# EXTENSIONS OF QUANTUM STOCHASTIC CALCULUS

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## I. ABSTRACT ITO CALCULUS ON FOCK SPACE

### I.1. Short notations

**I.1.1. The symmetric spaces.**

Let $\mathcal{P}$ denote the set of finite subsets of $\mathbb{R}^+$. That is, $\mathcal{P} = \biguplus_n \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of $n$ elements subsets of $\mathbb{R}^+$, $n \geq 1$. By ordering elements of a $\sigma = \{t_1, t_2, \dots, t_n\} \in \mathcal{P}_n$ we identify $\mathcal{P}_n$ with $\Sigma_n = \{0 \leq t_1 < t_2 < \cdots < t_n\} \subset (\mathbb{R}^+)^n$. This way $\mathcal{P}_n$ inherits the measured space structure of $(\mathbb{R}^+)^n$. By putting the Dirac measure $\delta_\emptyset$ on $\mathcal{P}_0$, we have defined a $\sigma$-finite measured space structure on $\mathcal{P}$ (which, I insist, is the $n$-dimensional Lebesgue measure on each $\mathcal{P}_n$) whose only atom is $\{\emptyset\}$. The elements of $\mathcal{P}$ are denoted with small Greek letters $\sigma, \omega, \tau, , \dots$, the associated measure is denoted $d\sigma$, $d\omega$, $d\tau, \dots$, (with, in mind, that $\sigma = \{t_1 < t_2 < \cdots < t_n\}$ and $d\sigma = dt_1 dt_2 \cdots dt_n$). It is now clear that $L^2(\mathcal{P})$ is isomorphic to the Fock space $\Phi$. Indeed, $L^2(\mathcal{P}) = \bigoplus_n L^2(\mathcal{P}_n)$ is isomorphic to $\bigoplus_n L^2(\Sigma_n)$ (with $\Sigma_0 = \{\emptyset\}$) that is $\Phi$. In order to be really clear, the isomorphism between $\Phi$ and $L^2(\mathcal{P})$ can be explicitly written as:

$$\begin{aligned} V : \Phi &\longrightarrow L^2(\mathcal{P}) \\ f &\longmapsto Vf \end{aligned}$$

where $f = \sum_n f_n$ and $[Vf](\sigma) = \begin{cases} f_0 & \text{if } \sigma = \emptyset \\ f_n(t_1, \dots, t_n) & \text{if } \sigma = \{t_1 < \cdots < t_n\}. \end{cases}$

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For example, a coherent vector $\varepsilon(u)$, seen in $L^2(\mathcal{P})$, satisfies

$$[\varepsilon(u)](\sigma) = \prod_{s\in\sigma} u(s) \quad \text{(where the empty product equals 1)} .$$

Let us fix some notations on $\mathcal{P}$.

If $\sigma \neq \emptyset$ we put $\vee\sigma = \max\sigma$, $\sigma- = \sigma \setminus \{\vee\sigma\}$.

If $t \in \sigma$ then $\sigma \setminus t$ denotes $\sigma \setminus \{t\}$.

If $\{t \notin \sigma\}$ then $\sigma \cup t$ denotes $\sigma \cup \{t\}$.

If $0 \leq s \leq t$ then
$$\begin{aligned} \sigma_{s)} &= \sigma \cap [0, s[ \\ \sigma_{(s,t)} &= \sigma \cap ]s, t[ \\ \sigma_{(t} &= \sigma \cap ]t, +\infty[ \; . \end{aligned}$$

$\mathbb{1}_{\sigma\leq t}$ means $\begin{cases} 1 & \text{if } \sigma \subset [0,t] \\ 0 & \text{otherwise.} \end{cases}$

If $0 \leq s \leq t$ then
$$\begin{aligned} \mathcal{P}^{s)} &= \{\sigma \in \mathcal{P}; \sigma \subset [0, s[\} \\ \mathcal{P}^{(s,t)} &= \{\sigma \in \mathcal{P}; \sigma \subset ]s, t[\} \\ \mathcal{P}^{(t} &= \{\sigma \in \mathcal{P}; \sigma \subset ]t, +\infty[\} \; . \end{aligned}$$

$\#\sigma$ is the cardinal of $\sigma$.

It is clear, with the notations of R. L. Hudson's course, that

$$\begin{aligned} \Phi_{s]} &\simeq L^2(\mathcal{P}^{s)}) \\ \Phi_{[s,t]} &\simeq L^2(\mathcal{P}^{(s,t)}) \\ \Phi_{[t} &\simeq L^2(\mathcal{P}^{(t}) \; . \end{aligned}$$

In the following we make several identifications:

- $\Phi$ is not distinguished from $L^2(\mathcal{P})$ (and the same holds for $\Phi_{s]}$ and $L^2(\mathcal{P}^{s)})$, etc...)
- $L^2(\mathcal{P}^{s)})$, $L^2(\mathcal{P}^{(s,t)})$ and $L^2(\mathcal{P}^{(t})$ are seen as subspaces of $L^2(\mathcal{P})$: the subspace of $f \in L^2(\mathcal{P})$ such that $f(\sigma) = 0$ for all $\sigma$ such that $\sigma \not\subset [0, s]$ (resp. $\sigma \not\subset [s, t]$, resp. $\sigma \not\subset [t, +\infty[$).

### I.1.2. Integral-sum lemma.

The following lemma is a very important and useful combinatoric result that we will use quite often in the sequel. What this lemma says is mainly the following: consider the Wick product on $\Phi$:

$$[f : g](\sigma) \stackrel{\text{def}}{=} \sum_{\alpha\subset\sigma} f(\alpha) g(\sigma \setminus \alpha)$$

then this product behaves like a convolution; in particular it maps isometrically $L^1(\mathcal{P}) \times L^1(\mathcal{P})$ to $L^1(\mathcal{P})$.

$\sum\!\!\!\!\int$-LEMMA. — *Let $f$ be a measurable positive (resp. integrable) function on $\mathcal{P} \times \mathcal{P}$. Define a function $g$ on $\mathcal{P}$ by*

$$g(\sigma) = \sum_{\alpha \subset \sigma} f(\alpha, \sigma \setminus \alpha) \ .$$

*Then $g$ is measurable positive (resp. integrable) and*

$$\int_{\mathcal{P}} g(\sigma) \ d\sigma = \int_{\mathcal{P} \times \mathcal{P}} f(\alpha, \beta) \ d\alpha \ d\beta \ .$$

*Proof.* — By density arguments one can restrict ourselves to the case where $f(\alpha, \beta) = h(\alpha)k(\beta)$ and where $h = \varepsilon(u)$ and $k = \varepsilon(v)$ are coherent vectors. In this case one has

$$\begin{aligned}\int_{\mathcal{P} \times \mathcal{P}} f(\alpha, \beta) \ d\alpha \ d\beta &= \int_{\mathcal{P}} \varepsilon(u)(\alpha) \ d\alpha \int_{\mathcal{P}} \varepsilon(v)(\beta) \ d\beta \\ &= e^{\int_0^\infty u(s) \ ds} e^{\int_0^\infty v(s) \ ds} \ (\text{take } u, v \in L^1 \cap L^2(\mathbb{R}^+))\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{P}} \sum_{\alpha \subset \sigma} f(\alpha, \sigma \setminus \alpha) \ d\sigma &= \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} \prod_{s \in \alpha} u(s) \prod_{s \in \sigma \setminus \alpha} v(s) \ d\sigma \\ &= \int_{\mathcal{P}} \prod_{s \in \sigma} (u(s) + v(s)) \ d\sigma = e^{\int_0^\infty u(s)+v(s) \ ds} \ . \qquad \blacksquare\end{aligned}$$

As we have seen in R. L. Hudson's course we have, for all $t$, an isomorphism between $\Phi$ and $\Phi_{t]} \otimes \Phi_{[t}$. In terms of this short notation, and with the help of the $\sum\!\!\!\!\int$-Lemma, the isomorphism describes nicely.

THEOREM I.1.1. — *The mapping:*

$$\begin{aligned}\Phi_{t]} \otimes \Phi_{[t} &\longrightarrow \Phi \\ f \otimes g &\longmapsto h\end{aligned}$$

*with $h(\sigma) = f(\sigma_{t)})g(\sigma_{(t})$ defines an isomorphism between $\Phi_{t]} \otimes \Phi_{[t}$ and $\Phi$.*

*Proof.*

$$\begin{aligned}\int_{\mathcal{P}} |h(\sigma)|^2 \ d\sigma &= \int_{\mathcal{P}} |f(\sigma_{t)})|^2 |g(\sigma_{(t})|^2 \ d\sigma \\ &= \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} \mathbb{1}_{\alpha \subset [0,t]} \mathbb{1}_{\sigma \setminus \alpha \subset [t, +\infty[} |f(\alpha)|^2 |g(\sigma \setminus \alpha)|^2 \ d\sigma\end{aligned}$$

$$= \int_{\mathcal{P}} \int_{\mathcal{P}} \mathbb{1}_{\alpha\subset[0,t]} \mathbb{1}_{\beta\subset[t,+\infty[} |f(\alpha)|^2 |g(\beta)|^2 \, d\alpha \, d\beta \quad \text{(by the } \textstyle\sum\text{-Lemma)}$$
$$= \int_{\mathcal{P}_{t)}} |f(\alpha)|^2 \, d\alpha \int_{\mathcal{P}_{(t}} |g(\beta)|^2 \, d\beta$$
$$= \|f \otimes g\|^2. \qquad \blacksquare$$

## I.2. Ito calculus on Fock space

We are now ready to define the main ingredients for developing our quantum stochastic calculus: several differential and integral operators on the Fock space.

### I.2.1. Conditional expectations.

For all $t > 0$ define the operator $P_t$ from $\Phi$ to $\Phi$ by

$$[P_t f](\sigma) = f(\sigma)\mathbb{1}_{\sigma\subset[0,t]} \ .$$

It is very easy to check that $P_t$ is actually the orthogonal projector from $\Phi$ onto $\Phi_{t]}$.

For $t = 0$ we define $P_0$ by

$$[P_0 f](\sigma) = f(\emptyset)\mathbb{1}_{\sigma=\emptyset}$$

which is the orthogonal projection onto $L^2(\mathcal{P}_0) = \mathbb{C}\mathbb{1}$ where $\mathbb{1}$ is the vacuum ($\mathbb{1}(\sigma) = \mathbb{1}_{\sigma=\emptyset}$).

### I.2.2. Adapted gradient.

For all $t \in \mathbb{R}^+$ and all $f$ in $\Phi$ define the following function on $\mathcal{P}$:

$$[D_t f](\sigma) = f(\sigma \cup t)\mathbb{1}_{\sigma\subset[0,t]}.$$

The first natural question is: for which $f$ does $D_t f$ lie in $\Phi$ that is, $L^2(\mathcal{P})$.

Proposition I.2.1. — *For all $f \in \Phi$, we have*

$$\int_0^\infty \int_{\mathcal{P}} |[D_t f](\sigma)|^2 \, d\sigma \, dt = \|f\|^2 - |f(\emptyset)|^2.$$

*Proof.* — This is again an easy application of the $\sum$-Lemma:

$$\begin{aligned}\int_0^\infty \int_{\mathcal P} |f(\sigma\cup t)|^2 \mathbb{1}_{\sigma\subset[0,t]}\, d\sigma\, dt &= \int_{\mathcal P}\int_{\mathcal P} |f(\alpha\cup\beta)|^2 \mathbb{1}_{\#\beta=1}\mathbb{1}_{\alpha\subset[0,\vee\beta]}\, d\alpha\, d\beta \\ &= \int_{\mathcal P}\sum_{\alpha\subset\sigma} |f(\alpha\cup\sigma\smallsetminus\alpha)|^2 \mathbb{1}_{\#(\sigma\smallsetminus\alpha)=1}\mathbb{1}_{\alpha\subset[0,\vee(\sigma\smallsetminus\alpha)]}\, d\sigma \\ &= \int_{\mathcal P\smallsetminus\mathcal P_0}\sum_{t\in\sigma} |f(\sigma)|^2 \mathbb{1}_{\sigma\smallsetminus t\subset[0,t]}\, d\sigma \quad \text{(this forces } t \text{ to be } \vee\sigma) \\ &= \int_{\mathcal P\smallsetminus\mathcal P_0} |f(\sigma)|^2\, d\sigma = \|f\|^2 - |f(\emptyset)|^2. \qquad \blacksquare\end{aligned}$$

This proposition implies the following: for all $f$ in $\Phi$, for almost all $t\in\mathbb{R}^+$ (the negligeable set depends on $f$), the function $D_t f$ belongs to $L^2(\mathcal P)$. So for *all* $f$ in $\Phi$, *almost all* $t$, $D_t f$ is an element of $\Phi$. Thought, $D_t$ is not a well-defined operator from $\Phi$ to $\Phi$. The only operators which can be well defined are either

$$\begin{aligned} D : L^2(\mathcal P) &\longrightarrow L^2(\mathcal P\times\mathbb{R}^+) \\ f &\longmapsto \big((\sigma,t)\mapsto D_t f(\sigma)\big)\end{aligned}$$

which is a partial isometry; or the regularised operators $D_h$, for $h\in L^2(\mathbb{R}^+)$:

$$[D_h f](\sigma) = \int_0^\infty h(t)[D_t f](\sigma)\, dt.$$

But, anyway, in this course we will treat the $D_t$'s as linear operators defined on the whole of $\Phi$. This, in general, poses no problem; one just has to be careful in some particular situations.

Note that $D_t$ is the adapted version of the well known Malliavin's gradient:

$$[\nabla_t f](\sigma) = f(\sigma\cup t).$$

The very important difference comes from the fact that $D_t f$ is defined for all $f$ (in the sense that for all $f$, $D_t f$ stays in $\Phi$), which is not the case for $\nabla_t$.

### I.2.3. Ito integral.

A family $(g_t)_{t\geq 0}$ of elements of $\Phi$ is said to be an *Ito integrable process* if the following holds:

*i)* $f\mapsto \|g_t\|$ is measurable

*ii)* $g_t\in\Phi_{t]}$ for all $t$

*iii)* $\int_0^\infty \|g_t\|^2\, dt < \infty$.

If $g_\cdot = (g_t)_{t\geq 0}$ is an Ito integrable process, define

$$[\mathcal{I}(g_\cdot)](\sigma) = \begin{cases} 0 & \text{if } \sigma=\emptyset \\ g_{\vee\sigma}(\sigma-) & \text{if } \sigma\neq\emptyset.\end{cases}$$

PROPOSITION I.2.2. — *For all Ito integrable process $g_{\cdot} = (g_t)_{t\geq 0}$ one has*

$$\int_{\mathcal{P}} |[\mathcal{I}(g_{\cdot})](\sigma)|^2 \; d\sigma = \int_0^\infty \|g_t\|^2 \; dt < \infty.$$

*Proof.* — Another application of the $\sum\!\!\!\!\!\!\int$-Lemma (Exercise). ■

So, for all Ito integrable process $g_{\cdot} = (g_t)_{t\geq 0}$, $\mathcal{I}(g_{\cdot})$ defines an element of $\Phi$, the *Ito integral* of the process $g_{\cdot}$.

One can notice that the Ito integral is just the restriction to Ito integrable processes of the well-known Skorohod integral:

$$[\mathcal{S}(g_{\cdot})](\sigma) = \sum_{s\in\sigma} g_s(\sigma \setminus s).$$

Recall the operator $D : L^2(\mathcal{P}) \to L^2(\mathcal{P} \times \mathbb{R}^+)$ from last section.

PROPOSITION I.2.3.

$$\mathcal{I} = D^*.$$

*Proof.*

$$\begin{aligned}
\langle f, \mathcal{I}(g_{\cdot})\rangle &= \int_{\mathcal{P}\setminus\mathcal{P}_0} \bar{f}(\sigma) g_{\vee\sigma}(\sigma-) \; d\sigma \\
&= \int_0^\infty \int_{\mathcal{P}} \bar{f}(\sigma \cup t) g_t(\sigma) \mathbb{1}_{\sigma\subset[0,t]} \; d\sigma \; dt \quad (\textstyle\sum\!\!\!\!\!\!\int\text{-Lemma}) \\
&= \int_0^\infty \int_{\mathcal{P}} [\overline{D_t f}](\sigma) g_t(\sigma) \; d\sigma \; dt \\
&= \int_0^\infty \langle D_t f, g_t\rangle \; dt.
\end{aligned}$$

■

**I.2.4. The Ito integral is really an integral.**

We are going to see that the Ito integral defined above can be interpreted as a true integral $\int_0^\infty g_t \; d\chi_t$ with respect to some particular process $(\chi_t)_{t\geq 0}$.

For all $t \in \mathbb{R}^+$, define the element $\chi_t$ of $\Phi$ by

$$\begin{cases} \chi_t(\sigma) = 0 & \text{if } \#\sigma \neq 1 \\ \chi_t(s) = \mathbb{1}_{[0,t]}(s). \end{cases}$$

This family of elements of $\Phi$ has some very particular properties. The main one is the following: not only $\chi_t \in \Phi_{t]}$ for all $t \in \mathbb{R}^+$, but also

$$\chi_t - \chi_s \in \Phi_{[s,t]} \quad \text{for all } s \leq t$$

which is very easy to check from the definition.

We will see later that, in some sense, $(\chi_t)_{t\geq 0}$ is the only process to satisfy this property.

For the moment, let us take an Ito integrable process $(g_t)_{t\geq 0}$ which is *simple* that is, constant on intervals:

$$g_t = \sum_i g_{t_i} \mathbb{1}_{[t_i,t_{i+1}[}(t).$$

Define $\int_0^\infty g_t d\chi_t$ to be $\sum_i g_{t_i} \otimes (\chi_{t_{i+1}} - \chi_{t_i})$ (recall that $g_{t_i} \in \Phi_{t_i]}$ and $\chi_{t_{i+1}} - \chi_{t_i} \in \Phi_{[t_i,t_{i+1}]} \subset \Phi_{(t_i})$. We have

$$\begin{aligned}
\left[\int_0^\infty g_t \; d\chi_t\right](\sigma) &= \sum_{n=1}^\infty [g_{t_i} \otimes (\chi_{t_{i+1}} - \chi_{t_i})](\sigma) \\
&= \sum_{n=1}^\infty g_{t_i}(\sigma_{t_i}))(\chi_{t_{i+1}} - \chi_{t_i})(\sigma_{(t_i}) \\
&= \sum_{n=1}^\infty g_{t_i}(\sigma_{t_i}))\mathbb{1}_{\#\sigma_{(t_i}=1}\mathbb{1}_{\vee\sigma_{(t_i}\in]t_i,t_{i+1}]} \\
&= \sum_{n=1}^\infty g_{t_i}(\sigma_{t_i})\mathbb{1}_{\sigma-\subset[0,t_i]}\mathbb{1}_{\vee\sigma\in]t_i,t_{i+1}]} \; .
\end{aligned}$$

If the partition $(t_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^+$ is fine enough to separate $\sigma-$ from $\vee\sigma$ (this can always been done by refining the partition and declaring the associated $g_{t_i}$'s to have the correct value), then the sum above contains one and only one non-vanishing term: the one for the only $i = i_0$ such that $\vee\sigma \in ]t_{i_0}, t_{i_0+1}]$ and $\sigma- \subset [0, t_{i_0}]$. We have

$$\left[\int_0^\infty g_t \; d\chi_t\right](\sigma) = g_{t_{i_0}}(\sigma_{t_{i_0}})) = g_{t_{i_0}}(\sigma-) = g_{\vee\sigma}(\sigma-).$$

Thus for simple Ito-integrable processes we have proved that

$$\mathcal{I}(g_\cdot) = \int_0^\infty g_t \; d\chi_t. \qquad (I.1)$$

But because of the isometry formula of Proposition I.2.2 we have

$$\|\mathcal{I}(g_\cdot)\|^2 = \|\int_0^\infty g_t \; d\chi_t\|^2 = \int_0^\infty \|g_t\|^2 \; dt.$$

So one can pass to the limit from simple Ito integrable processes to Ito integrable processes in general and extend the definition of this integral $\int_0^\infty g_t \; d\chi_t$. As a result, (I.1) holds for every Ito integrable process $(g_t)_{t\geq 0}$. So from now on we will denote the Ito integral by $\int_0^\infty g_t \; d\chi_t$.

### I.2.5. Fock space predictable representation property.

If $f$ belongs to $\Phi$, Proposition I.2.1 shows that $(D_t f)_{t\geq 0}$ is an Ito integrable process. So let us compute $\int_0^\infty D_t f \; d\chi_t$.

$$\left[\int_0^\infty D_t f \; d\chi_t\right](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ [D_{\vee\sigma} f](\sigma-) & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma - \cup\vee\sigma)\mathbb{1}_{\sigma-\subset[0,\vee\sigma]} & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma) & \text{otherwise} \end{cases}$$
$$= f(\sigma) - [P_0 f](\sigma).$$

This computation together with Propositions I.2.1 and I.2.2 give the following fundamental result.

THEOREM I.2.4 (Fock space predictable representation property). — *For all $f \in \Phi$ one has the representation*

$$f = P_0 f + \int_0^\infty D_t f \; d\chi_t \qquad (I.2)$$

*and*

$$\|f\|^2 = |P_0 f|^2 + \int_0^\infty \|D_t f\|^2 \; dt. \qquad (I.3)$$

*The representation (I.2) is unique; that is, $P_0 f$ and $(D_t f)_{t\geq 0}$ are respectively the unique constant, Ito integrable process, such that (I.2) holds.*

*The norm identity (I.3) polarises as follows*

$$\langle f, g\rangle = \overline{P_0 f} P_0 g + \int_0^\infty \langle D_t f, D_t g\rangle \; dt$$

*for all $f, g \in \Phi$.*

*Proof.* — The only thing that remains to prove is the uniqueness property. If $f = c + \int_0^\infty g_t \; d\chi_t$ then $P_0 f = P_0 c + P_0 \int_0^\infty g_t \; d\chi_t = c$. So $\int_0^\infty g_t \; d\chi_t = \int_0^\infty D_t f \; d\chi_t$ that is, $\int_0^\infty (g_t - D_t f) \; d\chi_t = 0$. This implies $\int_0^\infty \|g_t - D_t f\|^2 \; dt = 0$ thus the result. ■

### I.2.6. Fock space chaotic expansion property.

Let $h_1$ be an element of $L^2(\mathbb{R}^+) = L^2(\mathcal{P}_1)$, we can define

$$\int_0^\infty h_1(t) \; d\chi_t$$

in the sense $\int_0^\infty h_1(t)\mathbb{1}\ d\chi_t$. For $h_2{\in}L^2(\mathcal{P}_2)$ we want to define

$$\int_{0\le s_1\le s_2} h_2(s_1,s_2)\ d\chi_{s_1}\ d\chi_{s_2}\ .$$

This can be done in two ways:

- either by starting with simple $h_2$'s and defining the iterated integral above as being

$$\sum_{s_j}\sum_{t_i\le s_j} h_2(t_i,s_j)(\chi_{t_{i+1}}-\chi_{t_i})(\chi_{s_{j+1}}-\chi_{s_j}).$$

  One proves easily (exercise) that the norm$^2$ of the expression above is exactly

$$\int_{0\le s_1\le s_2} |h_2(s_1,s_2)|^2\ ds_1\ ds_2;$$

  so one can pass to the limit in order to define $\int_{0\le s_1\le s_2} h_2(s_1,s_2)\ d\chi_{s_1}\ d\chi_{s_2}$ for any $h_2\in L^2(\mathcal{P}_2)$.

- either one says that $g=\int_{0\le s_1\le s_2} h_2(s_1,s_2)\ d\chi_{s_1}\ d\chi_{s_2}$ is the only $g\in\Phi$ such that the continuous linear form

$$\begin{aligned}\lambda:\varphi &\longrightarrow \mathbb{C}\\ f &\longmapsto \int_{0\le s_1\le s_2} \bar f(\{s_1,s_2\})h_2(s_1,s_2)\ ds_1\ ds_2\end{aligned}$$

  is of the form $\lambda(f)=\langle f,g\rangle$.

  The two definitions coincide (exercise).

  In the same way, for $h_n\in L^2(\mathcal{P}_n)$ one defines

$$\int_{0\le s_1\le\cdots\le s_n} h_n(s_1,\ldots,s_n)\ d\chi_{s_1}\cdots d\chi_{s_n}.$$

We get

$$\begin{aligned}\Big\langle\int_{0\le s_1\le\cdots\le s_n} h_n(s_1,\ldots,s_n)\ d\chi_{s_1}\cdots d\chi_{s_n},\int_{0\le s_1\le\cdots\le s_m} k_m(s_1,\ldots,s_m)\ d\chi_{s_1}\cdots d\chi_{s_m}\Big\rangle\\ =\delta_{n,m}\int_{0\le s_1\le\cdots\le s_n} \bar h_n(s_1,\ldots,s_n)k_n(s_1,\ldots,s_n)\ ds_1\cdots ds_n\end{aligned}$$

For $f\in L^2(\mathcal{P})$ we define

$$\int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma = f(\emptyset)\mathbb{1}+\sum_n\int_{0\le s_1\le\cdots\le s_n} f(\{s_n,\ldots,s_n\})\ d\chi_{s_1}\cdots d\chi_{s_n}.$$

THEOREM I.2.5 (Fock space chaotic representation property). — *For all $f\in\Phi$ we have*

$$f=\int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma.$$

*Proof.* — For $g \in \Phi$ we have by definition

$$\langle g, \int_{\mathcal{P}} f(\sigma)\, d\chi_\sigma \rangle$$

$$= \overline{g(\emptyset)} f(\emptyset) + \sum_n \int_{0 \le s_1 \le \cdots \le s_n} \bar{g}(\{s_n, \ldots, s_n\}) f(\{s_n, \ldots, s_n\})\, ds_1 \cdots ds_n$$

$$= \langle g, f \rangle .$$

(Details are left to the motivated reader). ■

**I.2.7. $(\chi_t)_{t \ge 0}$ is the only independent increment process on $\phi$.**

We have seen that $(\chi_t)_{t \ge 0}$ is a process in $\Phi$ satisfying

*i)* $\chi_t \in \Phi_{t]}$ for all $t \in \mathbb{R}^+$;

*ii)* $\chi_t - \chi_s \in \Phi_{[s,t]}$ for all $0 \le s \le t$.

Are there any other processes $(Y_t)_{t \ge 0}$ in $\Phi$ satisfying these two properties?

If one takes $a(\cdot)$ to be a function on $\mathbb{R}^+$, and $h \in L^2(\mathbb{R}^+)$ then $Y_t = a(t)\mathbb{1} + \int_0^t h(s)\, d\chi_s$ clearly satisfies *i)* and *ii)*. This is the only possibility.

THEOREM I.2.6. — *If $(Y_t)_{t \ge 0}$ is a vector process on $\Phi$ satisfying i) and ii) then there exists $a : \mathbb{R}^+ \to \mathbb{C}$ and $h \in L^2(\mathbb{R}^+)$ such that*

$$Y_t = a(t)\mathbb{1} + \int_0^t h(s)\, d\chi_s .$$

*Proof.* — Let $a(t) = P_0 Y_t$. Then $\widetilde{Y}_t = Y_t - a(t)\mathbb{1}$, $t \in \mathbb{R}^+$, satisfies *i)* and *ii)* with $\widetilde{Y}_0 = 0$ (for $Y_0 = P_0 Y_0 = P_0(Y_t - Y_0) + P_0 Y_0 = P_0 Y_t$). We can now drop the $\sim$ symbol and assume $Y_0 = 0$. Now note that $P_s Y_t = P_s Y_t + P_s(Y_t - Y_s) = P_s Y_s = Y_s$. This implies easily (exercise) that the chaotic expansion of $Y_t$ is of the form:

$$Y_t = \int_{\mathcal{P}} \mathbb{1}_{\mathcal{P}^{t)}}(\sigma) y(\sigma)\, d\chi_\sigma \ .$$

If $\#\sigma \ge 2$, for example $\sigma = \{t_1 < t_2 < \cdots < t_n\}$, let $s < t$ be such that $t_1 < s < t_n < t$. Then

$$(Y_t - Y_s)(\sigma) = 0 \quad \text{for} \quad Y_t - Y_s \in \Phi_{[s,t]} \quad \text{and} \quad \sigma \not\subset [s,t] \ .$$

Furthermore

$$Y_s(\sigma) = P_s Y_s(\sigma) = \mathbb{1}_{\sigma \subset [0,s]} Y_s(\sigma) = 0 \ .$$

Thus $Y_t(\sigma) = 0$, for any $\sigma \in \mathcal{P}$ with $\#\sigma \ge 2$, any $t \in \mathbb{R}^+$. This means that $Y_t = \int_0^t y(s)\, d\chi_s$ . ■

## I.3. Probabilistic interpretations of Fock space

In this chapter we present the general theory of probabilistic interpretations of Fock space. This chapter is not necessary for the understanding of this series of lectures, but the ideas coming from these notions underly the whole work.

### I.3.1. Chaotic expansions.

Let us recall some of the definitions and properties we have seen in M. Emery's course. We consider a martingale $(x_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$. We take $(\mathcal{F}_t)_{t\geq 0}$ to be the natural filtration of $(x_t)_{t\geq 0}$ (the filtration is made complete and right continuous) and we suppose that $\mathcal{F} = \mathcal{F}_\infty \stackrel{\text{def}}{=} \bigvee_{t\geq 0} \mathcal{F}_t$. Such a martingale is called *normal* if $(x_t^2 - t)_{t\geq 0}$ is still a martingale for $(\mathcal{F}_t)_{t\geq 0}$. This is equivalent to saying that $\langle x, x\rangle_t = t$ for all $t \geq 0$, where $\langle \cdot\,, \cdot\rangle$ denotes the probabilistic angle bracket.

A normal martingale is said to satisfy the Predictable Representation Property (P.R.P.) if all $f \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$f = \mathbb{E}[f] + \int_0^\infty h_s \; dx_s$$

for a $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(h_t)_{t\geq 0}$. Recall that

$$\mathbb{E}[|f|^2] = |\mathbb{E}[f]|^2 + \int_0^\infty \mathbb{E}[|h_s|^2] \; ds$$

that is, in the $L^2(\Omega)$-norm notation:

$$||f||^2 = |\mathbb{E}[f]|^2 + \int_0^\infty ||h_s||^2 \; ds \; .$$

Recall that if $f_n$ is a function in $L^2(\Sigma_n)$, where $\Sigma_n = \{0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \in \mathbb{R}^n\} \subset (\mathbb{R}^+)^n$ is equipped with the restriction of the $n$-dimensional Lebesgue measure, one can define an element $I_n(f_n) \in L^2(\Omega)$ by

$$I_n(f_n) = \int_{0\leq t_1 < \cdots < t_n} f_n(t_1, \ldots, t_n) \; dx_{t_1} \cdots dx_{t_n}$$

which is defined, with the help of the Ito isometry formula, as an iterated stochastic integral and which satisfies

$$||I_n(f_n)||^2 = \int_{0\leq t_1 < \cdots < t_n} |f_n(t_1, \ldots, t_n)|^2 \; dt_1 \cdots dt_n \; .$$

It is also important to recall that

$$\langle I_n(f_n), I_m(f_m)\rangle = 0 \;\text{ if }\; n \neq m \; .$$

The *chaotic space* of $(x_t)_{t\geq 0}$, denoted $CS(x)$, is the sub-Hilbert space of $L^2(\Omega)$ made of the random variables $f \in L^2(\Omega)$ which can be written as

$$f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_{0\leq t_1<\cdots<t_n} f_n(t_1, \ldots, t_n)\ dx_{t_1} \cdots dx_{t_n} \qquad (I.4)$$

for some $f_n \in L^2(\Sigma_n)$, $n \in \mathbb{N}^*$, such that

$$\|f\|^2 = |\mathbb{E}[f]|^2 + \sum_{n=1}^{\infty} \int_{0\leq t_1<\cdots<t_n} |f_n(t_1, \ldots, t_n)|^2\ dt_1 \cdots dt_n < \infty\ .$$

When $CS(x)$ is the whole of $L^2(\Omega)$ one says that $x$ satisfies the *Chaotic Representation Property* (C.R.P.). The decomposition of $f$ as in (I.4) is called the *chaotic expansion* of $f$.

Note that the C.R.P. implies the P.R.P. for if $f$ can be written as in (1) then, by putting $h_t$ to be

$$h_t = f_1(t) + \sum_{n=1}^{\infty} \int_{0\leq t_1<\cdots<t_n} f_{n+1}(t_1, \ldots, t_n, t)\ dx_{t_1} \cdots dx_{t_n}$$

we have

$$f = \mathbb{E}[f] + \int_0^{\infty} h_t\ dx_t\ .$$

In the cases where $(x_t)_{t\geq 0}$ is the Brownian motion, the compensated Poisson process or the Azéma martingale with coefficient $\beta \in [-2, 0]$, we have examples of normal martingales which possess the C.R.P.

### I.3.2. Isomorphism with Fock space.

Let us consider a normal martingale $(x_t)_{t\geq 0}$ with the P.R.P. and its chaotic space $CS(x) \subset L^2(\Omega, \mathcal{F}, P)$.

By identifying a function $f_n \in L^2(\Sigma_n)$ with a symmetric function $\tilde{f}_n$ on $(\mathbb{R}^+)^n$, one can identify $L^2(\Sigma_n)$ with $L^2_{\text{sym}}((\mathbb{R}^+)^n) = L^2(\mathbb{R}^+)^{\odot n}$ (with the correct symmetric norm: $\|\tilde{f}_n\|^2_{L^2(\mathbb{R}^+)^{\odot n}} = n!\|\tilde{f}_n\|^2_{L^2(\mathbb{R}^+)^{\otimes n}}$ if one puts $\tilde{f}_n$ to be $\frac{1}{n!}$ times the symmetric expansion of $f_n$). It is now clear that $CS(x)$ is isomorphic to the symmetric Fock space

$$\Phi = \Gamma(L^2(\mathbb{R}^+)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^+)^{\odot n}\ .$$

The isomorphism can be explicitly written as follows

$$\begin{aligned} U_x : \Phi &\longrightarrow CS(x) \\ f &\longmapsto Uf \end{aligned}$$

where $f = \sum_n f_n$ with $f_n \in L^2(\mathbb{R}^+)^{\odot n}$, $n \in \mathbb{N}$, and

$$U_x f = f_0 + \sum_{n=1}^{\infty} n! \int_{0 \leq t_1 < \cdots < t_n} f_n(t_1, \ldots, t_n)\ dx_{t_1} \cdots dx_{t_n}\ .$$

If $f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_{0 \leq t_1 < \cdots < t_n} f_n(t_1, \ldots, t_n)\ dx_{t_1} \cdots dx_{t_n}$ is an element of $CS(x)$, then $U_x^{-1} f = \sum_n g_n$ with $g_0 = \mathbb{E}[f]$ and $g_n = \frac{1}{n!} f_n$ symmetrised.

### I.3.3. Structure equations, multiplications.

Let us recall M. Emery's course. If $(x_t)_{t \geq 0}$ is a normal martingale, with the P.R.P. and if $x_t$ belongs to $L^4(\Omega)$, for all $t$, then $([x,x]_t - \langle x,x \rangle_t)_{t \geq 0}$ is a $L^2(\Omega)$-martingale; so by the P.R.P. there exists a predictable process $(\psi_t)_{t \geq 0}$ such that

$$[x,x]_t - \langle x,x \rangle_t = \int_0^t \psi_s\ dx_s$$

that is,

$$[x,x]_t = t + \int_0^t \psi_s\ dx_s$$

or else

$$d[x,x]_t = dt + \psi_t\ dx_t\ . \qquad (I.5)$$

This equation is called a *structure equation* for $(x_t)_{t \geq 0}$. One has to be careful that, in general, there can be many structure equations describing the same solution $(x_t)_{t \geq 0}$; there also can be several solutions (in law) to some structure equations.

What can be proved is the following:

* when $\psi_t \equiv 0$ for all $t$ then the only solution (in law) of (I.5) is the Brownian motion;
* when $\psi_t \equiv c$ for all $t$ then the only solution (in law) of (I.5) is the compensated Poisson process with intensity $1/c^2$;
* when $\psi_t = \beta x_{t-}$ for all $t$, then the only solution (in law) of (I.5) is the Azéma martingale with parameter $\beta$.

The importance of structure equations appears when one considers products. Indeed, we have seen in Section I.2 that many operations in probabilistic interpretations of Fock space are independent of the choice of the interpretation and depends only on the Fock space structure: Ito integrals, Malliavin gradients, Skorohod integrals,...

The operation that differentiates two different probabilistic interpretations is the product of random variables. Let us be clearer. Let $f$, $g$ be two elements of $\Phi$. Let $U_w f$ and $U_w g$ be their interpretation in the Brownian motion interpretation $(w_t)_{t\geq 0}$. Make the product of the two random variables: $U_w f \cdot U_w g$. If the result is still in $L^2(\Omega)$ (for example if $f$ and $g$ are coherent vectors) then take it back to $\Phi : U_w^{-1}(U_w f \cdot U_w g)$. This operation defines an associative product on $\Phi$:

$$f *_w g = U_w^{-1}(U_w f \cdot U_w g)$$

called the *Wiener product*.

We could have done the same operations with the Poisson interpretation:

$$f *_p g = U_p^{-1}(U_p f \cdot U_p g),$$

this gives the *Poisson product* on $\Phi$.

You can also define an Azéma product,...

What I claim is that you are going to obtain two different products on $\Phi$. The point is that all probabilistic interpretations of $\Phi$ have the same angle bracket $\langle x, x\rangle_t = t$ but not the same square bracket: $[x, x]_t = t + \int_0^t \psi_s \, dx_s$. But the product of two random variables makes the square bracket appearing: if $f = \mathbb{E}[f] + \int_0^\infty h_s \, dx_s$ and $g = \mathbb{E}[g] + \int_0^\infty k_s \, dx_s$, if $f_s = \mathbb{E}[f|\mathcal{F}_s]$ and $g_s = \mathbb{E}[g|\mathcal{F}_s]$ for all $s \geq 0$ then one has

$$\begin{aligned} fg &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty f_s k_s \, dx_s + \int_0^\infty g_s h_s \, dx_s + \int_0^\infty h_s k_s ds \\ &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty f_s k_s \, dx_s + \int_0^\infty g_s h_s \, dx_s + \int_0^\infty h_s k_s ds + \int_0^\infty h_s k_s \psi_s \, dx_s \, . \end{aligned}$$

For example if one takes $\chi_t = \sum_n f_n^t \in \Phi$ with $f_n^t \equiv 0$ for $n \neq 1$ and $f_1^t(s) = \mathbb{1}_{[0,1t]}(s)$, we have

$$U_w \chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s) \, dw_s = w_t \quad \text{the Brownian motion itself}$$

and

$$U_p \chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s) \, dx_s = x_t \quad \text{the compensated Poisson process itself.}$$

So, as $w_t^2 = 2\int_0^t w_s \, dw_s + t$ and $x_t^2 = 2\int_0^t x_s \, dx_s + t + x_t$, we have

$$\chi_t *_w \chi_t = t + f_2^t \quad \text{with } f_2^t(u, v) = 2\mathbb{1}_{0\leq u\leq v\leq t}$$

and

$$\chi_t *_p \chi_t = t + f_1^t + f_2^t \text{ with } f_2^t \text{ the same as above and } f_1^t(u) = \mathbb{1}_{[0,t]}(u) \, .$$

So we get two different element of $\Phi$.

### I.3.4. Probabilistic interpretations of the Fock space calculus.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P, (x_t)_{t\geq 0})$ be a probabilistic interpretation of the Fock space $\Phi$. Via the isomorphism described in chapter I, the space $\Phi_{t]}$ interprets as the space of $f \in CS(x.)$ whose chaotic expansion contains only functions with support included in $[0,t]$; that is, the space $CS(x) \cap L^2(\mathcal{F}_t)$. So in case of C.R.P. we have $\Phi_{t]} \simeq L^2(\mathcal{F}_t)$ and thus $P_t$ is noting but $\mathbb{E}[\cdot\,|\mathcal{F}_t]$ (the conditional expectation) when interpreted in $L^2(\Omega)$.

The process $(\chi_t)_{t\geq 0}$ interprets as a process of random variables whose chaotic expansion is given by

$$\chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s)\ dx_s = x_t.$$

So, in any probabilistic interpretation $(\chi_t)_{t\geq 0}$ becomes the noise $(x_t)_{t\geq 0}$ itself (Brownian motion, compensated Poisson process, Azéma martingale,...). $(\chi_t)_{t\geq 0}$ is the "universal" noise, seen in the Fock space $\Phi$.

Thus, as we have proved that the Ito integral $\mathcal{I}(g.)$ on $\Phi$ is the $L^2$-limit of the Riemann sums $\sum_i g_{t_i}(\chi_{t_{i+1}} - \chi_{t_i})$, it is clear that in $L^2(\Omega)$, the Ito integral interprets as the usual Ito integral with respect to $(x_t)_{t\geq 0}$.

One remark is necessary here. When one writes the approximation of the Ito integral $\int_0^\infty g_s\ dx_s$ as $\sum_i g_{t_i}(x_{t_{i+1}} - x_{t_i})$ there are products $(g_{t_i}\cdot(x_{t_{i+1}} - x_{t_i}))$ appearing, so this notion seems to depend on the probabilistic interpretation of $\Phi$; it seems not being intrinsic to $\Phi$. But we have seen it is! The point in that the product $g_{t_i}\cdot(x_{t_{i+1}} - s_{t_i})$ is not really a product. By this I mean that the Ito formula for this product does not involve any bracket term:

$$g_{t_i}(x_{t_{i+1}} - x_{t_i}) = \int_{t_i}^{t_{i+1}} g_{t_i}\ dx_s$$

so it gives rise to the same formula whatever is the probabilistic interpretation $(x_t)_{t\geq 0}$. Actually this fact comes from the tensor product structure: $\Phi \simeq \Phi_{t_i]} \otimes \Phi_{[t_i}$; the product $g_{t_i}(x_{t_{i+1}} - x_{t_i})$ is actually a tensor product $g_{t_i} \otimes (x_{t_{i+1}} - x_{t_i})$ in this structure. But this tensor product structure is common to all the probabilistic interpretations.

So we have seen that $\int_0^\infty g_t\ d\chi_t$ interprets as the usual Ito integral $\int_0^\infty g_t\ d\chi_t$ in any probabilistic interpretation $(x_t)_{t\geq 0}$. Thus the representation

$$f = P_0 f + \int_0^\infty D_s f\ d\chi_x$$

of Theorem I.2.4 is just a Fock space intrinsic expression of the P.R.P. The process $(D_t f)_{t\geq 0}$ is then interpreted as the predictable process that represents $f$ in his

P.R.P. If you look at the formula (I.4) that gives this predictable representant $(h_t)_{t\geq 0}$ in terms of the chaotic expansion of $f$, it is not surprising that it should be intrinsic. A careful look at this formula shown clearly that $h_t$ has to be $D_t f$ (exercise).

# II. EXTENSION OF QUANTUM STOCHASTIC CALCULUS

## II.1. An heuristic approach to noise

### II.1.1. Adaptedness.

As we have seen in R. L. Hudson's course, the correct notion of time-adaptedness for operators on Fock space is the following: an operator $H$ on $\Phi$ is adapted at time $t$ if

*i)* $\mathrm{Dom}\, H \supset \mathcal{E}$ (the space of coherent vectors);

*ii)* $H\varepsilon(u_t) \in \Phi_{t]}$ for all $u \in L^2(\mathbb{R}^+)$ (where $u_{t]} = u\mathbb{1}_{[0,t]}$).

*iii)* $H\varepsilon(u) = [H\varepsilon(u_{t]})] \otimes \varepsilon(u_{[t})$ (where $u_{[t} = u\mathbb{1}_{[t,\infty[}$).

That is, roughly speaking $H = k \otimes I$ in the tensor product structure $\Phi \simeq \Phi_{t]} \otimes \Phi_{[t}$.

Why do we choose such a definition for adaptedness? This is motivated by several points:

*i)* this definition coincides with the usual definition of adaptedness in probabilistic interpretations of $\Phi$;

*ii)* just like in classical stochastic calculus, it is a definition that will allow to produce an integration theory.

Now the question is: what kind of process of operators $(X_t)_{t\geq 0}$ can we use to integrate adapted processes of operators $(H_t)_{t\geq 0}$? We want to form $\int_0^\infty H_s\, dX_s$ as a limit of Riemann sums:

$$\sum_i H_{t_i}(X_{t_{i+1}} - X_{t_i}).$$

If the process $(H_t)_{t\geq 0}$ is adapted then $H_{t_i}$ is of the form $k \otimes I$ in the tensor product $\Phi_{t_i]} \otimes \Phi_{[t_i}$. Or else, it is of the form $k \otimes I \otimes I$ in the tensor product $\Phi_{t_i]} \otimes \Phi_{[t_i,t_{i+1}]} \otimes \Phi_{[t_{i+1}}$. If the process $(X_t)_{t\geq 0}$ is also adapted we have that $X_{t_i}$ is of the form $k' \otimes I \otimes I$ and $X_{t_{i+1}}$ is of the form $k'' \otimes k''' \otimes I$, so the only thing one can say about $X_{t_{i+1}} - X_{t_i}$ is that it is of the form $k^{(4)} \otimes k^{(5)} \otimes I$. When one tries to compose $H_{t_i}$ with $X_{t_{i+1}} - X_{t_i}$ we will have to compose $k$ with $k^{(4)}$ on $\Phi_{t_i]}$, thus, except if we deal only with bounded operators, there is a big domain problem. Dealing with bounded operators only cannot be satisfactory as observables like energy (which are self-adjoint operators with unbounded spectrum) cannot be bounded operators. But, if by chance we have a process of integrators $(X_t)_{t\geq 0}$ which has the same independent increments property as $(\chi_t)_{t\geq 0}$, that is

$$X_{t_{i+1}} - X_{t_i} \quad \text{of the form} \quad I \otimes k^{(6)} \otimes I$$

we avoid the composition problem and we can consider the Riemann sums

$$\sum_i H_{t_i}(X_{t_{i+1}} - X_{t_i}) = \sum_i H_{t_i} \otimes (X_{t_{i+1}} - X_{t_i}) \ .$$

**II.1.2.There are only three noises.**

We will call "*noise*" (or better "*quantum noise*") adapted processes of operators on $\Phi$, say $(X_t)_{t\geq 0}$, such that, for all $t_i \leq t_{i+1}$, the operator $X_{t_{i+1}} - X_{t_i}$ acts as $I \otimes k \otimes I$ on $\Phi_{t_i]} \otimes \Phi_{[t_i,t_{i+1}]} \otimes \Phi_{[t_{i+1}}$.

Let us consider the operator $dX_t = X_{t+dt} - X_t$. It acts only on $\Phi_{[t,t+dt]}$. The chaotic representation property of Fock space (Theorem I.2.5) shows that this part of the Fock space is generated by the vacuum $\mathbb{1}$ and by $d\chi_t = \chi_{t+dt} - \chi_t$. So $dX_t$ is determined by its value on $\mathbb{1}$ and on $d\chi_t$. These values have to remain in $\Phi_{[t,t+dt]}$ and to be integrators also, that is $d\chi_t$ or $dt\mathbb{1}$ (denoted $dt$). So the only *irreductible* noises are:

| | $d\chi_t$ | $\mathbb{1}$ |
|---|---|---|
| $da_t^\circ$ | $d\chi_t$ | 0 |
| $da_t^-$ | $dt$ | 0 |
| $da_t^+$ | 0 | $d\chi_t$ |
| $da_t^\times$ | 0 | $dt$ |

There are four noises and not three as announced, but we will see later that $da_t^\times$ is just the usual $dt$.

## II.2. Extension of quantum stochastic integrals

### II.2.1. Heuristic approach.

Let us now consider a quantum stochastic integral

$$T_t = \int_0^t H_s da_s^\varepsilon$$

with respect to one of the four noises. Let it act on a vector process

$$f_t = P_t f = \int_0^t D_s f \ d\chi_s \quad \text{(we omit the expectation } P_0 f \text{ for the moment).}$$

The result is a process of vectors $(T_t f_t)_{t\geq 0}$ in $\Phi$. What can we expect from this process? R. L. Hudson showed you that when $T_t = A_t^+ + A_t$ it is the operator of multiplication by the Brownian motion $w_t$. In the same way, "any" multiplication operator by a classical martingale in a probabilistic interpretation of $\Phi$ can be represented as a quantum stochastic integral (in the sense of H.P.); so, at least, we should have $(T_t f_t)_{t\geq 0}$ satisfying the usual Ito integration by part formula:

$$\begin{aligned} d(T_t f_t) &= T_t df_t + (dT_t) f_t + (dT_t)(df_t) \\ &= T_t(D_t f \ d\chi_t) + (H_t da_t^\varepsilon) f_t + (H_t da_t^\varepsilon)(D_t f \ d\chi_t). \end{aligned}$$

In the tensor product structure $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ this writes

$$\begin{aligned} d(T_t f_t) &= (T_t \otimes I)(D_t f \otimes d\chi_t) + (H_t \otimes da_t^\varepsilon)(f_t \otimes \mathbb{1}) + (H_t \otimes da_t^\varepsilon)(D_t f \otimes d\chi_t) \\ &= T_t D_t f_t \otimes d\chi_t + H_t f_t \otimes da_t^\varepsilon \mathbb{1} + H_t D_t f \otimes da_t^\varepsilon \ d\chi_t. \end{aligned} \tag{II.1}$$

In the right hand side one sees three terms; the first one always remains and is always the same. The other two depend on the table shown above. Integrating (II.1) and using the table one gets

$$T_t f_t = \int_0^t T_s D_s f \ d\chi_s + \begin{cases} \int_0^t H_s D_s f \ d\chi_s & \text{if } \varepsilon = 0 \\ \int_0^t H_s P_s f \ d\chi_s & \text{if } \varepsilon = + \\ \int_0^t H_s D_s f \ ds & \text{if } \varepsilon = - \\ \int_0^t H_s P_s f \ ds & \text{if } \varepsilon = \times. \end{cases} \tag{II.2}$$

### II.2.2. A correct definition.

We want to exploit formula (II.2) as a definition of the quantum stochastic integrals $T_t = \int_0^t H_s da_s^\varepsilon$.

Let $(H_t)_{t\geq 0}$ be a given adapted process of operators on $\Phi$. Let $(T_t)_{t\geq 0}$ be another one. One says that (II.2) is *meaningful* for a given $f \in \Phi$ if

- $P_t f \in \operatorname{Dom} T_t$;
- $D_s f \in \operatorname{Dom} T_s$, $s \leq t$ and $\int_0^t \|T_s D_s f\|^2 \ ds < \infty$;
- $\begin{cases} \text{if } \varepsilon = 0,\ D_s f \in \operatorname{Dom} H_s,\ s \leq t \text{ and } \int_0^t \|H_s D_s f\|^2 \ ds < \infty \\ \text{if } \varepsilon = 0,\ P_s f \in \operatorname{Dom} H_s,\ s \leq t \text{ and } \int_0^t \|H_s P_s f\|^2 \ ds < \infty \\ \text{if } \varepsilon = -,\ D_s f \in \operatorname{Dom} H_s,\ s \leq t \text{ and } \int_0^t \|H_s D_s f\| \ ds < \infty \\ \text{if } \varepsilon = \times,\ P_s f \in \operatorname{Dom} H_s,\ s \leq t \text{ and } \int_0^t \|H_s P_s f\| \ ds < \infty. \end{cases}$

One says that (II.2) is *true* if the equality holds.

DEFINITIONS. — A subspace $\mathcal{D} \subset \Phi$ is called an *adapted domain* if for all $f \in \mathcal{D}$ and all (almost all) $t \in \mathbb{R}^+$, one has

$$P_t f \quad \text{and} \quad D_t f \in \mathcal{D} .$$

There are many examples of adapted domains. All the domains you will meet during this course are:

- $\mathcal{D} = \Phi$ is adapted;
- $\mathcal{D} = \mathcal{E}$ is adapted. And even $\mathcal{D} = \mathcal{E}(\mathcal{M})$ is adapted once $\mathbb{1}_{[0,t]}\mathcal{M} \subset \mathcal{M}$ for all $t$.
- The space of *finite particles* $\Phi_f = \{f \in L^2(\mathcal{P});\ f(\sigma) = 0$ for $\#\sigma > N$, for some $N \in \mathbb{N}\}$ is adapted.
- All the Fock scales $\Phi^{(a)} = \{f \in L^2(\mathcal{P});\ \int_{\mathcal{P}} a^{\#\sigma}|f(\sigma)|^2\ d\sigma < \infty\}$, for $a \geq 1$, are adapted.
- Maassen's space of test vectors: $\{f \in L^2(\mathcal{P});\ f(\sigma) = 0$ for $\#\sigma \not\subset [0,T]$,for some $T \in \mathbb{R}^+$, and $|f(\sigma)| \leq CM^{\#\sigma}$ for some $C, M\}$ is adapted.

(Exercises).

Let $(H_t)_{t\geq 0}$ be an adapted process of operators defined on an adapted domain $\mathcal{D}$. One says that a process $(T_t)_{t\geq 0}$ is the *stochastic integral* $T_t = \int_0^t H_s\ da_s^\varepsilon$ *on* $\mathcal{D}$, if (II.2) is meaningfull and true for all $f \in \mathcal{D}$.

THEOREM II.2.1. — *On the stable domains $\mathcal{E}(\mathcal{M})$ this definition is equivalent to Hudson-Parthasarathy's one.*

*Proof.* — We will first show that if $(H_t)_{t\geq 0}$ is an adapted process defined on $\mathcal{E}(\mathcal{M})$ such that, for all $t \geq 0$

$$\int_0^t |u(s)|^2 \|H_s \varepsilon(u_{s]})\|^2\ ds < \infty \quad \text{if} \quad \varepsilon = 0$$

$$\int_0^t \|H_s \varepsilon(u_{s]})\|^2\ ds < \infty \quad \text{if} \quad \varepsilon = +$$

$$\int_0^t |u(s)| \|H_s \varepsilon(u_{s]})\|\ ds < \infty \quad \text{if} \quad \varepsilon = -$$

$$\int_0^t \|H_s \varepsilon(u_{s]})\|\ ds < \infty \quad \text{if} \quad \varepsilon = \times$$

then the equation (II.2) admits a unique solution on $\mathcal{E}(\mathcal{M})$. We will prove it by the usual Picard method. Let us do it for the case $\varepsilon = 0$ and leave the three other cases to the reader. We want to solve

$$T_t\varepsilon(u_{t]}) = \int_0^t u(s)T_s\varepsilon(u_{s]})\ d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]})\ d\chi_s. \qquad \text{(II. 3)}$$

Let $x_t = T_t\varepsilon(u_{t]})$, $t \geq 0$. We have to solve

$$x_t = \int_0^t u(s)x_s\ d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]})\ d\chi_s.$$

Put $x_t^0 = \int_0^t u(s)H_s\varepsilon(u_{s]})\ d\chi_s$ and

$$x_t^{n+1} = \int_0^t u(s)x_s^n\ d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]})\ d\chi_s. \qquad \text{(II. 4)}$$

Let $y_t^0 = x_t^0$ and $y_t^{n+1} = x_t^{n+1} - x_t^n = \int_0^t u(s)y_s^n\ d\chi_s$. We have

$$\begin{aligned}
\|y_t^{n+1}\|^2 &= \int_0^t |u(s)|^2\|y_s^n\|^2\ ds \\
&= \int_0^t\int_0^{t_1} |u(t_1)|^2|u(t_2)|^2\|y_{t_2}^{n-1}\|^2\ dt_2\ dt_1 \\
&\ \ \vdots \\
&= \int_{0\leq t_1\leq\cdots\leq t_n\leq t} |u(t_1)|^2\cdots|u(t_n)|^2\|y_{t_1}^0\|^2\ dt_1\cdots dt_n \\
&= \int_{0\leq t_1\leq\cdots\leq t_n\leq t} |u(t_1)|^2\cdots|u(t_n)|^2\int_0^{t_1}|u(s)^2\|H_s\varepsilon(u_{s]})\|^2\ ds\ dt_1\cdots dt_n \\
&\leq \int_0^t |u(s)|^2\|H_s\varepsilon(u_{s]})\|^2\ ds\ \frac{\left(\int_0^t |u(s)|^2\ ds\right)^n}{n!}.
\end{aligned}$$

From this estimate one easily, sees that the sequences

$$x_t^n = \sum_{k=0}^n y_t^k, \quad n\in\mathbb{N}, \quad t\in\mathbb{R}^+$$

are Cauchy sequences in $\Phi$. Let us call $x_t = \lim_{n\to+\infty} x_t^n$. One also easily sees, from the same estimate, that

$$\int_0^t |u(s)|^2\|x_s\|^2\ ds < \infty \ \text{ for all } \ t\in\mathbb{R}^+.$$

Passing to the limit on equality (II.4), one gets

$$x_t = \int_0^t u(s)x_s\ d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]})\ d\chi_s.$$

Define operators $T_t$ on $\Phi_{t]}$ (more precisely on $\varepsilon\cap\Phi_{t]}$) by putting $T_t\varepsilon(u_{t]}) = x_t$. I leave to the reader to check that this defines (by linear extension) an operator on

$\varepsilon \cap \Phi_{t]}$ (use the fact that any finit family of coherent vectors is free). Extend the operator $T_t$ to $\varepsilon$ by adaptedness:

$$T_t\varepsilon(u) = T_t\varepsilon(u_{t]}) \otimes \varepsilon(u_{[t}).$$

We thus get a solution to (II.3).

Let us now prove that this solution satisfies Hudson-Parthasarathy's identity.

We have

$$\langle \varepsilon(v_{t]}), T_t\varepsilon(u_{t]})\rangle = \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{t]}), T_s\varepsilon(u_{t]})\rangle \; ds + \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s\varepsilon(u_{s]})\rangle \; ds.$$

Put $\alpha_t = \langle \varepsilon(v_{t]}), T_t\varepsilon(u_{t]})\rangle$, $t \in \mathbb{R}^+$. We have

$$\alpha_t = \int_0^t \bar{v}(s)u(s)\alpha_s \; ds + \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s\varepsilon(u_{s]})\rangle \; ds$$

that is,

$$\frac{d}{dt}\alpha_t = \bar{v}(t)u(t)\alpha_t + \bar{v}(t)u(t)\langle \varepsilon(v_{t]}), H_t\varepsilon(u_{t]})\rangle.$$

Or else

$$\begin{aligned}
\alpha_t &= e^{\int_0^t \bar{v}(s)u(s) \; ds} \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s\varepsilon(u_{s]})\rangle e^{-\int_0^s \bar{v}(k)u(k) \; dk} \; ds \\
&= \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s\varepsilon(u_{s]})\rangle e^{-\int_s^t \bar{v}(k)u(k) \; dk} \; ds \\
&= \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s\varepsilon(u_{s]})\rangle\langle \varepsilon(v_{[s,t]}), \varepsilon(u[s,t])\rangle \; ds \\
&= \int_0^t \bar{v}(s)u(s)\langle \varepsilon(v_{t]}), H_s\varepsilon(u_{t]})\rangle\langle \; ds \quad \text{(by adaptedness)}.
\end{aligned}$$

We have proved that our equation admits a solution on $\mathcal{E}$, that this solution coincides with Hudson-Parthasarathy's one on $\mathcal{E}$.

The converse is immediate from what we have already obtained. ■

The advantage of equation (II.2) on Hudson-Parthasarathy's setting is that we may have solutions of this equations on domains that are larger than $\mathcal{E}$, or even completely different. Let us see for example how equation (II.2) can provide a solution on $\Phi_f$, the space of finite particles. I still take the example $T_t = \int_0^t H_s \; da_s^\circ$ (the reader may check the other three cases). I make the computation algebraically, without caring about integrability or domain problems. We have the equation

$$T_t f_t = \int_0^t T_s D_s f \; d\chi_s + \int_0^t H_s D_s f \; d\chi_s.$$

Let $f = \mathbb{1}$. This implies (as $D_t\mathbb{1} = 0$ for all $t$)

$$T_t\mathbb{1} = 0\ .$$

Let $f = \int_0^\infty f_1(s)\ d\chi_s$ for $f_1 \in L^2(\Sigma_1)$. We have

$$\begin{aligned}T_t f_t &= \int_0^t T_s f_1(s)\mathbb{1}\ d\chi_s + \int_0^t H_s f_1(s)\mathbb{1}\ d\chi_s \\ &= 0 + \int_0^t f_1(s) H_s\mathbb{1}\ d\chi_s.\end{aligned}$$

Let $f = \int_{0\le t_1\le t_2} f_2(t_1,t_2)\ d\chi_{t_1}\ d\chi_{t_2}$ for $f_2 \in L^2(\Sigma_2)$. We have

$$\begin{aligned}T_t f_t &= \int_0^t T_s \int_0^s f_2(t_1,s)\ d\chi_{t_1}\ d\chi_s + \int_0^t H_s \int_0^s f_2(t_1,s)\ d\chi_{t_1}\ d\chi_s \\ &= \int_0^t \int_0^s f_2(u,s) H_u\mathbb{1}\ d\chi_u\ d\chi_s + \int_0^t H_s \int_0^s f_2(u,s)\ d\chi_u\ d\chi_s.\end{aligned}$$

You see that, by induction on the chaoses, we can derive the action of $T_t$ on $\Phi_f$.

By the way, notice that Theorem II.2.1 shows that the noises we have heuristically derived correspond to the ones introduced by R. L. Hudson:

$$\begin{cases} a^+ = A_t^+ \\ a_t^- = A_t \\ a_t^\circ = \Lambda_t \\ a_t^\times = tI. \end{cases}$$

So we have a definition of quantum stochastic integrals which coincides with Hudson-Parthasarathy's one on $\mathcal{E}$, but which extends it to many other domains. Let us see a very useful result that says under which conditions a quantum stochastic integral, which is defined on $\mathcal{E}$, can be extended (in our sense) to larger domains.

Extension Theorem. — *If $(T_t)_{t\ge 0}$ is an adapted process of operators on $\Phi$ which admits an integral representation on $\mathcal{E}(\mathcal{M})$ and such that the adjoint process $(T_t^*)_{t\ge 0}$ admits an integral representation on $\mathcal{E}(\mathcal{M}')$. Then the integral representations of $(T_t)_{t\ge 0}$ and $(T_t^*)_{t\ge 0}$ can be extended everywhere equation (II.2) is meaningful.*

Before proving this theorem, we shall maybe be clear about what it means.

The hypothesis are that:

- $T_f = \int_0^t H_s^\circ\ da_s^\circ + \int_0^t H_s^+\ da_s^+ + \int_0^t H_s^-\ da_s^- + \int_0^t H_s^\times\ da_s^\times$ on $\mathcal{E}(\mathcal{M})$, where $\mathcal{M}$ is a dense subspace of $L^2(\mathbb{R}^+)$, stable under $\mathbb{1}_{[0,t]}$ for all $t$. This in particular means that

  $$\int_0^t |u(s)|^2 \|H_s^\circ \varepsilon(u_{s]})\|^2 + \|H_s^+ \varepsilon(u_{s]})\|^2 + |u(s)|\ \|H_s^- \varepsilon(u_{s]})\| + \|H_s^\times \varepsilon(u_{s]})\|\ ds$$

is finite for all $t \in \mathbb{R}^+$, all $u \in \mathcal{M}$.

- The assumption on the adjoint simply means that

$$\int_0^t |u(s)|^2 \|H_s^{0*}\varepsilon(u_{s]})\|^2 + \|H_s^{-*}\varepsilon(u_{s]})\|^2 + |u(s)|\ \|H_s^{+*}\varepsilon(u_{s]})\| \\ + \|H_s^{\times *}\varepsilon(u_{s]})\|\ ds < \infty$$

for all $t \in \mathbb{R}^+$ and all $u$ in a $\mathcal{M}'$ dense in $L^2(\mathbb{R}^+)$.

The conclusion is that for all $f \in \Phi$, such that equation (II.2) is meaningful (for $(T_t)_{t\geq 0}$ or for $(T_t^*)_{t\geq 0}$), the equality (II.2) will be valid.

Let us take an example:

Let $J_t\varepsilon(u) = \varepsilon(-u_{t]}) \otimes \varepsilon(u_{[t})$. It is an adapted process of operators on $\Phi$ which is made of unitary operators, and $J_t^2 = I$.

*Exercises.*

- Check that $B_t = \int_0^t J_s\ da_s^-$ is well defined on $\mathcal{E}$ and that $B_t^* = \int_0^t J_s da_s^+$ is well defined on $\mathcal{E}$ and is the adjoint of $B_t$ (on $\mathcal{E}$);
- Show that $J_t = I - 2\int_0^t J_s\ da_s^\circ$;
- Show that if $X_t = -2\int_0^t X_s\ da_s^\circ$ then $X_t \equiv 0$ for all $t$.
- Use this to show that

$$B_tJ_t + J_tB_t = 0\ ;$$

- Conclude that $B_tB_t^* + B_t^*B_t = tI$.

The last identity shows that $B_t$ is a bounded operator with norm smaller that $\sqrt{t}$.

Now, we know that, for all $f \in \mathcal{E}$ we have

$$B_tf_t = \int_0^t B_sD_sf\ d\chi_s + \int_0^t J_sD_sf\ ds\ ; \qquad (II.5)$$

we know that the adjoint of $B_t$ can be represented as a Quantum stochastic integral on $\mathcal{E}$. So we are in the hypothesis of the Extension Theorem.

For which $f \in \Phi$ do we have all the terms of equation (II.5) being well defined? The results above easily show that for *all* $f \in \Phi$ we have $B_tf_t$, $\int_0^t B_sD_sf\ d\chi_s$, $\int_0^t J_sD_sf\ ds$ to be well defined. So the extension theorem says that equation (II.5) is valid for all $f \in \Phi$! The same holds for $B_t^*$. The integral representation of $(B_t)_{t\geq 0}$ (and $(B_t^*)_{t\geq 0}$) is valid on *all* $\Phi$, in the extended sense.

*Proof of the Extension Theorem.* — Let $f \in \Phi$ be such that all the terms of equation (II.5) are meaningful. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathcal{M})$ which converges to $f$. Let $g \in \mathcal{E}(\mathcal{M}')$. We have

$$\begin{aligned}
&\Big|\langle g, T_t f_t - \int_0^t T_s D_s f \; d\chi_s - \int_0^t H_s^\circ D_s f \; d\chi_s \\
&\qquad\qquad - \int_0^t H_s^+ P_s f \; d\chi_s - \int_0^t H_s^- D_s f \; ds - \int_0^t H_s^\times P_s f \; ds\rangle\Big| \\
&\leq \; |\langle g, T_t P_t (f - f_n)\rangle| + \Big|\langle g, \int_0^t T_s D_s (f - f_n) \; d\chi_s\rangle\Big| \\
&\quad + \Big|\langle g, \int_0^t H_s^\circ D_s (f - f_n) \; d\chi_s\rangle\Big| + \Big|\langle g, \int_0^t H_s^\times P_s (f - f_n) \; ds\rangle\Big| \\
&\quad + |\langle g, \int_0^t H_s^- D_s (f - f_n) \; ds\rangle| + |\langle g, \int_0^t H_s^\times P_s (f - f_n) \; ds\rangle| \\
&\leq \; \|T_t^* g\| \; \|f - f_n\| + \int_0^t |\langle T_s^* D_s g, D_s (f - f_n)\rangle| \; ds \\
&\quad + \int_0^t |\langle H_s^{\circ *} D_s g, D_s (f - f_n)\rangle| \; ds + \int_0^t |\langle H_s^{+ *} D_s g, P_s (f - f_n)\rangle| \; ds \\
&\quad + \int_0^t |\langle H_s^{- *} g, D_s (f - f_n)\rangle| \; ds + \int_0^t |\langle H_s^{\times *} g, P_s (f - f_n)\rangle| \; ds \\
&\leq \; \Big[\|T_t^* g\| + \int_0^t \|T_s^* D_s g\|^2 \; ds + \int_0^t \|H_s^{\circ *} D_s g\|^2 \; ds + \int_0^t \|H_s^{+ *} D_s g\| \; ds \\
&\quad + \int_0^t \|H_s^{- *} g\|^2 \; ds + \int_0^t \|H_s^{\times *} g\| \; ds\Big] \; \|f - f_n\| \; .
\end{aligned}$$

■

## II.3. Back to probabilistic interpretations

### II.3.1. Multiplication operators.

Let us take a probabilistic interpretation $(\Omega, \mathcal{F}, P, (x_t)_{t\geq 0})$ of the Fock space, which is described by the structure equation

$$d[x, x]_t = dt + \psi_t \; dx_t.$$

The operator $M_{x_t}$ on $\Phi$ of multiplication by $x_t$ (for this interpretation) is a particular operator on $\Phi$. It is adapted at time $t$. The process $(M_{x_t})_{t\geq 0}$ is an adapted process of operators on $\Phi$. Can we represent this process as quantum stochastic integrals?

If one denotes by $M_{\psi_t}$ the operator of multiplication by $\psi_t$ (for the $(x_t)_{t\geq 0}$-product again) we have the following:

THEOREM II.3.1.

$$M_{x_t} = a_t^+ + a_t^- + \int_0^t M_{\psi_t}\ da_t^\circ .$$

*Proof.* — Let us be clear about domains: the domain of $M_{x_t}$ is exactly the space of $f \in \Phi$ such that $x_t \cdot U_x f$ belongs to $L^2(\Omega)$, where you recall that $U_x$ is the isomorphism $U_x : \Phi \to L^2(\Omega)$.

Let us go to the proof of the result:

$$x_t f = \int_0^\infty x_s D_s f\ dx_s + \int_0^t P_s f\ dx_s + \int_0^t D_s f\ ds + \int_0^t \psi_s D_s f\ dx_s$$

by the usual Ito formula. That is, on $\Phi$

$$M_{x_t} f = \int_0^\infty M_{x_s} D_s f\ d\chi_s + \int_0^t P_s f\ d\chi_s + \int_0^t D_s f\ ds + \int_0^t M_{\psi_s} D_s f\ d\chi_s$$

which is exactly equation (II.2) for the quantum stochastic process $X_t = a_t^+ + a_t^- + \int_0^t M_{\psi_t}\ da_t^\circ$. ■

We recover that:

- Multiplication by Brownian motion is $a_t^+ + a_t^-$;
- Multiplication by compensated Poisson process is $a_t^+ + a_t^- + a_t^\circ$;
- Multiplication by the $\beta$-Azéma martingale is the unique solution of $X_t = a_t^+ + a_t^- + \int_0^t \beta X_s\ da_s^\circ$.

### II.3.2. Extension of some classical operators.

There are plenty other operators coming from classical calculus that give rise to operators on the Fock space. Let us take, for example, the Brownian interpretation of $\Phi$ (we could have taken any other). We don't look closely to domain problems in the following (though this could be done easily). Let $(h_t)_{t\geq 0}$ be a predictable process, let $m_t = \int_0^t \dot{m}_s\ dw_s$ be a martingale, and $n_t = \int_0^t \dot{m}_s\ dw_s$ be another one, let $v_t = \int_0^t k_s\ ds$. Define the following operators on $L^2(\Omega)$:

$$I_h^t : f \longmapsto \int_0^t h_s\ df_s \quad \text{where} \quad f_s = P_s f$$

$$J_m^t : f \longmapsto \int_0^t f_s\ dm_s$$

$$K_n^t : f \longmapsto \langle f_\cdot, n_\cdot \rangle_t$$

$$T_v^t : f \longmapsto \int_0^t f_s\ dv_s .$$

These are operators from $L^2(\Omega)$ into itself. That is they are operators on Fock space. Do they have an integral representation?

THEOREM II.3.2. — *Let $T_t = I_h^t + J_m^t + K_n^t + T_v^t$, made $t$-adapted. Then*

$$T_t = \int_0^t (M_{h_s} - T_s)\ da_s^\circ + \int_0^t M_{\dot m_s}\ da_s^+ + \int_0^t M_{\dot n_s}\ da_s^- + \int_0^t M_{k_s}\ da_s^\times .$$

*Proof.*

$$\begin{aligned} T_t f_t &= \int_0^t h_s D_s f\ dw_s + \int_0^t \dot m_s P_s f\ dw_s + \int_0^t \dot n_s D_s f\ ds + \int_0^t k_s P_s f\ ds \\ &= \int_0^t M_{h_s} D_s f\ dw_s + \int_0^t M_{\dot m_s} P_s f\ dw_s + \int_0^t M_{\dot n_s} D_s f\ ds + \int_0^t M_{k_s} P_s f\ ds. \end{aligned}$$

So on $\Phi$:

$$\begin{aligned} T_t f_t = \int_0^t T_s D_s f\ d\chi_s &+ \int_0^t (M_{h_s} - T_s) D_s f\ d\chi_s + \int_0^t M_{\dot m_s} P_s f\ d\chi_s \\ &+ \int_0^t M_{\dot n_s} D_s f\ ds + \int_0^t M_{k_s} P_s f\ ds. \end{aligned}$$

■

There result, if you forget the $\int_0^t -T_s\ da_s^\circ$ term, shows a bijection between the four basic operators $I$, $J$, $K$, $T$ and the four types of quantum stochastic integrals. A general process of the form

$$T_t = \sum_{\varepsilon = 0,+,-,\times} \int_0^t H_s^\varepsilon\ da_s^\varepsilon$$

acts on $\Phi$ in the same way as $I + J + K + T$ but where multiplication operators are replaced by general operators. The quantum stochastic integrals are the non commutative analogue of these four classical operators.

# III. THE ALGEBRA OF REGULAR QUANTUM SEMIMARTINGALES

## III.1. Everywhere defined quantum stochastic integrals

### III.1.1. A true Ito formula.

With our definition of quantum stochastic integrals defined on any stable domain, we may meet quantum stochastic integrals that are defined on the whole of $\Phi$. Let us recall it, An adapted process of bounded operators $(T_t)_{t\geq 0}$ on $\Phi$ is said to have the integral representation

$$T_t = \sum_{\varepsilon=\{0,+,-,\times\}} \int_0^t H_s^\varepsilon \, da_s^\varepsilon$$

on the *whole of* $\Phi$ if, for all $f \in \Phi$ one has

$$\int_0^t \|T_s D_s f\|^2 + \|H_s^\circ D_s f\|^2 + \|H_s^+ P_s f\|^2 + \|H_s^- D_s f\| + \|H_s^\times P_s f\| \, ds < \infty$$

for all $t \in \mathbb{R}^+$ (the $H_t^\varepsilon$ are bounded operators) and

$$T_t P_t f = \int_0^t T_s D_s f \, d\chi_s + \int_0^t H_s^\circ D_s f \, d\chi_s + \int_0^t H_s^+ P_s f \, d\chi_s + \int_0^t H_s^- D_s f \, ds + \int_0^t H_s^\times P_s f \, ds.$$

If one has two such processes $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ one can compose them, and wonder if the result $(S_t T_t)_{t\geq 0}$ is also representable on the whole of $\Phi$. The answer is yes. And you will not be surprised to recover the quantum Ito formula presented by R. L. Hudson; but this time for true compositions of operators.

THEOREM III.1.1. — *If $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, da_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon \, da_s^\varepsilon$ are everywhere defined quantum stochastic integrals, then $(S_t T_t)_{t\geq 0}$ is everywhere representable as quantum stochastic integrals:*

$$S_t T_t = \int_0^t (S_s H_s^\circ + K_s^\circ T_s + K_s^\circ H_s^\circ) \, da_s^\circ + \int_0^t (S_s H_s^+ + K_s^+ T_s + K_s^\circ H_s^+) \, da_s^+ + \int_0^t (S_s H_s^- + K_s^- T_s + K_s^- H_s^\circ) \, da_s^- + \int_0^t (S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+) \, da_s^\times .$$

Before proving this theorem we will need the following:

LEMMA III.1.2. — *Let $g_t = \int_0^t v_s\, ds$ be an adapted process of vectors of $\Phi$, with $\int_0^t \|v_s\|\, ds < \infty$ for all $t$. Let $(S_t)_{t\geq 0}$ be as in Theorem III.1.1. Then*

$$S_t g_t = \int_0^t S_s v_s\, ds + \int_0^t K_s^+ g_s\, d\chi_s + \int_0^t K_s^\times g_s\, ds.$$

*Proof of Lemma III.1.2.* — As $S_t$ is bounded we have

$$\begin{aligned}
S_t g_t &= S_t \int_0^t v_s\, ds = \int_0^t S_t v_s\, ds \quad \text{(Exercise)}\\
&= \int_0^t S_t(P_0 v_s + \int_0^s D_u v_s\, d\chi_u)\, ds\\
&= \int_0^t S_t P_0 v_s\, ds + \int_0^t \Big[\int_0^s S_u D_u v_s\, d\chi_u + \int_0^s K_u^\circ D_u v_s\, d\chi_u\\
&\qquad + \int_0^s K_u^- D_u v_s\, ds + \int_0^t K_u^+ P_u \int_0^s D_v v_s\, d\chi_v\, d\chi_u\\
&\qquad + \int_0^t K_u^\times P_u \int_0^s D_v v_s\, d\chi_v\, du\Big]\, ds\\
&= \int_0^t S_t P_0 v_s\, ds + \int_0^t \Big[S_s \int_0^s D_u v_s\, d\chi_u + \int_s^t K_u^+ \int_0^s D_v v_s d\chi_v\, d\chi_u\\
&\qquad + \int_s^t K_u^\times \int_0^s D_v v_s\, d\chi_v\, du\Big]\, ds\\
&= \int_0^t S_s v_s\, ds + \int_0^t \int_s^t K_u^+ v_s\, d\chi_u\, ds + \int_0^t \int_s^t K_u^\times v_s\, du\, ds\\
&= \int_0^t S_s v_s\, ds + \int_0^t \int_0^u K_u^+ v_s\, ds\, d\chi_u + \int_0^t \int_0^u K_u^\times v_s\, ds\, du\\
&\qquad\qquad \text{(kind of Fubini; exercise)}\\
&= \int_0^t S_s v_s\, ds + \int_0^t K_u^+ \int_0^u v_s\, ds\, d\chi_u + \int_0^t K_u^\times \int_0^u v_s\, ds\, du\\
&= \int_0^t S_s v_s\, ds + \int_0^t K_u^+ g_u\, d\chi_u + \int_0^t K_u^\times g_u\, du.
\end{aligned}$$

This proves the Lemma.

*Proof of Theorem III.1.1.* — Let us just compute the composition, using Lemma III.1.2:

$$\begin{aligned}
T_t f_t = \int_0^t T_s D_s f\, d\chi_s + \int_0^t H_s^\circ D_s f\, d\chi_s + \int_0^t H_s^+ P_s f\, d\chi_s\\
+ \int_0^t H_s^- D_s f\, ds + \int_0^t H_s^\times P_s f\, ds
\end{aligned}$$

so

$$
\begin{aligned}
S_tT_tf_t &= \int_0^t S_s\big[T_sD_sf + H_s^\circ D_sf + H_s^+P_sf\big]\ d\chi_s \\
&\quad + \int_0^t K_s^\circ\big[T_sD_sf + H_s^\circ D_sf + H_s^+P_sf\big]\ d\chi_s + \int_0^t K_s^-\big[T_sD_sf + H_s^\circ D_sf + H_s^+P_sf\big]\ ds \\
&\quad + \int_0^t K_s^+\Big[\int_0^s T_uD_uf\ d\chi_u + \int_0^s H_u^\circ D_uf\ d\chi_u + \int_0^s H_u^+P_uf\ d\chi_u\Big]\ d\chi_s \\
&\quad + \int_0^t K_s^\times\Big[\int_0^s T_uD_uf\ d\chi_u + \int_0^s H_u^\circ D_uf\ d\chi_u + \int_0^s H_u^+P_uf\ d\chi_u\Big]\ du \\
&\quad + \int_0^t S_s\big[H_s^-D_sf + H_s^\times P_sf\big]\ ds + \int_0^t K_s^+\Big[\int_0^s H_u^-D_uf + \int_0^s H_u^\times P_uf\ du\Big]\ d\chi_s \\
&\quad + \int_0^t K_s^\times\Big[\int_0^s H_u^-D_uf + \int_0^s H_u^\times P_uf\ du\Big]\ ds \\
&= \int_0^t S_sT_sD_sf\ d\chi_s + \int_0^t \big[S_sH_s^\circ + K_s^\circ T_s + K_s^\circ H_s^\circ\big]D_sf\ d\chi_s \\
&\quad + \int_0^t \big[S_sH_s^+ + K_s^+T_s + K_s^\circ H_s^+\big]P_sf\ d\chi_s + \int_0^t \big[S_sH_s^- + K_s^-T_s + K_s^-H_s^\circ\big]D_sf\ ds \\
&\quad + \int_0^t \big[S_sH_s^\times + K_s^\times T_s + K_s^-H_s^+\big]P_sf\ ds\ . \qquad \blacksquare
\end{aligned}
$$

### III.1.2. A family of examples.

We have seen $B_t = \int_0^t J_s\ da_s^-$ as an example of everywhere defined quantum stochastic integrals. This example belongs to a larger family of examples which is going to be fundamental in the sequel.

Let $\mathcal{S}$ be the set of *bounded* adapted processes of operators $(T_t)_{t\geq 0}$ on $\Phi$ such that

$$T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon\ da_s^\varepsilon \quad \text{on} \quad \mathcal{E}(\mathcal{M})$$

with all the operators $H_s^\varepsilon$ being bounded and

$$\begin{cases} t \mapsto \|H_t^\circ\| \in L^\infty_{\text{loc}} \\ t \mapsto \|H_t^+\| \in L^2_{\text{loc}} \\ t \mapsto \|H_t^-\| \in L^2_{\text{loc}} \\ t \mapsto \|H_t^\times\| \in L^1_{\text{loc}}\ . \end{cases}$$

With these conditions, we are going to see that $t \mapsto \|T_t\|$ has to be in $L^\infty_{\text{loc}}$. Indeed, the operator $\int_0^t H_s^\times\ da_s^\times$ satisfies $\int_0^t H_s^\times\ da_s^\times f = \int_0^t H_s^\times f\ ds$ (exercise), so it is a bounded operator, with norm less than $\int_0^t \|H_s^\times\|\ ds$, which is locally bounded in $t$. The difference $M_t = T_t - \int_0^t H_s^\times\ da_s^\times$ is thus a martingale of bounded operators.

But as $M$ is a martingale we have $\|M_s f_s\| = \|P_s M_t f_s\| \leq \|M_t f_s\|$ for $s \leq t$. So $t \mapsto \|M_t\|$ is locally bounded. Thus, so is $t \mapsto \|T_t\|$.

With all these informations, it is easy to check (exercise) that the integral representation of $(T_t)_{t\geq 0}$, as well as the one of $(T_t^*)_{t\geq 0}$, can be extended on the whole of $\Phi$ by the extension theorem.

## III.2. The algebra of regular quantum semimartingales

### III.2.1. It is an algebra.

As all elements of $\mathcal{S}$ are everywhere defined quantum stochastic integrals, one can compose them and use the extended quantum Ito formula.

THEOREM III.2.1. — *$\mathcal{S}$ is a $*$-algebra for the adjoint and composition operations*

*Proof.* — The adjoint process $(T_t^*)_{t\geq 0}$ is given by

$$T_t^* = \int_0^t H_s^{0*}\, da_s^\circ + \int_0^t H_s^{-*}\, da_s^+ + \int_0^t H_s^{+*}\, da_s^- + \int_0^t H_s^{\times *}\, da_s^\times \ .$$

It is straightforward to check that it belongs to $\mathcal{S}$. The Ito formula for the composition of two elements of $\mathcal{S}$ gives

$$\begin{aligned} S_t T_t = &\int_0^t \left[S_s H_s^\circ + K_s^\circ T_s + K_s^\circ H_s^\circ\right]\, da_s^\circ + \int_0^t \left[S_s H_s^+ + K_s^+ T_s + K_s^\circ H_s^+\right]\, da_s^+ \\ &+ \int_0^t \left[S_s H_s^- + K_s^- T_s + K_s^- H_s^\circ\right]\, da_s^- + \int_0^t \left[S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+\right]\, da_s^\times . \end{aligned}$$

From the conditions on the maps $t \mapsto \|S_t\|$, $t \mapsto \|T_t\|$, $t \mapsto \|K_r^\varepsilon\|$ and $t \mapsto \|H_t^\varepsilon\|$, it is easy to check that the coefficients in the representation of $(S_t T_t)_{t\geq 0}$ are bounded operators that satisfy the norm conditions for being in $\mathcal{S}$. For example, the coefficients of $da_t^\times$ satisfy

$$\begin{aligned} &\int_0^t \|S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+\|\, ds \\ &\qquad \leq \sup_{s\leq t} \|S_s\| \int_0^t \|H_s^\times\|\, ds + \sup_{s\leq t} \|T_s\| \int_0^t \|K_s^\times\|\, ds \\ &\qquad\quad + \Big(\int_0^t \|K_s^-\|^2\, ds\Big)^{1/2} \Big(\int_0^t \|H_s^+\|^2\, ds\Big)^{1/2} \end{aligned}$$

so it is locally integrable. ∎

Thus we have a nice space of quantum semimartingales that we can compose without bothering about any domain problem, we can pass to the adjoint, we can use formula (II.2) on the whole of $\Phi$.

As a consequence we can immediately think of looking at polynomial functions of elements of $\mathcal{S}$ and compute a Ito formula for it. We'll see that later.

### III.2.2. A characterization.

The problem with $\mathcal{S}$ is its definition! It is in general difficult to know if a process of operators is representable as quantum stochastic integrals; it is even more difficult to know the regularity of its coefficients. We know that $\mathcal{S}$ is not empty, as it contains $B_t = \int_0^t J_s \, da_s^-$ that we have met above. But how big is it? Can we have a characterization of $\mathcal{S}$ that depends only on the process $(T_t)_{t\geq 0}$?

One says that a process $(T_t)_{t\geq 0}$ of *bounded* adapted operators is a *regular quantum semimartingale* is there exists a locally integrable function $h$ on $\mathbb{R}$ such that for all $r \leq s \leq t$, all $f \in \mathcal{E}$ one has (where $f_r = P_r f$)

*i)* $||T_t f_r - T_s f_r||^2 \leq ||f_r||^2 \int_s^t h(u) \, du$;

*ii)* $||T_t^* f_r - T_s^* f_r||^2 \leq ||f_r||^2 \int_s^t h(u) \, du$;

*iii)* $||P_s T_t f_r - T_s f_r|| \leq ||f_r|| \int_s^t h(u) \, du$.

THEOREM III.2.4. — *A process $(T_t)_{t\geq 0}$ of bounded adapted operators is a regular quantum semimartingale if and only if it belongs to $\mathcal{S}$.*

*Proof.* — Showing that elements of $\mathcal{S}$ satisfy the three estimates that define regular quantum semimartingales is straightforward. We leave it as an exercise.

The interesting part is to show that a regular quantum semimartingale is representable as quantum stochastic integrals and belongs to $\mathcal{S}$. We will only sketch it, as the details are rather long and difficult.

Let $x_t = T_t f_r$ for $t \geq r$ ($r$ is fixed, $t$ varies). It is an adapted process of vectors on $\Phi$. It satisfies

$$||P_s x_t - x_s|| \leq ||f_r|| \int_s^t h(u) \, du.$$

This condition is a Hilbert space analogue of a condition in classical probability that defines particular semimartingales: the quasimartingales. O. Enchev has provided a Hilbert space extension of this result and we can deduce from his result that $(x_t)_{t\geq r}$ can be written

$$x_t = m_t + \int_0^t k_s \; ds$$

where $m$ is a martingale in $\Phi$ ($P_s m_t = m_s$) and $h$ is an adapted process in $\Phi$ such that $\int_0^t \|k_s\| \; ds < \infty$.

Thus $P_s x_t - x_s = \int_s^t P_s k_u \; du$ and we have

$$\left\| \int_0^t P_s k_u \; du \right\| \leq \|f_r\| \int_0^t h(u) \; du, \quad \text{for all} \quad r \leq s \leq t.$$

Actually $k_u$ depend linearly on $f_r$. The inequality above implies (difficult exercise) that

$$\|k_u(f_r)\| \leq \|f_r\| h(u) \; .$$

So $k_u$ is a bounded operator on $\Phi_{u]}$, we extend it as a bounded adapted operator $H_u^{\times}$.

Let $M_t = T_t - \int_0^t H_u^{\times} \; da_u^{\times}$, $t \in \mathbb{R}^+$. It is easy to check, from what we have already done, that $(M_t)_{t\geq 0}$ is a martingale of bounded operator (Hint: compute $P_s M_t f_r - M_s f_r$). It is easy to check that $(M_t)_{t\geq 0}$ also satisfies the conditions *i)* and *ii)* of the definition of regular quantum semimartingales, with another function $h$, say $h'$.

Now, let $(y_t)_{t\geq r}$ be $(M_t f_r)_{t\geq r}$. It is a martingale of vectors in $\Phi$. Thus it can be represented as

$$y_t - y_s = \int_s^t \xi_u \; d\chi_u.$$

$\xi_u$ depends linearly on $f_r$ and we have

$$\int_0^t \|\xi_u(f_r)\|^2 \; du \leq \|f_r\|^2 \int_s^t h'(u) \; du \quad (\text{by } i)).$$

Thus $\xi_u$ extends to a unique adapted operator $H_u^+$ on $\Phi$. Doing the same with $(M_t^* f_r)_{t\geq r}$ gives an adapted process of operators (bounded): $(H_u^-)_{u\geq 0}$.

Let $f \in \Phi$, let $f_t = P_t f$ and define

$$X_t f_t = T_t f_t - \int_0^t T_s D_s f \; d\chi_s - \int_0^t H_s^+ P_s f \; d\chi_s - \int_0^t H_s^- D_s f \; ds - \int_0^t H_s^{\times} P_s f \; ds.$$

One easily check that each $X_t$ commutes with all the $P_u$'s, $u \in \mathbb{R}^+$. Let us consider a bounded operator $H$ on $\Phi$ such that $P_u H = H P_u$ for all $u \in \mathbb{R}^+$. Notice that for (almost all $t$, all $a \leq t \leq b$, all $f$ one has

$$D_t H(P_b f - P_a f) = D_t P_b H f - D_t P_a H f = D_t H f$$

for $D_t P_s = \begin{cases} D_t & \text{if } t \leq s \\ 0 & \text{if } t > s. \end{cases}$

Define $\widetilde{H}^\circ_t$ by

$$\widetilde{H}^\circ_t f_t = D_t \int_a^b P_u f \; d\chi_u - H f_t \quad \text{for any} \quad a \leq t \leq b \; .$$

By computing $\int_a^b \|\widetilde{H}^\circ_t f_t\|^2 \; dt$ one easily check that $\widetilde{H}^\circ_t$ is bounded with locally bounded norm. And we have

$$Hg = \int_0^\infty H D_s f \; d\chi_s + \int_0^\infty \widetilde{H}^\circ_s D_s f \; d\chi_s \; .$$

That is exactly $H = \int_0^\infty \widetilde{H}^\circ_s \; da^\circ_s$.

We have actually (almost) proved the following nice characterization:

THEOREM III.2.4. — *Let $T$ be a bounded operator on $\Phi$. The following are equivalent:*

*i) $TP_t = P_t T$ for all $t \in \mathbb{R}^+$;*

*ii) $T = \lambda I + \int_0^\infty H_s \; da^\circ_s$ on the whole of $\Phi$.* ■

Applying this to $X_t$, we finally get, putting $H^\circ_s = \widetilde{H}^\circ_s + X_s$

$$T_t f_t = \int_0^t T_s D_s f \; d\chi_s + \int_0^t H^\circ_s D_s f \; d\chi_s + \int_0^t H^+_s P_s f \; d\chi_s + \int_0^t H^-_s D_s f \; ds + \int_0^t H^\times_s P_s f \; ds.$$

This is equation (II.2). ■

## III.3. Quantum brackets

### III.3.1. Definitions.

We are going to define the quantum analogue of the probabilistic angle and square brackets (cf. M. Emery's lectures).

Let $T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s \; da^\varepsilon_s$ and $S_t = \sum_\varepsilon \int_0^t K^\varepsilon_s \; da^\varepsilon_s$ be elements of $\mathcal{S}$. Define

$$[S,T]_t = \int_0^t K^\circ_s H^\circ_s \; da^\circ_s + \int_0^t K^\circ_s H^+_s \; da^+_s + \int_0^t K^-_s H^\circ_s \; da^-_s + \int_0^t K^-_s H^+_s \; da^\times_s \; ,$$

$$\langle S,T\rangle_t = \int_0^t K^-_s H^+_s \; da^\times_s \; ,$$

the *square bracket* (resp. *angle bracket*) of $S$ and $T$.

For the same $S_\cdot$ and $T_\cdot$ in $\mathcal{S}$ define

$$\int_0^t S_s\ dT_s = \sum_\varepsilon \int_0^t S_t H_s^\varepsilon\ da_s^\varepsilon$$

$$\int_0^t dS_s\ T_s = \sum_\varepsilon \int_0^t K_s^\varepsilon T_s\ da_s^\varepsilon\ .$$

The quantum Ito formula on $\mathcal{S}$ just writes:

THEOREM III.3.1. — *For all $S, T \in \mathcal{S}$ one has*

$$S_t T_t = \int_0^t S_s\ dT_s + \int_0^t dS_s\ T_s + [S,T]_t\ . \qquad \blacksquare$$

An important point has to be noticed. If $S, T \in \mathcal{S}$ then none of the processes $\int_0^\cdot S_s\ dY_s$, $\int_0^\cdot dS_s T_s$, $[S,T]_\cdot$ lie in $\mathcal{S}$ in general. Indeed, for example in the case of $[S,T]$, all the coefficients of the integral representation of $[S,T]$ satisfy the conditions that define $\mathcal{S}$, *but* the operators $[S,T]_t$ themselves have no reasons to be bounded!

We need to define a larger space. Let $\mathcal{S}'$ be the set of adapted processes of operators $(T_t)_{t\geq 0}$ on $\mathcal{E}$ such that $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon\ da_s^\varepsilon$ on $\mathcal{E}$, with the $H_s^\varepsilon$ being bounded and $t \mapsto \|H_t^\circ\| \in L^\infty_{\text{loc}}$, $t \mapsto \|H_t^\pm\| \in L^2_{\text{loc}}$, $t \mapsto \|H_t^\times\| \in L^1_{\text{loc}}$.

That is, $\mathcal{S}'$ has the same definition as $\mathcal{S}$ except that we do not ask the whole operators $T_t$ to be bounded. The integral representation of an element $(T_t)_{t\geq 0}$ of $\mathcal{S}'$ is *a priori* the exponential domain, but by the extension theorem we can extend this integral representation to any $f \in \bigcap_{t\geq 0} \operatorname{Dom} T_t$ such that $D_s f \in \operatorname{Dom} T_s$ for all $s$ and $\int_0^t \|T_s D_s f\|^2\ ds < \infty$ for all $t$.

Anyway, there is no reason for being anymore able to compose elements of $\mathcal{S}'$. But one easily check the following.

THEOREM III.3.2. — *$\mathcal{S}'$ is a $*$-algebra for the adjoint operation and for the square bracket as a product.* $\blacksquare$

What about the other operations: $(S,T) \mapsto \int dS\ T$, $\int S\ dT$, $\langle S,T\rangle$? One easily checks that $(S,T) \mapsto \int dS\ T$ or $\int T\ dS$ is well defined from $\mathcal{S}' \times \mathcal{S}$ to $\mathcal{S}'$. Whereas $(S,T) \mapsto \langle S,T\rangle$ goes from $\mathcal{S}' \times \mathcal{S}'$ to $\mathcal{S}$.

So, in the quantum Ito formula (Theorem III.3.1):

$$S_t T_t = \int_0^t S_s\ dT_s + \int_0^t dS_s T_s + [S,T]_t$$

we have that $ST$ belongs to $\mathcal{S}$, so does the sum of the three terms on the right hand side; but each of the terms can only be said to be in $\mathcal{S}'$. As an example let us consider the process $(J_t)_{t\geq 0}$ we have already met before:

$$J_t\varepsilon(u) = \varepsilon(-u_{t]}) \otimes \varepsilon(u_{[t}).$$

One checks easily that

$$J_t = I - 2\int_0^t J_s \, da_s^\circ.$$

As all the $J_t$'s are unitaries we have: $(J_t)_{t\geq 0} \in \mathcal{S}$. Let us compute $J_t^2$.

$$\begin{aligned} J_t^2 &= I + \int_0^t J_s \, dJ_s + \int_0^t dJ_s \, J_s + [J,J]_t \\ &= I - 2\int_0^t J_s^2 \, da_s^\circ - 2\int_0^t J_s^2 \, da_s^\circ + 4\int_0^t J_s^2 \, da_s^\circ. \end{aligned}$$

As $J_t^2 = I$ for all $t$ we have

$$I = J_t^2 = I - 2a_t^\circ - 2a_t^\circ + 4a_t^\circ.$$

It is clear that $J_t^2$ is bounded, but none of the three terms $\int J \, dJ$, $\int dJ \, J$, $[J,J]$ is.

### III.3.2. Properties.

Let us have a look to the main properties of these quantum brackets.

PROPOSITION III.3.3.

*i)* *If $S, T$ are martingales in $\mathcal{S}$ then $ST - [S,T]$ is a martingale and $ST - \langle S,T\rangle$ is a martingale,*

*ii)* $[S,[T,U]] = [[S,T,U]$ *for all* $S,T,U \in \mathcal{S}'$;

*iii)* $[S, \int dU \, T] = \int d[S,U] \, T$
$[\int T \, dU, S] = \int T d[U,S]$ *for all* $S,U \in \mathcal{S}'$, $T \in \mathcal{S}$;

*iv)* $[S,T]^* = [T^*, S^*]$ *for all* $S,T \in \mathcal{S}'$.

All the proof are straightforward from the definitions. We leave them as exercises. ∎

Because of the associativity property *ii)* we now write $[S,T,U]$ instead of $[S,[T,U]]$.

You can also easily check the following identity:

PROPOSITION III.3.4. — *If $S$ is a martingale in $\mathcal{S}'$ then*

$$S = [S, a^\circ] + [a^\circ, S] - [a^\circ, S, a^\circ]. \qquad \blacksquare$$

The main consequence of this identity is that the quantum square brackets of two quantum semimartingales can be as complicated as the semimartingale itself. Notice the difference with the classical case: in classical stochastic calculus the brackets of two semimartingales is always a finite variation process.

By the way, as we are speaking of classical probability, one can wonder what happens to these brackets when one considers a probabilistic interpretation of the Fock space.

THEOREM III.3.5. — *Let $(x_t)_{t\geq 0}$ be a probabilistic interpretation of $\Phi$. Let $(s_t)_{t\geq 0}$ and $(u_t)_{t\geq 0}$ be two semimartingales such that their multiplication operators $(\mathcal{M}_{s_t})_{t\geq 0}$ and $(\mathcal{M}_{u_t})_{t\geq 0}$ lie in $\mathcal{S}'$. Then we have*

$$[\mathcal{M}_{s.}, \mathcal{M}_{u\cdot}]_t = \mathcal{M}_{[s.,u.]_t}$$
$$\langle \mathcal{M}_{s.}, \mathcal{M}_{u\cdot}\rangle_t = \mathcal{M}_{\langle s.,u.\rangle_t}.$$

*Proof.* — Let $d[x,x]_t = dt + \psi_t\ dx_t$ be a structure equation that represents $x$. Let

$$s_t = \int_0^t \xi_s\ dx_s + \int_0^t h_s\ ds$$
$$u_t = \int_0^t \eta_s\ dx_s + \int_0^t k_s\ ds.$$

We have

$$\mathcal{M}_{s_t} = \int_0^t \mathcal{M}_{\xi_s}\ da_s^+ + \int_0^t \mathcal{M}_{\xi_s}\ da_s^- + \int_0^t \mathcal{M}_{\xi_s}\mathcal{M}_{\psi_s}\ da_s^\circ + \int_0^t \mathcal{M}_{h_s}\ ds$$
$$\mathcal{M}_{u_t} = \int_0^t \mathcal{M}_{\eta_s}\ da_s^+ + \int_0^t \mathcal{M}_{\eta_s}\ da_s^- + \int_0^t \mathcal{M}_{\eta_s}\mathcal{M}_{\psi_s}\ da_s^\circ + \int_0^t \mathcal{M}_{k_s}\ ds.$$

Thus

$$\begin{aligned}
[\mathcal{M}_s, \mathcal{M}_u]_t &= \int_0^t \mathcal{M}_{\xi_s}\mathcal{M}_{\psi_s}\mathcal{M}_{\eta_s}\mathcal{M}_{\psi_s}\ da_s^\circ + \int_0^t \mathcal{M}_{\xi_s}\mathcal{M}_{\psi_s}\mathcal{M}_{\eta_s}\ da_s^+ \\
&\quad + \int_0^t \mathcal{M}_{\xi_s}\mathcal{M}_{\eta_s}\mathcal{M}_{\psi_s}\ da_s^- + \int_0^t \mathcal{M}_{\xi_s}\mathcal{M}_{\eta_s}\ ds \\
&= \int_0^t \mathcal{M}_{\xi_s\eta_s\psi_s}(\mathcal{M}_{\psi_s}\ da_s^\circ + da_s^+ + da_s^-) + \int_0^t \mathcal{M}_{\xi_s\eta_s}\ ds \\
&= \mathcal{M}_{\int_0^t \xi_s\eta_s\psi_s\ dx_s + \int_0^t \xi_s\eta_s\ ds} \\
&= \mathcal{M}_{\int_0^t (\xi_s\ dx_s + ds)} \\
&= \mathcal{M}_{\int_0^t \xi_s\eta_s\ d[x,x]_s} \\
&= \mathcal{M}_{[s,u]_t}.
\end{aligned} \qquad \blacksquare$$

That is, the quantum brackets of multiplication operators are the multiplication operators by the classical brackets.

One very important point remains to be studied. It is well known that the square bracket of two classical semimartingale is a limit of quadratic variations:

$$[x, y]_t = \lim \sum_i (x_{t_{i+1}} - x_{t_i})(y_{t_{i+1}} - y_{t_i})$$

where the limit is taken over a refining sequence of partitions of $[0, t]$, and is understood to be a limit in probability. For the angle bracket one gets:

$$\langle x, y\rangle_t = \lim \sum_i \mathbb{E}\big[(x_{t_{i+1}} - x_{t_i})(y_{t_{i+1}} - y_{t_i})/\mathcal{F}_{t_i}\big].$$

One can naturally wonder what happens in the case of the quantum brackets. Obtaining similar results for the quantum brackets is interesting for two reasons:

* it is the quantum analogue of the classical result;
* we have done quite a good job by trying to get a characterisation of $\mathcal{S}$ which depends only on the process $(T_t)_{t\geq 0}$; it is thus rather disappointing to get a definition of the quantum brackets which again depends on the integral representation. Obtaining the brackets as limits of quadratic variations will provide a definition of the brackets which depends only on the processes $(T_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ involved, and not the integral representation.

THEOREM III.3.6. — *Let $(S_t)_{t\geq 0}$, $(T_t)_{t\geq 0}$ be elements of $\mathcal{S}$. Then $[S, T]_t$ is the weak limit, on exponential vectors with locally bounded coefficients, of the expression*

$$\sum_i (S_{t_{i+1}} - S_{t_i})(T_{t_{i+1}} - T_{t_i});$$

*the angle bracket $\langle S, T\rangle_t$ is the weak limit, on all $\Phi$, of*

$$\sum_i P_{t_i}(S_{t_{i+1}} - S_{t_i})(T_{t_{i+1}} - T_{t_i})P_{t_i}.$$

The proof of this theorem is very long, it takes 10 pages of various norm estimates. We don't give it here. The interested reader will find it in [At2]. ∎

# IV. SOME RECENT DEVELOPMENTS

## IV.1. Functional quantum Ito formulae

### IV.1.1. Polynomial, analytic functions.

With this algebra $\mathcal{S}$ we can immediately think of computing a quantum Ito formula for polynomial functions of an element of $\mathcal{S}$. This can be easily obtained simply by iterating the quantum Ito integration by part formula:

$$S_t T_t = \int_0^t S_s \, dT_s + \int_0^t dS_s T_s + [S,T]_t.$$

I won't write the corresponding formula for $f(S_t)$ when $f$ is a polynomial function, as it is included in a more general work performed by G. Vincent-Smith. Indeed, he showed that $\mathcal{S}$ is much more than an algebra: it is stable under two types of functionals: analytic ones and $C^{2+}$ ones. We first have a look to the first type.

Let us recall a few notations. Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$. Let $\lambda$ belong to the resolvant set of $T$ (the complementary set of the spectrum of $T$), let $R_\lambda(T)$ be the resolvant of $T$ at $\lambda$ (that is, $(T-\lambda I)^{-1}$). Let $f$ be an analytic function on the disc $D(0,R)$ where $R > \|T\|$. Then the operator $f(T)$ is defined by

$$f(T) = \oint_\gamma f(\lambda) R_\lambda(T) \, d\lambda$$

where $\gamma$ is the circle $C(0,r)$ with $R > r > \|T\|$ and $\oint_\gamma$ is $\frac{1}{2\pi i}$ times the contour integral along $\gamma$.

THEOREM IV.1.1. — *Let $(T_t)_{t\geq 0}$ be an element of $\mathcal{S}$. Let $T \in \mathbb{R}^+$ be fixed. Let $\rho = \max\left\{\|T_t\|, \|T_t + H_t^\circ\|; t \leq T\right\}$ where $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, da_s^\varepsilon$. Let $f$ be an analytic function on $D(0,R)$ for a $R > \rho$. Then $(f(T_t))_{t\geq 0}$ is still an element of $\mathcal{S}$ and one has*

$$f(T_t) = f(0) + \sum_\varepsilon \int_0^t H_f^\varepsilon(s) \, da_s^\varepsilon$$

*with*

$$\begin{aligned}
H_f^\circ &= f(T_s + H_s^\circ) - f(T_s) \\
H_f^+(s) &= \oint_\gamma f(\lambda) R_\lambda(T_s) H_s^+ R_\lambda(T_s + H_s^\circ)\ d\lambda \\
H_f^-(s) &= \oint_\gamma f(\lambda) R_\lambda(T_s + H_s^\circ) H_s^- R_\lambda(T_s)\ d\lambda \\
H_f^\times(s) &= \oint_\gamma f(\lambda) R_\lambda(T_s) H_s^\times R_\lambda(T_s)\ d\lambda \\
&\qquad + \oint_\gamma f(\lambda) R_\lambda(T_s) H_s^- R_\lambda(T_s + H_s^\circ) H_s^+ R_\lambda(T_s)\ d\lambda.
\end{aligned}$$

■

We don't give the proof, see G. Vincent-Smith : "*The Ito formula for quantum semimartingales*", Proceedings of London Math. Soc. (1998).

As a corollary one recovers our formula for polynomials.

Theorem IV.1.2. — *Let $(T_t)_{t\geq 0} \in \mathcal{S}$. Let $n \in \mathbb{N}$, then*

$$T_t^n = \sum_\varepsilon \int_0^t H_n^\varepsilon(s)\ da^\varepsilon$$

*where*

$$\begin{aligned}
H_n^\circ(s) &= (T_s + H_s^\circ)^n - T_s^n \\
H_n^+(s) &= \sum_{p+q=n-1} T_s^p H_s^+ (T_s + H_s^\circ)^q \\
H_n^-(s) &= \sum_{p+q=n-1} (T_s + H_s^\circ)^p H_s^- T_s^q \\
H_n^\times(s) &= \sum_{p+q=n-1} T_s^p H_s^\times T_s^q + \sum_{p+q+r=n-2} T_s^p H_s^- (T_s + H_s^\circ)^q H_s^+ T_s^r.
\end{aligned}$$

■

### IV.1.2. $C^{2+}$ functionals.

What is even more remarkable in Vincent-Smith's work is that $\mathcal{S}$ is much more than stable under analytic functions. You remember from M. Emery's course that classical semimartingales are stable under $C^2$ functions. We are almost going to get the same for elements of $\mathcal{S}$. This shows that $\mathcal{S}$ really plays the role of a quantum semimartingale space.

Let $f$ be an integrable function from $\mathbb{R}$ to $\mathbb{R}$. Let $\hat{f}$ be its Fourier transform. For a self-adjoint operator $T$ one can define

$$f(T) = \int_{-\infty}^{+\infty} \hat{f}(\lambda) e^{i\lambda T} \; d\lambda \; .$$

Let $C^{2+} = \left\{ f \in L^1(\mathbb{R}); p \mapsto p^2 \hat{f}(p) \in L^1(\mathbb{R}) \right\}$.

THEOREM IV.1.3. — *Let $(T_t)_t \in \mathcal{S}$ with $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \; da_s^\varepsilon$ and $T_t$ being self-adjoint (in particular $H_s^\circ$ and $H_s^\times$ are self-adjoint and $H_s^+ = H_s^{-*}$). Let $f \in C^{2+}$. Then $(f(T_t))_{t \geq 0}$ still belongs to $\mathcal{S}$ and*

$$T_t = \sum_\varepsilon \int_0^t H_f^\varepsilon(s) \; da_s^\varepsilon \quad \text{with} \quad H_f^+ = H_f^{-*}$$

*and*

$$\begin{aligned}
H_f^\circ(s) &= f(T_s + H_s) - f(T_s) \\
H_f^-(s) &= \int_{\mathbb{R}} ip\hat{f}(p) \Big\{ \int_0^1 e^{ip(1-u)T_s} H_s^- e^{ipu(T_s + H_s^\circ)} \; du \Big\} \; dp \\
H_f^\times(s) &= \int_{\mathbb{R}} ip\hat{f}(p) \Big\{ \int_0^1 e^{ip(1-u)T_s} H_s^\times e^{ipuT_s} \; du \Big\} \; dp \\
&\quad + \int_{\mathbb{R}} ip\hat{f}(p) \Big\{ \int_0^1 \int_0^1 u e^{ip(1-u)T_s} H_s^- e^{ipu(1-v)(T_s + H_s^\circ)} H_s^{-*} e^{ipuvT_s} \; du \; dv \Big\} \; dp.
\end{aligned}$$

■

## IV.2. A remarkable transform of quantum processes

I want to show you the first properties of a very remarkable transform of processes of operators. We will see that it has some very nice properties and that it relates $\mathcal{S}$ to $\mathcal{S}'$. I am sure that this transform will play an important role in quantum stochastic calculus.

### IV.2.1. Definitions.

The idea is the following. When one compute formula (II.2) of a quantum semimartingale $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \; da_s^\varepsilon$ one gets

$$T_t f_t = \int_0^t T_s D_s f \; d\chi_s + \int_0^t H_s^\circ D_s f \; d\chi_s + \cdots \text{etc.}$$

there is always this annoying term appearing : $\int_0^t T_s D_s f \; d\chi_s$. Let us remove it.

Let $(T_t)_{t\geq 0}$ be an adapted process of operators on $\Phi$. Define $(\mathcal{D}_t(T_.))_{t\geq 0}$ to be another adapted process of operators on $\Phi$ defined by

$$\mathcal{D}_t(T_.)P_t f = T_f P_t f - \int_0^t T_s D_s f \; d\chi_s \; .$$

For the moment I don't care about domain problems (you can easily work out the correct conditions on the exponential domain); for the use we will make from this $\mathcal{D}$, there won't be any problem. I will compute everything algebraically, I leave to the very motivated reader to formulate everything in a good setting (!).

PROPOSITION IV.2.1. — *$X_t = \mathcal{D}_t(T_.)$ is the only solution to the equation*

$$X_t = T_t - \int_0^t X_s \; da_s^\circ, \quad t \in \mathbb{R}^+.$$

*Proof.* — Let $X_t = \mathcal{D}_t(T_.)$, $t \in \mathbb{R}^+$. Let $Y_t = X_t - T_t$, $t \in \mathbb{R}^+$. We have

$$\begin{aligned} X_t P_t f &= T_t P_t f - \int_0^t T_s D_s f \; d\chi_s \\ Y_t P_t f &= \int_0^t -T_s D_s f \; d\chi_s = \int_0^t (Y_s - Y_s - T_s) D_s f \; d\chi_s \\ &= \int_0^t Y_s D_s f \; d\chi_s - \int_0^t X_s D_s f \; d\chi_s. \end{aligned}$$

That is, exactly equation (II.2) for saying that

$$Y_t = -\int_0^t X_s \; da_s^\circ. \qquad \blacksquare$$

### IV.2.2. The inverse transform.

The first surprising result is that the mapping $\mathcal{D}_.$ is invertible.

For an adapted process of operators $(T_t)_{t\geq 0}$ define

$$\mathcal{D}_t^{-1}(T_.) = T_t + \int_0^t T_s \; da_s^\circ.$$

PROPOSITION IV.2.2. — *$X_t = \mathcal{D}_t^{-1}(T_.)$ is the only adapted process of operators on $\Phi$ such that*

$$X_t P_t f = T_t P_t f + \int_0^t X_s D_s f \; d\chi_s, \quad t \in \mathbb{R}^+ \; .$$

*Proof.* — If $X_t = \mathcal{D}_t^{-1}(T_\cdot)$ then

$$X_t P_t f = T_t P_t f + \Big(\int_0^t T_s \; da_s^\circ\Big) P_t f.$$

Let $Y_t = \int_0^t T_s \; da_s^\circ = X_t - T_t$. Then

$$\begin{aligned} X_t P_t f &= T_t P_t f + \int_0^t Y_s D_s f \; d\chi_s + \int_0^t T_s D_s f \; d\chi_s \\ &= T_t P_t f + \int_0^t X_s D_s f \; d\chi_s. \end{aligned}$$

To prove uniqueness, consider another such process $(X'_t)_{t\geq 0}$. We have

$$(X_t - X'_t) P_t f = \int_0^t (X_s - X'_s) D_s f \; d\chi_s. \qquad (IV.2.1)$$

If this equality holds on $\mathcal{E}(\mathcal{M})$ we have

$$(X_t - X'_t)\varepsilon(u_t) = \int_0^t u(s)(X_s - X'_s)\varepsilon(u_{s]}) \; d\chi_s$$

that is,

$$\|(X_t - X'_t)\varepsilon(u_{t]})\|^2 = \int_0^t |u(s)|^2 \|(X_s - X'_s)\varepsilon(u_{s]})\|^2 \; ds$$

thus, by Gronwall lemma $(X_t - X'_t)\varepsilon(u_{t]}) = 0$. (If identity (IV.2.1) occurs on a space which has noting to do with the exponential vectors one can also show that $X_t - X'_t = 0$). ■

PROPOSITION IV.2.3. — *For any adapted process $(T_t)_{t\geq 0}$ one has*

$$\mathcal{D}_t^{-1}(\mathcal{D}_\cdot(T_\cdot)) = \mathcal{D}_t(\mathcal{D}_\cdot^{-1}(T_\cdot)) = T_t, \quad \textit{for all} \;\; t \in \mathbb{R}^+.$$

*Proof.* — Let $X_t = \mathcal{D}_t^{-1}(\mathcal{D}_\cdot(T_\cdot))$, $t \in \mathbb{R}^+$. We have

$$X_t = \mathcal{D}_t(T_\cdot) + \int_0^t \mathcal{D}_s(T_\cdot) \; da_s^\circ$$

that is $\mathcal{D}_t(T_\cdot) = X_t - \int_0^t \mathcal{D}_s(T_\cdot) \; da_s^\circ$.

By Proposition IV.2.1 this implies $\mathcal{D}_t(T_\cdot) = \mathcal{D}_t(X_\cdot)$ or else $\mathcal{D}_t(T_\cdot - X_\cdot) = 0$ for all $t \geq 0$.

But if a process $(Y_t)_{t\geq 0}$ is such that $\mathcal{D}_t(Y_\cdot) = 0$ for all $t \geq 0$, this means $Y_t P_t f = \int_0^t Y_s D_s f \; d\chi_s$, so $Y_t = 0$ by the same argument as in Proposition IV.2.2.

Let $Z_t = \mathcal{D}_t(\mathcal{D}_\cdot^{-1}(T_\cdot))$, then $Z_t = \mathcal{D}_t^{-1}(T_\cdot) - \int_0^t Z_s \; da_s^\circ$ thus $\mathcal{D}_t^{-1}(T_\cdot) = Z_+ + \int_0^t Z_s \; da_s^\circ = \mathcal{D}_t^{-1}(Z_\cdot)$. Or else $\int_0^t (T_s - Z_s) \; da_s^\circ = 0$. By uniqueness of integral representations for closable operators we get $T_\cdot = Z_\cdot$ (we assume all the operators to be closable).

### IV.2.3. The bijection.

The main property with $\mathcal{D}$ and $\mathcal{D}^{-1}$ is the way they apply to the spaces $\mathcal{S}$ and $\mathcal{S}'$.

THEOREM IV.2.4. — *The transform $\mathcal{D}$ is well defined on $\mathcal{S}'$, the transform $\mathcal{D}^{-1}$ is well defined on $\mathcal{S}$.*

*The transforms $\mathcal{D}$ and $\mathcal{D}^{-1}$ realise a bijection between $\mathcal{S}$ and $\mathcal{S}'$:*

$$\mathcal{S} \quad \underset{\mathcal{D}}{\overset{\mathcal{D}^{-1}}{\rightleftarrows}} \quad \mathcal{S}'.$$

*Proof.* — Let $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, da_s^\varepsilon$ be an element of $\mathcal{S}$. Then

$$\mathcal{D}_t^{-1}(T_\cdot) = \sum_\varepsilon \int_0^t K_s^\varepsilon \, da_s^\varepsilon \quad \text{with} \quad \begin{cases} K_s^\circ = H_s^\circ + T_s & \\ K_s^\varepsilon = H_s^\varepsilon & \text{for } \varepsilon = +, -, \times. \end{cases}$$

As $t \mapsto \|T_t\| \in L^\infty_{\text{loc}}$ we clearly have $(\mathcal{D}_t^{-1}(T_\cdot))_{t\geq 0} \in \mathcal{S}'$. Now let $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \, da_s^\varepsilon$ be an element of $\mathcal{S}'$. We put $X_t = \mathcal{D}_t(T_\cdot)$ and we have

$$\begin{aligned}
X_t f_t &= T_f f_t - \int_0^t T_s D_s f \, d\chi_s \\
&= \int_0^t H_s^\circ D_s f \, d\chi_s + \int_0^t H_s^+ P_s f \, d\chi_s + \int_0^t H_s^- D_s f \, ds + \int_0^t H_s^\times P_s f \, ds
\end{aligned}$$

$$\begin{aligned}
\|X_t f_t\|^2 &\leq 4 \left[ \int_0^t \|H_s^\circ D_s f\|^2 \, ds + \int_0^t \|H_s^+ P_s f\|^2 \, ds + \left[ \int_0^t \|H_s^- D_s f\| \, ds \right]^2 \right. \\
&\qquad \left. + \left[ \int_0^t \|H_s^\times P_s f\| \, ds \right]^2 \right] \\
&\leq 4 \left[ \sup_{s\leq t} \|H_s^\circ\|^2 \int_0^t \|D_s f\|^2 \, ds + \|P_t f\|^2 \int_0^t \|H_s^+\|^2 \, ds \right. \\
&\qquad \left. + \int_0^t \|H_s^-\|^2 \, ds \int_0^t \|D_s f\|^2 \, ds + \|P_t f\|^2 \left[ \int_0^t \|H_s^\times\| \, ds \right]^2 \right] \\
&\leq 4 \left[ \sup_{s\leq t} \|H_s^\circ\|^2 \, ds + \int_0^t \|H_s^+\|^2 \, ds + \int_0^t \|H_s^-\|^2 \, ds \right. \\
&\qquad \left. + \left[ \int_0^t \|H_s^\times\| \, ds \right]^2 \right] \|f_t\|^2.
\end{aligned}$$

Thus $X_t$ is a bounded operator, with locally bounded norm. Furthermore

$$X_t = \int_0^t H_s^\circ - X_s \, da_s^\circ + \sum_{\varepsilon=+,-,\times} \int_0^t H_s^\varepsilon \, da_s^\varepsilon$$

that is $(X_t)_{t\geq 0} \in \mathcal{S}$. ■

This theorem has many consequences. The first one is that $\mathcal{S}$ is a large space: if you are given any quadruple $(H^\varepsilon_\cdot,\ \ \varepsilon = 0, +, -, \times)$ of adapted processes of bounded operators with the norm conditions: $t \mapsto \|H^\circ_t\| \in L^\infty_{\text{loc}}$, $t \mapsto \|H^\pm_t\| \in L^2_{\text{loc}}$, $t \mapsto \|H^\times_t\|$ then you produce an element of $\mathcal{S}$ by putting $Y_t = \sum\limits_\varepsilon \int_0^t H^\varepsilon_s \, da^\varepsilon_s$ (on $\mathcal{E}$) and $T_t = \mathcal{D}_t(Y_\cdot)$.

Furthermore, two different quadruples $(H^\varepsilon_\cdot,\ \ \varepsilon = 0, +, -, \times)$ gives two different elements of $\mathcal{S}$.

As an example, let us consider some simple quadruples:

1) $H^\circ_s = I$, $H^\varepsilon_s = 0$, $\varepsilon = +, -, \times$ then $Y_t = \sum\limits_\varepsilon \int_0^t H^\varepsilon_s \ da^\varepsilon_s = a^\circ_t \in \mathcal{S}'$ and $T_t = \mathcal{D}_t(a^\circ_\cdot)$ acts as follows: $T_t f_t = \int_0^t D_s f \ d\chi_s = f_t - P_0 f$. Thus $T_t = I - P_0$.
2) $H^+_s = I$, $H^\varepsilon_s = 0$ otherwise, $Y_t = a^+_t$ and $T_t = \mathcal{D}_t(a^+_\cdot)$ acts as $T_t f_t = \int_0^t P_s f \ d\chi_s$. So $T_t$ is the operator of Ito integration with respect to $d\chi$ (recall Theorem II.3.2).
3) $H^-_s = I$, $H^\varepsilon_s = 0$ otherwise, $Y_t = a^-_t$ and $T_t = \mathcal{D}_t(a^-_\cdot)$ acts as $T_t f_t = \int_0^t D_s f \ d\chi_s = \langle f_\cdot, \chi_\cdot \rangle_t$ the angle bracket of $(f_t)_{t\geq 0}$ with $(\chi_t)_{t\geq 0}$.
4) $H^\times_s = I$, $H^\varepsilon_s = 0$ otherwise, $Y_t = tI$ and $T_t = \mathcal{D}_t(Y_\cdot)$ acts as $T_t f_t = \int_0^t P_s f \ ds$ the adapted time-integration.

The operations $\mathcal{D}$ and $\mathcal{D}^{-1}$ have also many nice algebraic properties (at least formally). I won't develop them here. Another point to be noticed after Theorem IV.2.4; the operation $\mathcal{D}$ has the property of bounding operators which were not, at least all those of $\mathcal{S}'$. What are all the processes $(T_t)_{t\geq 0}$ that get bounded by $\mathcal{D}$? We don't know the full answer. Many other problems remain open about these transforms.

## About references

CHAPTER I. — The short notations (symmetric measures) are due to Guichardet [Gui]. The Ito calculus on Fock space is developed in [A-L]. Structure equations have been defined and studied by Emery [Eme].

CHAPTER II. — A rigorous proof for the existence of only 3 quantum noises is in [Coq]. Hudson-Parthasarathy's approach of quantum stochastic calculus is developed in [H-P]. The approach of section II.2 comes from [A-M]. The correspondence between quantum stochastic calculus and probabilistic interpretations is developed in [At1].

Chapter III. — All the theory of quantum semimartingale algebras and quantum brackets comes from [At2].

Chapter IV. — The functional quantum Ito formula for polynomials is to be found in [At2]. The formulae for analytic or $C^{2+}$ functions are due to Vincent-Smith [ViS]. The theory of the remarkable transform of quantum processes is developed in [At3].

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