Why are there only three quantum noises?

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ABSTRACT. — In quantum stochastic calculus on the symmetric Fock space over $L^2(\mathbb{R}_+)$, adapted processes of operators are integrated with respect to creation, annihilation and number processes. The main property which allows this integration is that: the increments of integrators between s and t act only on Fock space over $L^2([s,t])$. In this article, we prove that there are no other process of closable operators on coherent vectors with this property. Then the only possible integrators in quantum stochastic calculus are the creation, annihilation and number processes.

I. Introduction

All the construction of quantum stochastic integrals on the symmetric Fock space over $L^2(\mathbb{R}_+)$, denoted $\Gamma\big(L^2(\mathbb{R}_+)\big)$, consists in integrating processes of operators with respect to the creation, annihilation and number processes, denoted respectively $(a_t^+)_{t\geq 0}$, $(a_t^-)_{t\geq 0}$ and $(a_t^0)_{t\geq 0}$. See [Hu-Par], [Bel], [Lin], [Att-Mey], [Att] for the different constructions and definitions.

The classical definition ([Hu-Par]) uses simple adapted processes, that is $H_t = \sum_i H_{t_i} \mathbb{1}_{[t_i,t_{i+1}]}(t)$ where $t_0 < \cdots < t_n$ and $H_{t_i} = H \otimes \mathrm{Id}$ on $\Gamma(L^2([0,t_i])) \otimes \Gamma([t_i,+\infty[) \cong \Gamma\big(L^2(\mathbb{R}_+)\big)$. So one defines $\int_0^{+\infty} H_s dT_s = \sum_i H_{t_i}\big(T_{t_{i+1}} - T_{t_i}\big)$ and extends this definition to other adapted operators. An important domain problem appears in the composition $H_{t_i}\big(T_{t_{i+1}} - T_{t_i}\big)$ as one usually deals with unbounded operators. But when $(T_t)_{t\geq 0}$ is one of the processes $(a_t^\varepsilon)_{t\geq 0}$, $\varepsilon \in \{-,0,+\}$; one has the following property:

if $s < t$, then $T_t - T_s = \mathrm{Id} \otimes K \otimes \mathrm{Id}$ on

$$\Gamma\big(L^2(\mathbb{R}_+)\big) \cong \Gamma\big(L^2([0,s])\big) \otimes \Gamma\big(L^2([s,t])\big) \otimes \Gamma\big(L^2([t,+\infty[)\big),$$

where K is an operator on $\Gamma\big(L^2([s,t])\big)$. So one can define $H_{t_i}\big(T_{t_{i+1}} - T_{t_i}\big)$ as being $H_{t_i} \otimes \big(T_{t_{i+1}} - T_{t_i}\big)$ on $\Gamma\big(L^2([0,t_i])\big) \otimes \Gamma\big(L^2([t_i,+\infty[)\big)$. We will say that a process with this property is a "noise".

We don't say that such a process is an independent increment process (in relation with the independent increments property of brownian motion or Poisson process) because this family of process already exists. Schürmann in [Schür], in another context, say that a family $(F_t)_{t\in\mathbb{R}_+}$ of symmetric operators on a prehilbert space $\mathcal{D}$ has independent increments if:

- for $s < t < s' < t'$, $F_t - F_s$ and $F_{t'} - F_{s'}$ commutes;
- for $t_1 < \cdots < t_n < t_{n+1}$ and for polynomials $a_1, \ldots, a_n$ in components of $F_{t_2} - F_{t_1}, \ldots, F_{t_{n+1}} - F_{t_n}$ resp., the expectation $\langle \Omega, a_1 \cdots a_n \Omega \rangle$ of $a_1 \cdots a_n$ is equal to the product $\langle \Omega, a_1 \Omega \rangle \cdots \langle \Omega, a_n \Omega \rangle$ where Ω is a cyclic unit vector in $\mathcal{D}$.

Schürmann proves that a non commutative stochastic process with independent and stationary additive increments, if it is continuous at the origin, can be embedded into a sum of annihilation, creation and second quantisation processes on a Fock space.

We can see that a "noise" has independent additive increments, but has not necessary stationary additive increments.

During a talk about quantum stochastic integral, S. Attal asked the following question: are there other "noises" on Fock space. He gave an heuristic answer which is developed in the fourth part of this article.

We will show in this article that under small regularity conditions, creation, annihilation and number processes are the only "noise" processes of closable operators defined on the coherent vector space.

This article is divided in four parts.

– The first one gives conventions and notations about the symmetric Fock space $\Gamma\big(L^2(\mathbb{R}_+)\big) = \Phi$.

– The second deals with curves $(x_t)_{t\geq 0}$ in Φ having the following "independent increments" property: if $s < t$, $x_t - x_s \in \Gamma\big(L^2([s,t])\big)$. When one considers the brownian or Poisson interpretation of the Fock space, the brownian motion or Poisson process are represented by the same curve, $(\chi_t)_{t\geq 0}$ in Φ. We prove that this curve $(\chi_t)_{t\geq 0}$ is essentially the only "independent increments" curve on Φ.

– The third part gives definitions of creation, annihilation and number operators on Fock space and some useful properties and formulas.

– The fourth part gives the proofs of the following two theorems: let $\mathcal{E}$ the space of coherent vectors and $\Phi_{t]} = \Gamma\big(L^2([0,t])\big)$, $\Phi_{[s,t]} = \Gamma\big(L^2([s,t])\big)$ and $\Phi_{[t} = \Gamma\big(L^2([t,+\infty[)\big)$.

Theorem 1. — *Let $(T_t)_{t\geq 0}$ a process of operators on $\mathcal{E}$ such that*

1) $\forall t > 0$, T_t is closable on $\mathcal{E}$.

2) $\forall s < t$, $T_t - T_s = \mathrm{Id} \otimes K_{s,t} \otimes \mathrm{Id}$ on $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ where $K_{s,t}$ is an operator on $\Phi_{[s,t]}$.

3) $\forall u \in L^2(\mathbb{R}_+)$, $\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ t & \longmapsto & T_t(\mathcal{E}(u)-\mathbb{1})(\emptyset) \end{array}$ *has finite quadratic variations on compact or is continuous.*

Then, there exist $A : \mathbb{R}_+ \to \mathbb{C}$, g *and* f *in* $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *and* k *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, dt)$ *such that:*

$$T_t = A(t)\,\mathrm{Id} + \int_0^t g(s)\, da_s^+ + \int_0^t f(s)\, da_s^- + \int_0^t k(s)\, da_s^0.$$

Theorem 2. — *Let* $(T_t)_{t\geq 0}$ *a process of operators on* $\mathcal{E}$ *such that:*

1) $\forall t > 0$, T_t^* *is defined on* $\mathcal{E}$.

2) $\forall s < t$, $T_t - T_s = \mathrm{Id} \otimes K_{s,t} \otimes \mathrm{Id}$ *on* $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ *where* $K_{s,t}$ *is an operator on* $\Phi_{[s,t]}$.

Then, there exist $A : \mathbb{R}_+ \to \mathbb{C}$, g *and* f *in* $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ *and* k *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, dt)$ *such that:*

$$T_t = A(t)\,\mathrm{Id} + \int_0^t g(s)\, da_s^+ + \int_0^t f(s)\, da_s^- + \int_0^t k(s)\, da_s^0.$$

Proposition 4.1 will give the general form of “noise” which does not satisfy the regularity condition.

II. Notations: the symmetric Fock space over $L^2(\mathbb{R}_+)$

Recall the definition of the symmetric Fock space, for details one can see [Mey]. Let $\mathcal{H}$ be a complex Hilbert space. We consider its n-fold Hilbert space tensor power $\mathcal{H}^{\otimes n}$, and define for $u_1, \ldots, u_n$ in $\mathcal{H}$

$$u_1 \circ u_2 \circ \cdots \circ u_n = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}, \tag{2.1}$$

S_n denoting the group of permutation of $\{1, \ldots, n\}$. The closed subspace of $\mathcal{H}^{\otimes n}$ generated by all vectors (2.1) is called the n-th symmetric power of $\mathcal{H}$, and denoted $\mathcal{H}_n$. We make the convention that $\mathcal{H}_0 = \mathbb{C}$ and the element $1 \in \mathbb{C}$ is called the vacuum vector and denoted by $\mathbb{1}$.

The symmetric Fock space $\Gamma(\mathcal{H})$ over $\mathcal{H}$ is the Hilbert space direct sum of all the symmetric chaos $\mathcal{H}_n$ with the following representation: $F = \sum_n \frac{f_n}{n!}$ with $f_n \in \mathcal{H}_n$ for all n and such that $\|F\|^2 = \sum_n \frac{\|f_n\|^2}{n!} < +\infty$. Given $h \in \mathcal{H}$, we define the exponential vector $\mathcal{E}(h)$

$$\mathcal{E}(h) = \sum_n \frac{h^{\otimes n}}{n!}. \tag{2.2}$$

In particular, the vacuum vector $\mathbb{1}$ is the exponential vector $\mathcal{E}(0)$. We have

$$\langle \mathcal{E}(g), \mathcal{E}(h)\rangle = e^{\langle g,h\rangle}. \tag{2.3}$$

The subspace generated by exponential vectors is dense in $\Gamma(\mathcal{H})$. We call it the exponential domain and denote it by $\mathcal{E}$.

The case where $\mathcal{H} = L^2(\mathbb{R}_+, dt)$.

An element of $\mathcal{H}_n$ is a class of symmetric functions in n variables and so is determined by its restriction to the increasing open simplex Δ_n of $\mathbb{R}_+^n$, *i.e.* the set of all n-uples $\{s_1 < \cdots < s_n\}$. We denote $\Phi = \Gamma\big(L^2(\mathbb{R}_+, dt)\big)$, $\Phi_{t]} = \Gamma\big(L^2([0,t], dt)\big)$, $\Phi_{[s,t]} = \Gamma\big(L^2(]s,t])\big)$ and $\Phi_{[t} = \Gamma\big(L^2(]t, +\infty[)\big)$.

If a Hilbert space $\mathcal{H}$ is split into a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$, the corresponding Fock space $\Gamma(\mathcal{H})$ appears as a tensor product $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2)$. Thus the "continuous sum" $L^2(\mathbb{R}_+) = L^2([0,s]) \oplus L^2([s,t]) \oplus L^2(]t, +\infty[)$ gives rise to a "continuous tensor product",

$$\Phi \cong \Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}\ . \tag{2.4}$$

We can consider that these subspaces are contained in Φ, $f \in \Phi_{t]}$ being identified with $f \otimes \mathbb{1} \in \Phi \cong \Phi_{t]} \otimes \Phi_{[t}$, the vacuum vector belongs to every $\Phi_{[s,t]}$.

If $u \in L^2(\mathbb{R}_+)$, we have by (2.4), if $s < t$

$$\mathcal{E}(u) = \mathcal{E}(u_{s]}) \otimes \mathcal{E}(u\mathbb{1}_{[s,t]}) \otimes \mathcal{E}(u\mathbb{1}_{[t,+\infty[}) \tag{2.5}$$

where $u_{s]} = u\mathbb{1}_{[0,s]}$.

We introduce now the "shortland notation" due to Guichardet [Gui]. Let $\mathcal{P}$ denote the finite power set of $\mathbb{R}_+$ that is the set of finite subsets of $\mathbb{R}_+$, and $\mathcal{P}_n$ the set of all subsets of $\mathbb{R}_+$ of cardinality n. There is a one to one correspondence between $\mathcal{P}_n$ and Δ_n. The $\mathcal{P}_n$ gets imbedded in $\mathbb{R}_+^n$, and acquires a natural σ-field and a natural measure dA, the n-dimensional volume element $ds_1 \cdots ds_n$. By letting $\emptyset \in \mathcal{P}$ be an atom of measure unity, we arrive at a σ-finite measure on $\mathcal{P} = \bigcup_{n\geq 0} \mathcal{P}_n$, the symmetric measure (or Guichardet measure) associated with Lebesgue measure on $\mathbb{R}_+$. Elements of $\mathcal{P}$ will always be denoted by lower case greek letters $\alpha, \ell, \sigma, \tau, \omega, \ldots$ and integration with respect to the symmetric measure on $\mathcal{P}$ will be written simply $\int_{\mathcal{P}} f(\sigma)\ d\sigma$. The cardinal of an element σ of $\mathcal{P}$ is denoted by $\#\sigma$.

We have then that the symmetric Fock space over $L^2(\mathbb{R}_+)$ is isomorphic with $L^2(\mathcal{P}, d\sigma)$, where we associate to $F = \sum_{n\geq 0} \frac{f_n}{n!} \in \Phi$, the function $F(\sigma) = f_n(t_1, \ldots, t_n)$ if $\sigma = \{t_1, \ldots, t_n\} \in \mathcal{P}_n$. In particular, we have, for $u \in L^2(\mathbb{R}_+)$ and $\sigma \in \mathcal{P}$

$$\mathcal{E}(u)(\sigma) = \prod_{s\in\sigma} u(s)\ . \tag{2.6}$$

If $s < t$, we denote $\mathcal{P}_{s]} = \{\sigma \in \mathcal{P} \mid \sigma \subset [0,s]\}$, $\mathcal{P}_{[s,t]} = \{\sigma \in \mathcal{P} \mid \sigma \subset [s,t]\}$ and $\mathcal{P}_{[t} = \{\sigma \in \mathcal{P} \mid \sigma \subset [t,+\infty[\}$. So $\Phi_{t]}$ is identified with $L^2(\mathcal{P}_{t]}, d\sigma)$ and the same for the others.

Brownian interpretation of the Fock space.

Let Ω be the set of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ and let B_s be the evaluation mapping $\omega \mapsto \omega(s)$. Let us provide Ω with the σ-fields $\mathcal{F}$, $\mathcal{F}_t$ generated by all mappings B_s with s arbitrary in the first case, $s \le t$ in the second case. One can show that $\mathcal{F}$ is the topological Borel field on Ω for uniform convergence topology over compacts sets. The Wiener measure is the only probability μ on Ω such that $B_0 = 0$ p.s. and the process (B_t) has centered Gaussian independent increments with variance $E[(B_t - B_s)^2] = t - s$.

A random variable $X \in L^2(\Omega, \mathcal{F}, \mu)$ has a representation

$$\begin{aligned} X = E[X] + \int_0^{+\infty} f_1(s)\, dB_s + \int_{\Delta_2} f(s_1, s_2)\, dB_{s_1}\, dB_{s_2} + \cdots \\ + \int_{\Delta_n} f_n(s_1, \ldots, s_n)\, dB_{s_1} \cdots dB_{s_n} + \cdots \\ = \sum_n J_n(f_n) \end{aligned}$$

where $f_n \in L^2(\Delta_n)$ and J_n is an isometric mapping from $L^2(\Delta_n)$ to $L^2(\Omega)$, the image of which is called the n-th Wiener chaos. So we have an isomorphism between $L^2(\Omega)$ and Φ. Let be $\mathcal{F}^t$, $\mathcal{F}_{s,t}$ the σ-field generated by the increments $X_u - X_t$; $u > t$ in the first case, and $X_u - X_s$, $s < u < t$ in the second case. The spaces $L^2(\Omega, \mathcal{F}_t, \mu)$, $L^2(\Omega, \mathcal{F}^t, \mu)$ and $L^2(\Omega, \mathcal{F}_{s,t}, \mu)$ are respectively isomorphic to $\Phi_{t]}$, $\Phi_{[t}$ and $\Phi_{[s,t]}$.

The fact that the brownian motion is a process with independent increments gives rise to the isomorphism (2.4).

Let be (χ_t) the curve in Φ defined by

$$\chi_t(\sigma) = \begin{cases} 1 & \text{if } \#\sigma = 1 \text{ and } \sigma \subset [0,t] \\ 0 & \text{else .} \end{cases} \tag{2.7}$$

Then $(\chi_t)_{t \ge 0}$ corresponds to the brownian motion by the preceding isomorphism.

III. "Independent increments" curves

We can construct on Φ an Ito integral. Let $(v_t)_{t \ge 0}$ be a curve on Φ. We say that $(v_t)_{t \ge 0}$ is an adapted curve if for all t, $v_t \in \Phi_{t]}$. If $(v_t)_{t \ge 0}$ is an adapted curve so that

$\int_0^{+\infty} \|v_t\|^2\, dt < +\infty$, we can define as in [Att], $\int_0^{+\infty} v_s\, d\chi_s$: suppose that $(v_t)_{t\geq 0}$ is a step process that is, there exists $t_0 < t_1 < \cdots < t_{n+1}$ in $\mathbb{R}^+$ such that $v_t = v_i$ for $t_i \leq t < t_{i+1}$ where $v_i \in \Phi_{t_i]}$. Then we define

$$\int_0^{+\infty} v_s\, d\chi_s = \sum_{i=0}^{n} v_i \otimes \left(\chi_{t_{i+1}} - \chi_{t_i}\right) \tag{3.1}$$

where the tensor product is the tensor product in $\Phi_{t_i]} \otimes \Phi_{[t_i,t_{i+1}]}$. It makes sense because $v_i \in \Phi_{t_i]}$ and $\chi_{t_{i+1}} - \chi_{t_i} \in \Phi_{[t_i,t_{i+1}]}$. We have then $\| \int_0^{+\infty} v_s\, d\chi_s\|^2 = \int_0^{+\infty} \|v_s\|^2\, ds$ and this isometric property allows us to extend the integral to adapted curves $(v_t)_{t\geq 0}$ such that $\int_0^{+\infty} \|v_s\|^2\, ds < +\infty$.

The two properties which have permitted this construction of the "Ito integral" are that:

$$\begin{cases} 1) \quad \chi_t - \chi_s \in \Phi_{[s,t]} & \text{for all } s < t \\ 2) \quad \|\chi_t - \chi_s\|^2 = t - s & \text{for all } s < t\,. \end{cases} \tag{3.2}$$

So if $(x_t)_{t\geq 0}$ is a curve in Φ such that $x_t - x_s \in \Phi_{[s,t]}$, $x_t(\emptyset) = 0$ and $\|x_t - x_s\|^2 = \mu([s,t[)$ for a measure μ on $\mathbb{R}_+$, we can define $\int_0^{+\infty} v_s\, dx_s$ by (3.1) for $(v_t)_{t\geq 0}$ a step process, and we have so $\| \int_0^{+\infty} v_s\, dx_s\|^2 = \int_0^{+\infty} \|v_s\|^2 \mu(ds)$.

A curve which verify (3.2), 1) is called an "independent increments" curve. What are the "independent increments curves" which permitted to define "Ito integrals"? The answer is given by the proposition 3.1. We denoted, if $g \in L^2_{\text{loc}}(\mathbb{R}_+)$, $\int_0^t g(s)\, d\chi_s$, the element of the first chaos $\int_0^{+\infty} g_{t]}(s) \mathbb{1}\, d\chi_s$, given by

$$\left(\int_0^t g(s) \mathbb{1}\, d\chi_s\right)(\sigma) = \begin{cases} g(u) & \text{if } \sigma = \{u\} \subset [0,t] \\ 0 & \text{else.} \end{cases}$$

Proposition 3.1. — *Let $(x_t)_{t\geq 0}$ a curve in Φ such that for all $s < t$, $x_t - x_s \in \Phi_{[s,t]}$; then it exists $g \in L^2_{\text{loc}}(\mathbb{R}_+)$ such that for all $t \geq 0$,*

$$x_t = x_t(\emptyset)\mathbb{1} + \int_0^t g(s)\, d\chi_s\,.$$

Proof. — Let $y_t = x_t - x_t(\emptyset)\mathbb{1}$, so $y_t(\emptyset) = 0$ and if $s < t$, $y_t - y_s \in \Phi_{[s,t]}$.

Let $t > 0$ and $\sigma \subset [0,t]$. We suppose that $\#\sigma \geq 2$, so it exists $0 < s < t$ such that $\sigma \not\subset [s,t]$ and $\sigma \not\subset [0,s]$. So we have $y_s(\sigma) = 0$ and $(y_t - y_s)(\sigma) = 0$ and thus $y_t(\sigma) = 0$. So y_t belongs to the first chaos.

Let $u \in \mathbb{R}_+$ and t, t' so that $u < t < t'$,

$$y_{t'}(\{u\}) = (y_{t'} - y_t)(\{u\}) + y_t(\{u\}) = y_t(\{u\})\,.$$

So we can define $g(u) = y_t(\{u\})$ if $t > u$ and so $\int_0^t |g(s)|^2\, ds = \int_0^t |y_t(\{s\})|^2\, ds \leq \|y_t\|^2$. In consequence, $g \in L^2_{\text{loc}}(\mathbb{R}_+)$ and $y_t = \int_0^t g(s)\, d\chi_s$. ■

Remarks.

1) If $t \mapsto x_t(\emptyset)$ has bounded variations on compact, we can so define for all adapted curves $(v_t)_{t\geq 0}$ such that $\int_0^{+\infty} \|v_s\|^2 g(s)\, ds < +\infty$ the integral

$$\int_0^{+\infty} v_s\, dx_s = \int_0^{+\infty} v_s dx_s(\emptyset) + \int_0^{+\infty} v_s g(s)\, d\chi_s.$$

2) In the Wiener space, the proposition 3.1 can be stated like that: if $(X_t)_{t\geq 0}$ belongs to $L^2(\Omega, \mathcal{F}_t, \mu)$ and satisfies $X_t - X_s$ is $\mathcal{F}_{s,t}$-adapted for all $s < t$, then $X_t - E[X_t]$ is a martingale of the first chaos, *i.e.* it exists g in $L^2_{\text{loc}}(\mathbb{R}_+)$ so that $X_t = E[X_t] + \int_0^t g(s)\, dB_s$.

IV. Creation, annihilation and number operators on symmetric Fock spaces

We keep the notations of II and [Mey]. For $h \in \mathcal{H}$, we define

$$\left\{ \begin{array}{l} a_h^+(u_1 \circ \cdots \circ u_n) = h \circ u_1 \circ \cdots \circ u_n \\ a_h^-(u_0 \circ \cdots \circ u_n) = \displaystyle\sum_{i=0}^{n} \langle h, u_i \rangle u_0 \circ \cdots \circ \hat{u}_i \circ \cdots \circ u_n \end{array} \right. \tag{4.1}$$

where a hat on a vector means that it is omitted. a_h^+ and a_h^- are called creation operator and annihilation operator.

If H is a bounded operator on $\mathcal{H}$, we define

$$\Lambda(H)(u_1 \circ \cdots \circ u_n) = \sum_{i=1}^{n} u_1 \circ \cdots \circ H u_i \circ \cdots \circ u_n \tag{4.2}$$

$\Lambda(H)$ is called a second quantification differential operator.

If $\mathcal{H} = L^2(\mathbb{R}_+, dt)$, we define $a_k^0 = \Lambda(M_k)$ where k is a bounded function and M_k is the operator of multiplication by k in $L^2(\mathbb{R}_+)$. a_h^+ and a_h^- are closable operators on the finite sum of chaos and are extended by closure to the largest possible domain. Each one is the adjoint of the other. Creation and annihilation operators are defined on the exponential domain, and we have

$$a_h^- \mathcal{E}(u) = \langle h, u \rangle \mathcal{E}(u) \;\text{ and }\; a_h^+ \mathcal{E}(u) = \frac{d}{d\lambda} \mathcal{E}(u + \lambda h) \mid \lambda = 0 \tag{4.3}$$

a_k^0 is also closable on the finite sum of chaos and the domain of its closure contains $\mathcal{E}$.

It's easy to show with (4.1) and (4.3) that

$$
(4.4)\qquad \begin{cases} \langle a_h^- \mathcal{E}(u), \mathcal{E}(v)\rangle = \displaystyle\int_0^{+\infty} \overline{u(s)} h(s)\; ds\, \langle \mathcal{E}(u), \mathcal{E}(v)\rangle \\ \langle a_h^+ \mathcal{E}(u), \mathcal{E}(v)\rangle = \displaystyle\int_0^{+\infty} v(s)\overline{h(s)}\; ds\, \langle \mathcal{E}(u), \mathcal{E}(v)\rangle \\ \langle a_h^+ \mathcal{E}(u), a_h^+ \mathcal{E}(v)\rangle = \left(\|h\|^2 + \overline{\langle h, u\rangle}\langle h, v\rangle\right) \langle \mathcal{E}(u), \mathcal{E}(v)\rangle \;. \end{cases}
$$

We have when we use "shortland notations", for f in the respective domain of a_h^+, a_h^- and a_k^+:

$$
(4.5)\qquad \begin{cases} (a_h^+ f)(\sigma) = \displaystyle\sum_{s\in\sigma} h(s) f(\sigma - \{s\}) \\ (a_h^- f)(\sigma) = \displaystyle\int_0^{+\infty} h(s) f(\sigma \cup \{s\})\; ds \\ (a_k^0 f)(\sigma) = \displaystyle\sum_{s\in\sigma} k(s) f(\sigma)\;. \end{cases}
$$

We denote $a_t^+ = a^+_{\mathbb{1}_{[0,t]}}$, $a_t^- = a^-_{\mathbb{1}_{[0,t]}}$ and $a_t^0 = a^0_{\mathbb{1}_{[0,t]}}$.

V. Noises

Hudson-Parthasarathy define stochastic integration for operators with respect to $(a_t^+)_{t\geq 0}$, $(a_t^-)_{t\geq 0}$ and $(a_t^0)_{t\geq 0}$.

Definition 5.1. — *A processus of operators $(H_t)_{t\geq 0}$ defined on $\mathcal{E}$ is adapted if, for all $t \geq 0$, for all $u \in L^2(\mathbb{R}_+)$:*

$$H_t\mathcal{E}(u) = \widetilde{H}_t\mathcal{E}(u_{t]}) \otimes \mathcal{E}(u\mathbb{1}_{[t,+\infty[}) \;\; \text{in } \Phi \cong \Phi_{t]} \otimes \Phi_{[t}$$

where $\widetilde{H}_t$ is an operator on $\Phi_{t]}$ (we say that H_t is t-adapted).

Let $(T_t)_{t\geq 0}$ be $(a_t^\varepsilon)_{t\geq 0}$ for $\varepsilon \in \{+,-,0\}$. We see by using (4.5) for example that for all $s < t$, it exists $K_{s,t}$, operator on $\Phi_{[s,t]}$ such that

(5.1) for all $u \in L^2(\mathbb{R}_+)$, $(T_t - T_s)\mathcal{E}(u) = \mathcal{E}(u_{s]}) \otimes K_{s,t}\mathcal{E}(u\mathbb{1}_{[s,t]}) \otimes \mathcal{E}(u\mathbb{1}_{[t,+\infty[})$ on $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$.

Let $(H_t)_{t\geq 0}$ be an elementary adapted process defined by $H_t = \sum_{i=0}^{p} H_{t_i}\mathbb{1}_{[t_i,t_{i+1}[}(t)$ where $t_0 < t_1 < \cdots < t_{p+1}$. One defines $\int_0^{+\infty} H_s\, dT_s = \sum_{i=0}^{p} H_{t_i}\left(T_{t_{i+1}} - T_{T_i}\right)$. This is possible because $H_{t_i}\left(T_{t_{i+1}} - T_{t_i}\right)$ is note a "real composition" of operators, indeed on $\Phi_{t_i]}\otimes\Phi_{[t_i,t_{i+1}]}\otimes\Phi_{[t_{i+1}}$, one have by (5.1),

$$H_{t_i}\left(T_{t_{i+1}} - T_{t_i}\right)\mathcal{E}(u) = H_{t_i}\mathcal{E}\left(u\mathbb{1}_{[0,t_i]}\right) \otimes \left(T_{t_{i+1}} - T_{t_i}\right)\mathcal{E}\left(u\mathbb{1}_{[t_i,t_{i+1}]}\right) \otimes \mathcal{E}\left(u\mathbb{1}_{[t_{i+1},+\infty[}\right)\;.$$

So it is this property (5.1) which permits to define stochastic integration of adapted processes with respect to creation, annihilation and number processes. A process of operators which satisfies (5.1) is called a “noise” process. During a talk about quantum stochastic integral, S. Attal asked the following question: are there other “noises”? He gave the following heuristic answer: let $(T_t)_{t\geq 0}$ be a process which verifies (5.1), then $dT_t = \mathrm{Id} \otimes dK_t \otimes \mathrm{Id}$ on $\Phi_{t]} \otimes \Phi_{[t,t+dt]} \otimes \Phi_{[t+dt}$. But in $\Phi_{[t,t+dt]}$, we have a base given by $\mathbb{1}$ and $d\chi_t$, so dK_t is an endomorphism on a space of dimension 2 and there is four possibilities. The proposition (5.1) will show that, in fact, $(T_t\mathcal{E}(u))(\sigma)$ will be known when $(T_t\mathcal{E}(u))(\emptyset)$ and $(T_t\mathcal{E}(u))(\{s\})$ were known.

Theorem 5.1. — *Let $(T_t)_{t\geq 0}$ a process of operators on $\mathcal{E}$ such that*

1) $\forall t > 0$, T_t is closable on $\mathcal{E}$;

2) $\forall s < t$, it exists $K_{s,t}$ on $\Phi_{[s,t]}$ such that

$$T_t - T_s = \mathrm{Id} \otimes K_{s,t} \otimes \mathrm{Id} \quad \text{on } \Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t};$$

3) $\forall u \in L^2(\mathbb{R}_+, dt)$, $\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{C} \\ t & \longmapsto & T_t(\mathcal{E}(u) - \mathbb{1})(\emptyset) \end{array}$ has finite quadratic variations on compact or is continuous.

Then, there exist $A : \mathbb{R}_+ \to \mathbb{C}$, g and f in $L^2_{\mathrm{loc}}(\mathbb{R}_+, dt)$, k in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, dt)$ such that $T_t = A(t)\,\mathrm{Id} + a^+_{g_{t]}} + a^-_{f_{t]}} + a^0_{k_{t]}}$.

Theorem 5.2. — *Let $(T_t)_{t\geq 0}$ a process of operators on $\mathcal{E}$ such that $(T_t^*)_{t\geq 0}$ is defined on $\mathcal{E}$ and such that the hypothesis 2) of Theorem 5.1 is satisfied, then the conclusion of Theorem 5.1 is true.*

Remark. — The condition 1) of Theorem 5.1 is necessary.

For example, if $T_t\mathcal{E}(u) = \left(\int_0^t u(s)^2\, ds\right)\mathcal{E}(u)$, then $(T_t)_{t\geq 0}$ satisfies 2) and 3) but is not closable on $\mathcal{E}$.

The proof of these two theorems needs a first common step:

Proposition 5.1. — *Let $(T_t)_{t\geq 0}$ a process of operators on $\mathcal{E}$ which satisfies the hypothesis 2) of Theorem 5.1. Then there exist $A : \mathbb{R}_+ \to \mathbb{C}$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, for all $u \in L^2(\mathbb{R}_+)$, for all $t \geq 0$, there exist $G_u(t) \in \mathbb{C}$ and $k_u \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ such that for a.a. $\sigma \in \mathcal{P}$, $\sigma \subset [0,t]$ we have*

$$(5.1)\quad (T_t\mathcal{E}(u))(\sigma) = A(t)(\mathcal{E}(u))(\sigma) + (a^+_{g_{t]}}\mathcal{E}(u))(\sigma) + G_u(t)\mathcal{E}(u)(\sigma) + \sum_{s\in\sigma} k_u(s)\mathcal{E}(u)(\sigma - \{s\}).$$

We have too, for all $t \in \mathbb{R}_+$, $a < b$,

$$\left\{ \begin{array}{l} G_u\mathbb{1}_{[a,b]}(t) = G_u(\min(b,t)) - G_u(\min(a,t)) \\ k_{u\mathbb{1}_{[a,b]}} = \mathbb{1}_{[a,b]} k_u. \end{array} \right. \tag{5.2}$$

Remark. — If $G_u : \mathbb{R}_+ \to \mathbb{C}$ and $k_u \in L^2_{\text{loc}}(\mathbb{R}_+)$ satisfy (5.2), then T_t defined by (5.1), verify the hypothesis of the proposition.

Proof of the Proposition 5.1.

1) Search of A and g.

Let $x_t = T_t\mathbb{1}$. Then we have $x_t - x_s = (T_t - T_s)\mathbb{1} = \mathbb{1} \otimes (K_{s,t}\mathbb{1}) \otimes \mathbb{1}$ in $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$, because $\mathbb{1} = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ in this decomposition. So $x_t - x_s \in \Phi_{[s,t]}$ and by the proposition 3.1, there exist $A : \mathbb{R}_+ \to \mathbb{C}$ and $g \in L^2_{\text{loc}}(\mathbb{R}_+)$ such that $x_t = A(t)\mathbb{1} + \int_0^t g(s)\, d\chi_s$. Let $S_t = T_t - A(t)\,\mathrm{Id} - a^+_{g_{t]}}$. We have $S_t\mathbb{1} = 0$.

2) Proof of the formulas (5.1) and (5.2).

Let for $u \in L^2(\mathbb{R}_+)$,

$$G_u(t) = S_t\mathcal{E}(u)(\emptyset) = T_t(\mathcal{E}(u) - \mathbb{1})(\emptyset) \tag{5.3}$$

For $s < t < t'$, for $u \in L^2(\mathbb{R}_+)$,

$$\begin{aligned} S_{t'}\mathcal{E}(u)(\{s\}) &= (S_{t'} - S_t)\mathcal{E}(u)(\{s\}) + S_t\mathcal{E}(u)(\{s\}) \\ &= u(s)(S_{t'} - S_t)\mathcal{E}(u)(\emptyset) + S_t\mathcal{E}(u)(\{s\}) \\ &= u(s)(G_u(t') - G_u(t)) + S_t\mathcal{E}(u)(\{s\}) \end{aligned}$$

so,

$$S_{t'}\mathcal{E}(u)(\{s\}) - u(s)G_u(t') = S_t\mathcal{E}(u)(\{s\}) - u(s)G_u(t)\,.$$

This number is thus independent of $t > s$ and we denoted it $k_u(s)$. So we have for all $s < t$,

$$S_t\mathcal{E}(u)(\{s\}) = u(s)G_u(t) + k_u(s) \tag{5.4}$$

and k_u belongs to $L^2_{\text{loc}}(\mathbb{R}^+)$.

We prove now the following equality by recurrence on $\#\sigma$: if $\sigma \subset [0,t]$,

$$(S_t\mathcal{E}(u))(\sigma) = G_u(t)\mathcal{E}(u)(\sigma) + \sum_{s\in\sigma} k_u(s)\mathcal{E}(u)(\sigma - \{s\}). \tag{5.5}$$

– (5.5) is true for $\#\sigma \leq 1$ by (5.4) and definition of G_u.

– We suppose now (5.5) true for $\#\sigma \leq n - 1$ and $n \geq 2$.

Let $\sigma \subset [0,t]$ and $\#\sigma = n$. It exists $s < t$ such that $\max\sigma \in [s,t]$ and $\sigma - \{\max\sigma\} \subset [0,s]$.

$$\begin{aligned}
S_t\mathcal{E}(u)(\sigma) &= (S_t - S_s)\mathcal{E}(u)(\sigma) + S_s\mathcal{E}(u)(\sigma)\\
&= \mathcal{E}(u\mathbb{1}_{[0,s]})(\sigma\cap[0,s])(S_t - S_s)\mathcal{E}(u\mathbb{1}_{[s,t]})(\sigma\cap[s,t])\\
&\quad + S_s\mathcal{E}(u_{s]})(\sigma\cap[0,s])\mathcal{E}(u\mathbb{1}_{[s,+\infty[})(\sigma\cap[s,+\infty[)\\
&= \mathcal{E}(u)(\sigma\cap[0,s])(S_t - S_s)\mathcal{E}(u)(\max\sigma) + u(\max\sigma)S_s\mathcal{E}(u)(\sigma\cap[0,s]).
\end{aligned}$$

We use (5.4) and (5.5) for $\#(\sigma\cap[0,s]) = n-1$

$$\begin{aligned}
S_t\mathcal{E}(u)(\sigma) &= \mathcal{E}(u)(\sigma\cap[0,s])(u(\max\sigma)G_u(t) + k_u(\max\sigma) - u(\max\sigma)G_u(s))\\
&\quad + u(\max\sigma)(G_u(s)\mathcal{E}(u)(\sigma\cap[0,s]) + \sum_{r\in\sigma\cap[0,s]} k_u(r)\mathcal{E}(u)(\sigma\cap[0,s] - \{r\})\\
&= G_u(t)\mathcal{E}(u)(\sigma) + \sum_{r\in\sigma} k_u(r)\mathcal{E}(u)(\sigma - \{r\}).
\end{aligned}$$

So we prove (5.1).

Let $a < b < t$ and $s \in]0,t[$,

$$\begin{aligned}
G_{u\mathbb{1}_{[a,b]}}(t) &= S_t\mathcal{E}(u\mathbb{1}_{[a,b]})(\emptyset)\\
&= (S_t - S_b)\mathcal{E}(u\mathbb{1}_{[a,b]})(\emptyset) + (S_b - S_a)\mathcal{E}(u\mathbb{1}_{[a,b]})(\emptyset) + S_a\mathcal{E}(u\mathbb{1}_{[a,b]})(\emptyset)
\end{aligned}$$

The first and third terms are equal to zero because $S_t\mathbb{1} = 0$, so

$$G_{u\mathbb{1}_{[a,b]}}(t) = (S_b - S_a)\mathcal{E}(u\mathbb{1}_{[a,b]})(\emptyset) = G_u(b) - G_u(a) \tag{5.6}$$

First case: $s \in]0,a[$,

$$k_{u\mathbb{1}_{[a,b]}}(s) = S_a\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) - (u\mathbb{1}_{[a,b]})(s)G_{u\mathbb{1}_{[a,b]}}(a) \text{ by (5.4).}$$

So, as $S_a\mathbb{1} = 0$, $k_{u\mathbb{1}_{[a,b]}}(s) = 0$.

Second case: $s \in]b,t[$,

$$\begin{aligned}
k_{u\mathbb{1}_{[a,b]}}(s) &= S_t\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) - (u\mathbb{1}_{[a,b]})(s)G_{u\mathbb{1}_{[a,b]}}(t)\\
&= (S_t - S_b)\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) + S_b\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\})\\
&= 0\,.
\end{aligned}$$

Third case: $s \in]a,b[$; if we use (5.6) and (5.4) we have

$$\begin{aligned}
k_{u\mathbb{1}_{[a,b]}}(s) &= S_b\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) - u(s)G_{u\mathbb{1}_{[a,b]}}(b)\\
&= (S_b - S_a)\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) + S_a\mathcal{E}(u\mathbb{1}_{[a,b]})(\{s\}) - u(s)(G_u(b) - G_u(a))\\
&= u(s)G_u(b) + k_u(s) - G_u(a)u(s) - u(s)(G_u(b) - G_u(a)) = k_u(s)\,.
\end{aligned}$$

So we have proved (5.2) and the proposition 5.1. ∎

Proof of theorem 5.2. — We verify easily with the hypothesis of theorem 5.2 that $(T_t^*)_{t\geq 0}$ satisfies also the hypothesis of the proposition 5.1. Let A^*, g^*, G_u^*, k_u^* associated by the proposition 5.1 with $(T_t^*)_{t\geq 0}$. So we have for all $t \geq 0$, for all $u, v \in L^2(\mathbb{R}_+)$,

$$\langle T_t \mathcal{E}(u_{t]}), \mathcal{E}(v_{t]})\rangle = \langle \mathcal{E}(u_{t]}, T_t^* \mathcal{E}(v_{t]})\rangle$$

so by using (5.1) and (4.4) :

$$A(t) + \int_0^t \overline{g(s)} v(s)\ ds + \overline{G_u(t)} + \int_0^t \overline{k_u}(s) v(s)\ ds$$
$$= A^*(t) + \int_0^t g^*(s)\overline{u(s)}\ ds + G_v^*(t) + \int_0^t k_v^*(s)\overline{u(s)}\ ds\,.$$

When we make $u = 0$ or $v = 0$ in the preceding formula ($G_0 = G_0^* = k_0 = k_0^* = 0$), we obtain :

$$\begin{cases} A(t) = A^*(t) \\ G_u(t) = \displaystyle\int_0^t \overline{g^*(s)} u(s)\ ds \\ G_v^*(t) = \displaystyle\int_0^t \overline{g(s)} v(s)\ ds \end{cases}$$

and for all u, v in $L^2(\mathbb{R}_+)$, $\int_0^t \overline{k_u(s)} v(s)\ ds = \int_0^t k_v^*(s)\overline{u(s)}\ ds$. Thus the application $\begin{array}{ccc} L^2([0,t], ds) & \longrightarrow & L^2([0,t], ds) \\ u & \longmapsto & k_u \end{array}$ is linear and closable, so it is bounded. And by (5.2), this operator commutes with $\mathbb{1}_{[a,b]}$ for all a, b in $[0, t]$, therefore it commutes with all the bounded functions ; so it exists $k \in L^\infty([0,t], ds)$ such that $k_u(s) = u(s)k(s)$. But by the definition of k_u, k is independent of t. So we prove theorem 5.2. ∎

Proof of theorem 5.1. — We prove first that $u \mapsto G_u$ and $u \mapsto k_u$ are linear. For simplification, we set $t = 1$. Let u, v in $L^2(\mathbb{R}_+, dt)$. Let $g_n = F_n(u+v) - F_n(u) - F_n(v)$ where

$$F_n(u) = \sum_{k=0}^{n-1} \left(\mathcal{E}(u\mathbb{1}_{\left[\frac{k}{n}, \frac{k+1}{n}\right]}) - \mathbb{1}\right).$$

LEMMA 5.1. — $g_n \xrightarrow{\Phi} 0$.

Proof. — We have by definition of $F_n(u)$, that $g_n(\emptyset) = 0$ and $g_n(\{s\}) = 0$ for all s. So

$$\|g_n\|^2 \leq 4\Big(\|F_n(u+v)\mathbb{1}_{\#\sigma\geq 2}\|^2 + \|F_n(u)\mathbb{1}_{\#\sigma\geq 2}\|^2 + \|F_n(v)\mathbb{1}_{\#\sigma\geq 2}\|^2\Big)$$

or by using (2.3),

$$\|F_n(u)\|^2 = \sum_{k=0}^{n-1}\left(e^{\int_{\frac{k}{n}}^{\frac{k+1}{n}} |u(s)|^2\ ds} - 1\right)$$

so

$$\|F_n(u)\mathbb{1}_{\#\sigma\geq 2}\|^2 = \sum_{k=0}^{n-1}\left(e^{\int_{\frac{k}{n}}^{\frac{k+1}{n}}|u(s)|^2\,ds} - 1 - \int_{\frac{k}{n}}^{\frac{k+1}{n}}|u(s)|^2\,ds\right)$$

and

$$\|F_n(u)\mathbb{1}_{\#\sigma\geq 2}\|^2 \leq e^{\|u\|^2}\sum_{k=0}^{n-1}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}}|u(s)|^2\,ds\right)^2$$

and this quantity tends to zero when n tends to infinity. ■

Recall that $S_t = T_t - A(t)\,\mathrm{Id} - a^+_{g_{t]}}$ and $S_t\mathbb{1} = 0$. So, using (5.5) and (5.2), we have that for $\sigma \in \mathcal{P}$

$$S_1F_n(u)(\sigma) = \sum_{k=0}^{n-1}\mathbb{1}_{\sigma\subset\left[\frac{k}{n},\frac{k+1}{n}\right[}\left\{\left(G_u\left(\frac{k+1}{n}\right) - G_u\left(\frac{k}{n}\right)\right)\mathcal{E}(u)(\sigma) + \sum_{s\in\sigma}k_u(s)\mathcal{E}(u)(\sigma-\{s\})\right\} \tag{5.7}$$

Lemma 5.2. — $a^+_{g_{1]}}(g_n) \xrightarrow{\Phi} 0$.

Proof. — We have that

$$a^+_{g_{1]}}F_n(u) = \sum_{k=0}^{n-1}\left(a^+_{g_{1]}}\left(\mathcal{E}(u\mathbb{1}_{\left[\frac{k}{n},\frac{k+1}{n}\right[})\right) - a^+_{g_{1]}}(\mathbb{1})\right).$$

So by (4.4), we have:

$$\|a^+_{g_{1]}}F_n(u)\|^2 = |\langle u,g\rangle|^2 + \sum_{k=0}^{n-1}\left(e^{\int_{\frac{k}{n}}^{\frac{k+1}{n}}|u(s)|^2\,ds} - \mathbb{1}\right)\|g\|^2 + \sum_{k=0}^{n-1}\left(e^{\int_{\frac{k}{n}}^{\frac{k+1}{n}}|u(s)|^2\,ds} - \mathbb{1}\right)\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}}u(s)g(s)\,ds\right|^2$$

but if $u, v \in L^2([0,t], dt)$,

$$\sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}}u(s)v(s)\,ds\right|^2 \underset{n\to+\infty}{\longrightarrow} 0$$

so

$$\|a^+_{g_{1]}}F_n(u)\|^2 \underset{n\to+\infty}{\longrightarrow} |\langle u,g\rangle|^2 + \|u\|^2\,\|g\|^2.$$

We have more that $a^+_{g_{1]}}F_n(u)(\emptyset) = a^+_{g_{1]}}F_n(u)(\{s\}) = 0$ for all $s \in [0,1]$ and

$$(a^+_{g_{1]}}F_n(u))(\{s_1;s_2\}) = g(s_1)u(s_2) + g(s_2)u(s_1)$$

if $\{s_1, s_2\} \subset [0,1]$.

So $\|a^+_{g_{1]}} F_n(u)\mathbb{1}_{\#\sigma\leq 2}\|^2 = \||\langle u, g\rangle|^2 + \|u\|^2\|g\|^2$ and therefore

$$\int_{\mathcal{P}} \mathbb{1}_{\#\sigma\geq 3}|a^+_{g_{1]}} F_n(u)|^2(\sigma)d\sigma \underset{n\to+\infty}{\longrightarrow} 0\,.$$

Or if $\#\sigma \leq 2$, $a^+_{g_{1]}}(g_n)(\sigma) = 0$, so we have lemma 5.2. ∎

LEMMA 5.3. — $S_1 g_n \xrightarrow{\Phi} \big(G_{u+v}(1) - G_u(1)(G_v(1)\big)\mathbb{1} + \int_0^1 \big(k_{u+v}(s) - k_u(s) - k_v(s)\big)\, d\chi_s$.

Proof. — By (5.7), $S_1 g_n(\emptyset) = G_{u+v}(1) - G_u(1) - G_v(1)$ and

$$S_1 F_n(u)(\{s\}) = k_u(s) + u(s)\sum_{k=0}^{n-1} \mathbb{1}_{s\in\left[\frac{k}{n},\frac{k+1}{n}\right]}\left(G_u\Big(\frac{k+1}{n}\Big) - G_u\Big(\frac{k}{n}\Big)\right)\,.$$

We have when $\left|G_u\left(\frac{k+1}{n}\right) - G_u\left(\frac{k}{n}\right)\right|^2$ is bounded uniformaly in k and n, that

$$\int_{\mathcal{P}} \mathbb{1}_{\#\sigma\geq 2}(S_1 F_n(u)(\sigma))^2\, d\sigma \xrightarrow{\Phi} 0\,.$$

This hypothesis is verify when we have 3) of theorem 5.1. We have too that

$$\begin{aligned}\int_0^1 &\left|u(s)\sum_{k=0}^{n+1} \mathbb{1}_{\left(s\in\left[\frac{k}{n},\frac{k+1}{n}\right[\right)}\left(G_u\Big(\frac{k+1}{n}\Big) - G_u\Big(\frac{k}{n}\Big)\right)\right|^2 ds\\ &= \sum_{k=0}^{n-1}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |u(s)|^2\, ds\right)\left|G_u\Big(\frac{k+1}{n}\Big) - G_u\Big(\frac{k}{n}\Big)\right|^2.\end{aligned}$$

This quantity tends to zero when the hypothesis 3) of theorem 5.1 is satisfied.

So

$$\int_{\mathcal{P}} \mathbb{1}_{\#\sigma\geq 2}(S_1 g_n(\sigma))^2\, d\sigma \xrightarrow{\Phi} 0$$

and

$$\begin{aligned}\int_{\mathcal{P}} \mathbb{1}_{\#\sigma\leq 1}(S_1 g_n)^2(\sigma)\, d\sigma \longrightarrow &\big(G_{u+v}(1) - G_u(1) - G_v(1)\big)\mathbb{1}\\ &+ \int_0^1 (k_{u+v} - k_u - k_v)(s)\, d\chi_s\,.\end{aligned}$$ ∎

We have proved by lemmas 5.1, 5.2, 5.3 that

$$\begin{cases} g_n \xrightarrow{\Phi} 0 \\ T_1 g_n \xrightarrow{\Phi} \big(G_{u+v}(1) - G_u(1) - G_v(1)\big)\mathbb{1} + \int_0^1 (k_{u+v} - k_u - k_v)(s)\, d\chi_s. \end{cases}$$

But T_1 is closable, so

$$\begin{cases} G_{u+v}(1) = G_u(1) + G_v(1) \\ k_{u+v} = k_u + k_v\,. \end{cases} \tag{5.8}$$

LEMMA 5.4. — *Let $(u_n)_{n\geq 0}$ in $L^2([0,1],dt)$ such that $u_n \to u$ in $L^2([0,1],dt)$ and* $\begin{cases} G_{u_n}(1) \xrightarrow{\mathbb{C}} \alpha \\ k_{u_n} \xrightarrow{L^2([0,1],dt)} w \end{cases}$*. Then we have $T_1\mathcal{E}(u_n) \xrightarrow{\Phi} f$ where for all $\sigma \in \mathcal{P}$,*

$$f(\sigma) = A(1)\mathcal{E}(u)(\sigma) + a^+_{g_{1]}}\mathcal{E}(u)(\sigma) + \alpha\mathcal{E}(u)(\sigma) + \sum_{s\in\sigma} w(s)\mathcal{E}(u)(\sigma - \{s\}) \,.$$

Proof of lemma 5.4. — It's easy to see by (4.4) that $a^+_{g_{1]}}\mathcal{E}(u_n) \xrightarrow{\Phi} a^+_{g_{1]}}\mathcal{E}(u)$.

$$\begin{aligned}
&\int_{\mathcal{P}} \left| \sum_{s\in\sigma} \Big(k_{u_n}(s)\mathcal{E}(u_n)(\sigma - \{s\}) - w(s)\mathcal{E}(u)(\sigma - \{s\}) \Big) \right|^2 d\sigma \\
&\quad \leq 2 \int_{\mathcal{P}} \left| \sum_{s\in\sigma} k_{u_n}(s)\big(\mathcal{E}(u_n)(\sigma - \{s\}) - \mathcal{E}(u)(\sigma - \{s\})\big) \right|^2 d\sigma \\
&\qquad\qquad + 2 \int_{\mathcal{P}} \left| \sum_{s\in\sigma} \big(k_{u_n}(s) - w(s)\big)\mathcal{E}(u_n)(\sigma - \{s\}) \right|^2 d\sigma \\
&\quad \leq 2 \Big\{ \|\mathcal{E}(u_n)\|^2 \Big(\|k_{u_n}\|^2 + |\langle k_{u_n}, u_n\rangle|^2 \Big) + \|\mathcal{E}(u)\|^2 \Big(\|k_{u_n}\|^2 + |\langle k_{u_n}, u\rangle|^2 \Big) \\
&\qquad\qquad - 2|\langle \mathcal{E}(u_n), \mathcal{E}(u)\rangle| \Big(\|k_{u_n}\|^2 + |\langle k_{u_n}, u_n\rangle|\,|\langle k_{u_n}, u\rangle| \Big) \\
&\qquad\qquad + \|\mathcal{E}(u)\|^2 \Big(\|k_{u_n} - w\|^2 + |\langle k_{u_n} - w, u\rangle|^2 \Big) \Big\} \,.
\end{aligned}$$

So, under the hypothesis this quantity tends to zero when $n \to +\infty$. ■

Let $\begin{array}{rccl} R : & L^2([0,1],dt) & \longrightarrow & \mathbb{C} \times L^2([0,1],dt) \\ & u & \longmapsto & (G_u(1), k_u) \end{array}$. The lemma 5.4 and the closability of T_1 prove that R is a linear operator of real vectorial space. In fact, with (5.8), for all $\lambda \in \mathbb{Q}$, $R(\lambda u) = \lambda R(u)$ and the density of $\mathbb{Q}$ in $\mathbb{R}$ gives us the result. By the same way, lemma 5.4 proves that R is closable on $L^2([0,1])$ so that by the closed graph theorem, R is a bounded operator. So it exists $M \in \mathbb{R}_+$ such that for all $u \in L^2([0,1])$

$$\begin{cases} |G_u(1)| \leq M\|u\| \\ \quad \|k_u\| \leq M\|u\| \,. \end{cases}$$

So there exist f in $L^2([0,1])$ such that $G_u(1) = \int_0^1 u(s)f(s)\,ds$ and as $u \mapsto k_u$ commutes with the indicatories of intervals, we conclude like in the proof of theorem 5.2.

Remarks.

1) We can prove the same result by similar methods for a process of operators defined on the space of finite sum of chaos and such that 1) and 2) of theorem 5.1, were verified.

2) We can see that we can change hypothesis 3) of theorem 5.1 in: $\forall t > 0$, $\begin{array}{rcl} L^2(\mathbb{R}_+, ds) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & T_t\mathcal{E}(u)(\emptyset) \end{array}$ is continuous at 0.

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