THE STRUCTURE OF THE QUANTUM SEMIMARTINGALE ALGEBRAS

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ABSTRACT. — In the theory of quantum stochastic calculus one disposes of two quantum semimartingale algebras $\mathcal{S}$ and $\mathcal{S}'$. The first one is an algebra for the composition of operators and it allows a quantum functional calculus for analytical functions. The second one is larger and is an algebra for the operations of quantum square and angle brackets. In this article we study the algebraic and analytic properties of these algebras. This study is mainly performed through a remarkable transform of quantum processes which, surprisingly, establishes a bijection in between these two algebras. This bijection allows to give norms on these algebras that equip them with Banach algebra structures.

INTRODUCTION

The quantum stochastic calculus on Fock space, defined by Hudson and Parthasarathy ([H-P]) is a non-commutative extension of the usual stochastic calculus. It deals with operators on the boson Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+))$ and allows to define quantum stochastic integrals

$$\int_0^t H_s \; da_s^\varepsilon$$

of adapted operator processes $(H_t)_{t\geq 0}$ with respect to the three basic quantum noises $(a_t^+)_{t\geq 0}$, $(a_t^-)_{t\geq 0}$, $(a_t^\circ)_{t\geq 0}$ (creation, annihilation and conservation processes) and with respect to the time process $(a_t^\times)_{t\geq 0}$. The resulting operator $\int_0^t H_s \; da_s^\varepsilon$ is only defined on a particular subspace of Φ, the space $\mathcal{E}$ of coherent vectors. This domain constraint prevents quantum stochastic integrals from being composed. A quantum Ito formula was nevertheless established in a weak sense. It was actually a quantum Ito integration by part formula that is, for the "composition" of two quantum stochastic integrals T_t and S_t. In their formulation all operator compositions HK were replaced by expressions of the form

$$\langle H^*\varepsilon(v), K\varepsilon(u)\rangle$$

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where $\varepsilon(v), \varepsilon(u)$ are arbitrary coherent vectors. Their formula admitted no extension to further functional calculus. For example, it is impossible to extract from this formulation a formula for the third power of a quantum stochastic integral.

In [A-M], the definition of quantum stochastic integrals is extended to arbitrary domains in Φ. As a consequence one may have bounded quantum stochastic integrals defined on the whole of Φ. This allows composition and a true quantum Ito integration by part formula. In [At1], a space $\mathcal{S}$ of quantum stochastic processes of the form

$$T_t = \lambda I + \sum_{\varepsilon=+,0,-,\times} \int_0^t H_s^\varepsilon \, da_s^\varepsilon,$$

defined on all Φ is obtained, and it is proved that $\mathcal{S}$ forms an algebra under operator composition (it is even a $*$-algebra through the adjoint mapping). This algebra $\mathcal{S}$ thus allows polynomial functional calculus. In [ViS] it is proved that the functional calculus on $\mathcal{S}$ can be extended to analytical function, and even to C^{2+} functions for the self-adjoint elements of $\mathcal{S}$. The algebra $\mathcal{S}$ deserves its name: the algebra of regular quantum semimartingales.

In [At1] a theory of quantum square and angle brackets is also described. These quantum brackets are the non-commutative extensions of the square and angle brackets of classical stochastic calculus ([Mey]). The integration by part formula on $\mathcal{S}$ then takes the following familiar form:

$$S_tT_t = S_0T_0 + \int_0^t S_s dT_s + \int_0^t dS_s T_s + [S,T]_t. \tag{1}$$

If $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ are in $\mathcal{S}$ then so is $(S_tT_t)_{t\geq 0}$, but in general none of the three processes appearing in the right hand side of (1) belongs to $\mathcal{S}$. Indeed, they satisfy the same kind of properties as the elements of $\mathcal{S}$, but they are not in general made of bounded operators. This remarks brought to define another space $\mathcal{S}'$, which is larger than $\mathcal{S}$ and which always contains the processes $(\int_0^t S_s dT_s)_{t\geq 0}$, $(\int_0^t dS_s T_s)_{t\geq 0}$, $([S,T]_t)_{t\geq 0}$ for $S,T \in \mathcal{S}$.

The space $\mathcal{S}'$ happens also to be a $*$-algebra but for the square bracket product :

$$\begin{aligned} \mathcal{S}' \times \mathcal{S}' &\longrightarrow \mathcal{S}' \\ (S_\cdot, T_\cdot) &\longmapsto ([S_\cdot, T_\cdot]_t)_{t\geq 0}. \end{aligned}$$

In this article we give a deep study of $\mathcal{S}$ and $\mathcal{S}'$, their relations and their algebraic properties. We study some norms on them and prove that they are Banach algebras.

The main point is the definition of a transform $\mathcal{D}$ which maps quantum stochastic processes to quantum stochastic processes, which is invertible and which establishes a perfect bijection between $\mathcal{S}$ and $\mathcal{S}'$. This result is surprising for the following reasons:

- it is very simply described,
- it proves that the algebra $\mathcal{S}$ is large,
- the transform $\mathcal{D}$ has the property to make bounded operator processes which were not.

II. ELEMENTS OF QUANTUM STOCHASTIC CALCULUS

II.1. The Fock space

The Fock space $\Phi = \Gamma(L^2(\mathbb{R}^+))$ is the direct sum $\bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^+)^{\otimes n}$ on the n-th symmetric tensor products of $L^2(\mathbb{R}^+)$ (cf. [Par] for complements). A good way to understand Φ is to use Guichardet's notations ([Gui]). Let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$, then $\mathcal{P} = \cup_n \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n-elements subsets of $\mathbb{R}^+$. Identifying $\mathcal{P}_n$ with the increasing simplex $\Sigma_n = \{0 < t_1 < \cdots < t_n\}$ one can equip $\mathcal{P}_n$ with the Lebesgue measure structure. By putting the Dirac mass $\delta_\emptyset$ on $\mathcal{P}_0$, this altogether gives a structure of σ-finite measured space to $\mathcal{P}$, whose only atom is $\{\emptyset\}$. It is not difficult to see that $L^2(\mathcal{P})$ is naturally isomorphic to Φ by identifying n-variable symmetric functions on $\mathbb{R}^+$ to functions on Σ_n. In this article the Fock space Φ will always be understood as $L^2(\mathcal{P})$. Elements of $\mathcal{P}$ are denoted by small Greek letters σ, τ, ω,... and the corresponding volume element is denoted by $d\sigma, d\tau, d\omega, \ldots$

Let us set some notations and recall some basic results in this context. For all $\sigma \in \mathcal{P}$, we denote by $\vee\sigma$ the maximum of σ (if $\sigma = \emptyset$) and by σ- the set $\sigma \setminus \{\vee\sigma\}$. For all $t \in \mathbb{R}^+$ and all $\sigma \in \mathcal{P}$ let $\sigma_{t)} = \sigma \cap [0, t[$, $\sigma_{(t} = \sigma \cap]t, +\infty[$, $\sigma \cup t = \sigma \cup \{t\}$. Let $\mathcal{P}_{t)}$ (resp. $\mathcal{P}_{(t}$) be the set of $\sigma \in \mathcal{P}$ such that $\sigma \subset [0, t]$ (resp. $\sigma \subset [t, +\infty[$). The space $L^2(\mathcal{P}_{t)})$ (resp. $L^2(\mathcal{P}_{(t)})$) is identified with the subspace of $f \in L^2(\mathcal{P})$ such that $f(\sigma) = 0$ whenever $\sigma \not\subset [0, t]$ (resp. $\sigma \not\subset [t, +\infty[$). One also writes $\Phi_{t]} = L^2(\mathcal{P}_{t)})$ and $\Phi_{[t} = L^2(\mathcal{P}_{(t)})$.

For all $u \in L^2(\mathbb{R}^+)$ let $\varepsilon(u)$ be the element of Φ such that $[\varepsilon(u)](\sigma) = \prod_{s \in \sigma} u(s)$ (with the empty product being equal to 1). The vectors $\varepsilon(u)$ are called *coherent vectors* on Φ. The space $\mathcal{E}$ generated by the coherent vectors is dense in Φ. The *vaccum vector* is the vector $\mathbb{1} = \varepsilon(0)$ that is $\mathbb{1}(\sigma) = \delta_\emptyset(\sigma)$. Note that, for all $u \in L^2(\mathbb{R}^+)$ one has $\|\varepsilon(u)\|^2 = e^{\|u\|^2}$. For $u \in L^2(\mathbb{R}^+)$ and $t \in \mathbb{R}^+$ one writes $u_{t]} = u\mathbb{1}_{[0,t]}$ and $u_{[t} = u\mathbb{1}_{[t,+\infty[}$. The mapping

$$\begin{aligned} \Phi &\longrightarrow \Phi_{t]} \otimes \Phi_{[t} \\ \varepsilon(u) &\longmapsto \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t}) \end{aligned}$$

extends to an isomorphism between Φ and $\Phi_{t]} \otimes \Phi_{[t}$. This property is the so-called *continuous tensor product structure* of Φ.

II.2. Calculus on Φ

We are now going to define some useful operators on Φ. Let $t \in \mathbb{R}^+ \setminus \{0\}$. Let P_t be the orthogonal projection from Φ onto $\Phi_{t]}$. That is,

$$[P_t f](\sigma) = f(\sigma)\mathbb{1}_{\mathcal{P}_{t)}}(\sigma).$$

We define $[P_0 f](\sigma) = \delta_\emptyset(\sigma) f(\emptyset)$. The vector $P_0 f$ is often seen as a scalar namely, $f(\emptyset)$ instead of $f(\emptyset)\mathbb{1}$.

For a $f \in \Phi$ and $t \in \mathbb{R}^+$ define

$$[D_t f](\sigma) = f(\sigma \cup t)\mathbb{1}_{\mathcal{P}_{t)}}(\sigma)$$

for all $\sigma \in \mathcal{P}$. One can easily check the following (cf. [At2]).

Lemma 1. — *For all $f \in \Phi$ one has*

$$\int_0^\infty \int_{\mathcal{P}} |[D_t f](\sigma)|^2 d\sigma dt = \|f\|^2 - |f(\emptyset)|^2.$$

Thus, for all $f \in \emptyset$, almost all $t \in \mathbb{R}^+$, $D_t f$ is an element of Φ.

A family $(g_t)_{t \geq 0}$ of elements of Φ is said to be an *Ito integrable process* if:

i) $(t, \sigma) \mapsto g_t(\sigma)$ is measurable on $\mathbb{R}^+ \times \mathcal{P}$,

ii) $g_t \in \Phi_{t]}$ for all t,

iii) $\int_0^\infty \|g_t\|^2 dt < \infty$.

If $(g_t)_{t \geq 0}$ is an Ito integrable process define $\int_0^\infty g_s d\chi_s$ by

$$\Big[\int_0^\infty g_s d\chi_s\Big](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g_{\vee\sigma}(\sigma-) & \text{otherwise.} \end{cases}$$

See [At2] for the proof of the following.

LEMMA 2. — *For all Ito integrable process $(g_t)_{t\geq 0}$ one has*

$$\int_{\mathcal{P}} \left| \left[\int_0^\infty g_s d\chi_s \right](\sigma) \right|^2 d\sigma = \int_0^\infty \|g_s\|^2 ds < \infty.$$

Thus $\int_0^\infty g_s d\chi_s$ belongs to Φ.

From all these definitions and lemmas, one easily deduces the following

THEOREM 3 (Fock space predictable representation property). — *For all $f \in \Phi$, the process $(D_t f)_{t\geq 0}$ is Ito integrable. One has the unique representation*

$$f = P_0 f + \int_0^\infty D_s f d\chi_s \tag{2}$$

with

$$\|f\|^2 = |P_0 f|^2 + \int_0^\infty \|D_s f\|^2 ds \tag{3}$$

and

$$\langle g, f\rangle = \overline{P_0 g} P_0 f + \int_0^\infty \langle D_s g, D_s f\rangle ds \tag{4}$$

for all $g \in \Phi$.

We denote by $\int_a^b g_s d\chi_s$ the Ito integral $\int_0^\infty g_s \mathbb{1}_{[a,b]}(s) d\chi_s$.

II.3. Adaptedness

An operator H on Φ is said to be *adapted at time t* (or *t-adapted*) if H is of the form $H_t \otimes I$ in the tensor product structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$, for some operator H_t on $\Phi_{t]}$. If the operator H is defined on the coherent vector domain $\mathcal{E}$, the t-adaptedness writes as follows:

i) $H\varepsilon(u_{t]}) \in \Phi_{t]}$,

ii) $H\varepsilon(u) = [H\varepsilon(u_{t]})] \otimes \varepsilon(u_{[t})$

for all $u \in L^2(\mathbb{R}^+)$. An *adapted process of operators* on Φ is a family $(H_t)_{t\geq 0}$ of operators defined on a domain $\mathcal{D}$ such that

i) $t \mapsto H_t f$ is measurable for all $f \in \mathcal{D}$,

ii) H_t is t-adapted, for all $t \in \mathbb{R}^+$.

II.4. Quantum stochastic integrals

Let us recall the generalized definition of quantum stochastic integrals as defined in [A-M].

Let $\mathcal{D}$ be a domain on Φ such that $f \in \mathcal{D}$ implies $P_t f \in \mathcal{D}$ and $D_t f \in \mathcal{D}$ for a.a $t \in \mathbb{R}^+$. Such a domain $\mathcal{D}$ is called *adapted domain.*

Let $(H_t^\varepsilon)_{t\geq 0}$, $\varepsilon = +, -, \circ, \times$, be four adapted processes of operators on $\mathcal{D}$ satisfying

$$\int_0^t \|H_s^\circ D_s f\|^2 ds + \int_0^t \|H_s^+ P_s f\|^2 ds + \int_0^t \|H_s^- D_s f\| ds + \int_0^t \|H_s^\times P_s f\| ds < \infty \quad (5)$$

for all $f \in \mathcal{D}$, all $t \in \mathbb{R}^+$.

An adapted process of operators $(T_t)_{t\geq 0}$ is said to be the *quantum stochastic integral* process

$$T_t = \sum_{\varepsilon=+,\circ,-,\times} \int_0^t H_s^\varepsilon \, da_s^\varepsilon, \quad t \in \mathbb{R}^+, \qquad (6)$$

on the domain $\mathcal{D}$ if

i) $\mathcal{D} \subset \mathrm{Dom}\, T_t$,

ii) $\int_0^t \|T_s D_s f\|^2 ds < \infty$ for all $f \in \mathcal{D}$, all $t \in \mathbb{R}^+$,

iii) for all $t \in \mathbb{R}^+$, all $f \in \mathcal{D}$ one has

$$T_t P_t f = \int_0^t T_s D_s f d\chi_s + \int_0^t H_s^\circ D_s f d\chi_s + \int_0^t H_s^+ P_s f d\chi_s + \int_0^t H_s^- D_s f ds + \int_0^t H_s^\times P_s f ds. \quad (7)$$

Theorem 4 ([A-M]). — *On the adapted domain $\mathcal{E}$ and under the condition (5), the equation (7) admits a unique solution $(T_t)_{t\geq 0}$ on $\mathcal{E}$ which is determined by the identity*

$$\langle \varepsilon(v), T_t \varepsilon(u) \rangle = \sum_{\varepsilon=+,\circ,-,\times} \int_0^t h^\varepsilon(s) < \varepsilon(v), H_s^\varepsilon \varepsilon(u) > ds$$

for all $u, v \in L^2(\mathbb{R}^+)$, all $t \in \mathbb{R}^+$ and where

$$h^\varepsilon(s) = \begin{cases} \overline{v}(s) u(s) & \text{if } \varepsilon = \circ \\ \overline{v}(s) & \text{if } \varepsilon = + \\ u(s) & \text{if } \varepsilon = - \\ 1 & \text{if } \varepsilon = \times. \end{cases}$$

THEOREM 5 ([A-M]) (Quantum Ito formula). — *Let* $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \; da_s^\varepsilon$ *and* $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon \; da_s^\varepsilon$, $t \in \mathbb{R}^+$, *be quantum stochastic integrals on the whole of* Φ. *Then* $(S_tT_t)_{t\geq 0}$ *is a quantum stochastic integral process on the whole of* Φ *and*

$$S_tT_t = \sum_\varepsilon \int_0^t S_s H_s^\varepsilon \; da_s^\varepsilon + \sum_\varepsilon \int_0^t K_s^\varepsilon T_s \; da_s^\varepsilon + \int_0^t K_s^\circ H_s^\circ da_s^\circ + \int_0^t K_s^- H_s^\circ da_s^- + \int_0^t K_s^\circ H_s^+ da_s^+ + \int_0^t K_s^- H_s^+ da_s^\times .$$

III. THE ALGEBRAS OF QUANTUM SEMIMARTINGALES

III.1. The algebra $\mathcal{S}$

Let $\mathcal{S}$ be the space of quantum stochastic integral processes $(T_t)_{t\geq 0}$ made of bounded operators and such that $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ on $\mathcal{E}$, with all the operators H_s^ε being bounded and $t \mapsto \|H_t^\varepsilon\| \in L^{f(\varepsilon)}_{\text{loc}}(\mathbb{R}^+)$ with $f(0) = +\infty$, $f(+) = f(-) = 2$, $f(\times) = 1$.

THEOREM 6 ([At1]). — *The stochastic integral representation of each element* $(T_t)_{t\geq 0}$ *of* $\mathcal{S}$ *can be extended to the whole of* Φ. *The mapping* $t \mapsto \|T_t\|$ *belongs to* $L^\infty_{\text{loc}}(\mathbb{R}^+)$. *The space* $\mathcal{S}$ *is a* $*$*-algebra for the adjoint mapping and the composition of operators.*

The algebra $\mathcal{S}$ admits another characterization which expresses only in terms of $(T_t)_{t\geq 0}$ (and not of its coefficients H_s^ε). This characterization can be found in [At1], but will not be used here.

The interesting point with $\mathcal{S}$ is that Theorem 6 allows to perform a polynomial functional calculus on $\mathcal{S}$ and to derive associated quantum Ito formulae ([At1]). This functional calculus can even be extended to analytical functions (and to C^{2+} functions in the case of self-adjoint elements of $\mathcal{S}$), cf. [ViS]. These results show that $\mathcal{S}$ really behaves like a space of quantum semimartingales.

III.2. Quantum brackets, the algebra $\mathcal{S}'$

For $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon da_s^\varepsilon$, elements of $\mathcal{S}$ define

$$\int_0^t S_s dT_s = \sum_\varepsilon \int_0^t S_s H_s^\varepsilon da_s^\varepsilon$$

$$\int_0^t dS_s T_s = \sum_\varepsilon \int_0^t K_s^\varepsilon T_s da_s^\varepsilon$$

$$[S,T]_t = \int_0^t K_s^\circ H_s^\circ da_s^\circ + \int_0^t K_s^- H_s^\circ da_s^- + \int_0^t K_s^\circ H_s^+ da_s^+ + \int_0^t K_s^- H_s^+ ds$$

$$\langle S,T\rangle_t = \int_0^t K_s^- H_s^+ ds.$$

The last two expressions are respectively called *quantum square bracket* and *quantum angle bracket* of S and T.

The quantum Ito formula on $\mathcal{S}$ then writes

$$S_t T_t = \int_0^t S_s dT_s + \int_0^t dS_s T_s + [S,T]_t.$$

In general, none of the processes $(\int_0^t S_s dT_s)_{t\geq 0}$, $(\int_0^t dS_s T_s)_{t\geq 0}$ and $([S,T]_t)_{t\geq 0}$ is in $\mathcal{S}$; only their sum is. These three process actually belong to a larger space. Let $\mathcal{S}'$ be the space of adapted processes $(T_t)_{t\geq 0}$ such that $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ on $\mathcal{E}$, with all the operators H_s^ε being bounded and $t \mapsto \|H_t^\varepsilon\| \in L^{f(\varepsilon)}_{\mathrm{loc}}(\mathbb{R}^+)$ with $f(0) = +\infty$, $f(+) = f(-) = 2$, $f(\times) = 1$.

The definition of $\mathcal{S}'$ is exactly the same as the one of $\mathcal{S}$ excepted that one does not ask the operators T_t to be bounded.

Clearly $\mathcal{S}'$ contains $\mathcal{S}$. The inclusion is strict, for $(a_t^\circ)_{t\geq 0}$, $(a_t^-)_{t\geq 0}$, $(a_t^+)_{t\geq 0}$ belong to $\mathcal{S}'$ and not to $\mathcal{S}$.

THEOREM 7 ([At1]). — *The mapping $(S,T) \mapsto [S,T]$ is well-defined from $\mathcal{S}' \times \mathcal{S}'$ to $\mathcal{S}'$. The space $\mathcal{S}'$ is a $*$-algebra for the adjoint mapping and the product $(S,T) \mapsto [S,T]$.*

The mapping $(S,T) \mapsto \langle S,T\rangle$ is well-defined from $\mathcal{S}' \times \mathcal{S}'$ to $\mathcal{S}$.

The mappings $(S,T) \mapsto \int_0^\cdot dS_s T_s$ and $(S,T) \mapsto \int_0^\cdot T_s dS_s$ are well-defined from $\mathcal{S}' \times \mathcal{S}$ to $\mathcal{S}'$.

One has to keep in mind the different possible roles of $\mathcal{S}'$: it is a space which extends $\mathcal{S}$, but it is also an algebra for a product which is different from the one of $\mathcal{S}$. As spaces we have $\mathcal{S} \subset \mathcal{S}'$, as algebras they have nothing to do together.

We have set all the preliminaries about these algebras, we can start the study of their algebraic and analytic structures.

IV. A REMARKABLE TRANSFORM OF QUANTUM PROCESSES

IV.1. Definition, characterization

Let $(T_t)_{t\geq 0}$ be an adapted process of operators defined on $\mathcal{E}$. Assume that for all $f \in \mathcal{E}$, all $t \in \mathbb{R}^+$ we have

$$\int_0^t \|T_s D_s f\|^2 ds < \infty. \tag{8}$$

Define $\big(\mathcal{D}_t(T.)\big)_{t\geq 0}$ to be the adapted process of operators defined by the following:

$$\mathcal{D}_t(T.)P_t f = T_t P_t f - \int_0^t T_s D_s f d\chi_s \tag{9}$$

for $f \in \mathcal{E}$, and $\mathcal{D}_t(T.)$ is extended t-adaptedly on $\mathcal{E}$.

Lemma 8. — *If $(T_t)_{t\geq 0}$ satisfies (8) on $\mathcal{E}$, then so does $\big(\mathcal{D}_t(T.)\big)_{t\geq 0}$.*

Proof. — We have, for all $f \in \mathcal{E}$, say $f = \varepsilon(v)$,

$$\begin{aligned}
\int_0^t \|\mathcal{D}_s(T.)D_s f\|^2 ds &= \int_0^t \|T_s D_s f - \int_0^s T_u D_u D_s f d\chi_u\|^2 ds \\
&\leq 2\int_0^t \|T_s D_s f\|^2 ds + 2\int_0^t \int_0^s \|T_u D_u D_s f\|^2 du ds \\
&\leq 2\int_0^t \|T_s D_s f\|^2 ds + 2\int_0^t \int_0^s |v(s)|^2 |v(u)|^2 \|T_u \varepsilon(v_{u]})\|^2 du ds \\
&\leq 2\int_0^t \|T_s D_s f\|^2 ds + 2\Big(\int_0^t (v(s))^2 ds\Big)\Big(\int_0^t \|T_u D_u f\|^2 du\Big) \\
&< \infty.
\end{aligned}$$

■

Proposition 9. — *The process $\big(\mathcal{D}_t(T.)\big)_{t\geq 0}$ is the unique solution $(X_t)_{t\geq 0}$ of the equation*

$$X_t = T_t - \int_0^t X_s da_s^\circ \quad \text{on } \mathcal{E}. \tag{10}$$

Proof. — If $X_t = \mathcal{D}_t(T_\cdot)$ for all $t \in \mathbb{R}^+$, then $\int_0^t X_s da_s^\circ$ is well-defined on $\mathcal{E}$, by Lemma 8. Furthermore, we have

$$\begin{aligned} X_t P_t f &= T_t P_t f - \int_0^t T_s D_s f d\chi_s \\ &= T_t P_t f - \int_0^t (X_s - T_s) D_s f d\chi_s - \int_0^t X_s D_s f d\chi_s. \end{aligned}$$

That is, the process $(Y_t)_{t\geq 0} = (X_t - T_t)_{t\geq 0}$ satisfies

$$Y_t P_t f = \int_0^t Y_s D_s f d\chi_s - \int_0^t X_s D_s f d\chi_s \quad \text{on} \quad \mathcal{E}.$$

By Theorem 4 and equation (7) this exactly means

$$Y_t = -\int_0^t X_s da_s^\circ.$$

If $(X'_t)_{t\geq 0}$ is another solution of (10) then the process $(Z_t)_{t\geq 0} = (X_t - X'_t)_{t\geq 0}$ satisfies

$$Z_t = -\int_0^t Z_s da_s^\circ \quad \text{on} \quad \mathcal{E}.$$

By (7) this means that for all $f \in \mathcal{E}$

$$\begin{aligned} Z_t P_t f &= \int_0^t Z_s D_s f d\chi_s - \int_0^t Z_s D_s f d\chi_s \\ &= 0. \end{aligned}$$

■

IV.2. The inverse transform

Let $(T_t)_{t\geq 0}$ be an adapted process of operators defined on $\mathcal{E}$. Suppose that, for all $f \in \mathcal{E}$, all $t \in \mathbb{R}^+$ we have

$$\int_0^t \|T_s D_s f\|^2 ds < \infty.$$

That is, the same condition as (8). Define $\left(\mathcal{D}_t^{-1}(T_\cdot)\right)_{t\geq 0}$ to be the following adapted process of operators on $\mathcal{E}$:

$$\mathcal{D}_t^{-1}(T_\cdot) = T_t + \int_0^t T_s da_s^\circ. \tag{11}$$

By Theorem 4, one easily proves the following.

Lemma 10. — *If $(T_t)_{t\geq 0}$ satisfies (8) on $\mathcal{E}$, then so does $\left(\mathcal{D}_t^{-1}(T_\cdot)\right)_{t\geq 0}$.*

PROPOSITION 11. — $\left(\mathcal{D}_t^{-1}(T_\cdot)\right)_{t\geq 0}$ *is the only process* $(X_t)_{t\geq 0}$ *on* $\mathcal{E}$ *such that*

$$X_t P_t f = T_t P_t f + \int_0^t X_s D_s f d\chi_s \tag{12}$$

for all $f \in \mathcal{E}$.

Proof. — We have

$$\begin{aligned}
\mathcal{D}_t^{-1}(T_\cdot) P_t f &= T_t P_t f + \int_0^t T_s da_s^\circ P_t f \\
&= T_t P_t f + \int_0^t \Big(\int_0^s T_u da_u^\circ D_s f \Big) d\chi_s + \int_0^t T_s D_s f d\chi_s \\
&= T_t P_t f + \int_0^t \big(\mathcal{D}_s^{-1}(T_\cdot) - T_s \big) D_s f d\chi_s + \int_0^t T_s D_s f d\chi_s \\
&= T_t P_t f + \int_0^t \mathcal{D}_s^{-1}(T_\cdot) D_s f d\chi_s .
\end{aligned}$$

If X' is another process satisfying (12) then $(Z_t)_{t\geq 0} = (X_t - X_t')_{t\geq 0}$ satisfies

$$Z_t P_t f = \int_0^t Z_s D_s f d\chi_s \quad \text{for all} \quad f \in \mathcal{E}.$$

Thus

$$Z_t \varepsilon(u_{t]}) = \int_0^t u(s) Z_s \varepsilon(u_{s]}) d\chi_s \quad \text{for all} \quad u \in L^2(\mathbb{R}^+)$$

and

$$\| Z_t \varepsilon(u_{t]}) \|^2 = \int_0^t |u(s)|^2 \| Z_s \varepsilon(u_{s]}) \|^2 ds.$$

So by Gronwall's lemma $Z_t \varepsilon(u_{t]}) = 0$ for all $u \in L^2(\mathbb{R}^+)$. ■

PROPOSITION 12. — *The transforms* $\mathcal{D}$ *and* $\mathcal{D}^{-1}$ *are inverse of each other. That is, if* $(T_t)_{t\geq 0}$ *is any adapted process of operators on* $\mathcal{E}$ *such that* $\int_0^t \|T_s D_s f\|^2 ds < \infty$ *for all* $f \in \mathcal{E}$*, all* $t \in \mathbb{R}^+$*, then*

$$\mathcal{D}_t\big(\mathcal{D}_\cdot^{-1}(T_\cdot)\big) = \mathcal{D}_t^{-1}\big(\mathcal{D}_\cdot(T_\cdot)\big) = T_t \quad \text{on} \quad \mathcal{E}.$$

Proof. — We have

$$\begin{aligned}
\mathcal{D}_t^{-1}\big(\mathcal{D}_\cdot(T_\cdot)\big) &= \mathcal{D}_t(T_\cdot) + \int_0^t \mathcal{D}_s(T_\cdot) da_s^\circ \\
&= T_t - \int_0^t \mathcal{D}_s(T_\cdot) da_s^\circ + \int_0^t \mathcal{D}_s(T_\cdot) da_s^\circ \\
&= T_t .
\end{aligned}$$

We also have

$$\begin{aligned}\mathcal{D}_t\big(\mathcal{D}_{\cdot}^{-1}(T_\cdot)\big)P_t f &= \mathcal{D}_t^{-1}(T_\cdot)P_t f - \int_0^t \mathcal{D}_s^{-1}(T_\cdot)D_s f\, d\chi_s \\ &= T_t P_t f + \int_0^t \mathcal{D}_s^{-1}(T_\cdot)D_s f\, d\chi_s - \int_0^t \mathcal{D}_s^{-1}(T_\cdot)D_s f\, d\chi_s \\ &= T_t P_t f.\end{aligned}$$

■

IV.3. The bijection

Note that $\mathcal{D}$ and $\mathcal{D}^{-1}$ are well-defined on $\mathcal{S}$ and $\mathcal{S}'$.

THEOREM 13. — *$\mathcal{D}^{-1}(\mathcal{S}) \subset \mathcal{S}'$ and $\mathcal{D}(\mathcal{S}') \subset \mathcal{S}$. That is, $\mathcal{D}$ and $\mathcal{D}^{-1}$ realize a bijection between $\mathcal{S}$ and $\mathcal{S}'$.*

Proof. — Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}$. Recall that in particular $t \mapsto \|T_t\|$ belongs to L^∞_{loc}. Suppose that the integral representation of $(T_t)_{t\geq 0}$ is $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$. Then

$$\begin{aligned}\mathcal{D}_t^{-1}(T_\cdot) &= T_t + \int_0^t T_s da_s^\circ \\ &= \int_0^t (H_s^\circ + T_s)da_s^\circ + \int_0^t H_s^+ da_s^+ + \int_0^t H_s^- da_s^- + \int_0^t H_s^\times da_s^\times.\end{aligned}$$

Thus $\mathcal{D}_t^{-1}(T_\cdot) = \sum_\varepsilon \int_0^t K_s^\varepsilon da_s^\varepsilon$ on $\mathcal{E}$, with $t \mapsto \|K_s^\circ\| \in L^\infty_{\text{loc}}$, $t \mapsto \|K_s^\pm\| \in L^2_{\text{loc}}$, $t \mapsto \|K_s^\times\| \in L^1_{\text{loc}}$. So $\big(\mathcal{D}_t^{-1}(T_\cdot)\big)_{t\geq 0}$ belongs to $\mathcal{S}'$.

Conversely, let $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$, $t \in \mathbb{R}^+$, be an element of $\mathcal{S}'$. By equation (7) we have, for all $f \in \mathcal{E}$

$$\begin{aligned}\mathcal{D}_t(T_\cdot)P_t f &= T_t P_t f - \int_0^t T_s D_s f\, d\chi_s \\ &= \int_0^t H_s^\circ D_s f\, d\chi_s + \int_0^t H_s^+ P_s f\, d\chi_s + \int_0^t H_s^- D_s f\, ds + \int_0^t H_s^\times P_s f\, ds.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{D}_t(T_\cdot)P_t f\| \leq \Big\|\int_0^t H_s^\circ D_s f\, d\chi_s\Big\| + \Big\|\int_0^t H_s^+ P_s f\, d\chi_s\Big\| + \Big\|\int_0^t H_s^- D_s f\, ds\Big\| \\ + \Big\|\int_0^t H_s^\times P_s f\, ds\Big\|\end{aligned}$$

$$
\begin{aligned}
&\leq \Big(\int_0^t \|H_s^\circ D_s f\|^2 ds\Big)^{1/2} + \Big(\int_0^t \|H_s^+ P_s f\|^2 ds\Big)^{1/2} \\
&\quad + \int_0^t \|H_s^- D_s f\| ds + \int_0^t \|H_s^\times P_s f\| ds \\
&\leq \sup_{s\leq t} \|H_s^\circ\| \Big(\int_0^t \|D_s f\|^2 ds\Big)^{1/2} + \Big(\int_0^t \|H_s^+\|^2 ds\Big)^{1/2} \|P_t f\| \\
&\quad + \Big(\int_0^t \|H_s^-\|^2 ds\Big)^{1/2} \Big(\int_0^t \|D_s f\|^2 ds\Big)^{1/2} + \Big(\int_0^t \|H_s^\times\| ds\Big) \|P_t f\| \\
&\leq \Big[\sup \|H_s^\circ\| + \Big(\int_0^t \|H_s^+\|^2 ds\Big)^{1/2} + \Big(\int_0^t \|H_s^-\|^2 ds\Big)^{1/2} \\
&\quad + \int_0^t \|H_s^\times\| ds\Big] \|P_t f\|. \qquad (13)
\end{aligned}
$$

Thus $\mathcal{D}_t(T.)$ is a bounded operator on $\mathcal{E} \cap \Phi_{t]}$. As $\mathcal{D}_t(T.)$ is adapted at time t, it is bounded on $\mathcal{E}$ with the same norm. It extends to a bounded operator on Φ. Furthermore, the estimate (13) proves that $t \mapsto \|\mathcal{D}_t(T.)\|$ belongs to L^∞_{loc}.

Finally, by Proposition 11 we have

$$
\begin{aligned}
\mathcal{D}_t(T.) &= \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon - \int_0^t \mathcal{D}_s(T.) da_s^\circ \\
&= \int_0^t (H_s^\circ - \mathcal{D}_s(T.)) da_s^\circ + \int_0^t H_s^+ da_s^+ + \int_0^t H_s^- da_s^- + \int_0^t H_s^\times da_s^\times.
\end{aligned}
$$

This integral representation, the boundedness of $\mathcal{D}_t(T.)$, the estimate on $\|\mathcal{D}_t(T.)\|$ altogether prove that $\big(\mathcal{D}_t(T.)\big)_{t\geq 0}$ is an element of $\mathcal{S}$.

Proposition 12 shows that $\mathcal{D}$ and $\mathcal{D}^{-1}$ thus realize a bijection between $\mathcal{S}$ and $\mathcal{S}'$. ■

At this stage it is natural to wonder whether $\mathcal{D}$ is an algebra homomorphism between $\mathcal{S}'$ and $\mathcal{S}$. The following formula proves that the answer is negative.

Theorem 14. — *If $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon da_s^\varepsilon$ are elements of $\mathcal{S}'$ then*

$$
\begin{aligned}
\mathcal{D}_t^{-1}\big(\mathcal{D}.(T.)\mathcal{D}.(S)\big) = [T,S]_t &+ \int_0^t H_s^+ \mathcal{D}_s(S.) da_s^+ + \int_0^t H_s^\times \mathcal{D}_s(S.) da_s^\times \\
&+ \int_0^t \mathcal{D}_s(T.) K_s^- da_s^- + \int_0^t \mathcal{D}_s(T.) K_s^\times da_s^\times. \qquad (14)
\end{aligned}
$$

Proof. — Let $X_t = \mathcal{D}_t(T.)$ and $Y_t = \mathcal{D}_t(S.)$. We have

$$
X_t = T_t - \int_0^t X_s da_s^\circ, \quad Y_t = S_t - \int_0^t Y_s da_s^\circ.
$$

Thus

$$\begin{aligned}
X_tY_t &= \int_0^t X_s dY_s + \int_0^t dX_s Y_s + [X,Y]_t \\
&= \int_0^t X_s dS_s - \int_0^t X_s Y_s da_s^\circ + \int_0^t dT_s Y_s - \int_0^t X_s Y_s da_s^\circ \\
&\quad + [T,S]_t - \Big[\int_0^\cdot X_s da_s^\circ, S.\Big]_t + \Big[T., \int_0^\cdot Y_s da_s^\circ\Big]_t + \int_0^t X_s Y_s da_s^\circ \\
&= [T,S]_t + \int_0^t X_s dS_s + \int_0^t dT_s Y_s - \int_0^t X_s K_s^\circ da_s^\circ - \int_0^t X_s K_s^+ da_s^+ \\
&\quad - \int_0^t H_s^\circ Y_s da_s^\circ - \int_0^t H_s^- Y_s da_s^- - \int_0^t X_s Y_s da_s^\circ \\
&= [T,S]_t + \int_0^t X_s K_s^- da_s^- + \int_0^t X_s K_s^\times da_s^\times + \int_0^t H_s^+ Y_s da_s^+ \\
&\quad + \int_0^t H_s^\times Y_s da_s^\times - \int_0^t X_s Y_s da_s^\circ.
\end{aligned}$$

This means that

$$X_tY_t = \mathcal{D}_t\Big([T,S]. + \int_0^\cdot X_s K_s^- da_s^- + \int_0^\cdot X_s K_s^\times da_s^\times + \int_0^\cdot H_s^+ Y_s da_s^+ + \int_0^\cdot H_s^\times Y_s da_s^\times\Big). \blacksquare$$

V. BANACH ALGEBRA STRUCTURES

V.1. A norm on $\mathcal{S}'$

Actually it is false to claim that $\mathcal{S}$ and $\mathcal{S}'$ admit Banach algebra structures. They only admit locally convex algebra structure. The problem comes from the fact that $\mathcal{S}$ and $\mathcal{S}'$ deal with processes indexed by $\mathbb{R}^+$ and that the norm conditions defining $\mathcal{S}$ and $\mathcal{S}'$ are only local. In order to get true Banach algebras one has to restrict to compact intervals of time. This restriction is not important for our uses.

Let A be a fixed in $\mathbb{R}^+$. We denote by $\mathcal{S}_A$ and $\mathcal{S}'_A$ the algebras of quantum semimartingales obtained by restricting the elements of $\mathcal{S}$ and $\mathcal{S}'$, respectively, to the time interval $[0, A]$.

Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}'$, with representation

$$T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon.$$

One defines

$$\|T.\|_{\mathcal{S}'_A} = \sup_{s\leq A} \|H^\circ_s\| + \Big(\int_0^A \|H^+_s\|^2 ds\Big)^{1/2} + \Big(\int_0^A \|H^-_s\|^2 ds\Big)^{1/2} + \int_0^A \|H^\times_s\| ds.$$

PROPOSITION 15.

a) *The mapping* $T. \mapsto \|T.\|_{\mathcal{S}'_A}$ *defines a norm on* $\mathcal{S}'_A$.

b) *If* $(A_n)_n$ *is an increasing sequence in* $\mathbb{R}^+$ *such that* $\lim_n A_n = +\infty$, *then the family* $(\|\cdot\|_{\mathcal{S}'_{A_n}})_{n\in\mathbb{N}}$ *is a separating family of seminorms on* $\mathcal{S}'$.

Proof.

a) The mapping $T. \mapsto \|T.\|_{\mathcal{S}'_A}$ identifies $\mathcal{S}'_A$ with

$$L^\infty\big([0,A];B(\Phi)\big) \oplus L^2\big([0,A];B(\Phi)\big) \oplus L^2\big([0,A];B(\Phi)\big) \oplus L^1\big([0,A];B(\Phi)\big)$$

as a normed vector space. The only point that needs to be developed is that $\|T.\|_{\mathcal{S}'_A} = 0$ if and only if $(T_t)_{t\in[0,A]}$ is the null process.

This is a consequence of the uniqueness theorem for quantum stochastic integrals ([At3]).

b) It is clear that $\|\cdot\|_{\mathcal{S}'_A}$ is a seminorm on $\mathcal{S}'$ and that $\|T\|_{\mathcal{S}'_{A_n}} = 0$ if and only if $T_t \equiv 0$ for all $t \in [0, A_n]$. It is thus clear that if $\lim_n A_n = +\infty$ then the family $(\|\cdot\|_{\mathcal{S}'_{A_n}})_{n\in\mathbb{N}}$ will be separating for $\mathcal{S}'$.

THEOREM 16.

a) *Equipped with the norm* $\|\cdot\|_{\mathcal{S}'_A}$, *the space* $\mathcal{S}'_A$ *is a Banach algebra.*

b) *Equipped with the family* $(\|\cdot\|_{\mathcal{S}'_n})_{n\in\mathbb{N}}$ *of seminorms, the space* $\mathcal{S}'$ *is a locally convex closed algebra.*

Proof. — Let us first check that $\|\cdot\|_{\mathcal{S}'_A}$ is a $*$-algebra norm for $\mathcal{S}'_A$. If $T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s da^\varepsilon_s$, $t \in [0, A]$, is an element of $\mathcal{S}'_A$, it is then clear that $\|T.^*\|_{\mathcal{S}'_A} = \|T.\|_{\mathcal{S}'_A}$. If $S_t = \sum_\varepsilon \int_0^t K^\varepsilon_s da^\varepsilon_s$, $t \in [0, A]$, is another element of $\mathcal{S}'_A$ then

$$[S,T]_t = \int_0^t K^\circ_s H^\circ_s da^\circ_s + \int_0^t K^-_s H^\circ_s da^-_s + \int_0^t K^\circ_s H^+_s da^+_s + \int_0^t K^-_s H^+_s da^\times_s$$

and

$$
\begin{aligned}
\|[S,T].\|_{\mathcal{S}'_A} &= \sup_{s\geq A}\|K_s^\circ H_s^\circ\| + \Big[\int_0^A \|K_s^- H_s^\circ\|^2 ds\Big]^{1/2} + \Big[\int_0^A \|K_s^\circ H_s^+\|^2 ds\Big]^{1/2} \\
&\qquad\qquad + \int_0^t \|K_s^- H_s^+\| ds \\
&\leq \sup_{s\leq A}\|K_s^\circ\| \sup_{s\leq A}\|H_s^\circ\| + \sup_{s\leq A}\|H_s^\circ\| \Big[\int_0^A \|K_s^-\|^2 ds\Big]^{1/2} \\
&\quad + \sup_{s\geq A}\|K_s^\circ\| \Big[\int_0^A \|H_s^+\|^2 ds\Big]^{1/2} + \Big[\int_0^t \|K_s^-\| ds\Big]^{1/2} \Big[\int_0^t \|H_s^+\|^2 ds\Big]^{1/2} \\
&\leq \|S.\|_{\mathcal{S}'_A} \|T.\|_{\mathcal{S}'_A}.
\end{aligned}
$$

Thus $\|\cdot\|_{\mathcal{S}'_A}$ is a $*$-algebra norm on $\mathcal{S}'_A$.

We have already noticed that, as a normed vector space, $\mathcal{S}'_A$ identifies with

$$
L^\infty\big([0,A];B(\Phi)\big) \oplus L^2\big([0,A];B(\Phi)\big) \oplus L^2\big([0,A];B(\Phi)\big) \oplus L^1\big([0,A];B(\Phi)\big).
$$

Thus if $T_t^n = \sum_\varepsilon \int_0^t H_s^{n,\varepsilon} da_s^\varepsilon$, $t \in [0,A]$, is a Cauchy sequence in $\mathcal{S}'_A$ then the coefficient $H_s^{n,\varepsilon}$ will converge to a s-adapted operator $H_s^\varepsilon \in L^{f(\varepsilon)}([0,A];B(\Phi))$, with $f(0) = +\infty$, $f(+) = f(-) = 2$, $f(\times) = 1$. The process $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ is thus an element of $\mathcal{S}'_A$, limit of $(T_.^n)_{n\in\mathbb{N}}$ in $\mathcal{S}'_A$. Thus $\mathcal{S}'_A$ is a Banach algebra.

Let us prove that $\mathcal{S}'$ is closed for the family of seminorms $\|\cdot\|_{\mathcal{S}'_n}$. If $(T_.^n)_n$ in a sequence in $\mathcal{S}'$ which is Cauchy in each $\mathcal{S}'_m$, $m \in \mathbb{N}$, then $(T_.^n)_{n\in\mathbb{N}}$ admits a limit $(T_{m,t})_{t\in[0,m]}$ in each $\mathcal{S}'_m$. But as $\mathcal{S}'_m \subset \mathcal{S}'_{m'}$ (as Banach algebras) for $m \leq m'$, we clearly have $T_{m,t} = T_{m',t}$ for $m < m'$ and $t \in [0,m]$. Thus there exists a process $(T_t)_{t\geq 0}$ such that $T_t = T_{m,t}$ for all $t \in [0,m]$, all $m \in \mathbb{N}$. Furthermore, if $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$, $t \in \mathbb{R}^+$, we know that

$$
\sup_{s\leq m}\|H_s^\circ\| + \Big[\int_0^m \|H_s^+\|^2 ds\Big]^{1/2} + \Big[\int_0^m \|H_s^-\|^2 ds\Big]^{1/2} + \int_0^m \|H_s^\times\| ds < \infty
$$

for all $m \in \mathbb{N}$. Thus $(T_t)_{t\geq 0} \in \mathcal{S}'$. ■

Note the following easy result.

Proposition 17. — *Equipped with $\|\cdot\|_{\mathcal{S}'_A}$, the space $\mathcal{S}'_A$ is a Banach algebra for the angle bracket product $(S.,T.) \mapsto \langle S.,T.\rangle$.*

V.2. Norms on $\mathcal{S}$

Now we aim to equip the algebra $\mathcal{S}$ with a norm that will give it a Banach algebra structure, in the same way as in $\mathcal{S}'$. We have seen that $\mathcal{S}'$ admits such a norm, namely $\|\cdot\|_{\mathcal{S}'}$; we also know that $\mathcal{S}$ is a subspace (not a subalgebra !) of $\mathcal{S}'$. The first natural question one can ask is: what happens to $\|\cdot\|_{\mathcal{S}'}$ when one restricts it to $\mathcal{S}$?

Let us see, with a counter-example that $\|\cdot\|_{\mathcal{S}'}$ is not an algebra norm for $\mathcal{S}$. (The author thanks A. Coquio from Institut Fourier, Grenoble, for finding this nice counter-example).

For $n \in \mathbb{N}$, let Π^n be the orthogonal projection onto the n first chaoses of Φ that is, on $\bigoplus_{k=0}^{n} L^2(\mathcal{P}_k)$.

Let $(\Pi^n_t)_{t\geq 0}$ be the operator martingale associated to Π^n that is,

$$\Pi^n_t \varepsilon(u) = \big[P_t \Pi^n \varepsilon(u_{t]})\big] \otimes \varepsilon(u_{[t}).$$

One can easily check that for $f \in \Phi_{t]}$ and $\sigma \in \mathcal{P}_{t]}$ we have

$$[\Pi^n_t f](\sigma) = \mathbb{1}_{\#\sigma\leq n}\ f(\sigma).$$

As each Π^n_t is a norm 1 operator, the quantum stochastic integral

$$T_t = \int_0^t \Pi^n_s \, da^\circ_s$$

is well defined, at least on $\mathcal{E}$.

One can check (cf. [A-L]) that, for $f \in \mathcal{E}$, $T_t f$ is given by the following formula

$$[T_t f](\sigma) = \sum_{\substack{s\in\sigma \\ s\leq t}} \big[\Pi^n_s D_s D_{\sigma_{(s}} f\big](\sigma_{s)})$$

where $D_\omega = D_{t_1}\dots D_{t_n}$ if $\omega = \{t_1 < \dots < t_n\}$.

Thus

$$\begin{aligned}[T_t f](\sigma) &= \sum_{\substack{s\in\sigma \\ s\leq t}} \mathbb{1}_{\#\sigma_{s)}\leq n} \big[D_s D_{\sigma_{(s}} f\big](\sigma_{s)}) \\ &= \sum_{\substack{s\in\sigma \\ s\leq t}} \mathbb{1}_{\#\sigma_{s)}\leq n}\ f(\sigma) \\ &= (n+1)\wedge(\#\sigma_{t)})\ f(\sigma).\end{aligned}$$

Thus $\int_{\mathcal{P}} |[T_t f](\sigma)|^2 d\sigma \leq (n+1)^2 \int_{\mathcal{P}} |f(\sigma)|^2 d\sigma$.

That is, T_t is a bounded operator (with norm $n+1$). Clearly $(T_t)_{t\geq 0}$ is an element of $\mathcal{S}$. We have $\|T_\cdot\|_{\mathcal{S}'_A} = \sup_{s\leq A} \|\Pi^n_s\| = 1$, for all A.

We have

$$T_t^2 = \int_0^t (T_s \Pi_s^n + \Pi_s^n T_s + \Pi_s^n) da_s^\circ$$

and $\|T_\cdot^2\|_{\mathcal{S}'_A} = \sup_{s\leq A} \|T_s\Pi_s^n + \Pi_s^n T_s + \Pi_s^n\|$.

Let us compute $\|T_t\Pi_t^n + \Pi_t^n T_t + \Pi_t^n\|$. We have, for $f \in \Phi_{t]}, \sigma \in \mathcal{P}_{t)}$

$$\begin{aligned}\big[(T_t\Pi_t^n + \Pi_t^n T_t + \Pi_t^n)f\big](\sigma) &= \mathbb{1}_{\#\sigma\leq n}[T_t f](\sigma) + [\Pi_t^n T_t f](\sigma) + \mathbb{1}_{\#\sigma\leq n}\, f(\sigma)\\ &= \mathbb{1}_{\#\sigma\leq n}\#\sigma f(\sigma) + \mathbb{1}_{\#\sigma\leq n}\,\#\sigma f(\sigma) + \mathbb{1}_{\#\sigma\leq n} f(\sigma)\\ &= (2\#\sigma+1)\mathbb{1}_{\#\sigma\leq n}\, f(\sigma).\end{aligned}$$

Thus $\|T_t\Pi_t^n + \Pi_t^n T_t + \Pi_t^n\| \leq 2n+1$. It is even equal to $2n+1$ by taking $f \in L^2(\mathcal{P}_n)$.

This finally gives $\|T_\cdot^2\|_{\mathcal{S}'_A} = 2n+1$ and clearly $\|T_\cdot^2\|_{\mathcal{S}'_A} > \|T\|^2_{\mathcal{S}'_A}$. This shows that $\|\cdot\|_{\mathcal{S}'}$ is not an algebra norm for $\mathcal{S}$.

Anyway it is possible to slightly modify $\|\cdot\|_{\mathcal{S}'}$ in order to produce an algebra norm for $\mathcal{S}$. This is performed through the transform $\mathcal{D}$.

Let $A \in \mathbb{R}^+$ be fixed. Let $\mathcal{S}_A$ be the restriction of $\mathcal{S}$ to processes indexed by $[0, A]$. On $\mathcal{S}_A$ define the following norm:

$$\|T_\cdot\|_{\mathcal{D}^{-1}} = \|\mathcal{D}^{-1}(T_\cdot)\|_{\mathcal{S}'_A}.$$

Proposition 18. — *$\|\cdot\|_{\mathcal{D}^{-1}}$ is a norm on $\mathcal{S}_A$ which makes it complete.*

Proof. — The fact that $\|\cdot\|_{\mathcal{D}^{-1}}$ is a norm on $\mathcal{S}_A$ comes easily from the linearity and injectivity of $\mathcal{D}^{-1}$. Let us check the completeness property. If $(T_\cdot^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}_A, \|\cdot\|_{\mathcal{D}^{-1}})$ then $(\mathcal{D}_\cdot^{-1}(T_\cdot^n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{S}'_A, \|\cdot\|_{\mathcal{S}'_A})$, thus it converges in $\mathcal{S}'_A$ to a process $(T_t)_{t\in[0,A]}$. The process $(\mathcal{D}_t(T_\cdot))_{t\in[0,A]}$ belongs to $\mathcal{S}_A$ and

$$\|T_\cdot^n - \mathcal{D}_\cdot(T_\cdot)\|_{\mathcal{S}_A} = \|\mathcal{D}_\cdot^{-1}(T_\cdot^n) - T_\cdot\|_{\mathcal{S}'_A} \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus $(\mathcal{D}_t(T_\cdot))_{t\in[0,A]}$ is the limit in $(\mathcal{S}_A, \|\cdot\|_{\mathcal{D}^{-1}})$ of $(T_\cdot^n)_{n\in\mathbb{N}}$. ∎

Unfortunately, $\|\cdot\|_{\mathcal{D}^{-1}}$ is not an algebra norm for $\mathcal{S}_A$ (this is not surprising as $\mathcal{D}$ is not an algebra morphism). Let us see a counter-example again. Let $A \in \mathbb{R}^+$ be fixed. Let $T_t = a_t^+, t \in [0, A]$ and $S_t = a_t^-, t \in [0, A]$. Let $X_t = \mathcal{D}_t(T_\cdot)$, $Y_t = \mathcal{D}_t(S_\cdot)$. Let us compute $\|X_t\|$ and $\|Y_t\|$ first. By equation (9) we have $X_t P_t f = \int_0^t P_s f d\chi_s$ and $Y_t P_t f = \int_0^t D_s f ds$. Thus

$$\|X_t P_t f\| = \Big(\int_0^t \|P_s f\|^2 ds\Big)^{1/2} \leq \sqrt{t}\|P_t f\|$$

$$\|Y_t P_t f\| = \int_0^t \|D_s f\| ds \leq \sqrt{t}\Big(\int_0^t \|D_s f\|^2 ds\Big)^{1/2} \leq \sqrt{t}\|P_t f\|.$$

Furthermore $X_tP_t\mathbb{1} = \int_0^t \mathbb{1} d\chi_s = \chi_t$ thus $\|X_tP_t\mathbb{1}\| = \sqrt{t} = \sqrt{t}\|P_t\mathbb{1}\|$ and $Y_tP_t\chi_t = \int_0^t \mathbb{1} ds = t$ thus $\|Y_tP_t\chi_t\| = t = \sqrt{t}\|P_t\chi_t\|$. This proves that $\|X_t\| = \|Y_t\| = \sqrt{t}$. Furthermore $\|X_\cdot\|_{\mathcal{D}^{-1}} = \|T\|_{\mathcal{S}'_A} = \sqrt{A}$ and $\|Y_\cdot\|_{\mathcal{D}^{-1}} = \|S\|_{\mathcal{S}'_A} = \sqrt{A}$.

By (14) we have

$$X_tY_t = \mathcal{D}_t\big(\int_0^\cdot Y_s da_s^+ + \int_0^\cdot X_s da_s^-\big)$$

thus

$$\begin{aligned}\|X_\cdot Y_\cdot\|_{\mathcal{D}^{-1}} &= \|\int_0^\cdot Y_s da_s^+ + \int_0^\cdot X_s da_s^-\|_{\mathcal{S}'_A}\\ &= \big(\int_0^A \|Y_s\|^2 ds\big)^{1/2} + \big(\int_0^A \|X_s\|^2 ds\big)^{1/2}\\ &= 2\big(\int_0^A s ds\big)^{1/2} = \sqrt{2}A.\end{aligned}$$

We do not have $\|X_\cdot Y_\cdot\|_{\mathcal{D}^{-1}} \leq \|X_\cdot\|_{\mathcal{D}^{-1}}\|Y\|_{\mathcal{D}^{-1}}$.

We have to define another norm on $\mathcal{S}$. Let A be fixed in $\mathbb{R}^+$. Let $(T_t)_{t\in[0,A]} \in \mathcal{S}_A$. Recall that $(T_t)_{t\in[0,A]}$ also belongs to $\mathcal{S}'_A$. Define

$$\|T_\cdot\|_{\mathcal{S}_A} = sup_{s\leq A}\|T_s\| + \|T_\cdot\|_{\mathcal{S}'_A}.$$

THEOREM 19. — *$\|\cdot\|_{\mathcal{S}_A}$ is a $*$-algebra norm on $\mathcal{S}_A$. It is equivalent to the norm $\|\cdot\|_{\mathcal{D}^{-1}}$ with:*

$$\|T_\cdot\|_{\mathcal{D}^{-1}} \leq \|T_\cdot\|_{\mathcal{S}_A} \leq 3\|T_\cdot\|_{\mathcal{D}^{-1}}, \tag{15}$$

the constants being optimal.

Thus $(\mathcal{S}_A, \|\cdot\|_{\mathcal{S}_A})$ is a Banach algebra.

Proof. — $\|\cdot\|_{\mathcal{S}_A}$ is clearly a norm on $\mathcal{S}_A$. Let us look at its behaviour with respect to the product in $\mathcal{S}_A$. Let $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon da_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon da_s^\varepsilon$ be elements of $\mathcal{S}_A$. Then

$$S_tT_t = \sum_\varepsilon \int_0^t S_sH_s^\varepsilon da_s^\varepsilon + \sum_\varepsilon \int_0^t K_s^\varepsilon T_s da_s^\varepsilon + [S_\cdot, T_\cdot]_t;$$

$$
\begin{aligned}
\|S.T.\|_{\mathcal{S}_A} = & \sup_{s\leq A}\|S_sT_s\| + \sup_{s\leq A}\|S_sH_s^\circ + K_s^\circ T_s + H_s^\circ K_s^\circ\| \\
& + \Big(\int_0^A \|S_sH_s^+ + K_s^+T_s + K_s^\circ H_s^+\|^2 ds\Big)^{1/2} \\
& + \Big(\int_0^A \|S_sH_s^- + K_s^-T_s + K_s^- H_s^\circ\|^2 ds\Big)^{1/2} \\
& + \int_0^A \|S_sH_s^\times + K_s^\times T_s + K_s^- H_s^+\| ds \\
\leq & \sup_{s\leq A}\|S_s\|\Big[\sup_{s\leq A}\|T_s\| + \sup_{s\leq A}\|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 ds\Big)^{1/2} \\
& + \Big(\int_0^A \|H_s^-\|^2 ds\Big)^{1/2} + \int_0^A \|H_s^\times\| ds\Big] \\
& + \sup_{s\leq A}\|K_s^\circ\|\Big[\sup_{s\leq A}\|T_s\| + \sup_{s\leq A}\|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 ds\Big)^{1/2}\Big] \\
& + \Big(\int_0^A \|K_s^+\|^2 ds\Big)^{1/2}\Big(\sup_{s\leq A}\|T_s\|\Big) \\
& + \Big(\int_0^A \|K_s^-\|^2 ds\Big)^{1/2}\Big[\sup_{s\leq A}\|T_s\| + \sup_{s\leq A}\|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 ds\Big)^{1/2}\Big] \\
& + \Big(\int_0^A \|K_s^x\| ds\Big)\Big(\sup_{s\leq A}\|T_s\|\Big) \\
\leq & \|S.\|_{\mathcal{S}_A}\|T.\|_{\mathcal{S}_A}.
\end{aligned}
$$

Thus $\|\cdot\|_{\mathcal{S}_A}$ is an algebra norm for $\mathcal{S}_A$. It is clearly a $*$-algebra norm.

Furthermore, the estimate (13) proves that for $T. \in \mathcal{S}'$

$$
\sup_{s\leq A}\|\mathcal{D}_s(T.)\| \leq \|T.\|_{\mathcal{S}'_A}. \tag{16}
$$

Let $T. \in \mathcal{S}_A$. We have

$$
\begin{aligned}
\|T.\|_{\mathcal{D}^{-1}} = \|\mathcal{D}.^{-1}(T.)\|_{\mathcal{S}'_A} & = \sup_{s\leq A}\|H_s^\circ + T_s\| + \Big(\int_0^A \|H_s^+\|^2 ds\Big)^{1/2} \\
& \quad + \Big(\int_0^A \|H_s^-\|^2 ds\Big)^{1/2} + \int_0^A \|H_s^\times\| ds \\
& \leq \sup_{s\leq A}\|T_s\| + \sup_{s\leq A}\|H_s^\circ\| + \Big(\int_0^A \|H_s^+\|^2 ds\Big)^{1/2} \\
& \quad + \Big(\int_0^A \|H_s^-\|^2 ds\Big)^{1/2} + \int_0^A \|H_s^\times\| ds \\
& = \|T\|_{\mathcal{S}_A}.
\end{aligned}
$$

This gives the first inequality.

Now, for $T_\cdot \in \mathcal{S}'_A$ we have

$$\begin{aligned}\|\mathcal{D}_\cdot(T)\|_{\mathcal{S}_A} &= \sup_{s\leq A}\|\mathcal{D}_s(T)\| + \sup_{s\leq A}\|H^\circ_s - \mathcal{D}_s(T_\cdot)\| + \Big(\int_0^A \|H^+_s\|^2 ds\Big)^{1/2} \\ &\quad + \Big(\int_0^A \|H^-_s\|^2 ds\Big)^{1/2} + \int_0^A \|H^\times_s\| ds \\ &\leq 2\sup_{s\leq A}\|\mathcal{D}_s(T)\| + \|T_\cdot\|_{\mathcal{S}'_A} \\ &\leq 3\|T_\cdot\|_{\mathcal{S}'}, \text{ by (17).}\end{aligned}$$

Thus

$$\|T_\cdot\|_{\mathcal{S}_A} = \|\mathcal{D}_\cdot \circ \mathcal{D}_\cdot^{-1}(T_\cdot)\|_{\mathcal{S}_A} \leq 3\|\mathcal{D}_\cdot^{-1}(T_\cdot)\|_{\mathcal{S}'_A} = 3\|T_\cdot\|_{\mathcal{D}^{-1}}.$$

This proves the second inequality and thus the equivalence of the norms. As a consequence, $\mathcal{S}_A$ is complete for $\|\cdot\|_{\mathcal{S}_A}$ (Proposition 18).

Let us now check that the constants in the inequality (15) are optimal.

Let $T_t = a^\times_t, t \in \mathbb{R}^+$; it is an element of $\mathcal{S}$. we have $T_t f = tf$ and thus $\|T_t\| = t$, $\|T_\cdot\|_{\mathcal{S}_A} = 2A$. Furthermore $\mathcal{D}_t^{-1}(T_\cdot) = tI + \int_0^t s\, da^\circ_s$ and $\|\mathcal{D}_\cdot^{-1}(T_\cdot)\|_{\mathcal{S}'_A} = 2A$.

This proves that the inequality $\|T_\cdot\|_{\mathcal{D}^{-1}} \leq \|T\|_{\mathcal{S}_A}$ is optimal.

Let $T_t = a^+_t, t \in [0, A]$. Then $\|T_\cdot\|_{\mathcal{S}'_A} = \sqrt{A}$. Let $X_t = \mathcal{D}_t(T_\cdot), t \in [0, A]$ that is $X_t P_t f = \int_0^t P_s f d\chi_s$. We know that $\|X_t\| = \sqrt{t}$. Thus

$$\|X_\cdot\|_{\mathcal{S}'_A} = \sup_{s\leq A}\|-X_s\| + \Big(\int_0^A \|I\|^2 ds\Big)^{1/2} = 2\sqrt{A}$$

and finally $\|X_\cdot\|_{\mathcal{S}_A} = 3\sqrt{A}$, whereas $\|X_\cdot\|_{\mathcal{D}^{-1}} = \|\mathcal{D}_\cdot^{-1}(X)\|_{\mathcal{S}'_A} = \|T_\cdot\|_{\mathcal{S}'_A} = \sqrt{A}$. This proves that the inequality $\|T_\cdot\|_{\mathcal{S}_A} \leq 3\|T_\cdot\|_{\mathcal{D}^{-1}}$ is optimal. ∎

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