QUANTUM STOCHASTIC CALCULUS
– A MAXIMAL FORMULATION

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Abstract

We describe a new formulation of quantum stochastic calculus which extends all the previous definitions, while preserving their most satisfactory components, and which identifies the maximal domain of quantum stochastic integrals. This formulation is based on a complete redefinition of the notion of operator adaptedness. We define adaptedness solely in terms of the orthogonal projections of the time filtration of Fock space, and sections of an adapted gradient operator. This way, we get freed from formulating adaptedness in terms of the coherent vectors and we can consider all sorts of domains in the Fock space. This leads to natural definitions of quantum stochastic integrals with minimal domain constraints. Coherent meaning is thereby given to operator products of quantum stochastic integrals, and their representation as sums of such integrals through quantum Itô formulae. We show our definition to coincide with Hudson-Parthasarathy's one on coherent vectors, with Belavkin-Lindsay's one on the domain of the Malliavin gradient. The most satisfactory application appears when one tries to extend Attal-Meyer's formulation. We, indeed, show that our formulation completely solves their equations, proves uniqueness of the solution in any case and gives rise to the solution with maximal domain.

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I. Introduction

In the original formulation of quantum stochastic calculus ([HP1]), the homogeneity of exponential vectors with respect to the continuous tensor product structure of Fock space is a cornerstone. Thus, in the natural isomorphism $\mathcal{F} \cong \mathcal{F}_t \otimes \mathcal{F}^t$, where $\mathcal{F}$, $\mathcal{F}_t$ and $\mathcal{F}^t$ are respectively (symmetric) Fock space over $L^2(\mathbb{R}_+)$, $L^2([0,t])$ and $L^2([t,\infty[)$, the coherent vector $\varepsilon(\varphi)$ corresponds to

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$\varepsilon(\varphi_{t]}) \otimes \varepsilon(\varphi_{[t})$, where $\varphi_{t]} = \varphi 1\!\!1_{[0,t]}$ and $\varphi_{[t} = \varphi 1\!\!1_{[t,\infty[}$. Indeed, continuous linear extension of this correspondence provides a simple way to *define* the isomorphism. An operator H on $\mathcal{F}$ is t–adapted if its domain is a linear span of coherent vectors and $H \subset H_t \otimes I^t$ for some operator H_t on $\mathcal{F}_t$. Quantum stochastic integrals are constructed from families of such operators, and are therefore necessarily defined on coherent domains too. Operator multiplication of quantum stochastic integrals is thus inadmissible, strictly speaking, within this *coherent vector formulation*, since quantum stochastic integrals typically do not leave coherent domains invariant. Only inner products of quantum stochastic integrals acting on coherent vectors may be formed. Perhaps surprisingly, this limitation has not been felt until recently – a rich stock of quantum stochastic processes has been constructed through an effective theory of quantum stochastic differential equations. This limitation does make itself felt when one is interested in algebraic questions – for example the structure of the collection of bounded–operator–valued quantum semimartingales ([Att]).

One way in which quantum stochastic calculus has been extended beyond the coherent vectors is by means of the Hitsuda–Skorohod integral of anticipative processes ([Sko]), and the related gradient operator of Malliavin calculus ([G-T], [N-Z]). In this *noncausal formulation* ([Bel], [Lin]), the action of each of the quantum stochastic integrals is defined explicitly on vectors in Fock space, and the essential quantum Itô formula (in inner product form) is seen in terms of the *Skorohod isometry*. Neither coherent vector domains, nor adaptedness of the operator integrals is required. Set against these advantages, the domains of both the annihilation and number integrals in the noncausal formulation are still restricted to parts of $\mathrm{Dom}\,\sqrt{N}$ (where N is the number operator) – even when the resulting operator is bounded. This unnatural domain limitation again precludes operator composition of quantum stochastic integrals.

A second way in which the coherent vector formulation of quantum stochastic calculus has been extended is by means of an Ito calculus on Fock space ([A-M]). Specifically, this formulation allows all vectors of the Fock space to admit a predictable representation:

$$f = I\!\!E[f] + \int_0^\infty \xi_s \, d\chi_s,$$

which is a generalization of the predictable representation property for the probabilistic interpretations of Fock space (Brownian motion, Poisson process, Azéma martingales,...). Once this is done, one obtains a formula, for the action of quantum stochastic integrals, which makes sense for all vectors in $\mathcal{F}$. For example, if $X_t = \int_0^t H_s \, dA_s^\dagger$ and $f_t = \int_0^t \xi_s \, d\chi_s$, then $X_t f_t = \int_0^t (X_s \xi_s + H_s f_s) \, d\chi_s$. This leads to a definition of quantum stochastic integrals which agrees with the coherent vector formulation when restricted to coherent vector domains. In this *Itô calculus formulation*, operator composition of quantum stochastic integrals is admitted. Under some conditions, the domain of quantum stochastic integrals may be the whole of $\mathcal{F}$. This fact plays an important role in the theory of quantum semimartingale algebras, quantum square and angle brackets ([Att]). The main disadvantage of this formulation is that the defining equations for the quantum stochastic integrals only give them implicitly. In fact the definitions amount to a kind of classical stochastic differential equation and, in full generality, the existence and uniqueness of solutions for these equations has not been known. Moreover, the maximal domains of these quantum stochastic integrals have been far from clear.

The purpose of this work is to unify and extend all these formulations. We wish to understand the relationship between the noncausal and the Itô calculus formulations. We seek definitions which preserve the advantages of each formulation whilst removing its disadvantages. In other words, definitions which (a) give the action of quantum stochastic integrals explicitly in terms of the process being integrated, (b) contain no unnatural domain limitations, (c) settle the existence and uniqueness question for the stochastic differential equations arising in the Itô calculus formulation, and (d) permit operator composition of quantum stochastic integrals. In this article we develop a formulation of quantum stochastic calculus which satisfies each of these criteria.

The main idea in this work is to base the calculus on a finely–tuned formulation of operator adaptedness, which exploits an *adapted gradient* operator inspired from classical stochastic calculus. Immediate advantages include a much clearer picture of the relationship between quantum and classical Itô calculus, and an explicitness of the criterion for adaptedness which imposes no unnecessary domain constraints. For example, the domain of a t–adapted operator H should be all of $\mathcal{F}$ when H is bounded, and should not be limited to an algebraic tensor product $V_t \otimes V^t$ when H is unbounded. The adapted gradient D is an operator from $\mathcal{F}$ to $L^2(I\!R_+;\mathcal{F})$ which is used in concert with the *adapted projection* on $L^2(I\!R_+;\mathcal{F})$. The adapted projection is defined in terms of the orthogonal projections P_t of $\mathcal{F}$ onto $\mathcal{F}_t$, by $(Px)_t = P_t x_t$. The adapted gradient may be defined in terms of the gradient operator of Malliavin calculus ∇, by $Df = P\nabla f$, except that whereas ∇ is an unbounded operator, D is bounded – so that Df is defined for each $f \in \mathcal{F}$. If H is an operator on $\mathcal{F}$ which is s–adapted in the coherent vector formulation, then it is easily verified that H satisfies $P_s H = HP_s$ and $D_t H = HD_t$ for (a.a.) $t > s$, on its domain, where $D_t f$ denotes $(Df)_t$. We *define* an operator H on $\mathcal{F}$ to be s–adapted if each $f \in \operatorname{Dom} H$ satisfies: $P_s f \in \operatorname{Dom} H$ with $HP_s f = P_s Hf$ and $D_t f \in \operatorname{Dom} H$ with $HD_t f = D_t Hf$ for a.a. $t > s$. On coherent vector domains this coincides with the coherent vector formulation of s–adaptedness. The new definition frees us from any prescribed domains; moreover it leads to a definition of time s–conditional expectation for operators on $\mathcal{F}$ which satisfies all the algebraic properties one could hope for, given the vagaries of unbounded operators.

The above refinement of operator adaptedness is also the point of departure for new definitions of quantum stochastic integrals. In particular, the gradient operator (used in the noncausal formulation) is replaced by the adapted gradient. This overcomes the unnatural domain constraint while maintaining explicitness of action of the quantum stochastic integrals. The connection with the Itô calculus formulation is then seen through commutation relations between the Skorohod integral and the adapted gradient. We are able to show that our quantum stochastic integrals solve the stochastic differential equations arising in the Itô calculus formulation, and that as solutions they are unique with maximal domains. By uniqueness and maximality we mean that any process X which is, say, a creation integral of a measurable adapted Fock operator process H *in the Itô calculus formulation* must satisfy $X_t \subset A_t^\dagger(H)$, where $A_t^\dagger(H)$ is the creation integral of H *in our formulation.* Adapted operators and quantum stochastic integrals may be freely composed, and a quantum Itô formula results.

In order best to illuminate the structure of quantum stochastic calculus we have sought from the outset to impose only those domain constraints that appeared to be essential for a workable theory. The result is a formulation of great generality, but we hope that its essential simplicity is manifest too. Our intention is to unify and enrich the various formulations of the calculus, and to deepen its connection with classical stochastic calculus. We work exclusively with Guichardet's version of Fock space (described in the next section). For this reason the formal connection with the noncausal approach is most apparent. This is misleading, since our approach results from a true synthesis of the noncausal and Itô calculus formulations.

A brief preliminary account of this work has appeared in [A-L]. That paper contains a section demonstrating how Fermion field operators, defined as quantum stochastic integrals ([HP2]), achieve their natural domains – namely all of Fock space – when considered from the viewpoint advocated here.

II. Notations and conventions

Let Γ denote the *finite power set* of $\mathbb{R}_+$ that is the set of finite subsets of $\mathbb{R}_+$. Let $\mathcal{H}_0$ be a fixed separable complex Hilbert space. Since, for $n = 1, 2, \ldots$, $\{\underline{s} \in \mathbb{R}_+^n : s_1 < \cdots < s_n\}$ is in bijective correspondence with $\Gamma^{(n)}$, the set of n-elements subsets of $\mathbb{R}_+$, through the map $\underline{s} \mapsto \{s_1, \ldots, s_n\}$ taking a point in the Cartesian product to the collection of its coordinates, Lebesgue measure induces a measure on $\Gamma^{(n)}$. By letting $\emptyset \in \Gamma$ be an atom of measure unity, we arrive at a σ-finite measure on $\Gamma = \bigcup_{n\geq 0} \Gamma^{(n)}$ – the *symmetric measure* (or *Guichardet measure*) associated with Lebesgue measure on $\mathbb{R}_+$ ([Gui]). *Guichardet–Fock space* (or simply *Fock space*) is then the Hilbert space tensor product $\mathcal{F} = \mathcal{H}_0 \otimes L^2(\Gamma)$, which we may identify with the space of (classes of) square–integrable $\mathcal{H}_0$–valued maps on $\Gamma : L^2(\Gamma; \mathcal{H}_0)$, by continuous linear extension of the map $v \otimes k \mapsto k(\cdot)v$. Such vectors will be written simply vk. Elements of Γ will always be denoted by lower case greek letters α, β, σ, τ, ω, ..., and integration with respect to the symmetric measure on Γ will be written simply $\int_\Gamma f(\sigma)\, d\sigma$. The cardinal of an element σ of Γ is denoted by $\#\sigma$.

The following elementary identity is fundamental – a proof may be found in [L-P].

$\not\!\int$-Lemma – *Let g be a measurable nonnegative (resp. (Bochner) integrable) map* *from $\Gamma \times \Gamma$ to $\mathbb{R}$ (resp. $\mathcal{H}_0$). Let G be the function on Γ defined by*

$$G : \sigma \mapsto \sum_{\alpha \subset \sigma} g(\alpha, \sigma \backslash \alpha).$$

Then G is measurable nonnegative (resp. integrable) and

$$\int_\Gamma G(\sigma)\, d\sigma = \int_\Gamma \int_\Gamma g(\alpha, \beta)\, d\alpha\, d\beta.$$

□

For any function $\varphi : \mathbb{R} \to \mathbb{C}$, let $\varphi(N)$ be the operator on $\mathcal{F}$ given by

$$\varphi(N) f(\sigma) = \varphi(\#\sigma) f(\sigma);\ \operatorname{Dom}\varphi(N) = \Big\{ f \in \mathcal{F} : \int_\Gamma |\varphi(\#\sigma)|^2\, \|f(\sigma)\|^2\, d\sigma < \infty \Big\}.$$

In other words, $\varphi(N)$ is the operator obtained by applying the functional calculus to the *number operator* N on $\mathcal{F}$. We define the subspaces:

$$\mathcal{F}^{(a)} = \operatorname{Dom} a^N,\ \text{for } a \geq 1, \quad \mathcal{K} = \bigcap_{a \geq 1} \operatorname{Dom} a^N \qquad (2.1a)$$

$$\mathcal{F}_f = \Big\{ f \in \mathcal{F} : \operatorname{supp}(f) \subset \bigcup_{n \leq m} \Gamma^{(n)},\ \text{for some } m \Big\}. \qquad (2.1b)$$

Here is a list of set–theoretic notation and measure–theoretic convention that we shall adopt throughout. Let $s, t \in \mathbb{R}_+$ and $\omega, \sigma, \tau \in \Gamma$, then

$\omega_{t)} = \omega \cap [0, t[$, $\omega_{[t} = \omega \cap [t, \infty[$, etc...

${}_\vee\sigma = \max\{s : s \in \sigma\}$, $\sigma_- = \sigma \backslash \{{}_\vee\sigma\}$, ${}_\wedge\sigma = \min\{s : s \in \sigma\}$, for $\sigma \neq \emptyset$,

$\omega \cup s = \omega \cup \{s\}$, and for $s \in \sigma$, $\sigma \backslash s = \sigma \backslash \{s\}$,

"$\sigma < \tau$" means $s < t$ for all $s \in \sigma$, $t \in \tau$,

$\Gamma_s = \{\omega \in \Gamma : \omega \subset [0, s[\}$, $\Gamma^s = \{\omega \in \Gamma : \omega \subset [s, \infty[\}$,

"for a.a. $\tau > s$" means for almost all $\tau \in \Gamma^s$ (here s is fixed), whereas

"for a.a. $(\tau > s)$" means for almost all elements of $\{(\tau, s) \in \Gamma \times \mathbb{R}_+ : \tau > s\}$,

$\mathcal{F}_s = \mathcal{H}_0 \otimes L^2(\Gamma_s)$, $\mathcal{F}^s = L^2(\Gamma^s)$.

Fock space has a *continuous tensor product* structure, in the following sense: for each $s \geq 0$, the map

$$f \otimes g \mapsto \big(\omega \mapsto f(\omega_{s)}) g(\omega_{[s})\big)$$

extends uniquely to an isometric isomorphism from (the Hilbert space tensor product) $\mathcal{F}_s \otimes \mathcal{F}^s$ onto $\mathcal{F}$.

Associated with any function $\varphi : \mathbb{R}_+ \to \mathbb{C}$ is the corresponding *product function* $\pi(\varphi) : \Gamma \to \mathbb{C}$, given by $[\pi(\varphi)](\sigma) = \prod_{s \in \sigma} \varphi(s)$, with the usual convention that an empty product gives 1. If φ is (Lebesgue) integrable then $\pi(\varphi)$ is easily seen to be integrable, with $\int [\pi(\varphi)](\sigma)\, d\sigma = \exp\{\int \varphi(s)\, ds\}$, moreover the $\not\int$–identity is easily verified when $g(\alpha, \beta) = [\pi(\varphi)](\alpha)\, [\pi(\psi)](\beta)$ with φ and ψ integrable.

When $\varphi \in L^2(\mathbb{R}_+)$ we write $\varepsilon(\varphi)$ for the measure equivalence class of $\pi(\varphi)$, thus $\varepsilon(\varphi) \in L^2(\Gamma)$. We also write $\delta_\emptyset$ for $\varepsilon(0)$, which maps $\emptyset$ to 1, and all other σ to 0. These are called the *coherent* or *exponential vectors* of Fock space, partly due to the relation: $\langle \varepsilon(\varphi), \varepsilon(\psi) \rangle = \exp\langle \varphi, \psi \rangle$, and their normalized forms are known as coherent states in quantum physics. Exponential vectors are part of the very fabric of Fock space – indeed if $\mathcal{H}$ is a Hilbert space and $j : L^2(\mathbb{R}_+) \to \mathcal{H}$ a mapping whose image is total in $\mathcal{H}$ and which satisfies $\langle j(\varphi), j(\psi) \rangle = \exp\langle \varphi, \psi \rangle$, then $\mathcal{H}$ is isometrically isomorphic to $L^2(\Gamma)$ under an isomorphism which maps each $j(\varphi)$ to the corresponding exponential vector $\varepsilon(\varphi)$. Partly for this reason, Hudson and Parthasarathy, in their original treatment of quantum stochastic calculus, defined all operators only on (linear spans of) exponential vectors. The exponential vectors are linearly independent, and form an overcomplete family: if $\mathcal{M}$ is a dense subset of $L^2(\mathbb{R}_+)$, then $\mathcal{E}(\mathcal{M}) = \text{lin}\{\varepsilon(\varphi) : \varphi \in \mathcal{M}\}$ is dense in $L^2(\Gamma)$, if $n \geq 1$, $\varphi_1, \ldots, \varphi_n \in L^2(\mathbb{R}_+)$ are distinct, and $v_1, \ldots, v_n \in \mathcal{H}_0$, then $\sum_i v_i\, \varepsilon(\varphi_i) \neq 0$ unless $v_i = \cdots = v_n = 0$ ([HP1]). Note that, along with $\mathcal{F}_f$, the space $\mathcal{E} = \mathcal{E}(L^2(\mathbb{R}_+))$ is contained in the subspace $\mathcal{K}$. In the continuous tensor product structure of $\mathcal{F}$, the vector $v\varepsilon(\varphi) \in \mathcal{F}$ arises as $v\varepsilon(\varphi_{s]}) \otimes \varepsilon(\varphi_{[s})$, where $\varphi_{s]} = 1\!\!1_{[0,s]}\varphi$ and $\varphi_{[s} = 1\!\!1_{[s,\infty[}\varphi$.

Finally, except when explicitly stated otherwise, all tensor products will be algebraic; thus if $\mathcal{H}$ is the Hilbert space tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, and if U_i is a subspace of $\mathcal{H}_i$ $(i = 1, 2)$ then $U_1 \otimes U_2$ is the linear span of $\{u_1 \otimes u_2 : u_i \in U_i\}$ in $\mathcal{H}$, moreover if R_i is an operator on $\mathcal{H}_i$, then $R_1 \otimes R_2$ is the operator on $\mathcal{H}$ with domain $\text{Dom}\, R_1 \otimes \text{Dom}\, R_2$, and obvious action.

III. Calculus in Fock space

Our aim in this section is twofold. Firstly we construct part of Itô-calculus on Fock space, describing familiar probabilistic concepts in this unfamiliar language whilst emphasizing its universality. Secondly, we develop relationships between the components of this calculus (derivative, projection, integral). These will be applied later, once we have introduced noncommutative processes. In this section, $\mathcal{H}$ will denote an arbitrary separable Hilbert space — in practice $\mathcal{H}$ will be either $\mathbb{R}$ or $\mathbb{C}$ or our initial space $\mathcal{H}_0$, and $\mathcal{F}$ will denote $\mathcal{H} \otimes L^2(\Gamma) = L^2(\Gamma; \mathcal{H})$.

III.1. Integration

The measurable structure on $\Gamma \times \mathbb{R}_+$ is the completed product measure of the Guichardet measure on Γ and the Lebesgue measure on $\mathbb{R}_+$. We need a spectrum of integrability conditions for a Hilbert space valued map x from $\Gamma \times \mathbb{R}_+$.

Definition 3.1 – Let x be a map $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$. We denote by $x_s(\omega)$ the quantity $x(s, \omega)$.

(ai) x is *time-integrable* if

- for a.a. ω, the map $x_\cdot(\omega)$ is integrable $\mathbb{R}_+ \to \mathcal{H}$.
- The a.e. defined map $\int_0^\infty x_s\, ds : \omega \to \int_0^\infty x_s(\omega)\, ds$ is square-integrable.

(aii) x is *absolutely time-integrable* if
- x is measurable
- the map $(\omega, s) \to \|x_s(\omega)\|$ is time-integrable.

(aiii) x is *Bochner-integrable* if
- for a.a. s, the map x_s is square-integrable $\Gamma \to \mathcal{H}$
- the a.e. defined map $s \mapsto x_s$ is integrable $\mathbb{R}_+ \to \mathcal{F}$.

(bi) x is *Skorohod-integrable* if
- the map $\mathcal{S}(x) : \sigma \mapsto \sum_{s\in\sigma} x_s(\sigma \setminus s)$ is square-integrable $\Gamma \to \mathcal{H}$.

(bii) x *belongs to* $\operatorname{Dom}\mathcal{S}$ if
- x is square-integrable $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$
- x is Skorohod-integrable

(bii) x is *absolutely Skorohod-integrable* if
- x is measurable $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$
- the map $(\omega, s) \mapsto \|x_s(\omega)\|$ is Skorohod-integrable $\Gamma \times \mathbb{R}_+ \to \mathbb{R}$.

$\int_0^\infty x_s\,ds$ and $\mathcal{S}(x)$ are called the *time integral of* x, and *Skorohod integral of* x, respectively. We emphasise here that, for the definitions of both time-integrability and Skorohod-integrability, we assume *neither* the square-integrability of each x_s, *nor* the (joint) measurability of x. Note, however, that if x and x' are maps $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$ which agree a.e. then x' is time-integrable if and only if x is, in which case $\int_0^\infty x'_s\,ds = \int_0^\infty x_s\,ds$ – and similarly for the Skorohod integral. Therefore, although $\int_0^\infty x_s\,ds$ and $\mathcal{S}(x)$ are pointwise defined, we view both integrals as mappings from measurable equivalence classes (of not necessarily measurable maps) into $\mathcal{F}$. The definition of Bochner integrability is the standard one, rephrased here for easy comparison with the pointwise integrability conditions. The space $\operatorname{Dom}\mathcal{S}$ is merely the domain of the Skorohod integral viewed as an unbounded Hilbert space operator $L^2(\Gamma \times \mathbb{R}_+; \mathcal{H}_0) \to \mathcal{F}$.

Proposition 3.2 – *Let x be a map $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$.*

(ai) If x is square-integrable, then x is locally Bochner-integrable, and

$$\int_0^t \|x_s\|\,ds \le \sqrt{t}\left(\int_0^t \int_\Gamma \|x_s(\omega)\|^2\,d\omega\,ds\right)^{1/2}.$$

(aii) If x is Bochner-integrable then x is absolutely time-integrable, and

$$\left\|\int_0^\infty x_s\,ds\right\| \le \int_0^\infty ||x_s||\,ds.$$

(bi) If x is measurable then

$$\int_0^\infty \int_0^\infty \int_\Gamma \|x_s(\omega \cup t)\|\;\|x_t(\omega \cup s)\|\,d\omega\,dt\,ds \le \int_0^\infty \int_\Gamma \#\omega\,\|x_s(\omega)\|^2\,d\omega\,ds.$$

(bii) If x is square-integrable and the function $(\omega, s, t) \to \langle x_s(\omega \cup t), x_t(\omega \cup s)\rangle$ is integrable, then $x \in \operatorname{Dom}(\mathcal{S})$ and

$$\|\mathcal{S}(x)\|^2 = \int_0^\infty \|x_s\|^2\,ds + \int_0^\infty \int_0^\infty \int_\Gamma \langle x_s(\omega \cup t), x_t(\omega \cup s)\rangle\,d\omega\,dt\,ds. \qquad (3.1)$$

(biii) If x is absolutely Skorohod-integrable, then x is square-integrable and the function $(\omega, s, t) \mapsto \|x_s(\omega \cup t)\| \, \|x_t(\omega \cup s)\|$ is integrable.

Proof

(ai) is an immediate consequence of the Cauchy-Schwarz inequality. If x is Bochner-integrable then, by standard vector integration theory (e.g., [D-U]) x is jointly measurable. Moreover, by a continuous version of Minkowski's inequality ([Str]),

$$\left(\int_\Gamma \left[\int_0^\infty \|x_s(\omega)\|\right]^2 ds\, d\omega\right)^{1/2} \leq \int_0^\infty \left(\int_\Gamma \|x_s(\omega)\|^2 \, d\omega \, ds\right)^{1/2} = \int_0^\infty \|x_s\| \, ds,$$

which establishes (aii). (bi–iii) follows from straightforward applications of the $\not\int$-Lemma – see [Lin] for further details.

□

Identity (3.1) will be referred to as the *Skorohod isometry.* We shall exploit the following polarized forms of the above.

Proposition 3.3 – *Let $f \in \mathcal{F}$ and let x and y be maps $\Gamma \times I\!R_+ \to \mathcal{H}$.*

(ai) If x is absolutely time-integrable, then the function

$$(\omega, s) \mapsto \langle f(\omega), x_s(\omega)\rangle$$

is integrable, and its integral is $\langle f, \int_0^\infty x_s \, ds\rangle$.

(aii) If x is absolutely Skorohod-integrable, then the function

$$(\omega, s) \mapsto \langle f(\omega \cup s), x_s(\omega)\rangle$$

is integrable, and its integral is $\langle f, \mathcal{S}(x)\rangle$.

(bi) If x and y are absolutely time-integrable, then the function

$$(\omega, s, t) \mapsto \langle x_s(\omega), x_t(\omega)\rangle$$

is integrable, and its integral is $\langle \int_0^\infty x_s \, ds, \int_0^\infty y_s \, ds\rangle$.

(bii) If x is absolutely time-integrable and y is absolutely Skorohod-integrable, then the function

$$(\omega, s, t) \mapsto \langle x_s(\omega \cup t), y_t(\omega)\rangle$$

is integrable, and its integral is $\langle \int_0^\infty x_s \, ds, \mathcal{S}(y)\rangle$.

(bii) If x and y are absolutely Skorohod-integrable, then the functions

$$(\omega, s) \mapsto \langle x_s(\omega), y_s(\omega)\rangle \quad \text{and} \quad (\omega, s, t) \mapsto \langle x_s(\omega \cup t), y_t(\omega \cup s)\rangle$$

are integrable, and the sum of their integrals is $\langle \mathcal{S}(x), \mathcal{S}(y)\rangle$.

□

Definition 3.4 – Let x be a map $\Gamma \times I\!R_+ \to \mathcal{H}$.

(a) x is *adapted* if $x_s(\omega) = 0$ unless $\omega \in \Gamma_s$.

(b) x is *Itô-integrable* if x is adapted and the map

$$[\mathcal{I}(x)](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \\ x_{\vee\sigma}(\sigma_-) & \text{otherwise} \end{cases}$$

is square-integrable in σ.

$\mathcal{I}(x)$ is then called the *Itô-integral of* x. Like $\int_0^\infty x_s\,ds$ and $\mathcal{S}(x)$, it will be viewed as an element of $\mathcal{F}$.

As with the time and Skorohod integrals, Itô-integrability depends only on the measure equivalence class of x, and the Itô-integral lifts to a mapping from measure equivalence classes into $\mathcal{F}$. In contrast to time integrals and Skorohod integrals, Itô-integrable maps are necessarily measurable.

Definition 3.5 – A *vector process* is a family $x = (x_s)_{s\geq 0}$ in $\mathcal{F}$. It is *adapted* if $x_s \in \mathcal{F}_s$ for each s, and *measurable* if the map $s \mapsto x_s$ is (strongly) measurable.

For a measurable vector process x, there is a measurable map $\widetilde{x} : \Gamma \times I\!\!R_+ \to \mathcal{H}$ such that $\widetilde{x}_s(\cdot)$ is a version of x_s for each s. If x is adapted, then $\widetilde{x}$ may be chosen to be adapted in the sense of Definition 3.4. The measure equivalence class of $\widetilde{x}$ is unique, and we shall therefore abuse notation by using x for the process as well as the map.

Proposition 3.6 – *Let x be an adapted map $\Gamma \times I\!\!R_+ \to \mathcal{H}$. Then the following are equivalent:*

(a) x is Itô-integrable; (b) x is Skorohod-integrable; (c) x is square-integrable.

Moreover, in any of these cases we have $\mathcal{I}(x) = \mathcal{S}(x)$ and

$$\|\mathcal{I}(x)\|^2 = \int_0^\infty \|x_s\|^2\,ds. \tag{3.2}$$

Proof

Since x is adapted,

$$\mathcal{S}(x)(\sigma) = \sum_{s\in\sigma} x_s(\sigma\backslash s) = \begin{cases} 0 & \text{if } \sigma = \emptyset, \\ x_{\vee\sigma}(\sigma_-) & \text{otherwise} \end{cases}$$
$$= \mathcal{I}(x)(\sigma).$$

This gives the equivalence of (a) and (b). If x is Itô-integrable then

$$x_s(\omega) = 1\!\!1_{\Gamma_s}(\omega)\,[\mathcal{I}(x)](\omega \cup s),$$

and since the map $(\omega, s) \mapsto \omega \cup s$ is measurable, x is necessarily measurable. If x is adapted and measurable, then the $\sum\!\!\!\!\int$-Lemma gives the identities

$$\int_0^\infty \int_\Gamma \|x_s(\omega)\|^2\,ds\,d\omega = \int_\Gamma \sum_{s\in\sigma} \|x_s(\sigma\backslash s)\|^2\,d\sigma = \int_\Gamma \|x_{\vee\sigma}(\sigma_-)\|^2\,d\sigma,$$

so that (b) and (c) are equivalent, and (3.2) holds.

$\square$

We call the identity (3.2) *Itô-isometry*. Comparison with (3.1) shows that Skorohod isometry extends Itô-isometry beyond adapted maps. Another way of expressing Itô-integrability is in terms of

$$\Gamma_{\rm ad} = \{(\omega, s) \in \Gamma \times \mathbb{R}_+ : \omega < s\}. \tag{3.3}$$

The collection of equivalence classes of Itô-integrable maps may be identified with $L^2(\Gamma_{\rm ad}; \mathcal{H})$. The *adapted projection* on $L^2(\Gamma \times \mathbb{R}_+; \mathcal{H})$ is the orthogonal projection onto the closed subspace $L^2(\Gamma_{\rm ad.}; \mathcal{H})$:

$$Px = \mathbb{1}_{\Gamma_{\rm ad}} x : (\omega, s) \mapsto \mathbb{1}_{\Gamma_s}(\omega)\, x_s(\omega). \tag{3.4}$$

It is worth noticing that the Ito integral operator $\mathcal{I}(\cdot)$ actually corresponds to an integration with respect to some curve in $\mathcal{F}$. Indeed, for all $t \in \mathbb{R}_+$ define the vector $\chi_t \in \mathcal{F}$ by

$$\begin{cases} \chi_t(\sigma) = 0 & \text{if } \ \#\sigma \neq 1, \\ \chi_t(\{s\}) = \mathbb{1}_{[0,t]}(s). \end{cases}$$

It is clear that χ_t belongs to $\mathcal{F}_t$ for all $t \in \mathbb{R}_+$ but one can check that $(\chi_t)_{t\geq 0}$ is furthermore an "independent increment" vector process that is, for $s \leq t$ the increment $\chi_t - \chi_s$ belongs to $\mathcal{F}_{[s,t]} = \{f \in \mathcal{F} : f(\sigma) = 0 \text{ unless } \sigma \subset [s,t]\}$.

Let $(x_t)_{t\geq 0}$ be an Itô-integrable process. Suppose first that $(x_t)_{t\geq 0}$ is a step process that is, there exists an increasing sequence $(t_n)_{n\in \mathbb{N}}$ in $\mathbb{R}_+$ converging to $+\infty$ and such that $x_t = x_{t_n}$ for $t_n \leq t < t_{n+1}$. Then define a mapping $\int_0^\infty x_t \otimes d\chi_t$ on $\mathcal{P}$ by

$$\Big[\int_0^\infty x_t \otimes d\chi_t\Big](\sigma) = \sum_n \big[x_{t_n} \otimes (\chi_{t_{n+1}} - \chi_{t_n})\big](\sigma)$$

(recall that x_{t_n} belongs to $\mathcal{F}_{t_n]}$ and $\chi_{t_{n-1}} - \chi_{t_n}$ belongs to $\mathcal{F}_{[t_n}$). It can be easily computed that

$$\Big[\int_0^\infty x_t \otimes d\chi_t\Big](\sigma) = [\mathcal{I}(x.)](\sigma).$$

Thus, for step Ito integrable processes $(x_t)_{t\geq 0}$ the vector $\int_0^\infty x_t \otimes d\chi_t$ coincides with the Ito integral $\mathcal{I}(x.)$. By the Itô-isometry formula (3.2) we get

$$\Big|\Big|\int_0^\infty x_t \otimes d\chi_t\Big|\Big|^2 = ||\mathcal{I}(x.)||^2 = \int_0^\infty ||x_t||^2\, dt.$$

Let $n \in \mathbb{N}$ be fixed. For all $t \in \mathbb{R}_+$ define $t_n = E[2^n t]2^{-n}$, where $E[\,\cdot\,]$ denotes the integer part function. Let x be an Itô-integrable process. Define $x_t^n(\sigma) = \mathbb{1}_{\Gamma_{t_n}}(\sigma)(t_{n+1} - t_n)^{-1} \int_{t_n}^{t_{n+1}} x_s(\sigma)\, ds$ for all t, all σ. The process x^n thus defined is clearly Itô-integrable and $||\mathcal{I}(x) - \mathcal{I}(x^n)||$ converges to 0 when n tends to $+\infty$. But x^n is a step process as above and $\mathcal{I}(x^n) = \int_0^\infty x_t^n \otimes d\chi_t$. This proves that the integral $\int_0^\infty x_t \otimes d\chi_t$ extends to all Ito-integrable processes $(x_t)_{t\geq 0}$ and coincides with the Ito integral $\mathcal{I}(x)$. From now on $\mathcal{I}(x)$ is denoted $\int_0^\infty x_t\, d\chi_t$. Note that we have dropped the $\otimes$ symbol.

We end this subsection with some notations we shall employ later. For all $a < b \in \mathbb{R}_+ \cup \{+\infty\}$, we denote by $\int_a^b x_t\, d\chi_t$ (*resp.* $\int_a^b x_t\, dt$, $\mathcal{S}_a^b(x)$) the Itô-integral (*resp.* time-integral, Skorohod integral) of the process $z_t = x_t \mathbb{1}_{[a,b[}(t)$, $t \in \mathbb{R}_+$.

Note that the Skorohod integrability of x *does not imply* the Skorohod integrablity of $1\!\!1_{[a,b[}(\cdot)x_{\cdot}$. The same goes for time-integrability. This is an essential feature of the integrals, however it does not arise for absolute time integrals, absolute Skorohod integrals, or for Itô integrals.

III.2. Differentiation and Projection

Definition 3.7 – For a map f with source space Γ, let ∇f and Df be the maps with source space $\Gamma \times I\!\!R_+$ given by

$$\nabla f(\omega, s) = f(\omega \cup s); \;\; Df(\omega, s) = 1\!\!1_{\Gamma_s}(\omega) f(\omega \cup s).$$

The target space will be either $\mathcal{H}$, $\mathbb{C}$ or $I\!\!R_+$. We exploit this freedom as follows. The operators ∇ and D *commute with* $\|\cdot\|_{\mathcal{H}}$ in the sense that if $k(\omega) = \|f(\omega)\|_{\mathcal{H}}$, where $f : \Gamma \to \mathcal{H}$, then $\nabla k(\omega, s) = \|\nabla f(\omega, s)\|$ (and the same for D).

Proposition 3.8 – *Let f be a measurable map $\Gamma \to \mathcal{H}$. Then ∇f and Df are measurable maps and*

$$\int_0^\infty \int_\Gamma \|\nabla f(\omega, s)\|^2 \, d\omega \, ds = \int_\Gamma \#\sigma \|f(\sigma)\|^2 \, d\sigma \tag{3.5 a}$$

$$\int_0^\infty \int_\Gamma \|Df(\omega, s)\|^2 \, d\omega \, ds = \int_\Gamma \|f(\sigma)\|^2 \, d\sigma - \|f(\emptyset)\|^2. \tag{3.5 b}$$

Proof

Straightforward application of the $\oint$–Lemma.

□

It follows that we may view ∇ and D as (measure equivalence) class mappings. When $f \in \mathcal{F}$, we call ∇f and Df the *stochastic gradient of f*, and the *adapted gradient of f*, respectively. Moreover, we write $\operatorname{Dom}\nabla$ for the domain of the stochastic gradient as an unbounded Hilbert space operator $\mathcal{F} \to L^2(\Gamma \times I\!\!R_+; \mathcal{H})$. Thus

$$\operatorname{Dom}\nabla = \{f \in \mathcal{F} : \nabla f \in L^2(\Gamma \times I\!\!R_+; \mathcal{H})\}.$$

Definition 3.9 – For $\sigma \in \Gamma$, $s \in I\!\!R_+$ and a map f with source space Γ, let $\nabla_\sigma f$, $D_\sigma f$ and $P_s f$ be the maps with source Γ, given by

$$\nabla_\sigma f(\omega) = f(\omega \cup \sigma); \quad D_\sigma f(\omega) = 1\!\!1_{\Gamma_{\wedge\sigma}}(\omega) f(\omega \cup \sigma); \quad P_s f = 1\!\!1_{\Gamma_s} f.$$

Thus, writing $D_s f$ for $D_{\{s\}} f$, we have

$$D_s f = Df(\cdot, s); \quad D_\emptyset f = f$$

and

$$D_\sigma f = D_{s_1} \cdots D_{s_n} f \quad \text{if} \quad \sigma = \{s_1 < \cdots < s_n\}.$$

The following algebraic relations are evident for $s < t$:

$$P_0 f = f(\emptyset)\delta_\emptyset; \quad P_s P_t f = P_t P_s f = P_s f; \tag{3.6 a}$$

$$D_t D_s f = D_t P_s f = 0; \tag{3.6 b}$$

$$D_s P_t f = P_t D_s f = D_s f; \tag{3.6 c}$$

as is the *reproducing relation* $D_\tau f(\sigma) = f(\sigma \cup \tau)$ for $\sigma < \tau$, with special cases:

$$f(\omega) = (D_\omega f)(\emptyset) = D_{\vee\omega} f(\omega_-) \tag{3.6 d}$$

when $\omega \neq \emptyset$.

III.3. Integro-differential and Adjoint Relations

First we relate Skorohod integration with stochastic differentiation, and give the adapted counterpart.

Proposition 3.10 – *Let $f \in \mathcal{F}$ and let $x : \Gamma \times I\!R_+ \to \mathcal{H}$ be Skorohod-integrable:*

(a) If the map $(\omega, s) \mapsto \langle x_s(\omega), f(\omega \cup s)\rangle$ is integrable, then

$$\langle \mathcal{S}(x), f\rangle = \int_0^\infty \int_\Gamma \langle x_s(\omega), \nabla_s f(\omega)\rangle \, d\omega \, ds.$$

(b) If x is Itô–integrable, then

$$\langle \int_0^\infty x_s \, d\chi_s, f\rangle = \int_0^\infty \langle x_s, D_s f\rangle \, ds.$$

Proof

More straightforward application of the $\not\int$–Lemma.

□

Next we summarize the Hilbert space properties of the stochastic and adapted gradients and the Skorohod and Itô–integrals. For further details see [Lin].

Theorem 3.11 – *Let $\sqrt{N \otimes I}$ denote the (self–adjoint multiplication operator on $L^2(\Gamma \times I\!R_+; \mathcal{H})$ given by $\sqrt{N \otimes I}\; x_s(\omega) = \sqrt{\#\omega}\; x_s(\omega)$.*

(a) $(\mathcal{S}, \mathrm{Dom}\, \mathcal{S})$ and $(\nabla, \mathrm{Dom}\, \nabla)$ are closed, densely defined operators.

(b) $\mathcal{S}^* = \nabla$ (and $\nabla^* = \mathcal{S}$).

(c) $\mathrm{Dom}\, \mathcal{S} \supset \mathrm{Dom}\, \sqrt{N \otimes I}$; $\mathrm{Dom}\, \nabla = \mathrm{Dom}\, \sqrt{N}$.

(d) The Itô–integral is an isometric operator $L^2(\Gamma_{\mathrm{ad}}; \mathcal{H}) \to \mathcal{F}$ with final space $[\delta_\emptyset]^\perp$, whose adjoint is the adapted gradient D:

$$D\mathcal{I} = \mathcal{I}^* = I \; i.e. \; D_t \int_0^\infty x_s \, d\chi_s = x_t \quad \textit{for all } x \in L^2(\Gamma_{\mathrm{ad}}; \mathcal{H}), \; \textit{a.a. } t \in I\!R_+;$$

$$Ker D = (Im \mathcal{I})^\perp = \mathbb{C}\delta_\emptyset;$$

$$\mathcal{I}D = \mathcal{I}\mathcal{I}^{\sigma(t} = P_0^\perp \; i.e. \; f = P_0 f + \int_0^\infty D_s f \, d\chi_s \quad \textit{for all } f \in \mathcal{F}.$$

(e) The Skorohod integral is an extension of the Itô integral:

$$\mathcal{I} = \mathcal{S}\big|_{L^2(\Gamma_{\mathrm{ad}}; \mathcal{H})}.$$

(f) The adapted gradient is the closure of the product of the adapted projection and the stochastic gradient:

$$D = \overline{P\nabla}.$$

□

III.4. Almost Everywhere Defined Operators

Our philosophy in this paper is to treat the maps D_s like operators on $\mathcal{F}$, exploiting the fact that D is a *bounded operator* on $\mathcal{F}$ so that, unlike ∇, it is defined on the whole of $\mathcal{F}$. With each $f \in \mathcal{F}$, $D_s f$ is a well–defined element of $\mathcal{F}$ for almost every s. Of course the null set depends on f, and for this reason D_s is not an operator on $\mathcal{F}$ in the usual sense — we shall speak of *almost everywhere defined operators* on $\mathcal{F}$. We take this viewpoint in order to exploit the relations (3.6). On measure equivalence classes of maps such as elements of $\mathcal{F}$, there are the a.e. relations

$$D_t D_s f = D_t P_s f = 0; \; D_s P_t f = P_t D_s f = D_s f \tag{3.7}$$

for a.a. $(s < t)$, and the *a.e. reproducing property*

$$D_\tau f(\sigma) = f(\sigma \cup \tau) \tag{3.8}$$

for a.a. $(\sigma < \tau)$.

III.5. Commutation Relations

In this subsection we describe the effect of the operators P_t and the a.e. defined operators D_t on Skorohod and time integrability. The relations we obtain will be applied to quantum stochastic integrals in VI.2. Note the a.e. properties

$$f \in \mathcal{F} \Rightarrow P_s f, D_s f \in \mathcal{F}_s; \quad f \in \mathcal{F}_s, s < t \Rightarrow D_t f = 0. \tag{3.9}$$

Proposition 3.12 – *Let x be a measurable map $\Gamma \times \mathbb{R}_+ \to \mathcal{H}$. If $P_t\, x_t$ is square integrable for almost every $t \geq 0$, then the following are equivalent:*

(a) x is Skorohod-integrable.

(b) (i) $1\!\!1_{[0,t[}(\cdot) D_t\, x.$ is Skorohod-integrable for a.a. t.

(ii) the map $t \mapsto \mathcal{S}_0^t(D_t\, x.) + P_t\, x_t$ is Itô-integrable.

In this case,

$$D_t\, \mathcal{S}(x) = \mathcal{S}_0^t(D_t\, x.) + P_t\, x_t \tag{3.10}$$

for a.a. t.

Proof

In view of the identity

$$1\!\!1_{\Gamma_t}(\sigma)\, \mathcal{S}(x)(\sigma \cup t) = \sum_{s \in \sigma} 1\!\!1_{\Gamma_t}(\sigma) 1\!\!1_{[0,t[}(s)\, x_s(\sigma \backslash s \cup t) + 1\!\!1_{\Gamma_t}(\sigma)\, x_t(\sigma),$$

we have

$$D_t\, \mathcal{S}(x)(\sigma) = \mathcal{S}(1\!\!1_{[0,t[}(\cdot) D_t\, x.)(\sigma) + (P_t\, x_t)(\sigma). \tag{3.11}$$

If x is Skorohod-integrable then, since $P_t\, x_t$ is square-integrable, $1\!\!1_{[0,t[}(\cdot) D_t\, x.$ is Skorohod-integrable and (3.10) holds for a.a. t; moreover, the a.e. defined map

$(\sigma,t)\mapsto \mathcal{S}_0^t(D_t\,x.)(\sigma)+P_t\,x_t)(\sigma)$ is adapted and square integrable, and thus Itô-integrable. Conversely, if x satisfies (b) then, since x is measurable and

$$\int_\Gamma \|\mathcal{S}(x)(\sigma)\|^2\,d\sigma = \int_0^\infty \int_\Gamma \|D_s\mathcal{S}(x)(\omega)\|^2\,d\omega\,ds,$$

x is Skorohod-integrable by (3.11).

□

Proposition 3.13 – *Let x be a measurable map $\Gamma\times I\!\!R_+\to\mathcal{H}$. If the map $x.(\emptyset)$ is integrable then the following are equivalent:*

(a) x is time integrable.

(b) (i) $D_t\,x.$ is time integrable for a.a. t

(ii) the map $t\mapsto\int_0^\infty D_t\,x_s\,ds$ is square–integrable.

In this case we have the a.e. identity

$$D_t\int_0^\infty x_s\,ds = \int_0^\infty D_t\,x_s\,ds. \tag{3.12}$$

Proof

Let x be time integrable. Then, for a.a. (ω,t), the map $s\mapsto 1\!\!1_{\Gamma_t}(\omega)x_s(\omega\cup t)$ is integrable and so, for a.a. t,

$$\begin{aligned} D_t\int_0^\infty x_s\,ds(\omega) &= 1\!\!1_{\Gamma_t}(\omega)\int_0^\infty x_s(\omega\cup t)\,ds = \int_0^\infty 1\!\!1_{\Gamma_t}(\omega)x_s(\omega\cup t)\,ds \\ &= \int_0^\infty (D_t\,x_s)(\omega)\,ds, \end{aligned}$$

for a.a. ω. Hence, for a.a. t, $D_t\,x.$ is time-integrable and (3.12) holds – in particular, the map $t\mapsto\int_0^\infty(D_t\,x_s)(\omega)\,ds$ is square-integrable.

Conversely, if (b) holds, then the map $x.(\omega)$ is either $x.(\emptyset)$ or $(D_{\vee\omega}\,x.)(\omega_-)$ and so is integrable for a.a. ω. Moreover the map

$$\alpha\in\Gamma\setminus\{\emptyset\}\mapsto\int_0^\infty x_s(\alpha)\,ds \tag{3.13}$$

is the composition of the measure isomorphism $\alpha\mapsto(\alpha_-,\vee\alpha)$ from $\Gamma\setminus\{\emptyset\}$ into Γ_{ad} and the square integrable map $(\omega,t)\mapsto\int_0^\infty(D_t\,x_s)(\omega)\,ds$. Hence (3.13) is square integrable, so that x is time integrable.

□

Proposition 3.14 – *Let x be a measurable map $\Gamma\times I\!\!R_+\to\mathcal{H}$, and let $t\ge 0$.*

(a) If x is time integrable then $P_t\,x.$ is time integrable and

$$\int_0^\infty P_t\,x_s\,ds = P_t\int_0^\infty x_s\,ds.$$

(b) If x is Skorohod integrable then $1\!\!1_{[0,t[}P_t x.$ is Skorohod integrable, and

$$\mathcal{S}_0^t(P_t\,x.) = P_t\,\mathcal{S}(x).$$

Moreover, if also $1\!\!1_{[t,\infty[}(\cdot)P_t\, x.$ *is Itô–integrable, then* $P_t\, x.$ *is Skorohod integrable, and*

$$\mathcal{S}(P_t\, x.) = P_t\, \mathcal{S}(x) + \int_t^\infty P_t\, x_s\, d\chi_s.$$

Proof

Straightforward.

□

Notice that each of the supplementary conditions in Propositions 3.12, 3.13 and 3.14 – namely square-integrability of $1\!\!1_{\Gamma_t} x_t$ for a.a. t, integrability of $x.(\emptyset)$ and Itô-integrability of $1\!\!1_{[t,\infty[}(\cdot)P_t\, x.$ – is a condition on the $\mathbb{R}_+$-valued map $(\omega, s) \mapsto \|x_s(\omega)\|$. In view of the fact that P_t and D_t "commute" with the norm $\|\cdot\|_{\mathcal{H}}$ (see the remark following Definition 3.7) each of these results also holds if time and Skorohod integrability are replaced by absolute time and absolute Skorohod integrability respectively.

III.6 Probabilistic interpretations

In this subsection we describe explicitly the connection between the objects we have introduced in Fock space $(P_t, D_t, \nabla_t, \mathcal{S}, \mathcal{I})$ and their actual (classical) probabilistic counterparts. While formally independent of the rest of the paper, the ideas here underly the whole work.

By a *probabilistic interpretation* of the Fock space, we mean a quintuplet $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P}, m)$ where $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ is the canonical filtered space of $m = (m_t)_{t\geq 0}$, and m is a *normal martingale* – that is a martingale for which $(m_t^2 - t)_{t\geq 0}$ is also a martingale – which has the chaotic representation property. Examples of such martingales include Brownian motion, the compensated Poisson process and some of the Azema martingales ([Eme]). The chaotic representation of random variables leads to a natural isomorphism Ψ between $\mathcal{F}$ and $\mathcal{X} = L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathcal{H}_0)$, which may suggestively be expressed $f \mapsto \int_\Gamma f(\sigma) dm_\sigma$ ([Me2]). Each of the operations P_t, D_t, ∇_t, $\mathcal{S}, \mathcal{I}$ and $\mathcal{L}$ have interpetations on $\mathcal{X}$ as well-known probabilistic operations.

The orthogonal projection P_t is $\Psi^{-1} \circ \mathbb{E}_t \circ \Psi$, where $\mathbb{E}_t$ is the time t conditional expectation of the process: $\mathbb{E}[\,\cdot\,|\mathcal{A}_t]$. Thus also $\mathcal{F}_t = P_t\mathcal{F}$ is $\Psi^{-1}(\mathcal{X}_t)$ where $\mathcal{X}_t = L^2(\Omega, \mathcal{A}_t, \mathbb{P}; \mathcal{H}_0)$. In particular, a square-integrable classical stochastic process in $\mathcal{X}$ is adapted if and only if its image under Ψ^{-1} is adapted in the sense of Definition 3.4. Since the martingale m has the chaotic representation property, it also possesses the predictable representation property. Any random variable f in $\mathcal{X}$ may therefore be expressed as $f = \mathbb{E}[f] + \int_0^\infty \xi_t(f)\, dm_t$, for some predictable process $(\xi_t(f))_{t\geq 0}$ in $\mathcal{X}$. Viewing $(\xi_t)_{t\geq 0}$ as a family of a.e. defined operators on $\mathcal{X}$ we see it as a probabilistic interpretation of $(D_t)_{t\geq 0}$: $D_t = \Psi^{-1} \circ \xi_t \circ \Psi$. Similarly, ∇ corresponds precisely to the gradient operator in Malliavin calculus, and ∇_t corresponds to the stochastic derivative, along the element $f : s \mapsto s \wedge t$ of the Cameron-Martin space, on $\mathcal{X}$ (see [N-Z], for example). By Theorem 3.11(b)

$\mathcal{S}$ is the adjoint of ∇ and therefore ([G-T]) corresponds to the Hitsuda-Skorohod integral with respect to the process m. Of course this may alternatively be seen directly. It also follows from Theorem 3.11(e) that $\mathcal{I}$, being the restriction of $\mathcal{S}$ to adapted (Fock vector) processes, corresponds to the Itô integral with respect to m. Theorem 3.11(d) includes an expression of the predictable representation property of m, and the isometry of Itô-integration with respect to m, on $\mathcal{F}$. Finally, Theorem 3.11(f) implies that $D_t = P_t \nabla_t$ (in the sense of a.e. defined operators), which corresponds to Clark's formula ([Cla]).

Thus each of the operations introduced in III.1-4 correspond to well-known operations of classical stochastic calculus once Fock space is interpreted as the *chaotic space* of some normal martingale. In fact one should rather think the other way around. Probabilistic operations such as Skorohod integration, stochastic differentiation, predictable representation and so on, may be expressed merely in terms of the chaotic expansion of random variables. They use no specific property of the particular martingale beyond chaotic representation and the form of Itô isometry. The normality of the martingale implies that its angle bracket $\langle m, m \rangle_t$ equals t, and so the formula for Itô isometry remains the same for each such martingale. Fock space is thereby seen as an abstract chaos space which encodes the chaotic representation property and the Itô isometry formula of normal martingales, and which carries simple intrinsic operations which perform the L^2-stochastic calculus of the martingale.

IV. Operator adaptedness

In this section we extend the class of domains for the operators and processes of quantum stochastic calculus beyond the exponential domains that have been used so far. We give three characterizations of a notion of adaptedness for Fock space operators. Hudson and Parthasarathy defined all operators and processes on domains consisting of exponential vectors and, in their definition of operator adaptedness, exploited the homogeneity of these vectors in the continuous tensor product structure of Fock space:

$$\varepsilon(\varphi) = \varepsilon(\varphi_{s]}) \otimes \varepsilon(\varphi_{[s}) \quad \text{for} \quad s > 0.$$

In our approach both the allowable domains and the criteria for adaptedness are described wholly in terms of the projection operators (P_t) and sections of the adapted gradient D. Fundamental for us are the a.e. relations:

$$D_s f = P_s D_s f; \; D_t P_s f = 0; \tag{4.1}$$

$$D_s P_t f = P_t D_s f = D_s f; \tag{4.2}$$

for $s < t$, and the a.e. reproducing property

$$f(\omega) = D_{\omega_{(s}} f(\omega_{s)}). \tag{4.3}$$

These hold for any measure equivalence class of map f with source Γ – in particular, for $f \in \mathcal{F}$.

The new condition for operator adaptedness is equivalent to the original one on exponential domains. However, the new condition frees us from exponential domains allowing us in particular to work on all of $\mathcal{F}$ for bounded operators, and to multiply unbounded operators. We show that the collection of s–adapted operators is closed under operator sums and products. Our definition gives a procedure for manufacturing an adapted operator from any Fock space operator. This is exploited in our discussion of conditional expectation in the next section; it is also relevant to maximality questions, for operator domains, considered later.

IV.1. Definitions and basic properties

Definition 4.1 – Let $s \geq 0$ be fixed. A subspace V of $\mathcal{F}$ is *s-adapted* if $f \in V$ implies

$$P_s f \in V \quad \text{and} \quad D_t f \in V \text{ for a.a. } t \geq s.$$

Differential definition of adaptedness – An operator H on $\mathcal{F}$ is *s-adapted* if

I.1 $\operatorname{Dom} H$ is an s-adapted subspace

I.2 For all $f \in \operatorname{Dom} H$

(i) $P_s H f = P_s H P_s f$;

(ii) $D_t H f = H D_t f$ for a.a. $t \geq s$.

We next give an equivalent definition.

Proposition 4.2 (Integral definition of adaptedness) – *An operator H on $\mathcal{F}$ is s-adapted if and only if it satisfies*

II.1 $\operatorname{Dom} H$ *is an s-adapted subspace*

II.2 For all $f \in \operatorname{Dom} H$,

(i) $H P_s f = P_s H P_s f$

(ii) $H D_t f \in \mathcal{F}_t$ *for a.a.* $t \geq s$;

(iii) $(H D_t f)_{t \geq s}$ *defines an Itô-integrable process, and*

$$H(f - P_s f) \ \left(= H \int_s^\infty D_t f \, d\chi_t\right) \ = \int_s^\infty H D_t f \, d\chi_t.$$

Proof

Suppose that H is s-adapted. Then, by (4.1) and (4.3),

$$\begin{aligned} HP_s f(\omega) &= (D_{\omega_{(s}} H P_s f)(\omega_{s)}) = (H D_{\omega_{(s}} P_s f)(\omega_{s)}) = 1\!\!1_{\Gamma_s}(\omega)(HP_s f)(\omega) \\ &= (P_s H P_s f)(\omega) \end{aligned}$$

for a.e. ω, so II.2 (i) holds. Moreover, by (4.1), I.2 (ii) implies II.2 (ii). By I.2 (ii) we also have, for a.a. $t > s$

$$H D_t f = D_t H f$$

so $(HD_t f)_{t\geq s}$ is an adapted square integrable process, and is therefore Itô-integrable. Finally,

$$\left[\int_s^\infty HD_t f\, d\chi_t\right](\omega) = \mathbb{1}_{[s,\infty[}(\vee\omega)(D_{\vee\omega}Hf)(\omega_-) = \mathbb{1}_{[s,\infty[}(\vee\omega)(Hf)(\omega)$$
$$= [(I-P_s)Hf](\omega) = [H(f-P_s f)](\omega)$$

for a.a. ω, so that H is s-adapted in the integral sense.

Conversely, suppose that H is s-adapted in the integral sense. Then I.2 (i) holds since the Itô integral of a process on $[s,\infty[$ is orthogonal to $\mathcal{F}_s$. Thus $HP_s = P_s H$ on $\operatorname{Dom} H$ and we have

$$HD_t f = D_t \int_s^\infty HD_u f\, d\chi_u = D_t H(f - P_s f) = D_t(I-P_s)Hf = D_t Hf,$$

so that H is s-adapted.

□

Note that, combining the above characterizations of adaptedness, we obtain the following.

Corollary 4.3 – *An operator H on $\mathcal{F}$ is s-adapted if and only if it has an s-adapted domain on which the commutation relations*

$$HP_s = P_s H, \quad \textit{and} \quad HD_t = D_t H \textit{ for a.a. } t > s, \tag{4.4}$$

are satisfied.

□

The above characterization of adaptedness is the most useful one, but it is sometime good to keep in mind the the commutation condition with P_s can be weakened.

Combining I.2 (i) with II.2 (i) and iterating I.2 (ii) we also see that an operator H on $\mathcal{F}$ is s-adapted if and only if it has an s-adapted domain on which the commutation relations

$$HP_s f = P_s Hf, \quad \text{and} \quad HD_\tau f = D_\tau Hf \text{ for a.a. } \tau \in \Gamma, \tau > s, \tag{4.5}$$

are satisfied.

In view of the relations

$$P_s(g\otimes h) = h(\emptyset) g\otimes \delta_\emptyset, \quad \text{and} \quad D_\tau(g\otimes h) = g\otimes D_\tau h \text{ for a.a. } \tau > s,$$

for $g\in\mathcal{F}_s$ and $h\in\mathcal{F}^s$, if V_s is any subspace of $\mathcal{F}_s$, then the algebraic tensor product $V_s\otimes\mathcal{F}^s$ is an s-adapted subspace. Moreover, if H_s is an operator on $\mathcal{F}_s$, then $H_s\otimes I^s$ with domain $\operatorname{Dom} H_s\otimes\mathcal{F}^s$ is an s-adapted operator.

We now come to our main operational definition.

Proposition 4.4 (Projective definition of adaptedness) – *An operator H on $\mathcal{F}$ is s-adapted if and only if it satisfies:*

III.1 $\operatorname{Dom} H$ *is an s-adapted subspace.*

III.2 For $f \in \operatorname{Dom} H$,

$$Hf(\omega) = (HP_s D_{\omega_{(s}} f)(\omega_{s)}).$$

for a.a. ω.

Proof

If H is s-adapted then, by (4.3) and I.2 (ii),

$$Hf(\omega) = (HD_{\omega_{(s}} Hf)(\omega_{s)}) = (HD_{\omega_{(s}} f)(\omega_{s)}) = (P_s HD_{\omega_{(s}} f)(\omega_{s)})$$

for a.a. ω, so III.2 follows from (4.4).

Conversely, suppose that H is s-adapted in the projective sense above. Then

$$(P_s Hf)(\omega) = 1\!\!1_{\Gamma_s}(\omega)(HP_s D_{\omega_{(s}} f)(\omega_{s)}) = 1\!\!1_{\Gamma_s}(\omega)(HP_s f)(\omega) = (P_s HP_s f)(\omega)$$

for a.a. ω, so that H satisfies I.2 (i). For a.a. ω and $t \geq s$ we have

$$\begin{aligned} HD_t f(\omega) &= (HP_s D_{\omega_{(s}} D_t f)(\omega_{s)}) \\ &= 1\!\!1_{\Gamma_t}(\omega)(HP_s D_{(\omega \cup t)_{(s}} f)((\omega \cup t)_{s)}) \\ &= 1\!\!1_{\Gamma_t}(\omega)(Hf)(\omega \cup t) = D_t Hf(\omega), \end{aligned}$$

so that H satisfies I.2 (ii) also, and H is therefore s-adapted.

□

From the projective definition it is easy to see that s-adapted operators satisfy

$$HP_t f = P_t Hf$$

for $t \geq s$, whenever both sides are defined: $f \in \operatorname{Dom} H \cap \operatorname{Dom} HP_t$. Therefore H is also t-adapted for each $t > s$ for which $\operatorname{Dom} H$ is a t-adapted subspace. It also follows from the projective definition of adaptedness that if two s-adapted operators H and H' agree on $\mathcal{F}_s \cap \operatorname{Dom} H \cap \operatorname{Dom} H'$, then they agree on $\operatorname{Dom} H \cap \operatorname{Dom} H'$.

Furthermore, let $\mathcal{D}$ be a s-adapted domain. Let H be an operator defined on $\mathcal{D} \cap \mathcal{F}_s$, with values in $\mathcal{F}_s$. The operator H', defined on $\mathcal{D}$ by

$$[H'f](\sigma) = [HP_s D_{\sigma_{(s}} f](\sigma_{s)})$$

is a s-adapted operator which coincides with H on $\mathcal{D} \cap \mathcal{F}_s$; it is unique with these properties. H' is called the *s-adapted extension* of H.

Proposition 4.5 – *Let H be an s-adapted operator on $\mathcal{F}$. Then for all $f \in \mathcal{F}$ and $g \in \operatorname{Dom} H$,*

$$\langle f, Hg \rangle = \int_\Gamma 1\!\!1_{\Gamma^s}(\beta) \langle P_s D_\beta f, HP_s D_\beta g \rangle \, d\beta.$$

Proof

This follows easily from the a.e. reproducing property and the $\not\int$–Lemma.

□

The adjoint of a densely defined s-adapted operator H may fail to be s-adapted as it stands. However we shall see in the next section that *conditioning* an operator which is *adjoint to* H yields an s-adapted operator adjoint to H. Let $\mathcal{A}_s$ denote the collection of all s-adapted operators on $\mathcal{F}$.

Proposition 4.6 – *$\mathcal{A}_s$ is a subset of the collection of all (not necessarily bounded) operators on $\mathcal{F}$ which is closed under operator products and linear combinations.*

Proof

Let H and H' be s-adapted operators on $\mathcal{F}$, and let $f \in \operatorname{Dom}(HH')$. Then $P_s f$ and $D_t f$ lie in $\operatorname{Dom}(H')$, $H'P_s f = P_s H' f$ and $H' D_t f = D_t H' f$ for a.a. $t > s$, since $f \in \operatorname{Dom}(H')$ and H' is s-adapted. But $H'f \in \operatorname{Dom} H$ and H is s-adapted, so $P_s H' f$ and $D_t H' f$ lie in $\operatorname{Dom} H$, $H(P_s H' f) = P_s H H' f$ and $H(D_t H' f) = D_t H H' f$ for a.e. $t > s$. This shows that $\mathcal{A}_s$ is closed under operator multiplication. Since an intersection of s-adapted subspaces is s-adapted, $\mathcal{A}_s$ is also closed under addition. It is obviously closed under scalar multiplication.

□

$\mathcal{A}_s$ fails to be an associative algebra only in the same sense in which the collection of all unbounded operators on $\mathcal{F}$ does, namely that an element H whose domain is not all of $\mathcal{F}$ fails to have an additive inverse, and scalar multiplication by 0 yields not the zero operator, but its restriction to $\operatorname{Dom} H$.

IV.2 Examples and previous definitions

We first show that when restricted to exponential domains our definition coincides with that of Hudson and Parthasarthy (HP). Let us fix $s \geq 0$. Let $\mathcal{M}$ be a subset of $L^2(\mathbb{R}_+)$ and let V_0 be a subspace of $\mathcal{H}_0$. The pair $(V_0, \mathcal{M})$ is *s-admissible* if

(i) $V_0 \otimes \mathcal{E}(\mathcal{M})$ is dense in $\mathcal{F}$;

(ii) $\varphi_{t]} \in \mathcal{M}$ whenever $\varphi \in \mathcal{M}$ and $t \geq s$.

An operator H on $\mathcal{F}$ is *HP-s-adapted with respect to* $(V_0, \mathcal{M})$, if

1. $(V_0, \mathcal{M})$ is an s-admissible pair, and $\operatorname{Dom} H = V_0 \otimes \mathcal{E}(\mathcal{M})$.
2. For each $u \in V_0$ and $\varphi \in \mathcal{M}$ we have
   (i) $C u\varepsilon[\varphi_{s]}] \in \mathcal{F}_s$;
   (ii) $Hu\varepsilon(\varphi) = \big(Hu\varepsilon(\varphi_{s]})\big) \otimes \varepsilon(\varphi_{[s})$.

Choices of $\mathcal{M}$ that have been found useful include $L^2(\mathbb{R}_+)$ itself; dense subspaces of $L^2(\mathbb{R}_+)$ such as $L^2(\mathbb{R}_+) \cap L^\infty_{loc}(\mathbb{R}_+)$; the set $\{\varphi \in (L^2 \cap L^\infty)(\mathbb{R}_+) : \|\varphi\|_2 \leq 1 \text{ and } \|\varphi\|_\infty \leq 1\}$ and, in view of a result of Partharasathy and Sunder ([P-S]), one could also use the collection of indicator functions $\{\mathbb{1}_B : B \subset \mathbb{R}_+$ Borel with finite Lebesgue measure$\}$.

Proposition 4.7 – *Let $V = V_0 \otimes \mathcal{E}(\mathcal{M})$ where V_0 is a subspace of $\mathcal{H}_0$ and $\mathcal{M}$ is a subset of $L^2(\mathbb{R}_+)$, such that V is dense in $\mathcal{F}$, and let H be an operator on $\mathcal{F}$ with domain V, then* *(a) V is an s-adapted subspace if and only if the pair $(V_0, \mathcal{M})$ is s-admissible;*

(b) H is an s-adapted operator if and only if H is HP-s-adapted with respect to the pair $(V_0, \mathcal{M})$.

Proof

(a) Since $P_t v\varepsilon(\varphi) = v\varepsilon(\varphi_{t]})$, and $D_t v\varepsilon(\varphi) = \varphi(t) v\varepsilon(\varphi_{t]})$ for a.a. t, this is immediate.

(b) In either case $Hv\varepsilon(\varphi_{s]}) \in \mathcal{F}_s$, so that for a.a. τ,

$$\begin{aligned}(Hv\varepsilon(\varphi_{s]})) \otimes \varepsilon(\varphi_{[s}))(\tau) &= (Hv\varepsilon(\varphi_{s]}))(\tau_{s]}) \prod_{t\in\tau_{[s}} \varphi(t) \\ &= (HP_s v\varepsilon(\varphi_{s\wedge\tau_{[s]}}))(\tau_{s]}) \prod_{t\in\tau_{[s}} \varphi(t) \\ &= (HP_s D_{\tau_{[s}} v\varepsilon(\varphi))(\tau_{s]}).\end{aligned}$$

Therefore $Hv\varepsilon(\varphi) = Hv\varepsilon(\varphi_{s]}) \otimes \varepsilon(\varphi_{[s})$ if and only if

$$(Hv\varepsilon(\varphi))(\tau) = (HP_s D_{\tau_{(s}} v\varepsilon(\varphi))(\tau_{s)})$$

for a.a. τ, so (b) holds.

□

Our definition thus specializes precisely to the usual one when restricted to exponential domains. We next show that, as well as these, all domains used previously for quantum stochastic calculus are *(fully) adapted* – that is they are s-adapted for every $s \geq 0$. Recall the subspaces of $\mathcal{F}$ defined in (2.1).

(a) $\mathcal{F}$ itself is obviously adapted. This is useful as it is the natural domain for all bounded-operator-valued processes.

(b) $\mathcal{K}$ is adapted. In fact each $\mathcal{F}^{(a)}$ is an adapted subspace, since P_s leaves $\mathrm{Dom}(\sqrt{a}^N)$ invariant and

$$\begin{aligned}\int_0^\infty \int_\Gamma a^{\#\omega} \|D_t f(\omega)\|^2 \, d\omega \, dt &= a^{-1} \int_0^\infty \int_\Gamma a^{\#(\omega\cup t)} \|f(\omega \cup t)\|^2 \mathbb{1}_{\Gamma_t}(\omega) \, d\omega \, dt \\ &= a^{-1} \|D\sqrt{a}^N f\|^2 \leq a^{-1} \|\sqrt{a}^N f\|^2.\end{aligned}$$

This is the natural domain for both non-causal (quantum) stochastic calculus and for integral-sum kernel operators on $\mathcal{F}$.

(c) $\mathcal{F}_f$ is adapted since if $\mathrm{supp}\, f \subset \Gamma^{N]} = \bigcup_{n\leq N} \Gamma^{(n)}$, then $\mathrm{supp}\, P_t f \subset \Gamma^{N]}$ and $\mathrm{supp}\, D_t f \subset \Gamma^{N-1]}$.

(d) $\mathrm{lin}\{\otimes^{(n)}\varphi : \varphi \in \mathcal{M}, n \geq 0\}$, the symmetric tensor algebra over a subspace $\mathcal{M}$ of $L^2(\mathbb{R}_+)$, is adapted provided only that $\mathbb{1}_{[0,t]}\varphi \in \mathcal{M}$ whenever $\varphi \in \mathcal{M}$ and $t \geq 0$, since $D_t(\otimes^{(n)}\varphi) = \varphi(t) \otimes^{(n-1)} (\varphi \mathbb{1}_{[0,t]})$.

(e) $\mathcal{K}_b := \{f \in \mathcal{F} : \exists\, T, C, K > 0 \text{ s.t. } \mathrm{supp}\, f \subset \Gamma_T \text{ and } \|f(\omega)\| \leq CK^{\#\omega}\}$, is adapted since both the support and boundedness properties are clearly undisturbed by both P_s and D_t: for example $\|D_t f(\omega)\| \leq C' K^{\#\omega}$, where $C' = CK$.

This is the original domain used by Maassen for expressing quantum stochastic integrals as integral-sum kernel operators ([Maa]).

V. Conditional expectation and operator processes

The projective definition of adaptedness leads to a natural way of defining conditional expectation for Fock space operators. When applied to any operator it yields an s-adapted operator; when applied to an operator which is already s-adapted, it yields an extension of the operator to a natural domain for the purposes of quantum stochastic calculus.

V.1. Conditioned spaces

The idea is to construct the domain of the conditioned operator so that it is maximal given the domain constraint of the unconditioned operator. Thus, for any subspace V of $\mathcal{F}$, its *time s-conditioned space* is the subspace

$$I\!D_s(V) := \{f \in \mathcal{F} : P_s D_\tau f \in V \text{ for a.a. } \tau > s\}.$$

Clearly $I\!D_s(V)$ is an s-adapted subspace. Here is a list of further properties of this construction.

Proposition 5.1 – *Let V and V' be subspaces of $\mathcal{F}$.*

(o) $I\!D_s(\mathcal{F}) = \mathcal{F}$, $I\!D_s(V \cap V') = I\!D_s(V) \cap I\!D_s(V')$.

(i) $I\!D_s(V)$ *is a t-adapted subspace for all* $t \geq s$.

(ii) If V is s-adapted then $I\!D_s(V) \supset V$.

(iii) For $s \leq t$, $I\!D_t(I\!D_s(V)) = I\!D_s(I\!D_t(V)) = I\!D_s(V)$.

(iv) $I\!D_s(V) \supset (V \cap \mathcal{F}_s) \otimes \mathcal{F}^s$.

(v) Let $s \leq a < b$. If $(x_t)_{t\in[a,b]}$ is a $I\!D_s(V)$-valued Itô-integrable vector process, then $\int_a^b x_s\, d\chi_s \in I\!D_s(V)$.

Proof

These are routine verifications. For example in (v), $\int_a^b x_s\, d\chi_s \in \mathcal{F}_b \ominus \mathcal{F}_a$ so that $P_s D_\tau \int_a^b x_s\, d\chi_s = P_s D_{\tau_-} x_{\vee\tau}$ if $\tau \in \Gamma_b \setminus \Gamma_a$, and 0 otherwise.

□

Thus the map $I\!D_s$ manufactures an s-adapted subspace from any subspace V, which moreover contains V if V is already s-adapted. (i) is a tower property of the maps, and (v) is a technical property which will be useful later.

V.2. Conditional expectation of operators

We come now to a central definition of our approach. To define conditional expectations of operators, we take our cue from the projective definition of adaptedness. Thus if H is an operator on $\mathcal{F}$ and $s \geq 0$, let V be the subspace of $\mathcal{F}$ consisting of those f in $I\!D_s(\operatorname{Dom} H)$ for which the a.e. defined map $\tau \mapsto 1\!\!1_{\Gamma^s}(\tau) P_s H P_s D_\tau f$ is square integrable $\Gamma \to \mathcal{F}$. For $f \in V$ there is a unique element of $\mathcal{F}$, denoted $I\!E_s[H]f$, satisfying the a.e. identity

$$(I\!E_s[H]f)(\omega) = (HP_s D_{\omega_{(s}} f)(\omega_{s)}). \tag{5.1}$$

The *time s-conditional expectation* of H is the resulting operator $I\!E_s[H]$ with domain V. Thus

$$\begin{aligned}\operatorname{Dom}(I\!E_s[H]) = \{f \in \mathcal{F} : P_s D_\tau f \in \operatorname{Dom} H \text{ for a.a. } \tau > s;\\ \tau \mapsto 1\!\!1_{\Gamma^s}(\tau) P_s H P_s D_\tau f \text{ issquare} - \text{integrable}\}.\end{aligned}$$

It is easily verified that $I\!E_s[H]$ is an s-adapted operator. Our first result on conditional expectation therefore includes an extension of Proposition 4.5.

Proposition 5.2 – *Let H be an operator on $\mathcal{F}$, and let $s \geq 0$.*

(a) $I\!E_s[H] = I\!E_s[HP_s] = I\!E_s[P_s H P_s]$.

(b) If $g \in \operatorname{Dom}\ I\!E_s[H]$, *then*

$$P_s I\!E_s[H]g = I\!E_s[H]P_s g = P_s H P_s g.$$

(c) If $g \in I\!E_s[H]$ *and* $f \in \mathcal{F}$, *then*

$$\langle f, I\!E_s[H]g\rangle = \int_\Gamma 1\!\!1_{\Gamma^s}(\beta)\langle P_s D_\beta f, HP_s D_\beta g\rangle \, d\beta.$$

(d) If $(H, H^\dagger)$ is an adjoint pair of operators on $\mathcal{F}$ then $(I\!E_s[H], I\!E_s[H^\dagger])$ is also an adjoint pair.

(e) If H and $I\!E_s[H]$ are densely defined, then

$$I\!E_s[H]^* \supset I\!E_s[H^*].$$

Proof

(a) and (b) are immediate consequences of the definition. (c) follows from (b) and Proposition 4.5. (d) follows from (c), and (e) from (d).

□

Notice that if H is s-adapted then the subspaces $\mathcal{F}_s \cap \operatorname{Dom} H$ and $\mathcal{F}_s \cap \operatorname{Dom}(I\!E_s[H])$ coincide, and $I\!E_s[H]g = Hg$ for g in this subspace. It follows that

$$I\!E_s[H]P_s f = HP_s f \quad \text{and} \quad I\!E_s[H]D_s f = HD_s f,$$

whenever $P_s f$ (respectively $D_s f$) belongs to $\operatorname{Dom} I\!E_s[H]$, equivalently belongs to $\operatorname{Dom} H$. We next give a list of the basic properties of time s-conditional expectation. A refinement of (d) of the above proposition is included.

Theorem 5.3 – *Let H and H' be operators on $\mathcal{F}$.*

(o) $I\!E_s[I] = I$, $I\!E_s[H + \lambda H'] \supset I\!E_s[H] + \lambda I\!E_s[H']$ *for* $\lambda \in \mathbb{C}$.

(i) $\mathbb{E}_s[H]$ *is* t*-adapted for every* $t \geq s$.

(ii) H *is* s*-adapted if and only if* $\operatorname{Dom} H$ *is an* s*-adapted subspace and* $H \subset \mathbb{E}_s[H]$.

(iii) For all $t \geq s \geq 0$, $\mathbb{E}_s \circ \mathbb{E}_t[H] = \mathbb{E}_t \circ \mathbb{E}_s[H] = \mathbb{E}_s[H]$.

(iv) $\mathbb{E}_s[H] \supset H_s \otimes I^s$, *with domain* $(\mathcal{F}_s \cap \operatorname{Dom} H) \otimes \mathcal{F}^s$, *where* $H_s g = P_s H g$.

(v) If H, $\mathbb{E}_s[H]$ *and* $\mathbb{E}_s[H]^*$ *are all densely defined, then* $\mathbb{E}_s[H]^*$ *is an* s*-adapted operator and* $\mathbb{E}_s[H]^* \supset \mathbb{E}_s[H^*]$.

(vi) If H *is bounded (with domain* $\mathcal{F}$*) then* $\mathbb{E}_s[H]$ *is bounded (with domain* $\mathcal{F}$*) too, with norm at most* $\|H\|$.

(vii) If H *is a non-negative operator, then so is* $\mathbb{E}_s[H]$.

(viii) If S *is an* s*-adapted operator, then* $\mathbb{E}_s[HS] \supset \mathbb{E}_s[H]S$.

(ix) If B *is an* s*-adapted and bounded operator, then* $\mathbb{E}_s[BH] \supset B\mathbb{E}_s[H]$.

(x) If $H = H_s \otimes H^s$, *where* H_s *is an operator on* $\mathcal{F}_s$ *and* H^s *is an operator on* $\mathcal{F}^s$, *then* $\mathbb{E}_s[H] = \langle \delta_\emptyset, H^s \delta_\emptyset \rangle H_s \otimes I^s$, *provided only that* $\delta_\emptyset \in \operatorname{Dom}(H^s)$.

Proof

Most of these properties follow from straightforward applications of the relations (4.1-4.3), the $\not{\!\!\int}$-Lemma and Propositions 5.1 and 5.2, to the definitions. For example (i) follows from Propositon 5.1(i) and the remark following 4.2. Parts (iii) and (v) are a little more delicate.

(iii) For $f \in \mathcal{F}$, $f \in \operatorname{Dom} \mathbb{E}_s \circ \mathbb{E}_t[H]$ if and only if $P_s D_\alpha f \in \operatorname{Dom} \mathbb{E}_t[H]$ for a.a. $\alpha > s$, and the map $\alpha \mapsto 1\!\!1_{\Gamma^s}(\alpha) P_s \mathbb{E}_t[H] P_s D_\alpha f$ is square-integrable. By Proposition 5.2, these hold if and only if

(a) $P_t D_\beta P_s D_\alpha f \in \operatorname{Dom} H$ for a.a. $\beta > t$ and a.a. $\alpha > s$.

(b) $\beta \mapsto 1\!\!1_{\Gamma^t}(\beta) P_t H P_t D_\beta P_s D_\alpha f$ is square-integrable for a.a. $\alpha > s$.

(c) $\alpha \mapsto 1\!\!1_{\Gamma^s}(\alpha) P_s H P_s D_\alpha f$ is square-integrable.

But since $t \geq s$, $P_t D_\beta P_s D_\alpha f = \delta_\emptyset(\beta) P_s D_\alpha f$ for a.a. β and (b) is vacuous, so $f \in \operatorname{Dom} \mathbb{E}_s \circ \mathbb{E}_t[H]$ if and only if $f \in \operatorname{Dom} \mathbb{E}_s[H]$, moreover

$$\mathbb{E}_s \circ \mathbb{E}_t[H] f(\omega) = (\mathbb{E}_t[H] P_s D_{\omega_{(s}} f)(\omega_{s)}) = (H P_s D_{\omega_{(s}} f)(\omega_{s)}) = \mathbb{E}_s[H] f(\omega)$$

for a.a. ω. Hence $\mathbb{E}_s \circ \mathbb{E}_t[H] = \mathbb{E}_s[H]$.

There remains to prove that $\mathbb{E}_t \circ \mathbb{E}_s[H] = \mathbb{E}_s[H]$. But $f \in \operatorname{Dom} \mathbb{E}_t \circ \mathbb{E}_s[H]$ means

(a) $P_s D_\beta P_t D_\alpha f \in \operatorname{Dom} H$ for a.a. $\beta > s$ and a.a. $\alpha > t$.

(b) $\beta \mapsto 1\!\!1_{\Gamma^s}(\beta) P_s H P_s D_\beta P_t D_\alpha f$ is square-integrable for a.a. $\alpha > t$.

(c) $\alpha \mapsto 1\!\!1_{\Gamma^t}(\alpha) P_t \mathbb{E}_s[H] P_t D_\alpha f$ is square-integrable.

That is,

(a) $P_s D_\beta f \in \operatorname{Dom} H$ for a.a. $\beta > s$.

(b) $\beta \mapsto 1\!\!1_{\Gamma^s}(\beta)P_s H P_s D_\beta f$ is square-integrable.

(c) $\alpha \mapsto 1\!\!1_{\Gamma^t}(\alpha)P_t D_\alpha \mathbb{E}_s[H]f$ is square-integrable.

The first two conditions exactely mean $f \in \operatorname{Dom} \mathbb{E}_s[H]$; condition (c) is trivial.

(v) From Proposition 5.2(d), all that remains to be proved is that $\mathbb{E}_s[]^*$ is s-adapted under the assumption that, along with H and $\mathbb{E}_s[H]$, it is densely defined. Thus let $f \in V^* := \operatorname{Dom} \mathbb{E}_s[H]^*$ and let $g \in V := \operatorname{Dom} \mathbb{E}_s[H]$. Then, by Proposition 5.2(a)

$$\langle P_s f, \mathbb{E}_s[H]g\rangle = \langle f, \mathbb{E}_s[H]P_s g\rangle = \langle P_s \mathbb{E}_s[H]^* f, g\rangle.$$

Thus $P_s f \in V^*$ and $\mathbb{E}_s[H]^* P_s f = P_s \mathbb{E}_s[H]^* f$. Next note that, for any $k, h \in \mathcal{F}$, $P.k$ is locally Itô-integrable, and $\langle D.k, h\rangle = \langle D.k, P.h\rangle$ is locally integrable with $\int_a^b \langle D_t k, h\rangle\, dt = \langle k, \int_a^b P_t h \, d\chi_t\rangle$. Using this, together with Proposition 4.5 and the fact that $P_t \mathbb{E}_s[H]f = \mathbb{E}_s[H]P_t f$ for $t \geq s$, we obtain

$$\int_s^b dt \langle D_t f, \mathbb{E}_s[H]g\rangle = \langle f, \mathcal{I}_s^b(\mathbb{E}_s[H]P.g)\rangle = \int_s^b dt \langle D_t \mathbb{E}_s[H]^* f, g\rangle,$$

for $b \geq s$. Since b is arbitrary, there is a null subset N_g of $[s, \infty[$ such that $\langle D_t f, \mathbb{E}_s[H]g\rangle = \langle D_t \mathbb{E}_s[H]^* f, g\rangle$ for $t \notin N_g$. Letting g run through a countable family in V, whose linear span is a core for the closure of $\mathbb{E}_s[H]$, we see that for a.a. $t > s$, $D_t f \in V^*$ and $\mathbb{E}_s[H]^* D_t f = D_t \mathbb{E}_s[H]^* f$. Hence $\mathbb{E}_s[H]^*$ is s-adapted. □

In view of property (ii) above, one says that an operator H is *maximally s-adapted* if $\mathbb{E}_s[H] = H$.

Property (v) expresses the sense in which conditional expectation commutes with the adjoint operation. When applied to already s-adapted operators, or maximally s-adapted operators, it gives us the following useful result.

Corollary 5.4 – *Let H be an operator on $\mathcal{F}$ which is densely defined and s–adapted. If $\mathbb{E}_s[H]$ is closable, then*

(a) $\mathbb{E}_s[H^*] = \mathbb{E}_s[H]^*$; (b) $\mathbb{E}_s\left[\overline{H}\right] = \overline{\mathbb{E}_s[H]}$.

In particular, the operators $\mathbb{E}_s[H]^*$ and $\overline{\mathbb{E}_s[H]}$ are maximally s-adapted.

Proof

By (v) and (ii), $\mathbb{E}_s(\mathbb{E}_s[H]^*) \supset \mathbb{E}_s[H]^* \supset \mathbb{E}_s[H^*]$. But $H \subset \mathbb{E}_s[H]$, so $H^* \supset \mathbb{E}_s[H]^*$, therefore $\mathbb{E}_s[H^*] \supset \mathbb{E}_s(\mathbb{E}_s[H]^*)$. Combining these we obtain $\mathbb{E}_s[H^*] \supset \mathbb{E}_s[H]^* \supset \mathbb{E}_s[H^*]$, which gives (a). Since $\mathbb{E}_s[H^*]$ is closed we may apply (a) to H^*:

$$\mathbb{E}_s[(H^*)^*] = \mathbb{E}_s[H^*]^* = (\mathbb{E}_s[H]^*)^*,$$

to obtain (b).

□

Thus, if H is densely defined, closable and maximally s-adapted, then both H^* and $\overline{H}$ are maximally s-adapted too. Let $\mathcal{A}_s^\#$ denote the collection of closed, densely defined and maximally s-adapted operators on $\mathcal{F}$. As complement to Proposition 4.6 we have:

Proposition 5.5 – *The collection $\mathcal{A}_s^\#$ is closed under the Hilbert space adjoint operation, and contains $\mathcal{B}(\mathcal{F}) \cap \mathcal{A}_s$ which is a strongly closed unital $*$-subalgebra of $\mathcal{B}(\mathcal{F})$ isomorphic to $\mathcal{B}(\mathcal{F}_s)$.*

Proof

The first part is contained in Corollary 5.4. Let $H \in \mathcal{B}(\mathcal{F}) \cap \mathcal{A}_s$, then for $u \in \mathcal{F}_s$, $H(u \otimes \delta_\emptyset) = HP_s(u \otimes \delta_\emptyset) = P_s H(u \otimes \delta_\emptyset) = u' \otimes \delta_\emptyset$, for some $u' \in \mathcal{F}_s$. Therefore, for $v \in \mathcal{F}^s$, $H(u \otimes v)(\sigma) = HP_s D_{\sigma_{(s}}(u \otimes v)(\sigma_{s)}) = v(\sigma_{(s)} H(u \otimes \delta_\emptyset)(\sigma_{s)}) = (u' \otimes v)(\sigma)$. It follows that $H = \overline{H_s \otimes I^s}$ for an operator H_s on $\mathcal{B}(\mathcal{F}_s)$. Conversely, if $H_s \in \mathcal{B}(\mathcal{F}_s)$ then $H_s \otimes I^s$ is s-adapted by (x) of Theorem 5.3, and $\overline{H_s \otimes I_s} = \mathbb{E}_s[\overline{H_s \otimes I_s}]$ by the Corollary, so $\overline{H_s \otimes I^s}$ is s–adapted. Clearly the map $H_s \mapsto \overline{H_s \otimes I^s}$ is an isomorphism.

□

V.3. Fock operator processes

A *Fock operator process* is a family of operators $H. = (H_s)_{s \geq 0}$ on $\mathcal{F}$. The *domain of a process* $H.$, denoted $\operatorname{Dom} H.$, is $\cap_{s \geq 0} \operatorname{Dom} H_s$. A Fock operator process is *measurable* (respectively, *continuous*) if $H.f$ is a measurable map (respectively, continuous map) $\mathbb{R}_+ \to \mathcal{F}$ for each $f \in \operatorname{Dom} H.$, and $H.$ is *adapted* if, for each $s \geq 0$, H_s is s-adapted. For a Fock operator process $H.$, we define a process $\widehat{H.}$, as follows:

$$\widehat{H}_s = \mathbb{E}_s[H_s], \quad s \geq 0. \tag{5.1}$$

Thus $\widehat{H.}$ is an adapted Fock operator process and when the original process $H.$ is adapted itself, $\widehat{H}_s \supset H_s$ for each s. Thus, when applied to adapted processes this procedure systematically extends the domains of the constituent operators of the process so that they become *maximally adapted.* This is helpful for dealing with unbounded-operator-valued processes – in particular for providing a robust definition of Fock operator martingale.

A *martingale* is an adapted Fock operator process $H.$ satisfying

$$\mathbb{E}_s[\widehat{H}_t] \subset \widehat{H}_s \text{ for } t \geq s \geq 0. \tag{5.2}$$

By the tower property of conditional expectations (5.2) may be written $\mathbb{E}_s[H_t] \subset \mathbb{E}_s[H_s]$. A martingale $H.$ is *complete with closure* H_∞ if H_∞ is an operator on $\mathcal{F}$ such that

$$\mathbb{E}_s[H_\infty] \subset \widehat{H}_s \text{ for } s \geq 0. \tag{5.3}$$

For any operator H on $\mathcal{F}$, the process $H.$ defined by $H_s = \mathbb{E}_s[H]$ is a complete martingale (by the tower property again), with closure H. Martingales of this form will be called *exact.* Note that closures are non-unique (every martingale has a truly trivial closure). Finally an *adjoint pair of* (adapted) Fock operator processes is a pair of (adapted) processes $(H., H^\dagger_.)$ such that, for each $s \geq 0$, $(H_s, H_s^\dagger)$ is an adjoint pair of (s-adapted) operators. As we have already remarked adaptedness of one of the pair *does not* entail adaptedness of the other. If $H.$ is a Fock operator process, and $\mathrm{Dom}\, H.$ is dense, then by Bessel's equality $H_s^* f = \sum_n \langle H_s e_n, f\rangle e_n$ for any Hilbert basis for $\mathcal{F}$ selected from $\mathrm{Dom}\, H.$, so $H^*_.$ is measurable if $H.$ is.

Let $\mathcal{A}$ denote the collection of adapted Fock operator processes, let $\mathcal{A}^{\#} = \{H. \in \mathcal{A} : H_s \in \mathcal{A}_s^{\#}$ for each $s\}$, let $\mathcal{A}^b = \{H. \in \mathcal{A} : H_s \in \mathcal{B}(\mathcal{F})$ for each $s\}$, and let $\mathcal{M}$ be the collection of Fock operator martingales. From Propositions 4.4 and 5.5 we have:

Proposition 5.6 – *$\mathcal{A}$ is closed under operator sums and products, $\mathcal{A}^{\#}$ is closed under operator adjoints and $\mathcal{A}^b$ is a unital $*$-algebra contained in $\mathcal{A}^{\#}$.*

□

Due to the (unavoidable) inclusion relations involved in the definition of martingales there is a dirth of algebraic properties of $\mathcal{M}$. However, the sum of two exact martingales is a complete martingale, and the collection of bounded-operator-valued martingales forms a linear space closed under the adjoint operation. Moreover the following $*$-subalgebra of $\mathcal{A}^b$ has been investigated in [Att]:

$$\begin{aligned}\mathcal{A}^r = \{H. \in \mathcal{A}^b : \exists \text{ Radon measure } \mu \text{ s.t. } &\|(H_t - H_s)f\|^2 + \|(H_t^* - H_s^*)f\|^2 \\ &+ \|(P_s H_t - H_s)f\| \le \mu([s,t[), \forall t > s > 0, f \in \mathcal{F}_s \text{ s.t. } \|f\| = 1\}.\end{aligned}$$

Elements of $\mathcal{A}^r$ are shown to be expressible as sums of quantum stochasticintegrals of processes in $\mathcal{A}^b$, and the resulting integrals are characterized.

VI. Quantum stochastic integrals

In this section we introduce new definitions of stochastic integrals, of adapted Fock operator processes, with respect to the basic martingales of quantum stochastic calculus. The technical core of the section is on commutation relations between the noncommutative stochastic integrals and sections of the adapted gradient operator. We verify that the integrals produce operator martingales, and that the martingales are complete.

In order to unify the notations, the time integral $\int_0^\infty g_t\, dt$ of a vectors process $(g_t)_{t\ge 0}$ will sometime be denoted $\mathcal{L}(g.)$.

VI.1. Definitions

Let $H.$ be an adapted Fock operator process. For $Q = P$ or D, let $V^Q(H.)$ denote the subspace of $\mathcal{F}$ consisting of those f which satisfy

- $Q_s D_\tau f \in \mathrm{Dom}\, H_s$ for a.a. $(s < \tau)$
- the a.e. defined map $(s,\tau) \mapsto 1\!\!1_{\Gamma^s}(\tau) H_s Q_s D_\tau f$ is measurable $I\!R_+ \times \Gamma \to \mathcal{F}$.

If $f \in V^Q(H.)$ then there is a measurable map $H^Q_. f : \Gamma \times I\!R_+ \to \mathcal{H}_0$, written $(\omega, s) \mapsto H_s^Q f(\omega)$, such that $1\!\!1_{\Gamma_s}(\cdot) H_s^Q f(\cdot \cup \tau)$ is a representative of $H_s Q_s D_\tau f$ for a.a. $(s < \tau)$. The map $H^Q_. f$ is uniquely defined up to a set of measure zero, and satisfies the defining a.e. identity

$$H_s^Q f(\omega) = (H_s Q_s D_{\omega_{(s}} f)(\omega_{s)}). \tag{6.1}$$

We emphasize here that, for each $s \geq 0$, while $H_s^Q f$ is a measurable map $\Gamma \to \mathcal{H}_0$, it need not be square-integrable – in other words, in general $H_s^Q f \notin \mathcal{F}$. Thus $H_\cdot^Q f$ should not be thought of as a Fock vector process – in general it isn't.

However, the following result describes subspaces on which the maps $H_\cdot^Q f$ simplify, and also gives conditions on $H.$ for the spaces $V^Q(H.)$ to have a simple description. Recall that the domain of the stochastic gradient ∇ coincides with that of $\sqrt{N}$ (Theorem 3.11). For any Fock operator process $H.$, define two associated subspaces:

$$V(H.) = \{f \in \operatorname{Dom} H. : H.f \text{ is measurable } \mathbb{R}_+ \to \mathcal{F}\},$$
$$V^\nabla(H.) = \{f \in \operatorname{Dom}\sqrt{N} : \nabla_s f \in \operatorname{Dom} H_s \text{ for a.a. } s; H.\nabla.f \text{ is measurable}\}.$$

Proposition 6.1 – *Let $H.$ be a Fock operator process:*
(a) If $H.$ is adapted then

$$f \in V(H.) \Rightarrow f \in V^P(H.), H_\cdot^P f = H.f;$$
$$f \in V^\nabla(H.) \Rightarrow f \in V^D(H.), H_\cdot^D f = H.\nabla.f.$$

(b) If $H.$ is measurable, then

(i) $V(H.) = \operatorname{Dom} H.$

(ii) If $H.$ is also adapted, then $V^P(H.) \supset \operatorname{Dom} H.$.

(c) If $\operatorname{Dom} H_s$ is dense for each s, $\operatorname{Dom} H_\cdot^*$ is dense, and $H_\cdot^*$ is measurable, then

(i) $V^\nabla(H.) = \{f \in \operatorname{Dom}\sqrt{N} : \nabla_s f \in \operatorname{Dom} H_s$ for a.a. $s\}$.

(ii) If $H.$ is also adapted, then $V^D(H.) = \{f \in \mathcal{F} : D_s D_\tau f \in \operatorname{Dom} H_s$ for a.a. $(s < \tau)\}$.

(d) If $H.$ is measurable and if, for each s, H_s is bounded (and everywhere defined), then

(i) $V^\nabla(H.) = \operatorname{Dom}\sqrt{N}$;

(ii) If $H.$ is also adapted, then $V^D(H.) = V^P(H.) = \mathcal{F}$.

Proof

(ai) This follows easily from the a.e. reproducing property.

(aii) Let $f \in V^\nabla(H.)$. Then $f \in \operatorname{Dom}\sqrt{N}$, for a.a. $(s < \tau)$ $P_s D_\tau \nabla_s f = P_s \nabla_s D_\tau f = D_s D_\tau f$, and for a.a. s, $\nabla_s f \in \operatorname{Dom} H_s$. Thus if $H.$ is adapted, $D_s D_\tau f \in \operatorname{Dom} H_s$ for a.a. $(s < \tau)$, and $1\!\!1_{\Gamma^s}(\tau) H_s D_s D_\tau f = 1\!\!1_{\Gamma^s}(\tau) P_s D_\tau H_s \nabla_s f$, which is a measurable function of (s, τ). Hence $f \in V^D(H.)$ and, for a.a. (s, ω) $H_s^D f(\omega) = H_s \nabla_s f(\omega)$ by the a.e. reproducing property.

(b) This is immediate.

(c) Let $H.$ satisfy the conditions of (c), and let (e_n) be a Hilbert basis for $\mathcal{F}$ selected from $\operatorname{Dom}(H_\cdot^*)$.

(i) If $f \in \operatorname{Dom}\sqrt{N}$ and $\nabla_s f \in \operatorname{Dom} H_s$ for a.a. s then, by Bessel's equality, for a.a. s, $H_s \nabla_s f = \sum_n \langle H_s^* e_n, \nabla_s f \rangle e_n$. But this is manifestly a measurable function of s, therefore $f \in V^\nabla(H.)$.

(ii) If $H.$ is adapted and $f \in \mathcal{F}$ satisfies $D_s D_\tau f \in \operatorname{Dom} H_s$ for a.a. $(s < \tau)$, then by another application of Bessel's equality, for a.a. $(s < \tau)$,

$$\mathbb{1}_{\Gamma^s}(\tau) H_s D_s D_\tau f = \mathbb{1}_{\Gamma^s}(\tau) \sum_n \langle H_s^* e_n, D_s D_\tau f \rangle$$

which is a measurable function of (s, τ). Thus $f \in V^D(H.)$.
(d) This is a special case of (c).

□

The *creation, number, annihilation and time integrals* of an adapted Fock operator process $H.$ are given respectively by the actions:

$$A^\dagger(H.)f = \mathcal{S}(H.^P f) : \omega \mapsto \sum_{s \in \omega} (H_s P_s D_{\omega_{(s}} f)(\omega_{s)})$$

$$N(H.)f = \mathcal{S}(H.^D f) : \omega \mapsto \sum_{s \in \omega} (H_s D_s D_{\omega_{(s}} f)(\omega_{s)})$$

$$A(H.)f = \mathcal{L}(H.^D f) : \omega \mapsto \int_0^\infty (H_s D_s D_{\omega_{(s}} f)(\omega_{s)})\, ds$$

$$T(H.)f = \mathcal{L}(H.^P f) : \omega \mapsto \int_0^\infty (H_s P_s D_{\omega_{(s}} f)(\omega_{s)})\, ds$$

with the following natural domains:

$\operatorname{Dom} A^\dagger(H.) = \{f \in V^P(H.) : H.^P f$ is Skorohod-integrable$\}$;
$\operatorname{Dom} N(H.) = \{f \in V^D(H.) : H^D f$ is Skorohod-integrable$\}$;
$\operatorname{Dom}(A(H.)) = \{f \in V^D(H.)) : H^D f$ is time-integrable$\}$;
$\operatorname{Dom}(T(H.)) = \{f \in V^P(H.) : H^P f$ is time-integrable$\}$;

Recall equation (5.1) defining the extension of an adapted Fock operator process $H.$ to its maximally adapted form $\widehat{H}.$. From the remarks following Proposition 5.2, it follows that $V^Q(H.) = V^Q(\widehat{H.})$ and $\widehat{H.^Q} f = H.^Q f$ for $f \in V^Q(H.)$ and $Q = P$ or D. Therefore each of the quantum stochastic integrals is unaffected by allowing the integrand to achieve its maximally adapted form:

$$\Lambda(H.) = \Lambda(\widehat{H.}) \tag{6.2}$$

for $\Lambda = A^\dagger$, N, A or T.

The following linear relations are clear form the definitions:

$V^Q(H. + K.) \supset V^Q(H.) \cap V^Q(K.); V^Q(\lambda H.) = V^Q(H.); V^Q(0) = \mathcal{F};$
$\Lambda(H. + K.) \supset \Lambda(H.) + \Lambda(K.); \Lambda(\lambda H.) = \lambda\Lambda(H.); \Lambda(0) = 0;$

where $H.$ and $K.$ are adapted Fock operator processes, $\lambda \in \mathbb{C} \setminus \{0\}$, $Q = P$ or D and $\Lambda = A^\dagger$, N, A or T. Multiplicative relations between the quantum stochastic integrals constitute the quantum Itô product formulae, to be described in the final section.

Notice that each of the quantum stochastic integrals is associated with either an adapted derivative or a projection, and with either Skorohod or time integration. It will considerably simplify the development of the basic theory if we forge a unified notation to describe the integrals. Thus to each quantum stochastic integrator Λ we associate $\mathcal{R}^\Lambda \in \{\mathcal{S}, \mathcal{L}\}$ as well as $Q^\Lambda \in \{P, D\}$ as follows: $(\Lambda, \mathcal{R}, Q)$ equals either

$$(A^\dagger, \mathcal{S}, P),\ (N, \mathcal{S}, D),\ (A, \mathcal{L}, D) \quad \text{or} \quad (T, \mathcal{L}, P). \tag{6.3}$$

Thus Λ is determined by the pair $(\mathcal{R}^\Lambda, Q^\Lambda)$ and vice versa. The definitions of the four quantum stochastic integrals are thereby unified:

$$\mathrm{Dom}(\Lambda(H_\cdot)) = \{f \in V^Q(H_\cdot) : H_\cdot^Q f \text{ is } \mathcal{R}\text{-integrable}\}$$
$$\Lambda(H_\cdot) = \mathcal{R}(H_\cdot^Q f)$$

where $Q = Q^\Lambda$, $\mathcal{R} = \mathcal{R}^\Lambda$ and $\Lambda = A^\dagger$, N, A or T. They are frequently used in the sequel.

The following identities are easily established.

Lemma 6.2 – *Let $H_\cdot$ be an adapted Fock operator process, and let $f \in V^Q(H_\cdot)$, where $Q = P$ or D. Then the following relations hold (a.e.):*

$$D_t f, P_t f \in V^Q(\mathbb{1}_{[0,t[} H_\cdot) \tag{6.4}$$
$$D_t H_s^Q f = \mathbb{1}_{[0,t[}(s) H_s^Q D_t f + D_t H_s Q_s f \tag{6.5}$$
$$P_t H_s^Q f = \mathbb{1}_{[0,t[}(s) H_s^Q P_t f + \mathbb{1}_{[t,\infty[}(s) P_t H_s Q_s f \tag{6.6}$$
$$P_t H_t^Q f = H_t Q_t f. \tag{6.7}$$

In particular, $\mathbb{1}_{[0,t[}(s)(D_t H_s^Q f)(\omega)$ gives a version of $\mathbb{1}_{[0,t[}(s)(H_s^Q D_t f)(\omega)$ which is jointly measurable in (s, t, ω).

VI.2. Commutation relations

We next apply the commutation relations defined in III.5 to quantum stochastic integrals. This allows us to deduce adaptedness and martingale properties in the next subsection. It is also the first step towards solving the problems raised by Itô calculus approach to quantum stochastic calculus (see VIII.3).

In the sequel, $\Lambda_t(H_\cdot)$ denotes $\Lambda(K_\cdot)$ with $K_s = \mathbb{1}_{[0,t]}(s) H_s$.

Theorem 6.3 – *Let $H_\cdot$ be an adapted Fock operator process, and let $f \in V^Q(H_\cdot)$, where $Q = Q^\Lambda$ and $\Lambda = A^\dagger$ or N. Then the following are equivalent:*

(a) $f \in \mathrm{Dom}\,\Lambda(H_\cdot)$

(b) (i) $D_t f \in \mathrm{Dom}\,\Lambda_t(H_\cdot)$ for a.a. t;

(ii) $t \mapsto \Lambda_t(H_\cdot) D_t f + H_t Q_t f$ is Itô-integrable.

When these hold, we have the a.e. identity

$$D_t \Lambda(H_\cdot) f = \Lambda_t(H_\cdot) D_t f + H_t Q_t f. \tag{6.8}$$

Proof

In view of (6.7), Proposition 3.12 applies to $H_\cdot^Q f$. If $f \in \mathrm{Dom}\,\Lambda(H_\cdot)$ then $H_\cdot^Q f$ is Skorohod-integrable so $\mathbb{1}_{[0,t[}(\cdot) D_t H_\cdot^Q f$ is Skorohod-integrable for a.a. t, and

$$D_t \mathcal{S}(H_\cdot^Q f) = \mathcal{S}_0^t(D_t H_\cdot^Q f) + P_t H_t^Q f,$$

which is square-integrable in t. By (6.5) and (6.7), f satisfies (b), and the a.e. identity (6.8) holds. Conversely, if f satisfies (b) then, since (6.14) implies that

$$\Lambda_t(H.)D_t f(\omega) = \mathcal{S}_0^t(H_{\cdot}^Q D_t f) = \mathcal{S}(D_t H_{\cdot}^Q f),$$

Proposition 3.12 gives the Skorohod-integrability of $H_{\cdot}^Q f$ - in other words $f \in \operatorname{Dom}\Lambda(H.)$.

□

Theorem 6.4 – *Let $H.$ be an adapted Fock operator process, and let $f \in V^Q(H.)$ be such that $(H.Q.f)(\emptyset)$ is integrable, where $Q = Q^\Lambda$ and $\Lambda = A$ or T. Then the conditions (a) and (b) are equivalent:*

(a) (i) $f \in \operatorname{Dom}\Lambda(H.)$; and (ii) $H.Q.f$ is time-integrable.

(b) (i) $D_t f \in \operatorname{Dom}\Lambda_a(H.)$ for a.a. t;

(ii) $D_t H.Q.f$ is time-integrable for a.a. t;

(iii) the maps $t \mapsto \Lambda_t(H.)D_t f$ and $t \mapsto \mathcal{L}(D_t H.Q.f)$ are Itô-integrable.

When these hold we have the a.e. identity,

$$D_t\Lambda(H.)f = \Lambda_t(H.)D_t f + \mathcal{L}(D_t H.Q.f). \tag{6.9}$$

Proof

In view of (6.7), $H_{\cdot}^Q f(\emptyset)$ is integrable and so Proposition 3.13 applies. If $f \in \operatorname{Dom}\Lambda(H.)$ and $H.Q.f$ is time-integrable then, by Proposition 3.13, both $D_t H_{\cdot}^Q f$ and $D_t H.Q.f$ are time-integrable for a.a. t, and

$$\mathcal{L}(D_t H_{\cdot}^Q f) = D_t\mathcal{L}(H_{\cdot}^Q f) = D_t\Lambda(H.)f;\ \mathcal{L}(D_t H.Q.f) = D_t\mathcal{L}(H.Q.f);$$

both of which are square-integrable in t. By (6.5) therefore, $D_t f \in \operatorname{Dom}\Lambda_t(H.)$, (6.9) holds and both $\Lambda_t(H.)D_t f$ and $\mathcal{L}(D_t H.Q.f)$ are square-integrable in t. Thus f satisfies (b). Conversely, if f satisfies (b) then, by Proposition 3.13, $H.Q.f$ is time-integrable and, by (6.5) $D_t H_{\cdot}^Q f = \mathbb{1}_{[0,t[}(\cdot)H_{\cdot}^Q D_t f + D_t H.Q.f$, which is time-integrable by (bi) and (bii), with time-integral $\Lambda_t(H.)D_t f + \mathcal{L}(D_t H.Q.f)$, which is Itô-integrable by (biii). Hence, using Proposition 3.13 once more, $H_{\cdot}^Q f$ is time-integrable – in other words $f \in \operatorname{Dom}(\Lambda(H.))$, so that (a) holds.

□

Proposition 6.5 – *Let H be an adapted Fock operator process, and let $t \geq 0$.*

(a) If $f \in \operatorname{Dom}\Lambda(H.)$, where $\Lambda \in A^\dagger$ or N, then $P_t f \in \operatorname{Dom}\Lambda_t(H.)$ and

$$\Lambda_t(H.)P_t f = P_t\Lambda(H.)f. \tag{6.10}$$

(b) If $f \in \operatorname{Dom}\Lambda(H.)$, where $\Lambda = A$ or T, then z^t is time-integrable and

$$\mathcal{L}(z^t) = P_t\Lambda(H.)f \tag{6.11}$$

where $z^t = (\mathbb{1}_{[0,t[}(\cdot)H_{\cdot}^Q P_t f + \mathbb{1}_{[t,\infty[}(\cdot)P_t H.Q.f)$, and $Q = Q^\Lambda$.

(c) If $f \in \mathcal{F}$ and $P_t f \in \mathrm{Dom}(\Lambda(H.))$, where $\Lambda = A^\dagger$, N or A, then $P_t f \in \mathrm{Dom}(\Lambda_t(H.))$ and

$$\Lambda_t(H.)P_t f = P_t \Lambda(H.)P_t f. \tag{6.12}$$

(d) If $\Lambda = N$ or A, then the subspaces $\mathcal{F}_t \cap \mathrm{Dom}\,\Lambda(H.)$ and $\mathcal{F}_t \cap \mathrm{Dom}\,\Lambda_t(H.)$ coincide, and

$$\Lambda_t(H.)P_t f = \Lambda(H.)P_t f \tag{6.13}$$

whenever $P_t f \in \mathrm{Dom}\,\Lambda_t(H.)$.

Proof

Each of these commutation relations follows easily from Proposition 3.14 by using (6.6).

□

VI.3 Adaptedness and martingale properties

The next two results show that our definitions synchronize satisfactorily.

Proposition 6.6 – *Let $H.$ be an adapted Fock operator process, and let $t \geq 0$. Then each of the operators $A_t^\dagger(H.)$, $N_t(H.)$, $A_t(H.)$ and $T_t(H.)$ is u-adapted for all $u \geq t$.*

Proof

Let $f \in \mathrm{Dom}\,\Lambda_t(H.)$ and let $u \geq t$. First note that, by (6.4), $P_u f, D_u f \in V^Q(\mathbb{1}_{[0,\infty[}(\cdot)H.)$. Since $\Lambda_u(\mathbb{1}_{[0,t[}(\cdot)H.) = \Lambda_t(H.)$ and $\mathbb{1}_{[u,\infty[}(\cdot)P_u \mathbb{1}_{[0,t[}(\cdot)H.Q.f = 0$. Theorem 6.4 implies that $P_u f \in \mathrm{Dom}\,\Lambda_t(H.)$ and $P_u \Lambda_t(H.)f = \Lambda_t(H.)P_u f$. Since $\mathbb{1}_{[0,t[}(u)H_u Q_u f = 0$, Theorem 6.3 implies that $D_u f \in \mathrm{Dom}\,\Lambda_t(H.)$ and $\Lambda_t(H.)D_u f = D_u \Lambda_t(H.)f$ for $\Lambda = A^\dagger$ or N. For $\Lambda = A$ or T, (6.5) implies that $D_u H_s^Q f = H_s^Q D_u f$ for $s < t$. Therefore, by Proposition 3.13, $\mathbb{1}_{[0,t[}(\cdot)H.^Q D_u f$ is time-integrable, so that $D_u f \in \mathrm{Dom}\,\Lambda_t(H.)$, and

$$\Lambda_t(H.)D_u f = \mathcal{L}_t(H.^Q D_u f) = \mathcal{L}_t(D_u H.^Q f) = D_u \mathcal{L}_t(H.^Q f) = D_u \Lambda_t(H.)f.$$

This completes the proof.

□

Theorem 6.7 – *Let $H.$ be an adapted Fock operator process. Then $(\Lambda_t(H.))_{t\geq 0}$ is a complete martingale with closure $\Lambda(H.)$, for $\Lambda = A^\dagger$, N or A.*

Proof

Let $t \geq 0$, let $u \in [t, \infty]$ and let $f \in \mathrm{Dom}(\mathbb{E}_t[\Lambda_u(H.)])$. Then $P_t D_\beta f \in \mathrm{Dom}(\Lambda_u(H.))$, the map $\beta \mapsto \mathbb{1}_{\Gamma^t}(\beta) P_t \Lambda_u(H.)P_t D_\beta f$ is square-integrable $\Gamma \to \mathcal{F}$, and $\mathbb{E}_t[\Lambda_u(H.)]f(\omega) = (P_t \Lambda_u(H.)P_u D_{\omega_{(t}} f)(\omega_{t)})$ for a.a. ω. Thus, by Theorem 6.5(c), $f \in \mathrm{Dom}(\mathbb{E}_t[\Lambda_t(H.)])$ and $\mathbb{E}_t[\Lambda_t(H.)]f = \mathbb{E}_t[\Lambda_u(H.)]f$. This shows that

$$\mathbb{E}_t[\Lambda_u(H.)] \subset \mathbb{E}_t[\Lambda_t(H.)], \tag{$*$}$$

and so $\Lambda.(H.)$ is a complete martingale, with closure $\Lambda(H.)$.

□

In view of Theorem 6.5(d), equality holds in the (complete) martingale inclusion relations (∗) in the cases $\Lambda = N$ or A:

$$\mathbb{E}_t[N(H.)] = \mathbb{E}_t[N_t(H.)]; \quad \mathbb{E}_t[A(H.)] = \mathbb{E}_t[A_t(H.)].$$

In other words the martingales $\mathbb{E}.[N.(H.)]$ and $\mathbb{E}.[A.(H.)]$ are exact.

VII. Restricted domains and adjoint relations

In Section VIII we shall see and exploit maximality of the domains of definition of the quantum stochastic integrals introduced above. In this section we introduce restricted domains for quantum stochastic integrals, which lead to good adjoint relations, and also Itô product relations to be proved in Section IX.

Let $H.$ be an adapted Fock operator process and recall the definitions (6.1) and (6.2). We define the *restricted quantum stochastic-integrals* ${}^R\Lambda(H.)$ as follows:

$$\operatorname{Dom} {}^R\Lambda(H) = \{f \in V^Q(H.) : H^Q f \text{ is absolutely } \mathcal{R}\text{-integrable}\},$$

where $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$ are as given by (6.3), and absolute $\mathcal{R}$-integrability is defined in Definition 3.1.

A simplifying feature of restricted quantum stochastic-integrals is the inclusions:

$$\operatorname{Dom} {}^R\Lambda_s(H.) \supset \operatorname{Dom} {}^R\Lambda_t(H.)$$

for $s \leq t$. Another is that the processes $t \mapsto {}^R\Lambda_t(H.)$ are continuous (see below).

Lemma 7.1 – *Let $f \in \operatorname{Dom} {}^R\Lambda(H.)$ where $H.$ is an adapted Fock operator process and Λ is a quantum stochastic-integrator, and let $Q = Q^\Lambda$. Then $H.Q.f$ is an adapted Fock vector process, which is*

- *Itô-integrable if $\Lambda = A^\dagger$ or N;*
- *absolutely time-integrable if $\Lambda = A$ or T.*

Proof

Since $f \in V^Q(H.)$, $s \mapsto H_s Q_s f = \mathbb{1}_{\Gamma^s}(\emptyset) H_s Q_s D_\emptyset f$ is a measurable map $\mathbb{R}_+ \to \mathcal{F}$. Since $H_s Q_s f = \mathbb{1}_{\Gamma_s} H^Q_s f$, $H.Q.f$ is (absolutely) Skorohod-integrable if $\Lambda = A^\dagger$ or N, and is absolutely time-integrable if $\Lambda = A$ or T. Since $H.Q.f$ is an adapted Fock vector process, the result follows.

□

Lemma 7.2 – *Let $X = {}^R\Lambda(H.)$ for some adapted Fock operator process $H.$ and quantum stochastic-integrator Λ. If $f \in \operatorname{Dom} X$ then*

(i) $P_t f \in \operatorname{Dom} X_t$ for all $t \geq 0$; (ii) $D_t f \in \operatorname{Dom} X_t$ for a.a. t.

Proof

Let $x(\omega, s) = \|H^Q_s f(\omega)\|$, so that x is $\mathcal{R}$-integrable, where $Q = Q^\Lambda$ and $\mathcal{R} = \mathcal{R}^\Lambda$ are given by (6.3). By Proposition 3.14, $\mathbb{1}_{[0,t[}(\cdot) P_t x.$ is $\mathcal{R}$-integrable, and by (6.5),

$$\mathbb{1}_{[0,t[}(s) P_t x_s(\omega) = \mathbb{1}_{[0,t[}(s) \|H^Q_s P_t f(\omega)\|$$

so that $P_t f \in \operatorname{Dom} {}^R\Lambda_t(H.)$ for each $t \geq 0$. Using Proposition 3.13 and (6.6) instead, the above argument yields (ii).

□

Reminder: We use the notations $P_t f$ and $D_t f$ for *any* maps f whose source space is Γ.

Proposition 7.3 – *Let $H.$ be an adapted Fock operator process, and let Λ be one of the quantum stochastic-integrators. Then*

(a) ${}^R\Lambda_t(H.)$ is u-adapted for each $u \geq t$;

(b) if $\Lambda = A^\dagger$, N or A then ${}^R\Lambda.(H.)$ is a complete martingale with closure ${}^R\Lambda(H.)$.

Proof

Let $X = {}^R\Lambda(H.)$ – both as operator and as process – let $(\mathcal{R}, Q)$ be the pair associated with Λ according to (6.3), and let $t \geq 0$.

(a) In view of Proposition 6.6 it suffices to show that $\operatorname{Dom}(X_t)$ is a u-adapted subspace, for each $u \geq a$. Let $f \in \operatorname{Dom}(X_t)$, then the map $k : (\omega, s) \mapsto \mathbb{1}_{[0,t[}(s)\|H_s^Q f(\omega)\|$ is $\mathcal{R}$-integrable. By (6.4), (6.5) and (6.6), if $v \geq u \geq t$ then $P_u f, D_v f \in V^Q(\mathbb{1}_{[0,t[}(\cdot)H.)$,

$$\mathbb{1}_{[0,t[}(s)\|H_s^Q P_u f(\omega)\| = (P_u k_s)(\omega) \leq k_s(\omega), \tag{1}$$

$$\mathbb{1}_{[0,t[}(s)\|(H_s^Q D_v f)(\omega)\| = (D_v k_s)(\omega), \tag{2}$$

for a.a. (ω, v). By (1), $P_u f \in \operatorname{Dom}(X_t)$, and by (2), together with Propositions 3.12 and 3.13, $D_v f \in \operatorname{Dom}(X_t)$ for a.a. v. Thus $\operatorname{Dom}(X_t)$ is u-adapted.

(b) By (a), $(X_s)_{s\geq 0}$ is an adapted Fock operator process so that, in view of Proposition 6.7, it suffices to show that $\operatorname{Dom}(\mathbb{E}_t[X.]) \subset \operatorname{Dom}(\mathbb{E}_t[X_t])$. Let $f \in \operatorname{Dom}(\mathbb{E}_t[X.])$, then $P_t D_\tau f \in \operatorname{Dom} X. \subset \operatorname{Dom}(X_t)$ and, since $\Lambda.(H.)$ is a complete martingale with closure $\Lambda(H.)$,

$$P_t X_t P_t D_\tau f = P_t \Lambda_t(H) P_t D_\tau f = P_t \Lambda(H) P_t D_\tau f = P_t X P_t D_\tau f$$

for a.a. $\tau > t$. Therefore $\tau \mapsto \mathbb{1}_{\Gamma^t}(\tau) P_t X_t P_t D_\tau f$ is square-integrable – in other words $f \in \operatorname{Dom} \mathbb{E}_t[X_t]$. This gives the required inclusion.

□

Proposition 7.4 – *Let $H.$ be an adapted Fock operator process, and let Λ be a quantum stochastic-integrator. Then the process ${}^R\Lambda.(H.)$ is continuous.*

Proof

Let $X. = {}^R\Lambda.(H.)$, let $\mathcal{R} = \mathcal{R}^\Lambda$ and $Q = Q^\Lambda$ according to (6.3), and let $f \in \operatorname{Dom} X.$ Writing k for the map $(\omega, s) \mapsto \|H_s^Q f(\omega)\|$, we have

$$\|X_u f - X_t f\| = \|\mathcal{R}_t^u(H_\cdot^Q f)\| \leq \mathcal{R}_t^u(k.).$$

Thus, if $\Lambda = A^\dagger$ or N,

$$\|X_u f - X_t f\|^2 \leq \int_t^u \int d\omega \{k_s(\omega)\}^2\, ds + \int_t^u \int_t^u \int_\Gamma k_s(\omega \cup t) k_t(\omega \cup s)\, d\omega\, dt\, ds,$$

which is finite by Proposition 3.3. If $\Lambda = A$ or T, then

$$\|X_u f - X_t f\| \leq \int_\Gamma \left\{ \int_t^u k_s(\omega)\, ds \right\}^2 d\omega < \infty.$$

Thus continuity follows in all four cases by Monotone Convergence.

□

Our next result is an extension of the *First Fundamental Formula* for quantum stochastic calculus ([Par]) beyond exponential domains.

Proposition 7.5 –

(a) Let $H.$ be an adapted Fock operator process. If $f \in \operatorname{Dom}{}^R\Lambda(H.))$ then for all $g \in \mathcal{F}$, the map

$$(s, \beta) \mapsto 1\!\!1_{\Gamma^s}(\beta)\langle H_s Q_s D_\beta f, R_s D_\beta g\rangle \tag{7.1}$$

is integrable and

$$\int_0^\infty \int_\Gamma 1\!\!1_{\Gamma^s}(\beta)\langle H_s Q_s D_\beta f, R_s D_\beta g\rangle\, d\beta\, ds = \langle \Lambda(H.) f, g\rangle. \tag{7.2}$$

Here $(Q, R) = (P, D), (D, D), (D, P)$ or (P, P), respectively, for $\Lambda = A^\dagger$, N, A or T.

(b) Let $(H., H.^\dagger)$ be an adjoint pair of adapted Fock operator processes. If $f \in \operatorname{Dom}\Lambda(H.)$ and $g \in \operatorname{Dom}\Lambda^\dagger(H^\dagger)$ (with $(A^dagger)^\dagger = A, N^\dagger = N, (A)^\dagger = A^\dagger$ and $T^\dagger = T$) and the map (7.1) is integrable, then

$$\langle \Lambda(H.) f, g\rangle = \langle f, \Lambda^\dagger(H^\dagger) g\rangle. \tag{7.3}$$

Proof

In case (a) straightforward calculation leads to the estimate

$$\int_0^\infty \int_\Gamma 1\!\!1_{\Gamma^s}(\beta)|\langle H_s Q_s D_\beta f, R_s D_\beta g\rangle|\, d\beta\, ds \leq \int_\Gamma h(\omega)\|g(\omega)\|,\, d\omega$$

where

$$h(\omega) = \sum_{s\in\omega} \|(H_s Q_s D_{\omega_{(s}} f)(\omega_{s)})\|$$

if $\Lambda = A^\dagger$ or N, and

$$h(\omega) = \int_0^\infty \|(H_s Q_s D_{\omega_{(s}} f)(\omega_{s)})\|\, ds$$

if $\Lambda = A$ or T. Similar calculation also reveals the identity (7.2).

(b) If (7.1) is integrable then

$$\langle (H_s Q_s D_{\omega_{(s}} f)(\omega_{s)}), (R_s D_{\omega_{(s}} g)(\omega_{s)})\rangle = \langle (Q_s D_{\omega_{(s}} f)(\omega_{s)}), (H_s^\dagger R_s D_{\omega_{(s}} g)(\omega_{s)})\rangle$$

is an identity of integrable functions of (ω, s), which integrates up to (7.3).

□

Corollary 7.6 – *Let $(H, H^\dagger)$ be an adjoint pair of adapted Fock operator processes.*

(a) $({}^R\Lambda.(H^\dagger_\cdot), {}^R\Lambda^\dagger_\cdot(H.))$ is also an adjoint pair of adapted Fock operator processes.

(b) If ${}^R\Lambda(H.)$ is densely defined, then

$$({}^R\Lambda(H.))^* \supset \Lambda^\dagger(H^\dagger_\cdot).$$

The next result is an Integration by Parts Lemma which contains the essential part of one form of the quantum Itô product formula described in the final section. It is an extension of the *Second Fundamental Formula* for quantum stochastic calculus ([Par]), beyond exponential domains.

Theorem 7.7 – *Let $F^i_\cdot$ be adapted Fock operator processes, let Λ^i be quantum stochastic-integrators, and let $X^i = {}^R\Lambda^i(F^i_\cdot)$, for $i = 1, 2$. If $f^i \in \operatorname{Dom} X^i_\cdot$, then*

$$\begin{aligned}\langle X^1 f^1, X^2 f^2\rangle = \int_0^\infty \int_\Gamma \mathbb{1}_{\Gamma^t}(\beta)\Big\{&\langle F^1_t Q^1_t D_\beta f^1, X^2_t R^1_t D_\beta f^2\rangle \\ &+ \langle X^1_t R^2_t D_\beta f^1, F^2_t Q^2_t D_\beta f^2\rangle + \varepsilon(R^1, R^2)\langle F^1_t Q^1_t D_\beta f^1, F^2_t Q^2_t D_\beta f^2\rangle\Big\}\, d\beta\, dt,\end{aligned}$$

where ε equals 1 if $\Lambda^1, \Lambda^2 \in \{A^\dagger, N\}$ ($R^1 = R^2 = D$), and equals 0 otherwise, and the pair $(R^i, Q^i) = (R^{\Lambda^i}, Q^{\Lambda^i})$ is given by (3.3).

Proof

First note that, since X^i is a restricted domain quantum stochastic-integral, $f^i \in \operatorname{Dom} X^i_u$ for each u and, by Lemma 7.1, $F^i_\cdot Q^i_\cdot f^i$ is Itô-integrable if $\Lambda^i = A^\dagger$ or N and is absolutely time-integrable if $\Lambda^i = A$ or T. Moreover, successive application of Theorems 6.3 and 6.4, together with Theorem 6.5 gives for $R = P$ or D,

$$D_u f^i \in \operatorname{Dom} X^i_u \quad \text{so} \quad D_\beta f^i \in \operatorname{Dom} X^i_{\wedge\beta} \quad \text{so} \quad R_t D_\beta f^i \in \operatorname{Dom}(X^i_t)$$

for a.a. u, β and $(t < \beta)$. Therefore, since also $f^i \in V^{Q_i}(F^i_\cdot)$, each of the expressions in the integrand is a.e. well-defined.

Let $\mathcal{R}^i = \mathcal{S}$ if $\Lambda^i = A^\dagger$ or N and $\mathcal{L}$ otherwise. For the rest of the proof we divide the possibilities into four cases, and it is convenient to remove the clutter of superscripts from the argument by substituting as follows:

$$\begin{gathered}F = F^1, G = F^2, X = X^1, Y = X^2, f = f^1, g = f^2, \\ Q = Q^1, Q' = Q^2, R = R^1, R' = R^2, \mathcal{R} = \mathcal{R}^1, \mathcal{R}' = \mathcal{R}^2.\end{gathered}$$

We therefore have to estblish the identities

$$\begin{aligned}\langle Xf, Yg\rangle = \int_0^\infty \int_\Gamma \mathbb{1}_{\Gamma^t}(\beta)\{&\langle F_t Q_t D_\beta f, Y_t R_t D_\beta g\rangle \\ &+ \langle X_t R'_t D_\beta f, G_t Q'_t D_\beta g\rangle + \mathcal{E}(R, R')\langle F_t Q_t D_\beta f, G_t Q'_t D_\beta g\rangle\}\, d\beta\, dt. \quad *\end{aligned}$$

We have

$$\langle Xf, Yg\rangle = \langle \mathcal{R}(F^Q_\cdot f), \mathcal{R}'(G^{Q'}_\cdot g)\rangle,$$

in which $F^Q_{\cdot} f$ is absolutely $\mathcal{R}$-integrable, and $G^{Q'} g$ is absolutely $\mathcal{R}'$-integrable.

Case (a): $\Lambda^1, \Lambda^2 \in \{A, T\}$. Then $R = R' = P$, $F^Q_{\cdot} f$ and $G^{Q'}_{\cdot} g$ are absolutely time-integrable, and Fubini's Theorem ensures both the integrability of the function $\Phi : (\omega, t, u) \mapsto \langle F^Q_t f(\omega), G^{Q'}_u(\omega)\rangle$, and that its integral is $\langle Xf, Yg\rangle$. Integrating Φ first over the region $\{t < u\}$ using the u-adaptedness of X_u, the a.e. reproducing property (3.8), (6.5) and (6.7), gives

$$\begin{aligned}
&\int_\Gamma \int_0^\infty \langle X_u f(\omega), G^{Q'}_u g(\omega)\rangle \, du \, d\omega \\
&\quad = \int_0^\infty \int_\Gamma \langle (X_u P_u D_{\omega_{(u}} f)(\omega_{u)}), (P_u G^{Q'}_u D_{\omega_{(u}} g)(\omega_{u)})\rangle \, d\omega \, du \\
&\quad = \int_0^\infty \int_\Gamma \mathbb{1}_{\Gamma^t}(\beta) \langle X_t R'_t D_\beta f, G_t Q'_t D_\beta g\rangle \, d\beta \, dt.
\end{aligned}$$

The integral of Φ over the region $\{u < t\}$ may be obtained by the same argument via complex conjugation, and the sum of the two agrees with $(*)$.

Case (b): $\Lambda^1 \in \{A^\dagger, N\}$, $\Lambda^2 \in \{A, T\}$. Then $R = D$ and $R' = P$; $F^Q_{\cdot} f$ is absolutely Skorohod-integrable and $G^{Q'} g$ is absolutely time-integrable; moreover, Fubini's Theorem together with the $\not{\int}$-Lemma ensure both the integrability of the function $\Psi : (\omega, t, u) \mapsto \langle F^Q_t f(\omega), G^{Q'}_u(\omega \cup t)\rangle$, and that the value of the integral is $\langle Xf, Yg\rangle$. Integrating Ψ over the region $\{t < u\}$, and arguing as in Case (a), gives

$$\begin{aligned}
&\int_0^\infty \int_\Gamma \sum_{t \in \alpha} \mathbb{1}_{[0,u[}(t) \langle F^Q_t f(\alpha \setminus t), G^{Q'}_u(\alpha)\rangle \, d\alpha \, du \\
&\quad = \int_0^\infty \int_\Gamma \langle X_u f(\alpha), G^{Q'}_u(\alpha)\rangle \, d\alpha \, du \\
&\quad = \int_0^\infty \int_\Gamma \mathbb{1}_{\Gamma^t}(\beta) \langle X_t R'_t D_\beta f, G_t Q'_t D_\beta g\rangle \, d\beta \, dt.
\end{aligned}$$

Integrating Ψ over the region $\{u < t\}$ we have, since $D_t D_\beta g \in \mathrm{Dom}(Y_t)$ for a.a. $(t < \beta)$,

$$\begin{aligned}
&\int_0^\infty \int_\Gamma d\omega \int_\Gamma \mathbb{1}_{[0,t[}(u) \langle F^Q_t f(\omega), (G^{Q'}_u D_t D_{\omega_{(t}} g)(\omega_{t)})\rangle \, d\beta \, d\omega \, dt \\
&\quad = \int_0^\infty \int_\Gamma \langle (P_t F^Q_t D_{\omega_{(t}} f)(\omega_{t)}), (Y_t D_t D_{\omega_{(t}} g)(\omega_{t)})\rangle \, d\omega \, dt \\
&\quad = \int_0^\infty \int_\Gamma \mathbb{1}_{\Gamma^t}(\beta) \langle F_t Q_t D_\beta f, Y_t R_t D_\beta g\rangle \, d\beta \, dt.
\end{aligned}$$

Therefore the result holds in this case.

Case (c): $\Lambda^1 \in \{A, T\}$ and $\Lambda^2 \in \{A^\dagger, N\}$. This is simply the complex conjugate of Case (b).

Case (d): $\Lambda^1, \Lambda^2 \in \{A^\dagger, N\}$. Then $R = R^1 = D$ so that $\varepsilon = 1$, $F^Q_{\cdot} f$ and $G^{Q'}_{\cdot} g$ are Skorohod integrable and the Skorohod isometry (3.1) ensures that both of the maps $\Phi : (\omega, t) \mapsto \langle F^Q_t f(\omega), G^{Q'}_t g(\omega)\rangle$ and $\Psi : (\omega, t, u) \mapsto \langle F^Q_t f(\omega \cup u), G^{Q'}_u g(\omega \cup t)\rangle$

are integrable, and also that the sum of their integrals is $\langle Xf, Yg\rangle$. The integral of Φ is simply $\int_0^\infty \int_\Gamma 1\!\!1_{\Gamma^t}(\beta)\langle F_t Q_t D_\beta f, G_t Q'_t D_\beta g\rangle \, d\beta \, dt$ by the $\oint$-Lemma. Since $D_u D_\gamma f \in \operatorname{Dom} X_b$ for a.a. $(u < \gamma)$, the integral of Ψ over the region $\{t < u\}$ is

$$\begin{aligned}
&\int_0^\infty \int_\Gamma \int_0^\infty 1\!\!1_{[0,u[}(t)\langle (F_t^Q D_u D_{\omega_{(u}} f)(\omega_{u)}), Q_u^{Q'} g(\omega \cup t)\rangle \, dt \, d\omega \, du \\
&\quad = \int_0^\infty \int_\Gamma \sum_{t \in \alpha_{u)}} 1\!\!1_{[0,u[}(t)\langle (F_t^Q D_u D_{\alpha_{(u}} f)(\alpha_{u)} \setminus t), (G_u^{Q'} D_{\alpha_{(u}} g)(\alpha_{u)})\rangle \, d\alpha \, du \\
&\quad = \int_0^\infty \int_\Gamma \langle (X_u D_u D_{\alpha_{(u}} f)(\alpha_{u)}), (P_u G_u^{Q'} D_{\alpha_{(u}} g)(\alpha_{u)})\rangle \, d\alpha \, du \\
&\quad = \int_0^\infty 1\!\!1_{\Gamma^t}(\beta)\langle X_t D_t D_\beta f, G_t Q'_t D_\beta g\rangle \, d\beta \, dt.
\end{aligned}$$

Again the integral of Ψ over the region $\{u < t\}$ is given by symmetry, and yields the first term in $(*)$. Thus the result holds in this final case too.

□

VIII. Relation to previous formulations

In this section we show that the integrals defined in the previous two sections are consistent with previous formulations. Specifically we prove that our integrals extend the Hudson-Parthasarthy integrals beyond exponential domains, while agreeing with them on these domains. We also show that the noncausal QS integrals, defined in [Bel] and [Lin], yield restrictions of our integrals when applied to adapted Fock operator processes. Specifically our use of the adapted gradient overcomes the domain constraint on annihilation and number integrals in the *noncausal formulation*, namely that they are only defined on (part of) Dom $\sqrt{N}$ – even when the integrals yield bounded operators. The other aspect of the extension from the noncausal to our formulation is the use of more refined integrability requirements in the definitions. Finally, and most importantly, we show that three questions immediately arising from the approach to quantum stochastic calculus through classical stochastic calculus ([A-M]) are all solved by the our formulation. In the *Itô calculus formulation*, each quantum stochastic integral is defined only implicitly via a system of abstract stochastic differential equations. The questions begged are: is there a solution, if so is it unique, and what is the natural (maximal) domain of the resulting operator? In [A-M], some sufficient conditions were given, but the answers in full generalty were not know up to now. A consequence of our solution to these problems is that the implicit definitions, which have already proved useful in elucidating the structure of quantum semimartingales, now have a far greater scope of applicability.

The exponential vector formulation of quantum stochastic integrals is subsumed by each of the noncausal and the Itô calculus formulations. Nevertheless, since the exponential vector formulation is still currently the one most used, we begin by showing how it is covered by the our formulation.

VIII.1. Exponential vector formulation

Let $(V_0, \mathcal{M})$ be an admissible pair – that is s-admissible for each $s \geq 0$ (see IV.2). An adapted $(V_0, \mathcal{M})$-process, in the sense of Hudson and Parthasarthy ([HP1], [Par]), is precisely an adapted Fock operator process $H_\cdot$, in our sense, which is measurable and has the domain of each of its constituent operators prescribed: $\operatorname{Dom} H_s = V_0 \otimes \mathcal{E}(\mathcal{M})$ for each $s \geq 0$. If $F_\cdot$, $G_\cdot$, $H_\cdot$ and $K_\cdot$ are four such processes which satisfy the integrability conditions

$$\int_0^t \{\|F_s v\varepsilon_\varphi\|^2 + |\varphi(s)|^2 \|G_s v\varepsilon_\varphi\|^2 + |\varphi(s)|\ \|H_s v\varepsilon_\varphi\| + \|K_s v\varepsilon_\varphi\|\}\, ds < \infty \quad (8.1)$$

for all $v \in V_0$, $\varphi \in \mathcal{M}$ and $t \geq 0$, then there is a unique $(V_0, \mathcal{M})$-process $X_\cdot$, denoted

$$X_t = \int_0^t F_s\, dA_s + \int_0^t G_s\, dN_s + \int_0^t H_s\, dA_s + \int_0^t K_s\, ds, \quad (8.2)$$

satisfying

$$\langle u\varepsilon_\psi, X_t v\varepsilon_\varphi\rangle = \int_0^t \langle u\varepsilon_\psi, \{\overline{\psi(s)} F_s + \overline{\psi(s)}\varphi(s) G_s + \varphi(s) H_s + K\} v\varepsilon_\varphi\rangle\, ds$$

for all $u \in \mathcal{H}_0$, $\psi \in L^2(\mathbb{R}_+)$, $v \in V_0$, $\varphi \in \mathcal{M}$ and $t \geq 0$. In particular, $\big(\int_0^t K_s\, ds\big) v\varepsilon_\psi$ and $\big(\int_0^t H_s\, dA_s\big) v\varepsilon_\varphi$ are the Bochner integrals $\int_0^t K_s v\varepsilon_\varphi\, ds$ and $\int_0^t \varphi(s) H_s v\varepsilon_\varphi\, ds$, respectively.

Theorem 8.1 – *Let $F_\cdot$, $G_\cdot$, $H_\cdot$ and $K_\cdot$ be adapted $(V_0, \mathcal{M})$-processes satisfying the local integrability conditions (8.1), and let $X_\cdot$ be the adapted $(V_0, \mathcal{M})$-process given by (8.2). Then, for each $t \geq 0$,*

$$X_t \subset (A_t^\dagger(F_\cdot) + N_t(G_\cdot) + {}^R A_t(H_\cdot) + {}^R T_t(K_\cdot)).$$

Proof

Let $f = v\varepsilon_\varphi$, where $v \in V_0$ and $\varphi \in \mathcal{M}$. By Proposition 6.1, any adapted $(V_0, \mathcal{M})$ process $Y_\cdot$, satisfies $V_0 \otimes \mathcal{E}(\mathcal{M}) = V(Y_\cdot) \cap V^\nabla(Y_\cdot) \subset V^P(Y_\cdot) \cap V^D(Y_\cdot)$, $Y_\cdot^P f = Y_\cdot f$, and $Y_\cdot^D = Y_\cdot \nabla_\cdot f = \varphi(\cdot) Y_\cdot f$. Since Bochner-integrability implies absolute time-integrability (Proposition 3.2), $f \in \operatorname{Dom} {}^R T_t(K_\cdot) \cap \operatorname{Dom} {}^R A_t(H_\cdot))$, and

$$\Big(\int_0^t K_s\, ds\Big) f = \int_0^t K_s f\, ds = \mathcal{L}_0^t(K_\cdot^P f) = T_t(K_\cdot) f;$$

$$\Big(\int_0^t H_s\, dA_s\Big) f = \int_0^t \varphi(s) H_s f\, ds = \mathcal{L}_0^t(H_\cdot^D f) = A_t(H_\cdot) f.$$

If $a > 1$ then

$$\int_\Gamma \Big\{a^{-\#\sigma} \sum_{s\in\sigma} 1\!\!1_{[0,t[}(s) \|F_s f(\sigma \setminus s)\|\Big\}^2 d\sigma$$

$$\leq \int_\Gamma a^{-2\#\sigma} \#\sigma \sum_{s\in\sigma} 1\!\!1_{[0,t[}(s) \|F_s f(\sigma \setminus s)\|^2\, d\sigma$$

$$= \int_0^t \int_\Gamma (1+\#\omega)a^{-2(1+\#\omega)} \|F_s f(\omega)\|^2 \, d\omega \, ds$$
$$\le C_a \int_0^t \|F_s f\|^2 \, ds$$

where $C_a = \max_n (1+n)a^{-2(1+n)} < \infty$, so that $f \in \mathrm{Dom}\, {}^R A^\dagger(a^{-(N+1)}F.)$. Putting $x(\sigma) = a^{-\#\sigma} \sum_{s\in\sigma} \mathbb{1}_{[0,t[}(s)(F_s f)(\sigma \setminus s)$ and $y = a^{-N} \int_0^t F_s \, dA_s^* f$, we have $x, y \in \mathcal{F}$ and

$$\langle u\varepsilon_\psi, y\rangle = \Big\langle u\varepsilon_{a^{-1}\psi}, \int_0^t F_s \, dA_s^\dagger f \Big\rangle$$
$$= \int_0^t a^{-1}\overline{\psi(s)} \int_\Gamma \langle u\varepsilon_{a^{-1}\psi}(\omega), F_s f(\omega)\rangle \, d\omega \, ds$$
$$= \int_0^t \int_\Gamma \langle u\varepsilon_\psi(\omega \cup s), a^{-(1+\#\omega)} F_s f(\omega)\rangle \, d\omega \, ds$$
$$= \int_\Gamma \sum_{s\in\sigma} \mathbb{1}_{[0,t[}(s) \langle u\varepsilon_\psi(\sigma), a^{-\#\sigma}(F_s f)(\sigma \setminus s) \, d\sigma$$
$$= \langle u\varepsilon_\psi, x\rangle$$

for all $u \in \mathcal{H}_0$ and $\psi \in L^2(I\!R_+)$. Hence, by the density of $\mathcal{H}_0 \otimes \mathcal{E}$ in $\mathcal{F}$, $x = y$. It follows that

$$\sum_{s\in\sigma} \mathbb{1}_{[0,t[}(s)(F_s f)(\sigma \setminus s) = \Big(\int_0^t F_s \, dA_s^\dagger f\Big)(\sigma)$$

for a.a. σ. Therefore $\mathbb{1}_{[0,t[}(\cdot)F^P_\cdot f = \mathbb{1}_{[0,t[}(\cdot)F_\cdot f$ is Skorohod-integrable, which means that $f \in \mathrm{Dom}\, A_t^\dagger(F.)$, moreover

$$A_t^\dagger(F.)f = \mathcal{S}_0^t(F.f) = \int_0^t F_s \, dA_s^\dagger f.$$

Replacing $F.$ by φG in the above argument gives $f \in \mathrm{Dom}\, N_t(G)$ and $N_t(G.)f = \int_0^t G_s \, dN_s f$ too.

□

VIII.2 Noncausal formulation

The introduction of noncausal quantum stochastic integrals ([Bel], [Lin]) demonstrated the close connection between quantum stochastic calculus and the classical stochastic calculus arising from extensions of Itô calculus to deal with nonadapted processes ([Hit], [Sko]), together with its relations to Malliavin calculus ([G-T]). When restricted to adapted Fock operator processes these integrals extend the original quantum stochastic integrals beyond exponential domains. We show that these restrictions are in turn extended by our integrals.

Let $H.$ be a Fock operator process. Recall the notation at the beginning of Section VI. The noncausal quantum stochastic integrals are defined as follows:

$$^{NC}A^\dagger(H.)f = \mathcal{S}(H.f); \ \ ^{NC}N(H.)f = \mathcal{S}(H.\nabla.f);$$
$$^{NC}A(H.)f = \int_0^\infty H_s \nabla_s f \, ds; \ \ ^{NC}T(H.)f = \int_0^\infty H_s f \, ds;$$

with respective domains,

$$\begin{aligned}
\mathrm{Dom}({}^{NC}A^\dagger(H.)) &= \{f \in V(H.) : H.f \in \mathrm{Dom}\,\mathcal{S}\}\\
\mathrm{Dom}({}^{NC}N(H.)) &= \{f \in V^\nabla(H.) : H.\nabla.f \in \mathrm{Dom}\,\mathcal{S}\}\\
\mathrm{Dom}({}^{NC}A(H.)) &= \{f \in V^\nabla(H.) : H.\nabla.f \text{ is Bochner-integrable}\}\\
\mathrm{Dom}({}^{NC}T(H.)) &= \{f \in V(H) : H.f \text{ is Bochner-integrable}\}.
\end{aligned}$$

The relationship between these integrals, when applied to an adapted process $H.$, and our integrals, is the same as that for the original quantum stochastic integrals (coherent vector-formulation).

Theorem 8.2 – *Let $H.$ be a Fock operator process. If $H.$ is adapted, then*

$$\begin{aligned}
{}^{NC}\Lambda(H.) &\subset \Lambda(H.) \text{ for } \Lambda = A^\dagger \text{ or } N;\\
{}^{NC}\Lambda(H.) &\subset {}^{R}\Lambda(H.) \text{ for } \Lambda = A \text{ or } T.
\end{aligned}$$

Proof

Since, for a map $x : \Gamma \times \mathbb{R}_+ \to \mathcal{H}_0$, x is Skorohod-integrable if $x \in \mathrm{Dom}\,\mathcal{S}$, and x is absolutely time-integrable if x is Bochner-integrable, the result follows immediate from Proposition 6.1.

□

VIII.3. Itô calculus formulation

Let $F.$, $G.$, $H.$ and $K.$ be four adapted Fock operator processes which are measurable and have common domain V, where V is a subspace of $\mathcal{F}$ containing $V_0 \otimes \mathcal{E}(\mathcal{M})$ for some admissible pair $(V_0, \mathcal{M})$, and which also satisfy the local integrability (and implied measurability) conditions:

$$\int_0^t \{\|F_s P_s f\|^2 + \|G_s D_s f\|^2 + \|H_s D_s f\| + \|K_s P_s f\|\}\, ds < \infty, \tag{8.3}$$

for all $f \in V$, $t \geq 0$. In [A-M], an adapted Fock operator process $X.$ with domain V, is denoted by

$$\int_0^t F_s\, dA_s^\dagger + \int_0^t G_s\, dN_s + \int_0^t H_s\, dA_s + \int_0^t K_s\, ds$$

provided that, for each $f \in V$, $t \geq 0$,

(i) $D_s f \in \mathrm{Dom}(X_s)$ for a.a. s; $\mathbb{1}_{[0,t[}(\cdot) X.D.f$ is square-integrable;

$$\begin{aligned}
\text{(ii)}\quad X_t f = &\int_0^\infty X_{s\wedge t} D_s f\, d\chi_s + \int_0^t \{F_s P_s f + G_s D_s f\}\, d\chi_s\\
&+ \int_0^t \{H_s D_s f + K_s P_s f\}\, ds.
\end{aligned} \tag{8.4}$$

When $V = V_0 \otimes \mathcal{E}(\mathcal{M})$, this is equivalent to X_t being the corresponding Hudson-Parthasarthy quantum stochastic-integral, and under various conditions the representation (8.4) is valid on larger domains V. This is exploited, in particular, in the QS-integral representability of regular semimartingales ([Att]). However, since the Fock operator process $X.$ appears on the right hand side, (8.4) represents a king of Fock space-valued stochastic differential equations. In other words, the Fock operator process $X.$ is only defined implicitly through (8.4). It was not known in general whether these equations had a solution; nor whether any solution it might have is unique; and moreover, it is not known what the correct (maximal) domain is for a Fock operator solution process. We shall see that our integrals completely solve all three of these problems.

For $\Lambda = A^\dagger, N, A$ or T, define $Q_t = P_t, D_t, D_t, P_t$ respectively (as in VI.1) and $r_t = \chi_t, \chi_t, t, t$ respectively. If $(H_t)_{t\geq 0}$ is an adapted process of operators on $\mathcal{F}$ then equation (8.4) for the process $X. = \Lambda.(H.)$ writes

$$X_t f = \int_0^\infty X_{s\wedge t} D_s f \, d\chi_s + \int_0^t H_s Q_s f \, dr_s. \qquad (\Lambda(H.))$$

We wish to give a meaning to the sentence "$X.$ is a solution to equation $(\Lambda(H.))$". A workable solution $X.$ would be such that

i) if any of the sides of equation $(\Lambda(H.))$ is well-defined for a $f \in \mathcal{F}$, then so is the other side;

ii) if i) holds for a f then equality $(\Lambda(H.))$ holds for this f.

That is, a process $X.$ is said to be a solution to $\Lambda(H.)$ if

a) $f \in \operatorname{Dom} X_t$ if and only if i) $D_s f \in \operatorname{Dom} X_{s\wedge t}$ for a.a. $s \geq 0$,
ii) $D_s f \in \operatorname{Dom} H_s$ for a.a. $s \leq t$,
iii) $\int_0^\infty ||X_{s\wedge t} D_s f||^2 \, ds < \infty$,
iv) $\int_0^t ||H_s Q_s f||^2 \, ds < \infty$ for $\Lambda = A^\dagger, N$,
$\int_0^t ||H_s Q_s f|| \, ds < \infty$ for $\Lambda = A, T$;

b) for all $f \in \operatorname{Dom} X_t$ equation $(N(H.))$ holds true.

Lemma 8.3 – *If $X.$ solves $(N(H.))$ then $X.$ is an adapted process of operators.*

Proof

Let $f \in \operatorname{Dom} X_t$. Let $g = P_t f$. We have $D_s g = 0$ for a.a. $s > t$ and $D_s g = D_s f$ for a.a. $s \leq t$; furthermore $P_s g = P_{s\wedge t} f$. It is thus clear that g satisfies the conditions a) i)-iv). That is, $g \in \operatorname{Dom} X_t$. Now, let $h = D_u f$ for some $u \geq t$. We have $h \in \operatorname{Dom} X_t$ by applying condition a)i) to f. we have proved that $\operatorname{Dom} X_t$ is a t-adapted domain.

Furthermore, by equation $(N(H.))$ we have

$$P_t X_t f = \int_0^t X_s D_s f \, d\chi_s + \int_0^t H_s Q_s f \, dr_s = X_t P_t f$$

and

$$D_u X_t f = X_t D_u f$$

for a.a. $u \geq t$. We have proved that X_t is a t-adapted operator.

□

The following theorem proves equation $(N(H.))$ always admits a solution, that the solution is always unique (up to domain restrictions) and that the maximal (in terms of domain) solution is our ${}^R\Lambda.(H.)$.

Theorem 8.4 – *For any adapted process $H.$, any integrator Λ, the process ${}^R\Lambda.(H.)$ solves equation $(N(H.))$.*

Any other solution $X.$ of $(N(H.))$ is such that $X_t \subset {}^R\Lambda_t(H.)$ for all t.

Proof

Let us first prove that ${}^R\Lambda.(H.)$ solves the equation $(N(H.))$. Let $X_t = {}^R\Lambda_t(H.)$, for all $t \in \mathbb{R}_+$. By Theorem 6.3, 6.4 and by Lemma 7.1 we have that $f \in \operatorname{Dom} X_t$ if and only if it satisfies conditions a)i)-iv). Furthermore, integrating equations (6.8) and (6.9) with respect to $d\chi_t$ show that $X_t f$ satisfies equation $(N(H.))$. This proves that $X.$ solves equation $(N(H.))$.

Conversly, suppose that $X.$ is an adapted process of operators which solves $(N(H.))$. Let $f \in \operatorname{Dom} X_t$. We have

$$[X_t f](\emptyset) = \begin{cases} 0 & \text{if } \Lambda = A^\dagger, N \\ \int_0^t [H_s Q_s f](\emptyset)\, ds & \text{if } \Lambda = A, T \end{cases}$$

and for a.a. $\sigma = \{t_1 < \ldots < t_n\} \neq \emptyset$

$$\begin{aligned}[X_t f](\sigma) =& [X_{t_n \wedge t} D_{t_n} f](t_1, \ldots, t_{n-1}) \\ &+ \begin{cases} [H_{t_n} Q_{t_n} f](t_1, \ldots, t_{n-1}) \mathbb{1}_{[0,t]}(t_n) & \text{if } \Lambda = A^\dagger, N \\ \int_0^t [H_s Q_s f](t_1, \ldots, t_n)\, ds & \text{if } \Lambda = A, T. \end{cases}\end{aligned}$$

But, we can now apply the same formula to $[X_{t_n \wedge t} D_{t_n} f](t_1, \ldots, t_{n-1})$:

$$\begin{aligned}[X_{t_n \wedge t} D_{t_n} f](t_1, \ldots, t_{n-1}) &= [X_{t_{n-1} \wedge t} D_{t_{n-1}} D_{t_n} f](t_1, \ldots, t_{n-2}) \\ &+ \begin{cases} [H_{t_{n-1}} Q_{t_{n-1}} D_{t_n} f](t_1, \ldots, t_{n-2}) \mathbb{1}_{[0,t]}(t_{n-1}) & \text{if } \Lambda = A^\dagger, N \\ \int_0^{t_n} [H_s Q_s D_{t_n} f](t_1, \ldots, t_{n-1})\, ds & \text{if } \Lambda = A, T. \end{cases}\end{aligned}$$

And so on, we finally get, putting $t_{n+1} = t$ and $t_0 = 0$,

$$\begin{aligned}[X_t f](\sigma) &= [X_{t_1 \wedge t} D_{t_1} \ldots D_{t_n} f](\emptyset) \\ &+ \begin{cases} \sum_{\substack{i=1,\ldots n \\ t_i \leq t}} [H_{t_i} Q_{t_i} D_{t_{i+1}} \ldots D_{t_n} f](t_1, \ldots t_i) & \text{if } \Lambda = A^\dagger, N \\ \sum_{i=1}^n \int_0^{t_{i+1}} [H_s Q_s D_{t_{i+1}} \ldots D_{t_n} f](t_1, \ldots t_i)\, ds & \text{if } \Lambda = A, T \end{cases} \\ &= \begin{cases} \sum_{\substack{s \in \sigma \\ s \leq t}} [H_s Q_s D_{\sigma_{(s}} f](\sigma_{s)}) & \text{if } \Lambda = A^\dagger, N \\ \sum_{i=0}^n \int_{t_i}^{t_{i+1}} [H_s Q_s D_{t_{i+1}} \ldots D_{t_n} f](t_1, \ldots t_i)\, ds & \text{if } \Lambda = A, T \end{cases} \\ &= \begin{cases} \sum_{\substack{s \in \sigma \\ s \leq t}} [H_s Q_s D_{\sigma_{(s}} f](\sigma_{s)}) & \text{if } \Lambda = A^\dagger, N \\ \int_0^t [H_s Q_s D_{\sigma_{(s}} f](\sigma_{s)})\, ds & \text{if } \Lambda = A, T. \end{cases}\end{aligned}$$

This proves that $X_t f = \Lambda_t(H.) f$ for all $f \in \operatorname{Dom} X_t$.

□

IX. Quantum Itô formula

In the previous section we saw how our quantum stochastic-integrals explicitly solve the problem of extending the domain of Hudson-Parthasarathy integrals, by means of classical Itô integration and stochastic differentiation. We also saw that the enlarged domains are maximal subject to a mild measurability condition. In this section we shall show that on their restricted domains at least, our quantum stochastic-integrals behave as we would wish under operator multiplication. Products of quantum stochastic-integrals are given by integration by parts with a "correction" term when Wick ordering of the integrators has been violated. We give two forms for the quantum Itô formula – one for when there is a correction, and another for the general case. The hypotheses are slightly different, moreover we give a probabilistic proof of the first and an analytic proof of the second. In fact we have already done the analysis.

Theorem 9.1 – *Let $X = {}^R\Lambda^1_\cdot(F)$ and $Y = {}^R\Lambda^2_\cdot(G)$ where F and G are adapted Fock operator processes, and the ordered pair (Λ^1, Λ^2) is either $(N, A^\dagger)$, (N, N), (A, N) or $(A, A^\dagger)$. If W is the adapted Fock operator process given by*

$$W_t = {}^R\Lambda^1_t(FY) + {}^R\Lambda^2_t(XG) + {}^R\Lambda_t(FG), \qquad (*)$$

where Λ is respectively $A^\dagger$, N, A or T, then $Z = XY - W$ is a restriction of the zero process.

Proof

By Theorem 8.3 it suffices to show that the equation

$$Z_t f - \int_0^t Z_s D_s f \, d\chi_s = 0 \qquad (0)$$

is well-defined and valid for all $f \in \mathcal{F}_t \cap \operatorname{Dom} Z_t$. Therefore let f be such a vector; let $\mathcal{R}$ denote time or Itô integration if respectively $\Lambda^1 = A$ or N; and let $Q = P$ or D if respectively, $\Lambda^2 = A^\dagger$ or N. By Corollary 8.6, W satisfies

$$W_t f - \int_0^t W_s D_s f \, d\chi_s = \int_0^t X_s G_s Q_s f \, d\chi_s + \mathcal{R}_0^t(F_\cdot Y_\cdot D_\cdot f) + \mathcal{R}_0^t(F_\cdot G_\cdot Q_\cdot f). \quad (1)$$

Moreover, by Corollary 8.6 and Theorem 6.3 we have

$$\begin{aligned} X_t Y_t f &= \int_0^t X_s D_s Y_t f) + \mathcal{R}_0^t(F_\cdot D_\cdot Y_t f) \\ &= \int_0^t X_s \{Y_s D_s f + G_s Q_s f\} \, d\chi_s + \mathcal{R}_0^t(F_\cdot \{Y_\cdot D_\cdot f + G_\cdot Q_\cdot f\}) \qquad (2) \end{aligned}$$

with all the implied domain conditions holding. But the Itô-integrability of the process $X.G.Q.f$ and the $\mathcal{R}$-integrability of $F.G.Q.f$ on $[0,t[$ imply that $Y_s D_s f \in \operatorname{Dom} X_s$ for a.a. $s \in [0,t[$, and that $X.Y.D.f$ is Itô-integrable on $[0,t[$, and allow us to write $X_t Y_t f$ as a sum of four integrals. Comparison of (1) and (2) therefore reveals that the equation (0) is indeed well-defined and valid.

□

th9.2*Let* $X = {}^R\Lambda^1(F)$ *and* $Y = {}^R\Lambda^2(G)$ *where* F *and* G *are adapted Fock operator processes and* Λ^1 *and* Λ^2 *are quantum stochastic-integrators, and let*

$$W = {}^R\Lambda^1(FY.) + {}^R\Lambda^2(X_.G) + \varepsilon\, {}^R\Lambda(FG).$$

Then, for all $g \in \mathrm{Dom}\, Y \cap \mathrm{Dom}\, W$,

$$g \in \mathrm{Dom}(X^\dagger)^* \quad and \quad (X^\dagger)^* Y g = W g,$$

provided that F *has an adapted adjoint process* $F^\dagger$ *such that* $X^\dagger = {}^R\Lambda^{1\dagger}(F^\dagger)$ *is densely defined. Here* $\varepsilon\Lambda$ *is* $A^\dagger$, N, A *or* T *if the ordered pair* $(\Lambda^{(1)}, \Lambda^{(2)})$ *is respectively* $(N, A^\dagger)$, (N, N), (A, N) *or* $(A, A^\dagger)$, *and is zero otherwise.*

Proof

If $g \in \mathrm{Dom}\, Y \cap \mathrm{Dom}\, W$ then, by Theorem 7.7, Proposition 7.5 and Corollary 7.6,

$$\langle X^\dagger f, Y g\rangle = \langle f, W g\rangle \quad \forall\ f \in \mathrm{Dom}(X^\dagger).$$

Since $X^\dagger$ is densely defined, this implies that $Yg \in \mathrm{Dom}(X^\dagger)^*$, and

$$(X^\dagger)^* Y g = (\Lambda^1(FY) + \Lambda^2(XG) + \Lambda(FG))g.$$

The result follows.

□

As a consequence of this theorem we have the quantum Ito formula

$${}^R\Lambda^1(F)^R\Lambda^2(G)g = ({}^R\Lambda^1(FY_.) + {}^R\Lambda^1(X_.G) + \varepsilon^R\Lambda(FG))g,$$

where $X_t = {}^R\Lambda^1_t(F)$, $Y_t = {}^R\Lambda^2_t(G)$ and $\varepsilon\Lambda$ is the Itô-correcting quantum stochastic integrator, whenever g lies in the domain of both left and right hand side operators.

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