Lambda'=\{(g,\eta)\in G_{\mathbb{R}}\times\mathfrak{g}_{\mathbb{R}}^{\perp};\ \exists(x,\xi)\in T^*X\ \eta=\mu(x,\xi)\ \}\ =G_{\mathbb{R}}\times(\mathfrak{g}_{\mathbb{R}}^{\perp}\cap\mu(T^*X)).$$

Up to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ through the Killing form, the image of T^*X by the moment map is the nilpotent cone, $\mathcal{N}$, of $\mathfrak{g}$; hence $\Lambda'=G_{\mathbb{R}}\times(\mathfrak{g}_{\mathbb{R}}^{\perp}\cap\mathcal{N})$. Let $\mathfrak{k}_{\mathbb{C}}$ be the Lie algebra of $K_{\mathbb{C}}$. With the identification $T^*G=G\times\mathfrak{g}^*$, we have $T^*_{g\cdot K_{\mathbb{C}}}G=g\cdot K_{\mathbb{C}}\times\mathfrak{k}_{\mathbb{C}}^{\perp}$ and the lemma is reduced to $\mathcal{N}\cap\mathfrak{k}_{\mathbb{C}}^{\perp}\cap\mathfrak{g}_{\mathbb{R}}^{\perp}=\{0\}$. □

Lemma 10.6. *With the notations of the proof of theorem 10.4 we have:*

$$\Lambda'_\psi\cap SS(F)\subset T^*_XX.$$

Proof. Recall that

$$\Lambda'_\psi=\{(x,\xi)\in T^*X;\ \exists(z,y)\in g\cdot K_{\mathbb{C}}\times X,\ z\cdot y=x\text{ and }{}^t({}_y\psi)'_z(\xi)=0\}.$$

Here ${}^t({}_y\psi)'_z:T^*_xX\to T^*_z(g\cdot K_{\mathbb{C}})$ is the composition of ${}^t({}_y\phi)'_z:T^*_xX\to T^*_zG$ and the projection $T^*_zG\to T^*_z(g\cdot K_{\mathbb{C}})$. With the identification $T^*_zG=\mathfrak{g}^*$, we have $T^*_z(g\cdot K_{\mathbb{C}})=\mathfrak{k}_{\mathbb{C}}^*$ and ${}^t({}_y\psi)'_z$ is the moment map $\mu_{K_{\mathbb{C}}}:T^*_xX\to\mathfrak{k}_{\mathbb{C}}^*$ with respect to the action of $K_{\mathbb{C}}$. Hence Λ'_ψ is a subset of $\mu_{K_{\mathbb{C}}}^{-1}(0)=T^*_{K_{\mathbb{C}}}X$. On the other hand, since F is $G_{\mathbb{R}}$-equivariant its micro-support is contained in $T^*_{G_{\mathbb{R}}}X$. Since the orbits of $K_{\mathbb{C}}$ and $G_{\mathbb{R}}$ are transversal (see for example [17], lemma 1.3) we have $T^*_{K_{\mathbb{C}}}X\cap T^*_{G_{\mathbb{R}}}X\subset T^*_XX$ and the lemma follows. □

Lemma 10.7 (lemma A.5 (3) of [7])**.** *Let $G_{\mathbb{R}}$ be a semi-simple connected Lie group and K a maximal compact subgroup. Let $\pi:G_{\mathbb{R}}\to\mathrm{End}(E)$ be an admissible representation of $G_{\mathbb{R}}$ with trivial infinitesimal character, with character χ. Let π^K be the restriction of π to K. Then for $g\in G_{\mathbb{R}}$, χ has a well-defined restriction to $g\cdot K$ and for a density, α, on K we have:*

$$<\chi|_{g\cdot K},l^*_{g^{-1}}\alpha>=\mathrm{tr}(\pi(g)\circ\pi^K_\alpha),$$

where $l_{g^{-1}}$ is multiplication on the left by g^{-1} and $\pi^K_\alpha(x)=\int_K\pi^K(k)(x)\cdot\alpha(k)$.

Proof. For ω a density with compact support on $G_{\mathbb{R}}$ we set as before $\pi_\omega(x) = \int_{G_{\mathbb{R}}} \pi(g)(x) \cdot \omega(g)$. If X and Y are two submanifolds of $G_{\mathbb{R}}$ such that the multiplication $X \times Y \to G$, $(x,y) \mapsto x \cdot y$, is a diffeomorphism and if α (resp. β) is a density on X (resp. Y) with compact support, we have by Fubini identity $\pi_{\alpha\otimes\beta} = \pi_\alpha \circ \pi_\beta$ (by abuse of notations we use the same notations for α, β, $\alpha \otimes \beta$ and their direct images on $G_{\mathbb{R}}$).

Let B be a subgroup of $G_{\mathbb{R}}$ such that $K \times B \to G_{\mathbb{R}}$, $(k,b) \mapsto k \cdot b$ is a diffeomorphism. Then the map $g \cdot K \times B \to G_{\mathbb{R}}$, $(k', b) \mapsto k' \cdot b$ is also a diffeomorphism. Let α be a $\mathcal{C}^\infty$-density on K such that π_α^K has finite rank (such α are dense among the densities on K). Let β_i be a sequence of $\mathcal{C}^\infty$-densities on B, with compact supports decreasing to $\{e\}$, such that $\int_B \beta_i = 1$. The distribution χ has a restriction to $g \cdot K$ and we have

$$< \chi|_{g\cdot K}, l^*_{g^{-1}}\alpha >= \lim_i < \chi, l^*_{g^{-1}}\alpha \otimes \beta_i > .$$

But

$$\begin{aligned} < \chi, l^*_{g^{-1}}\alpha \otimes \beta_i > &= \mathrm{tr}(\pi_{l^*_{g^{-1}}\alpha\otimes\beta_i}) \\ &= \mathrm{tr}(\pi_{l^*_{g^{-1}}\alpha} \circ \pi_{\beta_i}) \\ &= \mathrm{tr}(\pi(g^{-1}) \circ \pi_\alpha \circ \pi_{\beta_i}). \end{aligned}$$

Since β_i tends to the Dirac function at $\{e\}$, π_{β_i} tends to id_E and since π_α has finite rank $\mathrm{tr}(\pi(g^{-1}) \circ \pi_\alpha \circ \pi_{\beta_i})$ tends to $\mathrm{tr}(\pi(g^{-1}) \circ \pi_\alpha)$. By definition $\pi_\alpha = \pi_\alpha^K$ and the lemma is proved. □

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