STRUCTURE OF KÄHLER GROUPS, I : SECOND COHOMOLOGY

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0. Introduction

Fundamental groups of complex projective varieties are very difficult to understand. There is a tremendous gap between few computed examples and few general theorems. The latter all deal with either linear finite dimensional representations ([Sim]) or actions on trees ([Gr-Sch]); besides, one knows almost nothing.

This paper presents a new general theorem, partially settling a well-known conjecture of Carlson-Toledo ([CT]).

MAIN THEOREM. — *Let Γ be a fundamental group of a compact Kähler manifold. Assume Γ is not Kazhdan. Then $H^2(\Gamma, \mathbb{R}) \neq 0$.*

COROLLARY. — *Suppose Δ is not Kazhdan and $b_1(\Delta, \mathbb{Q}) = 0$. Let $\widetilde{\Delta}$ be the universal central extension $0 \to H_2(\Delta, \mathbb{Z}) \to \widetilde{\Delta} \to \Delta \to 1$. Then Γ is not a fundamental group of a compact Kähler manifold.*

Examples.

1. Any torsion group Δ of an intermediate growth is amenable, therefore not Kazhdan, besides, $b_1(\Delta, \mathbb{Q}) = 0$.

2. Any lattice in $SU(n, 1)$ is not Kazhdan; some of them have finite abelianization. However, in this case one can do better, see remark in section 8 below.

THEOREM 0.1. — *Let Γ be a fundamental group of a complex projective variety. Suppose Γ has a Zariski dense rigid representation in $SO(2, n)$, n odd. Then*

(i) $H^2(\Gamma, \mathbb{R}) \neq 0$.

Mots-clés : Kähler groups, property T.

(ii) Moreover, $H^2_b(\Gamma, \mathbb{R}) \neq 0$ and the canonical map $H^2_b \to H^2$ is not zero.

COROLLARY 0.1. — *Let Δ be a lattice in $SO(2,n)$, n odd, uniform or not. Let $\widetilde{\Delta}$ be a universal central extension of Δ. Then Δ is not a fundamental group of a complex projective variety.*

THEOREM 0.2. — *Let Γ be a fundamental group of a complex projective variety. Suppose Γ has a Zariski dense rigid representation in $Sp(4)$. Then*

(i) $H^2(\Gamma, \mathbb{R}) \neq 0$.

(ii) Moreover, $H^2_b(\Gamma, \mathbb{R}) \neq 0$ and the canonical map $H^2_b \to H^2$ is not zero.

COROLLARY 0.2. — *Let Δ be a lattice in $Sp(4)$. Then $\widetilde{\Delta}$ is not a fundamental group of a complex projective variety.*

COROLLARY 0.3. — *Lattices in $\mathrm{Spin}(2,n)$, n odd, and $\widetilde{Sp(4)}$ are not fundamental groups of a complex projective variety.*

1. A geometric picture for rigid representations

Let Y be a compact Kähler manifold. All rigid irreducible representations $\rho : \pi_1(Y) \to SL(N,\mathbb{C})$ are conjugate to representations landing in a $SU(m,n) \subset SL(N,\mathbb{C})$ with $m+n=N$ ([Sim 1]) and have a structure of complex variation of Hodge structure ([Sim 1]). Moreover, we can always arrange that this conjugate representation is defined over $\overline{\mathbb{Q}}$ (see e.g. [Re 1]). Relabeling , we assume that ρ itself is defined over $\overline{\mathbb{Q}}$. We assume that moreover, ρ is defined over $\mathcal{O}(\overline{\mathbb{Q}})$; by a conjecture of Carlos Simpson ([Sim 1]) this is always the case. Let $\{\rho_i\}$ be all the Galois twists of ρ, then ρ_i are rigid therefore land in $SU(m_i,n_i)$. The image of $\pi_1(Y)$ in $\prod_i SU(m_i,n_i)$ is discrete; call it Γ.

Coming back to ρ, consider a corresponding θ-bundle E ([Sim 1]). It has the following structure: $E = \bigoplus_{p+q=k} L^{p,q}$ and θ maps $L^{p,q}$ to $L^{p-1,q+1} \otimes \Omega^1$. Any θ-invariant subbundle of E has negative degree; in particular, the degree of $L^{k,0}$ is positive. A harmonic metric K in E is the unique metric satisfying the equation ([Hit]) $F_K = [\theta, \theta^*]$. The hermitian connection ∇_K leaves all $E^{p,q}$ invariant. The connection $\nabla_K + \theta + \theta^*$ is flat with monodromy ρ. Let V be the corresponding flat holomorphic bundle. In V, we have a flag of holomorphic subbundles $F^p = V^{k,0} \oplus \cdots \oplus V^{p,k-p}$, where $V^{p,q}$ are $E^{p,q}$ thought as C^∞-subbundles of V with its new holomorphic structure. We have therefore a ρ-equivariant map $\widetilde{Y} \xrightarrow{s} D$, where D is a corresponding Griffiths domain ([G], ch. I–II). Changing the sign of K alternatively on $V^{p,q}$ we obtain a flat pseudo-hermitian metric in V.

So if $m = \bigoplus_{p \text{ even}} \dim V^{p,q}$, $n = \bigoplus_{p \text{ odd}} \dim V^{p,q}$, then ρ lands in $SU(m,n)$. The Griffiths domain D carries a horizontal distribution defined by the condition that the derivative of F_p lies in F_{p+1}. The developing map s is horizontal. Differentiating this condition we obtain a second order equation ([Sim 1]) $[\theta,\theta] = 0$, in other words for $Z, W \in T_x Y$, $\theta(Z)$ and $\theta(W)$ commute.

Since the image of $\pi_1(Y)$ in $\prod_i SU(m_i, n_i)$ is discrete, we obtain

Proposition 1.1 (Geometric picture for rigid representations). — *Let $\rho : \pi_1(Y) \to SL(N,\mathbb{C})$ be a rigid irreducible representation, defined over $\mathcal{O}(\overline{\mathbb{Q}})$. Then there exist Griffiths domains $D_i = SU(m_i,n_i)/K_i$, a discrete group Γ in $\prod SU(m_i,n_i)$ and a horizontal holomorphic map*

$$S : Y \longrightarrow \prod D_i/\Gamma$$

which induces ρ and all its Galois twists.

Remark. — Though a Griffiths domain D is topologically a fibration over a hermitian symmetric space with fiber a flag variety, generally it does not have a $SU(m,n)$-invariant Kähler metric. So the complex manifold $\prod D_i/\Gamma$ is not Kähler.

Remark. — This proposition tells us that one cannot expect too many compact Kähler manifolds to have a nontrivial linear representation of their fundamental group, of finite dimension.

Lemma 1.2 (Superrigidity).

(1) Let $X = \Gamma \diagdown SU(m,n)/S(U(m) \times U(n)$ be a compact hermitian locally symmetric space of Siegel type I. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to SU(n,m)$ is rigid and Zariski dense, e.g. $f_* : \pi_1(Y) \to \Gamma$ an isomorphism, then either f is homotopic to a holomorphic map, or there exists a compact complex analytic space Y', $\dim Y' < \dim X$, and a holomorphic map $\varphi : Y \to Y'$ such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(2) Let $X = \Gamma \diagdown SO(2,n)/S(O(2) \times O(n))$ be of Siegel type IV. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to SO(2,n)$ is rigid and Zariski dense then either f is homotopic to a holomorphic map f_0, or n is even and there exists a compact complex analytic space Y', $\dim Y' < \dim X$, and a holomorphic map $\varphi : Y \to Y'$ such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(3) Let $X = \Gamma \diagdown Sp(2n)/U(n)$ be of Siegel type III. Let Y be compact Kähler and let $f : Y \to X$ be continuous. If $f_* : \pi_1(Y) \to Sp(2n)$ is rigid and Zariski dense, then either f is homotopic to a holomorphic map f_0, or there exists a compact complex analytic space Y', $\dim Y' < \dim Y$, and a holomorphic map $\varphi : Y \to Y'$, such that f is homotopic to a composition $Y \xrightarrow{\varphi} Y' \xrightarrow{f_1} X$.

(4) Let $X = \Gamma \diagdown Sp(4)/U(2)$ be a Shimura threefold. Let Y be compact Kähler and let $f : Y \to X$ be continuous. Then either f is homotopic to a holomorphic map f_0, or there exists a (singular) proper curve S, and a holomorphic map $\varphi : Y \to S$, such that f is homotopic to a composition $Y \xrightarrow{\varphi} S \longrightarrow X$.

Remarks.

1. I leave the case of Siegel type II to the reader (the proof is similar).

2. If f_* is *not* rigid, one has strong consequences for $\pi_1(Y)$, see 9.1.

3. The lemma should be viewed as a final (twistorial) version of the superrigidity theorem ([Si]).

2. Proof of the Superrigidity Lemma (1)

(1) Since we are given a continuous map $Y \xrightarrow{f} X = \Gamma \diagdown SU(m,n)/S(U(m) \times U(n))$ the map S of Proposition 1.1 is simply a holomorphic map $Y \to D/\Gamma$, where D is a Griffiths domain corresponding to the complex variation of Hodge structure, defined by $\rho = f_* : \pi_1(Y) \to SU(m,n)$. Suppose the Higgs bundle looks like $\oplus E^{p,q}$ where $E^{p,q}$ have dimensions $m_1, n_1, m_2, n_2, \ldots, m_s, k_s$ where k_s is possibly missing. Then $\sum m_i = m$, $\sum n_i = n$. Now, the dimension of the horizontal distribution is

$$m_1 \cdot n_1 + n_1 \cdot m_2 + m_2 \cdot n_2 + \cdots + m_s \cdot k_s \,.$$

We notice that this number is strictly less than $m \cdot n = \dim X$ except for the cases:

I) $s = 1$, *i.e.* $E = E^{1,0} \oplus E^{0,1}$

II) $s = 2$, $k_2 = 0$, *i.e.* $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$.

In the first case, D is the symmetric space, and $D/\Gamma = X$ so we arrive to a holomorphic map to. $Y \to X$. In the second case the second order equation reads $\theta_1(Z)\theta_2(W) - \theta_1(W)\theta_2(Z) = 0$ where $\theta_1 : TY \otimes E^{0,2} \to E^{1,1}$ and $\theta_2 : TY \otimes E^{1,1} \to E^{2,0}$ are the components of the (horizontal) derivative DS. So the image of DS is strictly less than $\mathrm{Hom}(E^{0,2}, E^{1,1}) \oplus \mathrm{Hom}(E^{1,1}, E^{1,0}) = m \cdot n = \dim X$. In other words, $\dim Y' < \dim X$ where $Y' = S(Y)$.

3. Variation of Hodge structure, corresponding to rigid representations to $SO(2,n)$

Let $\rho : \pi_1(Y) \to SO(2,n)$ be a Zariski dense rigid representation. Complexifying, we obtain a variation of Hodge structure $E = \oplus E^{p,q}$. Since ρ is defined over reals, we deal with real variation of Hodge structure ([Sim 1]) that is to say, $E^{p,q} = \overline{E^{p,q}}$ with respect to a flat complex conjugation. For $n \geq 3$, this leaves exactly two possibilities:

I) $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$, $\dim E^{1,1} = n$, $\dim E^{2,0} = \dim E^{0,2} = 1$.

II) n is even, $E = E^{2,0} \oplus E^{1,1} \oplus E^{0,2}$, $\dim E^{1,1} = 2$, $\dim E^{2,0} = \dim E^{0,2} = n/2$.

In case I) the Griffits domain is the symmetric spaces $SO(2,n)/S(O(2) \times O(n))$ so the harmonic metric viewed as a harmonic section of the flat bundle with fiber a symmetric space, is holomorphic. In the second case the second order equation implies that the rank of the derivative DS of the ρ-equivariant holomorphic map $\widetilde{Y} \to D$ is strictly less than n.

Proof of the Superrigidity Lemma (2). — This follows immediately from the previous discussion in the same manner as in (1).

4. Variations of Hodge structure, corresponding to rigid representation to $Sp(4)$

Let $\rho : \pi_1(Y) \to Sp(2n)$ be a Zariski dense rigid representation. Complexifying, we obtain a representation $\pi : \pi_1(Y) \to SU(n,n)$ and a real variation of Hodge structure $E = \bigoplus_{p+q=k} E^{p,q}$, $E^{p,q} = \overline{E}^{q,p}$ and k odd. For $n = 2$ this leaves two possibilities:

(I) $E = E^{1,0} \oplus E^{0,1}$, and both $E^{1,0}$ and $E^{0,1}$, or rather $V^{1,0}$ and $V^{0,1}$ viewed as C^∞-subbundles of the flat bundle V, are lagrangian with respect to the flat complex symplectic structure. This means first, that the Griffiths domain D is the symmetric space $SU(2,2)/S(U(2) \times U(2))$, second, that the image of the equivariant horizontal holomorphic map $S : \widetilde{Y} \to D$ lies in the copy of the Siegel upper half-space $Sp(4)/U(2)$ under the Satake embedding ([Sa]). In other words, the unique ρ-equivariant harmonic map $\widetilde{Y} \to Sp(4)/U(2)$ is holomorphic.

(II) $E = E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}$ and $\dim E^{p,q} = 1$. The second order equation for θ implies immediately that D_s has rank at most one everywhere on Y.

Proof of the Superrigidity Lemma (4). — Follows from the discussion above.

6. Variations of Hodge structure, corresponding to rigid representation to $Sp(2n)$, and proof of the Superrigidity Lemma (3)

In general, the Higgs bundle is $E = \bigoplus_{p+q=2s+1} E^{p,q}$, $E^{p,q} = \overline{E^{p,q}}$. The dimension of the horizontal distribution is

$$d = \sum_{p<s} \dim E^{p,q} \cdot \dim E^{p+1,q-1} = \frac{\dim E^{s,s+1} \cdot (\dim E^{s,s+1} + 1)}{2},$$

since $\theta : E^{s,s+1} \to E^{s+1,s}$ viewed as bilinear form, should be symmetric. Moreover, $\sum_{p \leq s} \dim E^{p,q} = n$. An elementary exercise shows that if $s > 1$, $d < \frac{n(n+1)}{2}$. If $s = 1$, we get a holomorphic map to the Siegel upper half-plane.

7. Regulators, I: proof of the Main Theorem

The reader is supposed to be familiar with the geometric theory of regulators ([Re 1], [Co]).

Let $\mathbb{H}$ be a complex Hilbert space. The constant Kähler form (dX, dX) is invariant under the affine isometry group $\mathrm{Iso}(\mathbb{H})$, and $\mathbb{H}$ is contractible, therefore there is a regulator class in $H^2(\mathrm{Iso}^{\delta}(\mathbb{H}), \mathbb{R})$. In fact, there is a class ℓ in $H^1(\mathrm{Iso}^{\delta}(\mathbb{H}), \mathbb{H})$ defined by a cochain $(x \mapsto Ux + b) \longmapsto b$. The regulator class in simply (ℓ, ℓ).

If $\pi_1(Y)$ does not have property T, then there exist a representation $\rho : \pi_1(Y) \to \mathrm{Iso}(\mathbb{H})$ and a holomorphic nonconstant section S of the associated flat holomorphic affine bundle with fiber $\mathbb{H}$ ([Ko-Sch]). It follows that the pull-back $\rho^*((\ell, \ell))$ of the regulator class to $H^2(\pi_1(Y), \mathbb{R})$ restricts to a cohomology class in $H^2(Y, \mathbb{R})$, given by a non-zero semi-positive $(1,1)$ form. Multiplying by the ω^{n-1}, where ω is a Kähler form, and $n = \dim Y$, and integrating over Y we get a positive number, therefore this cohomology class is non-zero. Therefore $H^2(\pi_1(Y), \mathbb{R}) \neq 0$.

Remark. — Historically, the first break through in this direction has been made in [JR], under assumption of having a nontrivial variation of a unitary representation. Compare Proposition 9.1 below.

Proof of the Corollary 0.2. — Since $\widetilde{\Delta} \to \Delta$ is surjective and Δ does not have property T, neither does $\widetilde{\Delta}$ ([HV]). Since $H_1(\Delta, \mathbb{Z})$ is finite, the Lyndon-Serre-Hochschild spectral sequence implies that $H^2(\widetilde{\Delta}, \mathbb{R}) = \emptyset$. So $\widetilde{\Delta}$ is not a Kähler group.

Remark. — Suppose $\pi_1(Y)$ does not have property T. Suppose moreover that that $\pi_1(Y)$ has a permutation representation in $\ell^2(B)$, where B is a countable set, and $H^1(\pi_1(Y),\ell^2(B)) \neq 0$. Then we actually proved that $H^2(\pi_1(Y),\ell^1(B)) \neq 0$. That is because the scalar product $\ell^2(B) \times \ell^2(B) \to \mathbb{C}$ factors through $\ell^1(B)$. Moreover, the canonical map $H^2(\pi_1(Y),\ell^1(B)) \to H^2(\pi_1(Y),\mathbb{C})$ is nonzero.

8. Regulators, II: proof of Theorems 0.1, 0.2

Let G be an isometry group of a classical symmetric bounded domain D. With the exception of $SO(2,2)$, $H^1(G,\mathbb{Z}) = \mathbb{Z}$. This defines a central extension $1 \to \mathbb{Z} \to \widetilde{G} \to G \to 1$ and an extension class $e \in H^2(G^\delta,\mathbb{Z})$. On the other hand, the Bergman metric on D is G-invariant, so it defines a regulator class $r \in H^2_{\text{cont}}(G,\mathbb{R})$. It is proved in [Re 2], [Re 3] that, first, these classes coincide up to a factor, and second, lie in the image of the bounded cohomology: $H^2_b(G^\delta,\mathbb{R}) \to H^2(G^\delta,\mathbb{R})$.

If Y is a compact Kähler manifold, $\rho : \pi_1(Y) \to G$ a representation , s a holomorphic nonconstant section of the associated flat D-bundle, then one sees immediately that $(\rho^*(r),\omega^{n-1}) > 0$, so $\rho^*(r), \rho^*(e) \neq 0$. Theorem 0.1 follows now from the analysis of VHS given in sections 3, 4. To prove Theorem 0.2 notice that the case when Y fibers over a curve is obvious, otherwise Y admits a holomorphic map to a quotient of the Siegel half-plane and the proof proceeds as before.

Remark. — By [CT], the result of Theorem 0.1 is true for lattices in $SU(n,1)$.

9. Nonrigid representations

PROPOSITION 9.1. — *Let Y be compact Kähler and let $\rho : \pi_1(Y) \to SL(n,\mathbb{C})$ be a nonrigid irreducible representation. Then $H^2(\pi_1(Y),\mathbb{R}) \neq 0$.*

Proof. — Let $\mathfrak{g} = \mathfrak{s}l(n,\mathbb{C})$ and let $\bar\rho$ be the adjoint representation. We know that $H^1(\pi_1(Y),\mathfrak{g}) \neq 0$. Therefore $H^1(Y,\underline{\mathfrak{g}}) \neq 0$ where $\underline{\mathfrak{g}}$ is the local system. By the Simpson's hard Lefschetz ([Sim 1]), the multiplication by ω^{n-1} gives an isomorphism $H^1(Y,\underline{\mathfrak{g}}) \to H^{2n-1}(Y,\underline{\mathfrak{g}})$, where ω is the polarization class and $n = \dim_{\mathbb{C}} Y$. The Poincaré duality implies that the Goldman's pairing $H^1(Y,\mathfrak{g}) \times H^1(Y,\mathfrak{g}) \to \mathbb{C}$ is nondegenerate. Let z be homology class in $H_2(Y)$, dual to ω^{n-1}, and $\bar z$ its image in $H_2(\pi_1(Y))$. It follows that the pairing $H^1(\pi_1(Y),\mathfrak{g}) \times H^1(\pi_1(Y),\mathfrak{g}) \to \mathbb{C}$ defined by $f,g \mapsto [(f,g),\bar z]$ is nondegenerate. Here $(f,g) \in H^2(\pi_1,\mathbb{C})$ is the pairing defined by the Cartan-Killing form. In particular, $\bar z \neq 0$.

Corollary 10.1. — *Let Y a compact Kähler manifold. If $\pi_1(Y)$ has a Zariski dense representation in either $Sp(4)$ or $SO(2,n)$, n odd, then $H^2(\pi_1(Y),\mathbb{R}) \neq 0$.*

Proof. — For rigid representations, this is proved in Theorems 0.1, 0.2. For non-rigid representations, this follows from Proposition 9.1.

Corollary 9.2. — *Let Γ be any overgroup of a Zariski dense countable subgroup of $Sp(4)$ or $SO(2,n)$, n odd. Suppose $b_1(\Gamma) = 0$. Then the universal central extensions $\widetilde{\Gamma}$ is not Kähler.*

10. Three-manifolds groups are not Kähler

In this section, based on the previous development, we will present a strong evidence in favour of the following:

Conjecture 10.1. — *Let M^3 be irreducible closed 3-manifold with $\Gamma = \pi_1(M)$ infinite. Then Γ is not Kähler.*

Proposition 10.2 (Seifert fibration case). — *A cocompact lattice in $\widetilde{SL(2,\mathbb{R})}$ is not Kähler.*

Proof. — Passing to a subgroup of finite index, we can assume that Γ is a central extension of a surface group:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(S) \longrightarrow 1$$

with a nontrivial extension class. In particular, $H^1(\Gamma,\mathbb{Q}) \simeq H^1(\pi_1(S),\mathbb{Q}))$, so the multiplication in $H^1(\Gamma,\mathbb{Q})$ is zero, which is impossible if Γ is Kähler.

Recall that "most" of closed three-manifolds admit a Zariski dense homomorphism $\pi_1(M) \xrightarrow{\rho} SL_2(\mathbb{C})$ ([CGLS], [Re 1]).

Theorem 10.3. — *Let M^3 be atoroidal. Suppose there exists a Zariski dense homomorphism $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. Then $\Gamma = \pi_1(M)$ is not Kähler.*

Proof. — By a theorem of [Zi] $\pi_1(M)$ does not have property T. By the Main Theorem, $H^2(\Gamma,\mathbb{R}) \neq 0$, hence by [Th], M is hyperbolic, which is impossible by [CT].

Alternatively, ρ is not rigid by [Sim 1], so $H^2(\Gamma,\mathbb{R}) \neq 0$ by Proposition 9.1, and then one procedes as before.

Remark. — In view of [CGLS], [Re 1], we obtain a huge number of groups which are not Kähler.

11. Central extensions of lattices in $PSU(2,1)$

We saw a general result, that, if $\Gamma \subset SU(n,1)$ a cocompact lattice and $[\omega] \in H^2(\Gamma, \mathbb{Z})$ is given by any ample line bundle, then a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

with the extension class $[\omega]$ is not Kähler. For $n = 2$ one can also prove:

THEOREM 11.1. — *Let $\omega \in H^2(B^2/\Gamma, \mathbb{Z}) \cap (H^{2,0} \oplus H^{0,2})$, $\omega \neq 0$. Then an extension*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\Gamma}_\omega \longrightarrow \Gamma \longrightarrow 1$$

with the extension class ω is not Kähler.

Remark. — $H^{2,0}$ becomes big on étale finite coverings of B^2/Γ by Riemann-Roch.

Proof. — Suppose $\widetilde{\Gamma}_\omega = \pi_1(Y)$. The representation $\pi_1(M) \to \Gamma \to SU(2,1)$ is rigid by the Lyndon-Serre-Hochschild spectral sequence. It follows that there exists a dominating holomorphic map $Y \to B^2/\Gamma$. But then the pullback map on $H^{2,0}$ is injective, a contradiction.

12. Smooth hypersurfaces in ball quotients which are not $K(\pi, 1)$

We saw that under various algebraic assumptions on $\Gamma = \pi_1(Y)$, there is a class in $H^2(Y, \mathbb{R})$ which vanishes on the Hurewitz image $\pi_2(Y) \to H_2(Y, \mathbb{Z})$, therefore defining a nontrivial element of $H^2(\Gamma, \mathbb{R})$. On the contrary, we will show now that there are hypersurfaces Y in ball quotients B^n/Γ, $n \geq 3$ with a surjective map $\pi_1(Y) \to \Gamma$ such that $\pi_i(Y) \neq 0$ for some i. The proof is very indirect and we don't know the exact value of i. The varieties Y were in fact introduced in [To] where it is proved that $\pi_1(Y)$ is not residually finite. We will show that $cd(\pi_1(Y)) \geq 2n - 1$, therefore Y is not $K(\pi, 1)$.

Let X^n be an arithmetic ball quotient and let $X_0 \subset X$ be a totally geodesic smooth hypersurface. Let $D = X - X_0$, then D is covered topologically by $\mathbb{C}^n$ minus a countable union of hyperplanes, so D is $K(\pi, 1)$. Let S be a boundary of a regular heighbourhood of X_0, so S is a circle bundle over X_0, in particular S is $K(\pi, 1)$ and $\pi_1(S)$ is a central extension

$0 \to \mathbb{Z} \to \pi_1(S) \to \pi_1(X_0) \to 1$ with a nontrivial extension class (this is because the normal bundle to X_0 is negative). Let V be a finite dimensional module over $\pi_1(X)$ with an invariant nondegenerate form $V \to V'$. We have an exact sequence

$$H^{2n-1}(\pi_1(X), V) \longrightarrow H^{2n-1}(\pi_1(X_0), V) \oplus H^{2n-1}(\pi_1(D), V) \longrightarrow H^{2n-1}(\pi_1(S), V) \longrightarrow H^{2n}(\pi_1(X), V) \longrightarrow \cdots$$

Now, we make a first assumption:

1) $H^0(\pi_1(X), V) = 0$.

It follows that $H_0(\pi_1(X), V) = 0$, so $H^{2n}(\pi_1(X), V) = 0$; we make a second assumption :

2) $H^1(\pi_1(X), V) = 0$.

It follows that $H^{2n-1}(\pi_1(X), V) = 0$. So we have (remember that X_0 has dimension $n-1$)

$$H^{2n-1}(\pi_1(D), V) \simeq H^{2n-1}(\pi_1(S), V) .$$

Now, in the E^2 of the Lyndon-Serre-Hochschild spectral sequence for $H^*(\pi_1(S), V)$ the term $H^{2n-2}(\pi_1(X_0), H^1(\mathbb{Z}, V))$ is not hit by any differential. Since $\mathbb{Z}$ acts trivially, this is just $H^{2n-2}(\pi_1(X_0), V) \simeq H_0(\pi_1(X_0), V)$. We now make a third assumption:

3) $H^0(\pi_1(X_0), V) \neq 0$.

Then we will have $H^{2n-1}(\pi_1(D), V) \neq 0$. Let Y be a generic hyperplane section of X/X_0, constructed in [To], then [GM], $\pi_1(Y) = \pi_1(D)$ and we are done.

Now, we take for V the adjoint module. The assumption 2) follows from Weil's rigidity. The assumption 3) is satisfied for standard examples of X_0 ([To]).

Remark. — The construction of [To] is given for lattices in $SO(2, n)$, but it applies verbatim here.

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