On the projective geometry of homogeneous varieties*

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Contents

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1 Introduction

This is the first in a planned series of papers establishing new relations between the representation theory of complex simple Lie groups and the algebraic and differential geometry of their homogeneous varieties.

Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety, where G is a complex simple Lie group, P is a maximal parabolic subgroup and V is the minimal G-module associated to P. Fix a point $x \in X$. We calculate the intrinsic structure of the tangent space T_xX, the variety of tangent directions to lines on X through x, and apply these calculations to determine unirulings of X and the varieties parametrizing the unirulings.

We show the variety of tangent directions to lines on X through x, which is the base locus of the second fundamental form $\mathbb{FF}^2_{X,x}$, is a Hermitian symmetric space if and only if P does not correspond to a short root (i.e., no arrow in the Dynkin diagram points towards the node of P).

In the case P does not correspond to a short root, we show that all linear spaces on X are G-homogeneous in the sense of Tits [29]. In particular, the spaces of unirulings are (the disjoint union of) G-homogeneous varieties, and can be determined pictorially.

If P corresponds to a short root, then $\text{Base}\,|\mathbb{FF}^2_{X,x}|$ is not G-homogeneous, and often not homogeneous for any group, and thus the linear spaces and the varieties which parametrize them must be determined by other methods. We describe the homogeneous varieties corresponding to the exceptional short roots and their unirulings using the octonions.

We also calculate the projective fundamental forms of X in many cases. In the case X is a Hermitian symmetric space, a strict prolongation property holds. Consequences of this strict prolongation property are a duality involving Koszul complexes and the appearance of secant varieties at the infinitesimal level.

Our study began with the observation that Freudenthal's magic chart could be derived from Zak's theorem on Severi varieties and standard geometric constructions (hyperplane sections and blowups). Our attempt to understand this observation led us to discover further connections between projective geometry and representation theory.

What follows is a brief description of some topics in the next two articles of the series.

In [22] we will give new interpretations of the chart, its distinguished spaces, and their connections with projective geometry.

Freudenthal constructed the exceptional Lie algebras in several different ways, which from a modern perspective are best understood in terms of the $\mathbb{Z}$ and $\mathbb{Z}_m$ gradings of the Lie algebra $\mathfrak{g}$. In [22] we will also explain how the projective fundamental forms naturally produce these constructions and gradings.

Our study of the magic chart led us to discover algorithms which are valid for more general classes of Lie groups.

In [23] we present algorithms that construct all Hermitian symmetric spaces and all adjoint varieties from $\mathbb{P}^1$ using elementary geometric procedures.

We plan to prove the algorithms work without relying on the known classification theorems. Such proofs would provide the classifications of Hermitian symmetric varieties and complex simple Lie groups via elementary projective geometry rather than root systems.

We plan to determine algorithms that construct all homogeneous varieties $X = G/P$ where P is maximal. This generalization will give geometric interpretations to Kac's classification of type I θ-groups operating irreducibly on a vector space [16].

What is particularly interesting about our algorithms is that when one varies them slightly, one can construct (nonhomogeneous) varieties that are quite similar to homogeneous varieties.

For example, in [23], we construct (nonhomogeneous) varieties that have the same second fundamental form as the adjoint varieties at general points. (Note that there are no nonhomogeneous varieties having the same second fundamental form of $G(2,n)$ for $n \geq 6$, $\mathbb{P}^k \times \mathbb{P}^l$ for $k,l \geq 2$, or $\mathbb{OP}^2$ at general points, see [21]). We will describe these new examples of projective varietes in detail.

Overview

In §2.1 we collect basic information about the differential invariants of projective varieties relevant for our study. Let $X \subset \mathbb{P}V$ be a projective variety, let $x \in X$ be a smooth point, let $\mathbb{FF}^k{}_{X,x} \in S^k T^*_x X \otimes N_x X$ denote its k-th fundamental form and let $\sigma_p(X)$ denote its p-th secant variety, the closure of the union of all secant $\mathbb{P}^{p-1}$'s to X (see §2.1 for more information).

While secant varieties are usually studied at the macroscopic level, we show they also play a role in the infinitesimal geometry:

Proposition 2.1. Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and let $x \in X$ be a general point. Then

$$\mathrm{Base}\,|\mathbb{FF}^k_{X,x}| \supseteq \sigma_{k-1}(\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|).$$

In §2.2 we begin our infinitesimal study of homogeneous varieties. We establish the basic connection between representation theory and projective geometry: if V is a G-module with smallest orbit $X = G/P \subset \mathbb{P}V$, the action of the associated graded algebra to the universal envelopping algebra $U(\mathfrak{g})_k/U(\mathfrak{g})_{k-1}$ on the P-filtration of V corresponds to $\mathbb{FF}^k_{X,x}$.

Proposition 2.7. Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety and let $x \in X$. Let $\hat{T}^{(k)}_x X$ denote the cone over the k-th osculating space at x and let $N_k = \hat{T}^{(k)}_x X/\hat{T}^{(k-1)}_x X$ be the k-th normal space twisted by $\mathcal{O}_X(-1)$. Then

$$\hat{T}^{(k)}_x X = V^{(k)}_\lambda, \qquad N_k = V^{(k)}_\lambda / V^{(k-1)}_\lambda.$$

Moreover, there is a commutative diagram

$$\begin{array}{rcc} S^k\mathfrak{g} & = & U(\mathfrak{g})_k/U(\mathfrak{g})_{k-1} \\ \downarrow & & \downarrow \\ {}^t\mathbb{FF}^k_{X,x} : S^k T & \longrightarrow & N_k, \end{array}$$

where the bottom horizontal map is the (twisted) k-th fundamental form at x.

Let $P = LP^u$ be a Levi decomposition of P, where P^u is unipotent and L is reductive. Let H denote the semi-simple part of L. In §2.3 we describe the decomposition of $T_x X$ as L and P modules. In particular, when P is maximal, there is a preferred L-module $T_1 \subset T_x X$ that is easy to read off the marked Dynkin diagram of G.

In §2.4 we begin our computations of the base loci of the second fundamental forms of the varieties $X = G/P$ where P is maximal. The closed orbit $Y_1 \subset \mathbb{P}T_1$ plays an important role. One of the main results of this paper is the following:

Theorem 2.19. Let $X = G/P$, with G simple and P a maximal parabolic subgroup. Embedd $X \subset \mathbb{P}V$ by its minimal G-equivariant embedding. Let H be the semi-simple part of P and let Y_1 be the closed H-orbit in $\mathbb{P}T_1$. Then

$$Y_1 \subseteq \mathrm{Base}\,|\mathbb{FF}^2_{X,x}|.$$

Equality holds if and only if α is not a short root (i.e. no arrow in $\mathcal{D}(G)$ points towards α).

If α is a short root, then $Y_2 \subset \mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$.

Fortunately, Y_1 is easy to understand:

We say a Hermitian symmetric space $X = G/P$ is *G-Hermitian symmetric* if G is the local automorphism group of X.

Theorem 2.17. Notations and hypothesis as above. Y_1 is a Hermitian symmetric space.

Moreover, $Y_1 \subset \mathbb{P}T_1$ is an H-Hermitian symmetric space in its minimal embedding except in the following cases:

1. C_n/P_k for $k < n$. Here $Y_1 = Seg(\mathbb{P}^{k-1} \times \mathbb{P}^{2n-2k-1})$, $H = SL_k \times Sp_{2n-2k} \subsetneq SL_k \times SL_{2n-2k}$.
2. F_4/P_4. Here $Y_1 = B_3/P_3$.
3. G_2/P_2. Here $Y_1 = v_3(\mathbb{P}^1)$ is the twisted cubic, which is A_2-Hermitian symmetric, but not in its minimal embedding.

For the homogeneous varieties corresponding to short roots, Base $|\mathbb{FF}^2_{X,x}|$ is not always homogeneous. We obtain the base loci for the other cases (propositions 6.1, 6.7, 6.13) by a case by case study. Our computations are guided by a *geometric folding principle* discussed below. For example, we show Base $|\mathbb{FF}^2_{F_4/P_4}|$ is a generic hyperplane section of the spinor variety D_5/P_5 because F_4/P_4 is a generic hyperplane section of E_6/P_1.

In section 3 we calculate all normal spaces and fundamental forms of the Hermitian symmetric spaces. The fundamental forms satisfy a *strict prolongation property* (see §2.1 for the definition of the prolongation property):

Theorem 3.2. Let $X = G/P_{\alpha_i} \subset \mathbb{P}(V_{\omega_i})$ be a Hermitian symmetric space in its fundamental embedding and let $x \in X$. Then for $k \geq 2$,

$$|\mathbb{FF}^{k+1}_{X,x}| = |\mathbb{FF}^2_{X,x}|^{(k-1)}.$$

This strict prolongation property does not hold for a general homogeneous variety, see proposition 2.20.

Strict prolongation for an arbitrary variety gives rise to Koszul complexes (see corollary 2.4). In the case of Hermitian symmetric spaces, these Koszul complexes have an interesting duality, see remark 3.9.

A closer study of the fundamental forms of Hermitian symmetric spaces yields a geometric description of the base loci (compare with Proposition 1.1 above):

Corollary 3.8. Let $X \subset \mathbb{P}V$ be a Hermitian symmetric space in its fundamental embedding, and let $x \in X$. Then

$$\text{Base}\,|\mathbb{FF}^k_{X,x}| = \sigma_{k-1}(\text{Base}\,|\mathbb{FF}^2_{X,x}|).$$

In section 4 we describe the adjoint case. This case has been studied previously [2, 15]. We recover and rephrase what is known in terms of our study.

In section 5 we compute the fundamental forms for all G/P with P maximal of the classical groups, and determine their unirulings and G-homogeneous unirulings.

In section 6 we calculate Base $(\mathbb{FF}^2_{G/P_\alpha})$ in the case G is exceptional and α short. We determine varieties parametrizing their unirulings and offer geometric interpretations in terms of octonionic geometry.

In section 7 we apply the results of the preceeding sections.

In §7.1, we take homogeneous varieties corresponding to extremal nodes of $\mathcal{D}(G)$ as building blocks and describe the other spaces in terms of them. The corresponding irreducible representations are called *elementary representations*. We let X_{end} be the homogeneous variety corresponding to an end node and X_{end-k} the variety corresponding to the parabolic a distance k from the end on the branch of X_{end}. For $k = 1$, our main result is:

Proposition 7.7. Let $X_{end} = G/P \subset \mathbb{P}V$ be a homogeneous variety in its standard embedding, where P is the parabolic corresponding to an end in $\mathcal{D}(G)$. Let $\mathcal{Y}_1 \subset TX$ be the distribution defined by $(\mathcal{Y}_1)_x = \hat{Y}_1$. Let $Z_1^{\mathcal{Y}}$ denote the set of lines on X_{end} that are integral manifolds for the distribution $\mathcal{Y}_1$. Then

$$X_{end-1} = Z_1^{\mathcal{Y}}.$$

We have a similar (but slightly more technical) statement for $k > 1$, see theorem 7.8. A special case of theorem 7.8 is

Corollary 7.8. Let G be a simple group, let $\alpha_{end} = \alpha_1$ be a root that is not short, corresponding to an end node on $\mathcal{D}(G)$ with branch of length p. Label the roots on the branch $\alpha_1, ..., \alpha_p$ and let $X_{end-k} = G/P_{\alpha_{k+1}}$. Then X_{end-k} is the space of k-planes on X_{end}.

When α_{end} is a short root, we interpret the clas of linearly embedded $\mathbb{P}^k$'s parametrized by X_{end-k} from the perspectives of representation theory and projective geometry.

In §7.2 we prove Theorem 1.3.

In §7.3 we study the varieties parametrizing the unirulings of X and their relations to Tits geometries and geometric folding. Tits geometries, geometric folding, and their relations to our results are discussed at the end of this introduction.

Theorem 7.18. Let G be a simple group, let α be a simple root, let $P = P_\alpha$ be the corresponding parabolic and let $X = G/P \subset \mathbb{P}V$ be the corresponding homogeneous variety in its minimal homogeneous embedding.

1. If α is not short (i.e., if no arrow in $\mathcal{D}(G)$ points towards α), then for all k, the variety parametrizing the linearly embedded $\mathbb{P}^k$'s on X is the disjoint union of homogeneous varieties $G/P_{\Sigma\beta_j}$ where $\{\beta_j\} \subset \Delta_+$ is a reduced set such that the component of $\mathcal{D}(G)\backslash\{\beta_j\}$ containing α is isomorphic to $\mathcal{D}(A_k)$ or $\mathcal{D}(B_k)$ and α is an extremal node of this component.
2. If $G/P_\alpha = B_n/P_n, G_2/P_1$ or C_n/P_1, then the space of linearly embedded $\mathbb{P}^k$'s on X is not G-homogeneous, but $\tilde{G}$-homogeneous, where $\mathcal{D}(G)$ is the fold of $\mathcal{D}(\tilde{G})$.
3. If $G/P_\alpha = C_n/P_k$, $1 < k < n, F_4/P_4$ or F_4/P_3, then the space of linearly embedded $\mathbb{P}^1$'s on X is not homogeneous.

Using theorem 1, we explain a quick way to determine the dimension of the largest linear space on X if α is not a short root:

Corollary 7.19. Let G be a simple Lie group and let P_α be a maximal parabolic with α not short (i.e. such that no arrow in $\mathcal{D}(G)$ points towards P). Let $X = G/P$ be the corresponding homogeneous variety in its fundamental embedding. Suppose that the longest of the A or B chains in $\mathcal{D}(G)$ beginning at α has length n. Then the largest linear space on X is a $\mathbb{P}^n$.

In §7.4, we explain how to recover the marked Dynkin diagram $\mathcal{D}^*(G)$ associated to X from the base locus of the second fundamental form of X.

Remark. We originally made our calculations by several different methods, using moving frames, using Verma modules, and using an analog of Griffith's transversality combined with restrictions coming from the prolongation property. The calculations we present are hybrids of these methods.

Remark on Intrinsic v.s. extrinsic geometry. Athough we phrase our results in terms of a homogeneous variety embedded in a projective space, the embedding comes from a natural line bundle on the homogeneous variety. Thus a "minimal degree rational curve" on X (e.g. [15]) is a line on X in its minimal embedding. Similarly the "cone of tangents to minimal degree rational curves" is Base $|\mathbb{FF}^2_{X,x}|$.

Relations to Tits geometries.

Tits gives geometric interpretations of homogeneous varieties G/P in terms of incidence geometries (see, e.g. [28]), generalizing the characterization of projective space as the space where linear spaces of complementary dimension must intersect. From Tits' perspective, a point of G/P is a parabolic conjugate to P. The intersection of two such is another parabolic subgroup of G and thus an element of a different G-homogeneous variety. Tits deduces all the possible incidence relations axiomatically from the geometry of four building blocks:

Classically, these are the four possible angles between simple roots. Tits' perspective is that these are the building blocks of the homogeneous varieties; that the homogeneous varieties of these four diagrams should be taken as the basic geometric objects, and that other homogeneous varieties should be considered as spaces of such spaces, just as we think of $G(k+1, n+1)$ as the space of $\mathbb{P}^k$'s on $\mathbb{P}^n$. One can also think of the four blocks as corresponding to the four division algebras.

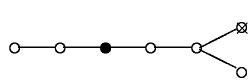
From the perspective of projective geometry, the four incidence geometries correspond to the four homogeneous mappings of $A_2/P_1 = \mathbb{P}^1$ such that the target space has a finite number of $A_2 = SL_3$ orbits: the Veronese embeddings $v_0(\mathbb{P}^1) \subset \mathbb{P}^0$, $v_1(\mathbb{P}^1) \subset \mathbb{P}^1$, $v_2(\mathbb{P}^1) \subset \mathbb{P}^2 = \mathbb{P}(S^2\mathbb{C}^2)$ and $v_3(\mathbb{P}^1) \subset \mathbb{P}^3 = \mathbb{P}(S^3\mathbb{C}^2)$. (One could also describe this condition without reference to group orbits.) These four varieties are the base loci of the fundamental forms of the homogeneous varieties associated to the short roots in the four building blocks.

Let G be a simple Lie group, let $\{\alpha_j\}, \{\beta_l\}$ be two subsets of the sets of positive roots of G, let P, P' be the respective parabolics and let $X = G/P \subset \mathbb{P}V$ and $X' = G/P' \subset \mathbb{P}V'$ be the corresponding homogeneous varieties in their minimal embeddings. For simplicity assume that $\{\alpha_j\} \cap \{\beta_l\} = \emptyset$. Consider the diagram

$$\begin{array}{ccc} & G & \\ & {}^{\pi}\swarrow \quad \searrow^{\pi'} & \\ X = G/P & & X' = G/P' \end{array}$$

Let $x' \in X'$ and consider $Y := \pi({\pi'}^{-1}(x')) \subset X$. Then X is covered by such varieties Y. Tits shows that $Y = H/Q$ where $\mathcal{D}(H)$ consists of the components of $\mathcal{D}(G)\backslash\{\alpha_j\}$ that contain some β_l, and $Q \subset H$ is the parabolic subgroup corresponding to the $\{\beta_l\}$'s, but he does *not* address the question of how Y is embedded. We show that Y is not always embedded by its minimal embedding (see §6.3).

Example 1.1. For $X = D_n/P_3$ and $X' = D_n/P_n$, we read on the diagram below that $Y = G(3, n)$.



We will call such subvarieties Y of X,

G-homogeneous subvarieties and Y the (P', P)-*Tits transform* of x'. We will assume that $\{\beta_j\}$ is *reduced*, in the sense that no subset of the $\{\beta_j\}$ has the same variety Y as Tits transform.

Now consider the special case where Y is a linearly embedded $\mathbb{P}^k \subset X$. Then X is uniruled by $\mathbb{P}^k$'s and X' is the variety that parametrizes this uniruling. (Although there may be a larger uniruling by $\mathbb{P}^k$'s containing X'.) A word of caution in applying Tits geometries:

Proposition 1.2. Let $X = G/P_\alpha \subset \mathbb{P}V$ be a minimaly embedded homogeneous variety. Let $Y = \mathbb{P}^k$ be the Tits transform of some $x' \in X' = G/P'$.

If α is not short, then $Y \subset \mathbb{P}V$ is linearly embedded.

If α is short, then Y may be embedded by a Veronese embedding. For example, the pair $(X, X') = (F_4/P_4, F_4/P_1)$ is such that the Tits transform of x' is the Veronese $v_2(\mathbb{P}^5)$.

A consequence of 1 is:

Proposition 1.3. Let $X = G/P_\alpha \subset \mathbb{P}V$ be a minimaly embedded homogeneous variety.

If α is not short. Then all linear spaces on X are G-homogeneous.

If α is short, then the G-homogeneous linear spaces on X form a proper subset of all linear spaces on X.

Now consider the subcase where P is maximal and $k = 1$. A moment's play with marked Dynkin diagrams shows that there is a unique variety X' of G-homogeneous lines on X, namely P' corresponds to marking all the nodes adjacent to the node of P. Thus part 1 of theorem 1 can be rephrased as stating that all lines on $X = G/P$ are G-homogeneous if and only if P does not correspond to a short root.

When α is short, there can be spaces parametrizing unirulings of X that are homogeneous varieties of G, but the rulings are not G-homogeneous in the sense of Tits (i.e. they do not arise via Tits transforms). This phenomenon occurs for $X = B_n/P_n$, see §5.3.

Some further relations between our perspective and Tits' are as follows:

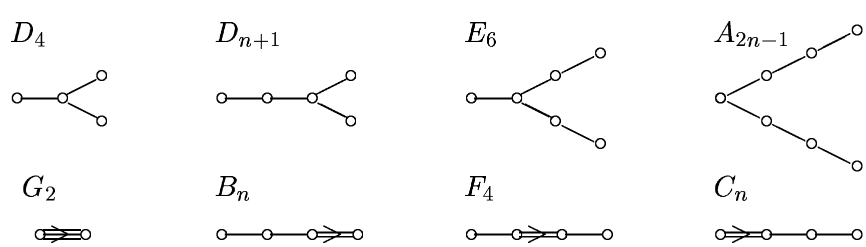
Proposition 1.4. Let $X = G/P_\alpha$ be a homogeneous variety and let X' be the space of G-homogeneous lines on X. Then

1. X' is the space of lines tangent to the distribution defined by Y_1.
2. Y_1 is isomorphic to the (P, P')-Tits transform of a point $x \in X$.

Note that one can use our perspective to prove new results about Tits geometries. For example, if P is maximal and $X' = G/P'$ is the variety of G-homogenous lines on X, then the (P, P')-Tits transform of $x \in X$ is a Hermitian symmetric space.

Geometric folding.

Dynkin diagrams with multiple bonds arise from folding simply laced diagrams along symmetries. The double (or triple) bond occurs next to the hinge with the arrow facing away from the hinge. (This was apparently first observed by Cartan, see [8].)



Let $X = G/P_\alpha$ be a homogeneous variety such that there is an arrow in $\mathcal{D}(G)$ pointing towards α. Consider the group $\tilde{G}$ such that $\mathcal{D}(G)$ is the fold of $\mathcal{D}(\tilde{G})$ and $\tilde{X} = G/P_{\alpha', s(\alpha')}$ where s is the symmetry and $\alpha', s(\alpha')$ fold to α. We observe that X is a subvariety of the flag variety $\tilde{X}$ satisfying certain incidence relations. Moreover, X is often isomorphic to a (possibly trivial) linear section of $\tilde{G}/\tilde{P}_{\alpha'}$. We discuss geometric folding in more detail in [22].

Notation

Alternating products of vectors will be denoted with a wedge, and symmetric products will not have any symbol (e.g. $\omega \circ \beta$ will be denoted $\omega\beta$).

T_xX denotes the holomorphic tangent space to X at x and $\tilde{T}_xX$ the embedded tangent projective space. We often supress reference to the base point of X so T should be read as T_xX for a general $x \in X$, N as N_xX etc... If $Y \subset \mathbb{P}^m$ then $\hat{Y} \subset \mathbb{C}^{m+1}$ will be used to denote the cone over Y. We will often ignore twists in bundles so T will be used to denote both T_xX and $T_xX(1) = \hat{T}/\hat{x}$. If $A \in \mathbb{C}^{n+a+1}$, its projection to $\mathbb{P}^{n+a}$ will be denoted $[A]$.

$\mathbb{CP}^k$ will be denoted $\mathbb{P}^k$. $\mathbb{FF}^k_{X,x}$ denotes the k-th projective fundamental form of X at x. $\sigma_d(X)$ denotes *d-th secant variety of X*, the closure of the union of the span of all d-uples of points on X.

If V is a vector space, $S_{a_1,\dots,a_p}V$ denotes the Schur functor obtained by the Young diagram whose j-th row has length a_j.

If G is a Lie group, we denote its Lie algebra by $\mathfrak{g}$. We let α_i, $1 \leq i \leq \operatorname{rank} G$ denote a base of the roots with respect to some Cartan subalgebra, ω_i denote the fundamental weights, and $\tilde{\alpha}$ denotes the largest root. Δ denotes the set of all roots, Δ_+ the set of positive roots, $R = R(\mathfrak{g})$ the root lattice and $\mathcal{D}(G)$ the Dynkin diagram of G. If $\beta \in \Delta$, we write $\beta = \sum_i m_i(\beta)\alpha_i$ to define the functions m_i. We often say "an arrow pointing towards ω, P, α" by which we mean an arrow pointing towards the corresponding node in $\mathcal{D}(G)$. We use the ordering of the roots as in [4]. We say a root is *not short* if it is either long, or all roots of $\mathfrak{g}$ have the same length.

2 Under the microscope

2.1 Fundamental forms of projective varieties

If one works locally in projective geometry, the most important differential invariant at a point $x \in X \subset \mathbb{P}V$ is the *projective second fundamental form*, which is the first in series of fundamental forms. The *k-th projective fundamental form of X at x*, $\mathbb{FF}^k = \mathbb{FF}^k_{X,x}$ is a map

$$\mathbb{FF}^k : N_k^* \to S^kT_x^*X$$

where T_xX denotes the tangent space, and $N_k = N_{k,x}X$ is the k-th normal space. The notation is such that $T_xX = N_1$.

To define $\mathbb{FF}^k$ and N_k, consider the mapping that takes a curve $\hat{c}(t) \subset \hat{X} \subset V$ such that $[c(0)] = x$ to $c''(0)$. Let $\hat{T}_x{}^{(2)}X$ denote the image of this mapping. To have a linear map, we consider the vector $c'(0) \in \hat{T}_xX$ and its image, which is well defined as an element of $N_2 = N_{2x}X = \hat{T}_x{}^{(2)}X/\hat{T}_xX$. The mapping quotients to a map $T_xX \to N_2$ and this mapping is quadratic. Polarizing it, on obtains the second fundamental form. By considering the k-th deriviative of $\hat{c}$, one obtains $\hat{T}_x{}^{(k)}X$ and by polarizing the degree k map $T_xX \to N_k$, the corresponding k-th fundamental form.

Note that for each $x \in X_{sm}$, the spaces $\hat{T}_x{}^{(k)}X$ determine a flag of V,

$$0 \subset \hat{x} \subset \hat{T}_xX \subset \hat{T}_x{}^{(2)}X \subset \dots \subset \hat{T}_x{}^{(f)} = V.$$

More generally, given a mapping $\phi : Y \to \mathbb{P}V$, one defines the fundamental forms of the mapping ϕ, $\mathbb{FF}^2_\phi$, in the same manner. $\mathbb{FF}^2_{\phi,x}$ quotiented by $\ker \phi_{*x}$ is isomorphic to the second fundamental form of the image, $\mathbb{FF}^2_{\phi(Y),\phi(x)}$. See [19] for details.

In what follows, we will slightly abuse notation by ignoring twists. We will use N_k^* to denote both N_k^* and $N_k^* \otimes \mathcal{O}(-1)$ and use a twist of $\mathbb{FF}^k$. This abuse of notation will be harmless for this paper.

We let $|\mathbb{FF}^k|_x \subset \mathbb{P}S^kT^*_xX$ denote the image of $\mathbb{FF}^k_x$ and Base $|\mathbb{FF}^k| \subset \mathbb{P}T_xX$ denote its *base locus*, the intersection of a $\dim N_k$ dimensional family of hypersurfaces of degree k (see [18] for definitions).

In particular, Base $|\mathbb{FF}^2_{X,x}|$ is an algebraic set defined by quadratic equations. If one hopes to use Base $|\mathbb{FF}^2_{X,x}|$ to study X, it should be a simpler object than X itself. We will show that this is often the case for homogeneous varieties, that in fact, Base $|\mathbb{FF}^2_{X,x}|$ is usually Hermitian symmetric (see Propositions 2.15 and 2.17).

Let V be a vector space and let $A \subset S^dV^*$ be a linear subspace. We define

$$A^{(l)} := (A \otimes S^lV^*) \cap S^{d+l}V^*,$$

the *l-th prolongation* of A, and $Jac(A) := \{v \lrcorner P | v \in V, P \in A\} \subseteq S^{d-1}V^*$, the *Jacobian ideal of* A. Note that $A^{(1)} = \{P \in S^{d+1}V^* | Jac(P) \subset A\}$.

A basic fact about fundamental forms, due to Cartan ([6], p 377) (and rediscovered by Griffiths and Harris), is that if $x \in X$ is a general point, then the *prolongation property* holds:

$$|\mathbb{FF}^k_{X,x}| \subseteq |\mathbb{FF}^{k-1}_{X,x}|^{(1)}$$

(see [13, 18] for modern proofs). We will say *strict prolongation* holds if $|\mathbb{FF}^k_{X,x}| = |\mathbb{FF}^{k-1}_{X,x}|^{(1)}$ for all $k > 2$.

Define the *k-th secant variety of* X by:

$$\sigma_k(X) := \overline{\bigcup_{x_1,\dots,x_k \in X} \mathbb{P}_{x_1,\dots,x_k}}$$

where if $x_1, \dots, x_k \subset \mathbb{P}V$, then $\mathbb{P}_{x_1,\dots,x_k}$ is the projective space they span (a $\mathbb{P}^{k-1}$ when they are linearly independent). A geometric consequence of the prolongation property is as follows:

Proposition 2.1. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety and $x \in X$ a general point. Then*

$$\text{Base}\,|\mathbb{FF}^k_{X,x}| \supseteq \sigma_{k-1}(\text{Base}\,|\mathbb{FF}^2_{X,x}|).$$

Proposition 2.1 is a consequence of the following lemma:

Lemma 2.2. *Let $A \subset S^2V^*$ be a system of quadrics and let $B(A) \subset V$ denote the cone over its base locus: $B(A) = \{v \in V |\ Q(v,v) = 0\ \forall Q \in A\}$. Then for $k \geq 2$,*

$$\sigma_k(B(A)) \subseteq B(A^{(k-1)}).$$

Proof. Let $P \in A^{(1)}$ and let $x, y \in B(A)$. Consider

$$\begin{aligned} P(x+ty, x+ty, x+ty) &= P(x,x,x) + 3tP(x,x,y) + 3t^2P(x,y,y) + t^3P(y,y,y) \\ &= P_x(x,x) + 3t^2P_x(y,y) + 3tP_y(x,x) + t^3P_y(y,y) \quad = 0, \end{aligned}$$

where $P_x \in S^2V^*$ denotes the derivative of P with respect to x. Indeed, saying $P \in A^{(1)}$ is equivalent to demanding all derivatives of P lie in A. This proves our claim for $k = 2$, and the generalization is clear. □

A consequence of the strict prolongation property is the appearance of Koszul complexes:

Proposition 2.3. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety such that strict prolongation holds and let $x \in X$ be a general point. Then there are natural maps $N^*_i \otimes N^*_j \to N^*_{i+j}$.*

Proof. The maps are the restrictions of the symmetrization maps $S^iT^* \otimes S^jT^* \to S^{i+j}T^*$ and the image is assured to lie in N^*_{i+j} by the strict prolongation property. □

Corollary 2.4. *Let $X^n \subset \mathbb{P}^{n+a}$ be a variety such that strict prolongation holds and let $x \in X$ be a general point. Then there is a Koszul complex:*

$$\cdots \longrightarrow N^*_{j-1} \otimes \Lambda^{k+1} T^* \longrightarrow N^*_j \otimes \Lambda^k T^* \longrightarrow N^*_{j+1} \otimes \Lambda^{k-1} T^* \longrightarrow \cdots$$

induced by the maps $T^* \otimes N^*_j \to N^*_{j+1}$ (recall that $T^* = N^*_1$).

For homogeneous spaces of classical groups, this Koszul complex is closely related to the complex that computes, in terms of Koszul cohomology, the syzygies not of Base $|\mathbb{FF}^2_{X,x}|$, but of another variety Z which is in a kind of duality with Y (see remark 3.9).

An elementary fact about projective varieties is that if $X^n \subset \mathbb{P}^{n+a}$ is a variety whose ideal is generated in degree $\leq d$, and L a linear space osculating to order d at a smooth point $x \in X$, then $L \subset X$. The ideal of a homogeneous variety is generated in degree two (see e.g. [24]).

Corollary 2.5. *If $X \subset \mathbb{P}V$ is homogenous, any line osculating to order two is contained in X. Moreover, if $y \in \tilde{T}_x X \cap X$ then the line $\mathbb{P}^1_{xy}$ is contained in X.*

Corollary 2.5 has consequences for the relative differential invariants which we describe in [23]. We will also use the following elementary fact (which follows, e.g., from [20], 3.10, case $k = 2$):

Proposition 2.6. *Let $X \subset \mathbb{P}V$ be a variety. Then the Veronese re-embeddings $v_d(X) \subset \mathbb{P}(S^d V)$ of X, contain no lines.*

2.2 Osculating spaces of homogeneous varieties

Let G be a simply connected complex semi-simple Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^{\otimes}$ the tensor algebra of $\mathfrak{g}$. The *universal envelopping algebra* $U(\mathfrak{g})$ is the quotient of $\mathfrak{g}^{\otimes}$ by the ideal generated by the elements $x \otimes y - y \otimes x - [x, y]$, $x, y \in \mathfrak{g}$. $U(\mathfrak{g})$ inherits a filtration from the natural grading of $\mathfrak{g}^{\otimes}$. Forming the associated graded algebra, we have an isomorphism

$$grad_k U(\mathfrak{g}) = U(\mathfrak{g})_k / U(\mathfrak{g})_{k-1} \simeq S^k \mathfrak{g},$$

where $S^k\mathfrak{g}$ denotes the k-th symmetric power of $\mathfrak{g}$. Fix a maximal torus T and a Borel subgroup B of G containing T. We adopt the convention that B is generated by the negative roots, and we write the corresponding root space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

where Δ_+ denotes the set of positive roots.

Let V_λ be an irreducible G-module with highest weight λ, and $v_\lambda \in V_\lambda$ a highest weight vector. The induced action of $\mathfrak{g}$ extends to the universal envelopping algebra, inducing a filtration of V_λ whose k-th term is

$$V_\lambda^{(k)} = U(\mathfrak{g})_k v_\lambda.$$

Let $x = [v_\lambda]$ be the line in V_λ generated by v_λ, and let $X = G/P \subset \mathbb{P}V_\lambda$ be its G-orbit. Here P is the stabilizer of x, it is a parabolic subgroup of G. The tangent bundle TX is a homogeneous bundle and we identify $T_x X$ with the associated P-module $\mathfrak{g}/\mathfrak{p}$. The osculating spaces and the fundamental forms of X have a simple representation-theoretic interpretation:

Proposition 2.7. *Let $X = G/P \subset \mathbb{P}V$ be a homogeneous variety and let $x \in X$. Let $\hat{T}_x^{(k)}X$ denote the cone over the k-th osculating space at x and let $N_k = \hat{T}_x^{(k)}X/\hat{T}_x^{(k-1)}X$ be the k-th normal space twisted by $\mathcal{O}(-1)$. Then*

$$\hat{T}_x^{(k)}X = V_\lambda^{(k)}, \qquad N_k = V_\lambda^{(k)}/V_\lambda^{(k-1)}.$$

Moreover, there is a commutative diagram

$$\begin{array}{ccc} S^k\mathfrak{g} & = & U(\mathfrak{g})_k/U(\mathfrak{g})_{k-1} \\ \downarrow & & \downarrow \\ {}^t\mathbb{FF}^k_{X,x} : S^kT & \longrightarrow & N_k, \end{array}$$

where the bottom horizontal map is the (twisted) k-th fundamental form at x.

Proof. A curve $\tilde{c}(t) \subset G$ projects to a curve $c(t) \subset X$. Assuming $\tilde{c}(0) = e$ and $x = [v_\lambda]$, the diagram above is the fundamental form of the mapping $\phi : G \to \mathbb{P}V$, where $\phi(G) = X$. (Recall that $T_eG = \mathfrak{g}$). □

Note that $V_\lambda^{(k)}$ has a natural P-module structure. Thus the osculating spaces of X at x correspond to an increasing filtration of P-modules

$$0 \subset \hat{x} \subset \hat{T}_xX = V_\lambda^{(1)} \subset V_\lambda^{(2)} \subset \cdots \subset V_\lambda^{(f)} = Res_P^G V_\lambda.$$

Our next goal is to understand the first quotient of this filtration, namely the structure of the tangent space as a P-module.

2.3 Decomposing the tangent space

In the rest of this paper we assume that P is a maximal parabolic subgroup of G, unless stated otherwise. There exists a simple root α_i such that, up to conjugation, P is generated by the fixed Borel subgroup B and the root groups U_{α_j} associated to the simple roots $\alpha_j \neq \alpha_i$. We will use the notation $P = P_{\alpha_i} = P_i$.

Let $P = LP^u$ be a Levi decomposition of P, where P^u is unipotent and L is reductive and contains the maximal torus T. The Killing form of $\mathfrak{g}$ induces a perfect duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}^u$, which coincides with the P-module structure on T_x^*X at any point $x \in X$.

If α is a positive root, let $\alpha = \sum_j m_j(\alpha)\alpha_j$ be its decomposition in terms of simple roots. Let $\Delta_X = \{\alpha \in \Delta_+ | m_i(\alpha) > 0\}$. We have the root space decompositions

$$\begin{array}{rcl} \mathfrak{p} & = & \mathfrak{t} \oplus (\bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_{-\alpha}) \oplus (\bigoplus_{\alpha\in\Delta_+\setminus\Delta_X} \mathfrak{g}_\alpha), \\ \mathfrak{l} & = & \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta_+\setminus\Delta_X} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \\ \mathfrak{p}^u & = & \bigoplus_{\alpha\in\Delta_X} \mathfrak{g}_{-\alpha}. \end{array}$$

Proposition 2.8. *Let G be simple, let $\tilde{\alpha}$ be the highest root of $\mathfrak{g}$, let α_i be a simple positive root, and let $P = P_{\alpha_i}$ be the associated maximal parabolic subgroup. For $1 \leq k \leq m_i(\tilde{\alpha})$, let*

$$T_k^* = \bigoplus_{m_i(\alpha)=k} \mathfrak{g}_{-\alpha}.$$

Then the sums $S_j^* = \bigoplus_{k\geq j} T_k^*$ define an increasing filtration of T^* by P-submodules, with irreducible quotients. In particular, if H denotes the semisimple part of P, each T_k^* is an irreducible H-module.

Proof. The fact that each S_j^* is a P-module is clear. The irreducibility of T_k^* is a special case of [30] 8.13.3 (which is attributed to Kostant). □

The highest weight ϕ_k of T_k^* is easy to obtain. The irreducibility of T_k^* implies that the set $\{\alpha \in R(\mathfrak{g}) \mid m_i(\alpha) = k\}$ has a unique minimal element (and a unique maximal element as well). Considered as a weight of L, this minimal element is $-\phi_k$. In particular, the highest weight of T_1^* is

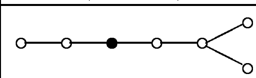
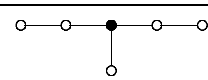
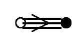
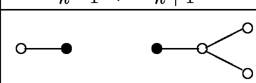
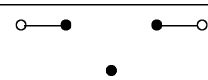
$$\phi_1 = -\alpha_i = -\sum_j n(\alpha_i, \alpha_j)\omega_j,$$

where $n(\alpha_i, \alpha_j)$ denotes the entries of the Cartan matrix. This weight is easy to read directly on the Dynkin diagram of G. Let H denote the semi-simple part of L. Note that L has a one dimensional center and that the Lie algebra of H is

$$\mathfrak{h} = \ker \omega_i \oplus \bigoplus_{\alpha \in \Delta_+ \backslash \Delta_X} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

The Dynkin diagram of H is therefore deduced from that of G by suppressing the node corresponding to the simple root α_i. If $\mathcal{D}(G)$ is simply laced or an arrow points towards the marked node, then ϕ_1 is the sum of the fundamental weights of H corresponding to the nodes of $\mathcal{D}(H)$ that where connected to α_i in $\mathcal{D}(G)$. In the case of a multiple bond with the arrow pointing away from the marked node, one takes the representation pointed at with the multiplicity of the bond. (These observations can be found in [8], they appear to have been already known to Cartan.)

Example 2.9. We illustrate this procedure with the classical roots of D_n, the root α_4 of E_6, and the long simple root of G_2. $\mathcal{D}^*(G)$ denotes the marked Dynkin diagram, that is, the Dynkin diagram of G with the node representing the simple root dual to ω marked in black. In $\mathcal{D}^*(H)$, the black nodes represent ϕ_1.

(G, ω)	(D_n, ω_k)	(E_6, ω_4)	(G_2, ω_2)
$\mathcal{D}^*(G)$			
H	$A_{k-1} \times D_{n-k}$	$A_2 \times A_1 \times A_2$	A_1
ϕ_1	$\omega_{k-1} + \omega_{k+1} =$	$\omega_2 + \omega_3 + \omega_5$	$3\omega_1$
$\mathcal{D}^*(H)$			

In the D_n case, if we label the H modules E^k and U^{2n-2k}, where superscripts denote dimension, then $T_1 = E \otimes U$, $T_2 = \Lambda^2 E$. For the calculation, see §5.

For (E_6, ω_4), the highest root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, so that $T_x^* E_6/P_{\alpha_4}$ has three irreducible components. The highest weight of T_1^* is $-\alpha_4 = \omega_2 + \omega_3 + \omega_5 - 2\omega_4$; the highest weight of T_2^* is $-(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) = \omega_1 + \omega_6 - \omega_4$ and that of T_3^* is $-(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) = \omega_2 - \omega_1$. Thus $\phi_1 = \omega_2 + \omega_3 + \omega_5$, $\phi_2 = \omega_1 + \omega_6$, $\phi_3 = \omega_2$ and the respective dimensions of these three components of T^* are 18, 9 and 2. If we label the H modules A^3, B^2, C^3 (superscripts denote dimensions), then

$$T_1 = A \otimes B \otimes C, \quad T_2 = A^* \otimes C^* = \Lambda^2 A \otimes \Lambda^2 C, \quad T_3 = B.$$

For (G_2, ω_2), we have $\tilde{\alpha} = 3\alpha_1 + 2\alpha_2$, so $T_x^* G_2/P_{\alpha_2}$ has two irreducible components, with respective highest weights $\phi_1 = 3\omega_1$, $\phi_2 = 0$, and respective dimensions 4 and 1. Here $T_1 = S^3 A^2$, $T_2 = \mathbb{C}$. This is an example of the adjoint case (see §4).

We summarize our discussion:

Proposition 2.10. *Let $X = G/P$ be a homogeneous variety with P a maximal parabolic and let H be the semi-simple part of P. Then T_1, the first irreducible component of T_xX as an H-module, is obtained by marking the nodes of $\mathcal{D}(H)$ adjacent to the node from $\mathcal{D}(G)$ that was removed. A node β is given multiplicty two (resp. three) if there is an arrow emanating from α towards β with a double (resp. triple) bond.*

Corollary 2.11. *Notations are as above. Let $Y_1 \subset \mathbb{P}(T_1)$ be the closed orbit. Then Y_1 is isomorphic to the (P, P')-Tits transform of a point $x \in X$, where $X' = G/P'$ is the space of G-homogeneous lines in X.*

The following proposition will be useful later:

Proposition 2.12. *If $X \subset \mathbb{P}V$ is homogeneous, then there are natural maps $T_i^* \otimes T_j^* \to T_{i+j}^*$. Moreover, the map $T_i^* \otimes T_i^* \longrightarrow T_{2i}^*$, is skew-symmetric and thus descends to a map $\Lambda^2 T_i^* \to T_{2i}^*$.*

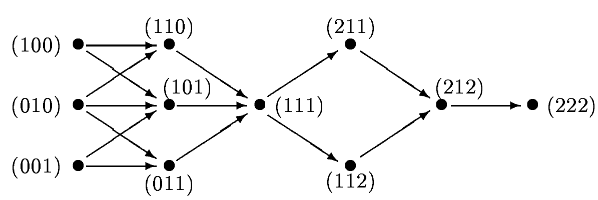
Proof. The map is the Lie bracket which comes from the natural inclusion of $T^* \simeq \mathfrak{p}^u \subset \mathfrak{g}$. □

Remark 2.13. There is a decomposition of the tangent space for arbitrary parabolics similar to the decomposition for maximal parabolics above. If P is an arbitrary parabolic subgroup, then, up to conjugacy, P is generated by a set I of simple positive roots, and there is an irreducible component of the L-module T^* for each choice of the coefficients of the positive roots on these simple roots. If we choose such a family of coefficients $a = (a_i)_{i \in I}$, and let

$$S_a^* = \bigoplus_{\alpha \in \Delta_+,\, m_i(\alpha) \geq a_i} \mathfrak{g}_{-\alpha} \quad \text{and} \quad T_a^* = \bigoplus_{\alpha \in \Delta_+,\, m_i(\alpha) = a_i} \mathfrak{g}_{-\alpha},$$

then S_a^* is a P-submodule of $\mathfrak{p}^u$, and T_a^* is an irreducible L-submodule of S_a^*. An important difference with the case of maximal parabolics is that the incidence relations between the non-zero S_a^*'s is no longer a simple chain of inclusion, but defines a partial order which can be encoded in a finite oriented graph: the vertices of this graph correspond to the families a of coefficients for which S_a^* is non zero, and there is an oriented edge from a to b if and only if there is some $i \in I$ such that $b_i = a_i + 1$, and $b_j = a_j$ for $j \neq i$.

Example 2.14. Consider $G = E_6$, and $P = P_{2,3,5}$ the parabolic subgroup generated by the set $\alpha_2, \alpha_3, \alpha_5$ of simple roots. The associated graph is as follows:



2.4 Second fundamental forms

We begin our calculations of the second fundamental forms of homogeneous varieties by describing an important subvariety of Base $|\mathbb{FF}^2_{X,x}|$.

Proposition 2.15. *Let $X = G/P_i \subset \mathbb{P}V_{\omega_i}$, with G simple and P_i a maximal parabolic subgroup. Let H be the semi-simple part of P_i and let Y_1 be the closed H-orbit in $\mathbb{P}T_1$. Then*

$$Y_1 \subseteq \text{Base}\, |\mathbb{FF}^2_{X,x}|.$$

Proof. $X_{\alpha_i} \in \mathfrak{g}$ verifies $X^2_{\alpha_i} v_{\omega_i} = 0$ by [7], lemme 7.2.4. Thus the line $(1 + tX_{\alpha_i})v_{\omega_i} = \exp(tX_{\alpha_i})v_{\omega_i}$ is contained in X and $X_{\alpha_i} v_{\omega_i} \in \mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$. But $X_{\alpha_i} v_{\omega_i}$ is the highest weight vector of T_1 and $\hat{Y}_1$ is its orbit. □

Given the importance of Y_1, we now study it:

Definition 2.16. *We say that $X = G/P$ is G-*Hermitian symmetric *if X is symmetric as a Riemannian manifold and G is the local automorphism group of X.*

The varieties G/P with G simple that are Hermitian symmetric but not G-Hermitian symmetric are: $C_n/P_{\alpha_1} = A_{2n-1}/P_1 = \mathbb{P}^{2n-1}$, $B_n/P_{\alpha_n} = D_{n+1}/P_{\alpha_{n+1}} = \mathbb{S}_{n+1}$, and $G_2/P_{\alpha_2} = B_3/P_1 = \mathbb{Q}^5$ ([1], Theorem 2 p. 75).

Proposition 2.17. *Let $X = G/P_i \subset \mathbb{P}V_{\omega_i}$, with G simple and P_i a maximal parabolic subgroup. Let H be the semi-simple part of P_i and let Y_1 be the closed H-orbit in $\mathbb{P}(T_1)$. Then Y_1 is a Hermitian symmetric space.*

Moreover, $Y_1 \subset \mathbb{P}(T_1)$ is an H-Hermitian symmetric space in its minimal embedding except in the following cases:

1. C_n/P_k for $k < n$. Here $Y_1 = Seg(\mathbb{P}^{k-1} \times \mathbb{P}^{2n-2k-1})$, $H = SL_k \times Sp_{2n-2k} \subsetneq SL_k \times SL_{2n-2k}$.

2. F_4/P_4. Here $Y_1 = B_3/P_{\alpha_3}$.

3. G_2/P_2. Here $Y_1 = v_3(\mathbb{P}^1)$ is the twisted cubic, which is A_1-Hermitian symmetric, but not in its minimal embedding.

Proof. Y_1 is the H-orbit of X_{α_i} in $\mathbb{P}T_1$, which implies $T_{[X_{\alpha_i} v]} Y_1 = [\mathfrak{h}, X_{\alpha_i}]/\mathbb{C}X_{\alpha_i}$, that is

$$T_{[X_{\alpha_i} v]} Y_1 = \bigoplus_{\beta \in \Delta_+,\, m_i(\beta)=1,\, \beta - \alpha_i \in \Delta_+} \mathfrak{g}_{-\beta}.$$

For a root β in the sum, $m_i(\beta - \alpha_i) = 0$, so that the set of such β is the disjoint union of root subsystems, one for each edge in $\mathcal{D}(G)$ through α_i. H is the product of the simple groups corresponding to each of these root subsystems, and Y_1 is the corresponding Segre product of homogeneous varieties. Because of corollary 3.1 below, we just need to check that there exists no root γ such that $m_i(\gamma) = 0$ and $m_j(\gamma) = 2$ for some simple root α_j connected to α_i, which is a straightforward verification. □

By propositions 2.17 and 2.11, we have the following new result on Tits geometries:

Corollary 2.18. *Let $X = G/P$ with P maximal, and $X' = G/P'$ the space of G-homogenous lines on X. Then the (P, P')-Tits transform of $x \in X$ is a Hermitian symmetric space.*

It turns out that in most cases, $Y_1 = \mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$. More precisely, one of the main results of this paper is the following:

Theorem 2.19. *Let $X = G/P_\alpha \subset \mathbb{P}(V)$ be a homogeneous variety and let $x \in X$. Then*

$$\mathrm{Base}\,|\mathbb{FF}^2_{X,x}| = Y_1$$

if and only if α is not a short root (i.e. no arrow in $\mathcal{D}(G)$ points towards α).

If α is a short root, then $Y_2 \subset \mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$.

The proof of this theorem, which uses results from sections 3-5, is postponed until the end of section 7.

We give explicit descriptions of $\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$ when $X = G/P_\alpha$, with α a short root. For the classical cases (symplectic Grassmannians and odd spinor varieties), see §5. For the three exceptional cases G_2/P_1, F_4/P_4 and F_4/P_3, see §6.

2.5 Y_2 and the third fundamental form

While Y_2 is not in the base locus of the second fundamental form in general, there is strong evidence that it is contained in the base locus of the third fundamental form. Here is a special case:

Proposition 2.20. *Let $X = G/P_{\alpha_i} \subset \mathbb{P}(V)$ be such that $m_i(\tilde{\alpha}) = 2$ (i.e., $T_xX = T_1 \oplus T_2$). Then*

$$Y_2 \subset \text{Base}\,|\mathbb{FF}^3_{X,x}|,$$

where Y_2 is the closed H-orbit in $\mathbb{P}(T_2)$. In particular, if $\text{Base}\,|\mathbb{FF}^2_{X,x}| = Y_1$, *then*

$$\sigma(\text{Base}\,|\mathbb{FF}^2_{X,x}|) \subsetneq (\text{Base}\,|\mathbb{FF}^3_{X,x}|).$$

Proof. Let v be a highest weight vector of V, and $X, Y \in T_1$ such that $X^2v = Y^2v = 0$. We prove that $[X,Y]^3v = 0$.

First note that since $T_3 = 0$, the vector $[X,Y]$, which is an element of T_2, commutes with T, and in particular with X and Y. So that in $U(\mathfrak{g})$, we have the relations

$$XY^2 + Y^2X - 2YXY = YX^2 + X^2Y - 2XYX = 0,$$

from which we deduce that $X^2YXv = Y^2XYv = 0$. Moreover,

$$\begin{aligned}
[X,Y]^2v &= (XY - YX)(XY - YX)v \\
&= (XYXY + YXYX - XY^2X - YX^2Y)v \\
&= -(XYXY + YXYX)v, \\
[X,Y]^3v &= (YX - XY)(XYXY + YXYX)v \\
&= 5(YXYXYX - XYXYXY)v.
\end{aligned}$$

Thanks to the relations in $U(\mathfrak{g})$, we can compute for example $XYXYXYv$ in two different ways:

$$\begin{aligned}
4XYXYXYv &= 2XY(X^2Y + YX^2)Yv \\
&= 2XY^2X^2Yv, \\
4XYXYXYv &= 2(X^2Y + YX^2)YXYv \\
&= 2YX^2YXYv \\
&= YX^2Y^2Xv.
\end{aligned}$$

Using the symmetric expressions for $YXYXYXv$, we get

$$4[X,Y]^3v = 10(YX^2Y^2X - XY^2X^2Y)v = 5(XY^2X^2Y - YX^2Y^2)v = 0.$$

In particular, we can choose $X = X_{\alpha_i}$, and $Y = X_\beta$ with β the highest root such that $m_i(\beta) = 1$. Then $-\beta$ is the lowest weight of T_1, and X_β is therefore H-conjugate to X_{α_i}. In particular, $X_\beta^2 v = 0$, as required. It follows that $X^3_{\alpha_i+\beta}v = 0$, which implies, since $\alpha_i + \beta$ is a positive root with coefficient two on α_i, that $\text{Base}\,|\mathbb{FF}^2_{X,x}| \cap \mathbb{P}(T_2) \neq \emptyset$, and therefore $Y_2 \subset \text{Base}\,|\mathbb{FF}^2_{X,x}|$. ☐

A geometric consequence is that, just as X contains lines tangent to Y_1, at least in this case, it also contains (possibly degenerate) conics tangent to Y_2.

Corollary 2.21. *Let $X = G/P_{\alpha_i} \subset \mathbb{P}(V)$ be such that $m_i(\tilde{\alpha}) = 2$ (i.e., $T_xX = T_1 \oplus T_2$). Then there are plane conics in X tangent to the directions of Y_2, where Y_2 is the closed H-orbit in $\mathbb{P}(T_2)$ at a point $x \in X$.*

In the next two sections we study two special types of our homogeneous spaces. We begin with G-Hermitian symmetric spaces G/P.

3 Hermitian symmetric spaces

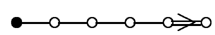
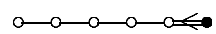
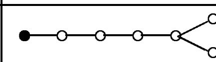
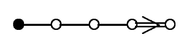
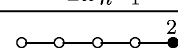
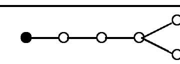
3.1 Their tangent spaces

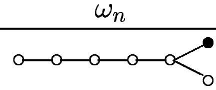
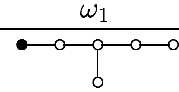
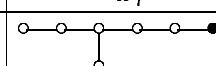
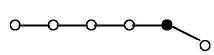
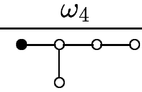
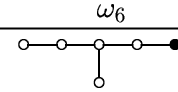
One of the possible characterisations of G-Hermitian symmetric spaces is that their tangent space are irreducible as P-modules ([17], proposition 8.2). Theorem 2.8 implies the following known statement:

Corollary 3.1. *Let G be a simple Lie group and $P = P_{\alpha_i}$ a maximal parabolic subgroup. The following are equivalent:*

1. *G/P is G-Hermitian symmetric,*
2. *$m_i(\tilde{\alpha}) = 1$ (the highest root $\tilde{\alpha}$ has coefficient one on the simple root α_i),*
3. *The fundamental weight ω_i is minuscule,*
4. *$\mathfrak{p}^u$ is an abelian subalgebra of $\mathfrak{g}$.*

Here is a table of the G-Hermitian symmetric spaces, there are four infinite series and two exceptional spaces. While these structures are known (e.g. [11]), we include the chart for completeness:

Name	*Grassmannian*	*Quadric*	*Lagrangian Grassm.*	*Quadric*
Notation	$G(k, n+1)$	$\mathbb{Q}^{2n-1}$	$G_{Lag}(n, 2n)$	$\mathbb{Q}^{2n-2}$
G	A_n	B_n	C_n	D_n
ω	ω_k	ω_1	ω_n	ω_1
$\mathcal{D}(G)$				
H	$A_{k-1} \times A_{n-k}$	B_{n-1}	A_{n-1}	D_{n-1}
ϕ_1	$\omega_{k-1} + \omega_{k+1}$	ω_1	$2\omega_{n-1}$	ω_1
$\mathcal{D}(H)$				
T	$E^* \otimes Q$	$E^* \otimes (E^\perp/E)$	$S^2 Q$	$E^* \otimes (E^\perp/E)$

Name	*Spinor variety*	*Cayley plane*	??
Notation	$\mathbb{S}_n$	$\mathbb{OP}^2$	$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$
G	D_n	E_6	E_7
ω	ω_n	ω_1	ω_7
$\mathcal{D}(G)$			
H	A_{n-1}	D_5	E_6
ϕ_1	ω_{n-2}	ω_4	ω_6
$\mathcal{D}(H)$			
T	$\Lambda^2 E^*$	$\mathcal{S}^+$	$\mathcal{J}_3(\mathbb{O})$

Here E and Q are the tautological and quotient vector bundles on the Grassmannian or their pullbacks to the varieties in question. $\mathcal{S}^+$ is the half spin representation of D_5, and $\mathcal{J}_3(\mathbb{O})$ is the space of 3×3 $\mathbb{O}$-Hermitian matrices, the representation V_{ω_1} for E_6 (see §6.2 for details). $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ is a cryptic advertisement for [22].

3.2 The strict prolongation property

Theorem 3.2. *Let $X = G/P_{\alpha_i} \subset \mathbb{P}V_{\omega_i}$ be an irreducible Hermitian symmetric space in its fundamental embedding. Then for $k \geq 2$,*

$$|\mathbb{FF}^{k+1}| = |\mathbb{FF}^2|^{(k-1)}.$$

Proof. Let T be the tangent space of X at its distinguished point. We denote by $R_k \subset S^kT$ the space of relations of degree k, that is, the space of homogeneous polynomials P_k of degree k in the X_α, with $\alpha \in \Delta_X$, such that if $v = v_{\omega_i} \in V_{\omega_i}$ is the highest weight vector, then $P_k.v \in \hat{T}^{(k-1)}_{[v]}$, the $(k-1)$-st osculating space. We have the following commutative diagram, where horizontal middle long sequence and the vertical short sequences are exact:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & R_{k-1}\otimes\Lambda^2T & \longrightarrow & R_k\otimes T & \longrightarrow & R_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & S^{k-1}T\otimes\Lambda^2T & \longrightarrow & S^kT\otimes T & \longrightarrow & S^{k+1}T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & N_{k-1}\otimes\Lambda^2T & \longrightarrow & N_k\otimes T & \longrightarrow & N_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Lemma 3.3. *$N^*_{k+1} = N^{*(1)}_k$ for all $k \geq 2$ if and only if the relations are generated in degree two, that is, the map $R_k\otimes T \to R_{k+1}$ is surjective for all $k \geq 2$.*

Proof. We first note that $N^*_{k+1} = N^{*(1)}_k$ holds if and only if the sequence

$$N^*_{k+1} \longrightarrow N^*_k\otimes T^* \longrightarrow N^*_{k-1}\otimes\Lambda^2T^*$$

is exact at the middle term. This is because, by definition, $N^{*(1)}_k = (N^*_k\otimes T^*) \cap S^{k+1}T^*$ and $S^{k+1}T^*$ is the kernel of the map $S^kT^*\otimes T^* \to S^{k-1}T^*\otimes\Lambda^2T^*$.

A diagram chase, using the above partially exact diagram, shows that the exactness of the dual sequence

$$N_{k-1}\otimes\Lambda^2T \longrightarrow N_k\otimes T \longrightarrow N_{k+1}$$

is equivalent to the surjectivity of the map $R_k\otimes T \to R_{k+1}$. ☐

Now we analyze the space of relations. By [7] (Lemme 7.2.5 p. 225), the relations all come from the identities

$$X^2_{\alpha_i}v = 0 \quad \text{and} \quad X_\beta v = 0 \ \text{ for } m_i(\beta) = 0.$$

More precisely, if P_k is a homogeneous relation of degree k, there must exist an identity of the following kind in $U(\mathfrak{n})$ (where $\mathfrak{n}$ is the subalgebra of $\mathfrak{g}$ generated by positive root vectors):

$$P_k + Q_{<k} + \sum_{m_i(\beta)=0} R_\beta X_\beta + SX^2_{\alpha_i} = 0,$$

where $Q_{<k}$ is a polynomial of degree less than k in the X_α, $\alpha \in \Delta_X$, and the R_β and S are polynomials in the X_γ, $\gamma \in \Delta_+$.

Now we fix an ordered basis of $\mathfrak{n}$, beginning first with the X_β, $\beta \neq \alpha_i$, such that $m_i(\beta) > 0$, then X_{α_i}, and then continuing with the X_γ for which $m_i(\gamma) = 0$. By the Poincare-Birkhoff-Witt theorem ([7], Théorème 2.1.11 p. 69), the monomials in the X_γ compatible with this order form a basis of $U(\mathfrak{n})$.

We will say that a polynomial expression in the X_γ, $\gamma \in \Delta_+$, is *well-ordered* if each of its monomials is compatible with our ordered basis. We may suppose that in the identity above, all the polynomials P_k, $Q_{<k}$, R_β and S are well-ordered. We may even suppose that the products $R_\beta X_\beta$ are well-ordered, as if they are not, reordering them will give a sum of expressions of the same type, since the space generated by the X_γ for which $m_i(\gamma) = 0$ is stable under the Lie bracket. However, and this is the crucial point, we cannot suppose *a priori* that the product $SX_{\alpha_i}^2$ is also well-ordered.

The conclusion of this analysis is that all relations appear in the following way: we first chose a well-ordered monomial $X_{\beta_1} \cdots X_{\beta_m}$, with $m_i(\beta_1) = \cdots = m_i(\beta_m) = 0$; we reorder its product with $X_{\alpha_i}^2$, which gives an expression of the form:

$$X_{\beta_1} \cdots X_{\beta_m} X_{\alpha_i}^2 = \sum_{\gamma\delta} c_{\gamma\delta} X_{\alpha_i+\gamma} X_{\alpha_i+\delta} + dX_{2\alpha_i+\beta_1+\cdots+\beta_m} + \sum_{m_i(\eta)=0} U_\eta X_\eta,$$

where U_η is some polynomial in the X_γ. We then multiply on the left by a monomial in the X_β with $m_i(\beta) > 0$ and reorder if necessary, then we make linear combinations, and finally, we only keep the homogeneous terms of maximal degree in the resulting expression.

This doesn't seem very enlightening, but when X is Hermitian symmetric the procedure is much simpler. By corollary 3.1, a positive root β can then only have $m_i(\beta) = 0$ or 1 in the Hermitian symmetric case, and an immediate consequence, as we observed, is that when $m_i(\beta) = m_i(\gamma) = 1$, then X_β and X_γ must commute. So the above relation will simplify to an expression of the form

$$X_{\beta_1} \cdots X_{\beta_m} X_{\alpha_i}^2 = \sum_{\gamma\delta} c_{\gamma\delta} X_{\alpha_i+\gamma} X_{\alpha_i+\delta} + \sum_{m_i(\eta)=0} U_\eta X_\eta.$$

Moreover, the relations are then obtained by multiplying the sums $\sum_{\gamma\delta} c_{\gamma\delta} X_{\alpha_i+\gamma} X_{\alpha_i+\delta}$ by monomials in the $X_{\alpha_i+\eta}$, which need no reordering; and finally, the resulting expression is necessarily homogeneous, since we can assume that all its monomials have the same total weight.

This means in particular that all relations are deduced from the degree two relations

$$\sum_{\gamma\delta} c_{\gamma\delta} X_{\alpha_i+\gamma} X_{\alpha_i+\delta} = 0$$

by simple polynomial multiplication in T. Thus the maps $R_2 \otimes S^{k-1}T \to R_{k+1}$ are surjective for $k \geq 2$, which implies surjectivity of $R_k \otimes T \to R_{k+1}$. □

3.3 Their normal spaces

An interesting property of G-Hermitian symmetric spaces is that the irreducibility of the tangent space turns out to imply the irreducibility of all normal spaces. Indeed, the normal spaces and fundamental forms of the Hermitian symmetric spaces are as follows:

Proposition 3.4. *The tangent space T, and the normal spaces N_j, with $2 \leq j \leq l$, of the classical irreducible Hermitian symmetric spaces X in their minimal embeddings, are given by the following table:*

X	$G(k,n)$	$G_{Lag}(n,2n)$	$\mathbb{S}_{2n}$	$\mathbb{Q}^n$
G	SL_n	Sp_{2n}	$Spin_{2n}$	SO_{n+2}
H	$SL_k \times SL_{n-k}$	SL_n	SL_n	SO_n
T	$W_{\omega_{k-1}+\omega_{k+1}} = E^* \otimes Q$	$W_{2\omega_1} = S^2U$	$W_{\omega_{n-2}} = \Lambda^2 U$	W_{ω_2}
N_j	$W_{\omega_{k-j}+\omega_{k+j}} \Lambda^j E^* \otimes \Lambda^j Q$	$W_{2\omega_j} = S_{2\ldots 2}U$	$W_{\omega_{n-2j}} = \Lambda^{2j}U$	$\mathbb{C}$
l	$\min(k, n-k)$	n	$[\frac{n}{2}]$	2

For the two exceptional irreducible Hermitian symmetric spaces, we have the following table:

X	G	H	T	N_2	N_3
$\mathbb{OP}^2$	E_6	$Spin_{10}$	W_{ω_2}	W_{ω_6}	0
$G_\omega(\mathbb{O}^3, \mathbb{O}^6)$	E_7	E_6	W_{ω_6}	W_{ω_1}	$\mathbb{C}$

Remark 3.5. let X, Y be projective varieties. One can compute $\mathbb{FF}^k_{v_d(X)}$, $\mathbb{FF}^k_{Seg(X\times Y)}$ etc... using the fundamental forms of X (and Y) (see [20]). Thus using proposition 3.4 one can calculate the fundamental forms of any Hermitian symmetric space in any homogeneous embedding.

The fundamental forms may be described explicitly as follows:

For a non-degenerate quadric $\mathbb{Q}^n$, the second fundamental form is a nondegenerate quadratic form with base locus a smooth quadric $\mathbb{Q}^{n-2}$.

For the respective cases $G(k,v), G_\omega(k,V), \mathbb{S}, G_\omega(\mathbb{O}^3, \mathbb{O}^6)$, T is a (subset) of a matrix space, respectively $T = E^*\otimes Q, S^2E^*, \Lambda^2E^*, \mathcal{J}_3(\mathbb{O})$. In all but $\mathbb{S}$, the last fundamental form is the set of maximal minors (the determinant for S^2E^* and $\mathcal{J}_3(\mathbb{O})$), and the lower fundamental forms are just the successive Jacobian ideals. For $\mathbb{S}$, the last form is the Pfaffian (since the determinant is a square) and the other forms are the successive Jacobian ideals, which are the Pfaffians of the minors centered about the diagonal.

For the $\mathbb{OP}^2$ case, let $V = \mathbb{C}^{10}$. Then $T = S_+(V)$ is a half-spin representation, and N_2 is the vector representation V. The half-spin representations S_+ and S_- can be constructed as the even and odd parts of the exterior algebra of a null 5-plane E in V (see e.g. [14]): S_+, S_- are dual to one another, the wedge product giving a perfect pairing $S_+ \otimes S_- \longrightarrow \Lambda^5 E = \mathbb{C}$. Moreover, the full exterior algebra of E is a module over the Clifford algebra of V. If F is a complementary null 5-plane of E, then E acts on S_+ by exterior multiplication, F by interior multiplication, and this action of $V = E\oplus F$ extends to the whole Clifford algebra. In particular, there is a natural map from V to $End(S_-, S_+) \simeq S_+\otimes S_+$. The transpose of the symmetric part of this morphism is the second fundamental form.

Alternatively, identifying $S_+(V) = \mathbb{O}\oplus\mathbb{O}$ (see [14]) with octonionic coordinates u, v, we have $|\mathbb{FF}^2_{X,x}| = \{u\overline{u}, u\overline{v}, v\overline{v}\}$ where, considering $\mathbb{O}$ as an eight dimensional vector space over $\mathbb{C}$, the middle equation is eight quadrics.

Remark 3.6. Note that in all cases, the *only* H-orbit closures in $\mathbb{P}T_1$ are the secant varieties. This actually characterizes the Hermitian symmetric spaces. We will have more to say about this in [23]. A special case of this phenomenon is observed in [27].

A corollary of these explicit computations is the following geometric description of the base loci of fundamental forms:

Corollary 3.7. *Let $X = G/P$ be an irreducible G-Hermitian symmetric space in its fundamental embedding. Then the base locus of the second fundamental form is the unique closed P-orbit inside the projectivized tangent space $\mathbb{P}T$.*

Proof of the corollary. In each case, S^2T^* only has two H-irreducible components (where H is the semi-simple part of P): the Cartan product $S^{(2)}T^*$, and the space of quadrics vanishing on Y. The base locus of the second fundamental form, being a proper and non empty subset of $\mathbb{P}T$ defined by an H-module of quadratic equations, is therefore Y. $\square$

Corollary 3.8. *Let X be a Hermitian symmetric space, and let $x \in X$. Then*

$$\text{Base}\,|\mathbb{FF}^k_{X,x}| = \sigma_{k-1}(\text{Base}\,|\mathbb{FF}^2_{X,x}|).$$

Moreover, $|\mathbb{FF}^k_{X,x}| = I_k(\text{Base}\,|\mathbb{FF}^k_{X,x}|)$.

Proof of the corollary. Immediate from our explicit descriptions of the fundamental forms. $\square$

Proof of the proposition. For each of these varieties, and each integer j, there is a unique irreducible H-module which is a component of both S^jT and of the restriction $Res^G_H V_{\omega_i}$. N_j is thus this H-module.

For an ordinary Grassmannian $G(k,n) = G(k,V)$, $T = E^* \otimes Q$, where E is the tautological subbundle and $Q = V/E$ the quotient bundle. Its symmetric powers are given by the Cauchy formula ([26], p. 33)

$$S^jT = \bigoplus_{|\lambda|=j} S_\lambda E^* \otimes S_\lambda Q,$$

the sum is over all partitions λ with the sum of its parts $|\lambda|$ equal to j. We have

$$Res^G_H \Lambda^k V = \Lambda^k(E \oplus Q) = \bigoplus_{h\geq 0} \Lambda^h E^* \otimes \Lambda^h Q = \bigoplus_{h \geq 0} W_{\omega_{k-h}+\omega_{k+h}}$$

since rank $(E) = k$. The only common component of these two decompositions is $\Lambda^j E^* \otimes \Lambda^j Q = W_{\omega_{k-j}+\omega_{k+j}}$. The case of Lagrangian Grassmannians is similar. Here $Q \simeq E^*$, $T = S^2E^*$ and we use the formula ([26], p. 45)

$$S^jT = \bigoplus_{|\lambda|=j} S_{2\lambda} E^*.$$

We compute the decomposition

$$Res^G_H V_{\omega_n} = \bigoplus_{h\geq 0} S_{\underbrace{2\ldots 2}_{h}} E^* = \bigoplus_{h\geq 0} W_{2\omega_h},$$

and the conclusion follows as above. On spinor varieties, $Q \simeq E^*$ again, $T = \Lambda^2 E^*$ and we use the formula ([26], p. 46)

$$S^jT = \bigoplus_{|\lambda|=j} S_{\lambda(2)} E^*,$$

where if $\lambda = (\lambda_1, \ldots, \lambda_m)$, then $\lambda(2) = (\lambda_1, \lambda_1, \ldots, \lambda_m, \lambda_m)$. Finally, the case of quadrics is immediate since they are hypersurfaces.

For exceptional symmetric spaces the same argument goes through, except that we use the LiE package [25], or Littelmann paths, instead of the above classical decomposition formulas. $\square$

Remark 3.9. As we noticed in §2, a consequence of the strict prolongation property is that the osculating spaces fit into complexes

$$\cdots \longrightarrow N^*_{j-1} \otimes \wedge^{k+1} T^* \longrightarrow N^*_j \otimes \wedge^k T^* \longrightarrow N^*_{j+1} \otimes \wedge^{k-1} T^* \longrightarrow \cdots$$

(see corollary 2.4). For example, the Koszul complex associated to the fundamental forms of a Grassmannian is

$$\cdots \longrightarrow \Lambda^j E \otimes \Lambda^j F \otimes \Lambda^k (E \otimes F) \longrightarrow \Lambda^{j+1} E \otimes \Lambda^{j+1} F \otimes \Lambda^{k-1} (E \otimes F) \longrightarrow \cdots$$

Recall that if N_j^* is replaced by the space of sections $\Gamma(X, \mathcal{O}_X(j))$ for a subvariety X of $\mathbb{P}(T)$, the homology of the corresponding Koszul complexes compute the syzygies of X [12].

For a classical Hermitian symmetric space X, there is a strange relation between the complexes constructed from their normal spaces, and the Koszul complexes computing the syzygies of another symmetric space Z. Indeed, we obtain this second family of complexes from the first, by a natural involution on the set of highest weights of irreducible L-modules.

For $L = GL_n$, this involution is defined in the following way: to the Schur power S_λ we associate S_{λ^*}, where λ^* is the conjugate partition of λ, obtained by symmetry along the main diagonal of its diagram (which actually defines a bijection between partitions inscribed in a $k \times (n-k)$ rectangle, and partitions inscribed in a $(n-k) \times k$ rectangle). The complex associated to our example above is therefore

$$\cdots \longrightarrow S^j E \otimes S^j F \otimes \Lambda^k (E \otimes F) \longrightarrow S^{j+1} E \otimes S^{j+1} F \otimes \Lambda^{k-1} (E \otimes F) \longrightarrow \cdots$$

In small degrees, for $X = G(k,n)$, $G_{Lag}(n,2n)$, $\mathbb{S}_n$, we obtain the Koszul complexes associated to $Z = \mathbb{P}^{n-k-1} \times \mathbb{P}^{k-1}$, $G(2,n)$, $v_2(\mathbb{P}^{n-1})$ respectively. (Note that $Y_1 = \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$, $v_2(\mathbb{P}^{n-1})$, $G(2,n)$ respectively.)

4 Adjoint varieties

Let G be any simple complex Lie group, $\mathfrak{g}$ its Lie algebra. The *adjoint variety* of $\mathfrak{g}$ is the (unique) closed G-orbit in $X \subset \mathbb{P}(\mathfrak{g})$. It is the projectivization of the minimal nilpotent orbit in the Lie algebra $\mathfrak{g}$.

Example 4.1. For $\mathfrak{g} = \mathfrak{sl}_n$, the adjoint variety X is the projectivization of the space of rank one and traceless matrices. Up to scalars, such a matrix is specified by its kernel, which is a hyperplane, and its image, which is a line contained in the kernel. Geometrically, $X = \mathbb{F}_{1,n-1}$ is the flag variety in its minimal embedding. For $\mathfrak{g} = \mathfrak{so}_n$, the adjoint variety X is the projectivized space of minimal rank skew symmetric matrices. Geometrically, $X = G_o(2,V)$ is the Grassmannian of isotropic 2-planes in its Plucker embedding. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, the adjoint variety is the projectivized space of rank one symmetric matrices. Geometrically $X = v_2(\mathbb{P}^{2n-1})$, the quadratic Veronese embedding of $\mathbb{P}^{2n-1}$.

4.1 Their tangent spaces

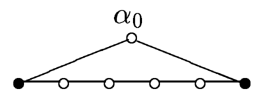
Adjoint varieties are not Hermitian symmetric, but their tangent space T is as close as possible to being irreducible:

Proposition 4.2. *Let $\tilde{\alpha}$ be the highest root of a simple Lie group G. ($\tilde{\alpha}$ is the highest weight of the adjoint representation.) Then the set S of simple roots β such that $\tilde{\alpha} - \beta$ is still a root has one or two elements, and as an H-module, the tangent space of the adjoint variety $X = G/P_{adj}$ is the sum of*

$$T_1 = \bigoplus_{\beta \in S} W_{-\beta} \quad \textit{and} \quad T_2 = \mathbb{C}.$$

Remark 4.3. The set S is directly seen on the *completed Dynkin diagramm* $\mathcal{D}^+(G)$ of G, which is the union of $\mathcal{D}(G)$ and a new vertex (corresponding to $\alpha_0 = \tilde{\alpha}$): S is the set of simple roots corresponding to vertices of $\mathcal{D}(G)$ joined to $\tilde{\alpha}$ in $\mathcal{D}^+(G)$. (The adjoint variety can be thought of as the space of G-homogeneous lines on the - non-existent - homogenous variety G/P_0. See [22] for such geometric descriptions.)

$\mathcal{D}^+(A_n)$



Proof. S is also the set of simple roots β such that $n(\tilde{\alpha}, \beta) \neq 0$. Moreover, by [4], prop. 25 p. 165, we know that for any positive root γ different from $\tilde{\alpha}$, $n(\gamma, \tilde{\alpha})$ is zero or one. In particular, $\beta \in S$ if and only if $n(\beta, \tilde{\alpha}) = 1$, that is, $(\tilde{\alpha}, \tilde{\alpha}) = 2(\beta, \tilde{\alpha})$. But if we write $\tilde{\alpha} = \sum_\gamma n_\gamma \gamma$, this implies that

$$2 = 2\sum_\beta n_\beta(\beta, \tilde{\alpha})/(\tilde{\alpha}, \tilde{\alpha}) = \sum_{\beta \in S} n_\beta.$$

In particular, S has one or two elements. If $S = \{\beta\}$, then $n_\beta = 2$, so that T^* has two irreducible components. Moreover, $\tilde{\alpha}$ is the only positive root with coefficient 2 on the simple root β, since $\tilde{\alpha} - \gamma$ is not a root when the simple root γ is different from β. This implies that $T_2^* = \mathbb{C}$, and $T_1^* = W_{-\beta}$. The proof is the same when S has two elements. $\square$

Definition 4.4. *Let W be a vector space with a symplectic form. A variety $Y \subset \mathbb{P}W$ is* Legendrian *if $\hat{Y} \subset W$ is Lagrangian, i.e., for all $x \in \hat{Y}\backslash 0$, $T_x\hat{Y} \subset W$ is a maximal isotropic subspace.*

The additional structure on the tangent space induced by the Lie bracket of $\mathfrak{g}$ (Proposition 2.12) is particularly interesting in the adjoint case. The following proposition has been observed in [2, 15]. We give an independent proof along the lines of our study:

Proposition 4.5. *The map $\Lambda^2 T_1 \longrightarrow T_2 = \mathbb{C}$ makes T_1 a symplectic H-module. Moreover, if the adjoint representation is fundamental, then Y_1 is a Legendrian subvariety of $\mathbb{P}T_1$.*

In particular, we recover the fact that an adjoint variety has a natural contact structure, induced from the symplectic structure of the corresponding nilpotent orbit.

Proof. A non zero invariant skew-symmetric form on an irreducible module is automatically nondegenerate, which proves that T_1 is a symplectic H-module.

If the adjoint representation is the i-th fundamental representation, that is $\tilde{\alpha} = \omega_i$, this implies in particular that for each positive root β such that $m_i(\beta) = 1$, there exists another such simple root γ with $\beta + \gamma = \tilde{\alpha}$. Note that β and γ cannot be equal, since the root system is reduced. Moreover, we have

$$\langle \beta, \alpha_i \rangle + \langle \gamma, \alpha_i \rangle = \langle \omega_i, \alpha_i \rangle = 1,$$

which implies that one of the integers $\langle \beta, \alpha_i \rangle$ and $\langle \gamma, \alpha_i \rangle$ is positive (let us say the first one), and the other negative or zero. But then $\beta - \alpha_i$ is a root (or zero), while $\gamma - \alpha_i$ is not a root (nor zero).

Thus we have two isomorphic subsets of the set of positive roots α such that $m_i(\alpha) = 1$, depending on whether $\alpha - \alpha_i \in \Delta_+ \cup \{0\}$ or not. Proposition 3.2 contains an explicit formula for the tangent space of Y_1 at $[X_{\alpha_1} v]$, which implies the tangent space at $X_{\alpha_1} v$ of $\hat{Y}_1$ is given by the formula

$$T_{X_{\alpha_1} v}\hat{Y}_1 = \bigoplus_{\substack{\beta \in \Delta_+, m_i(\beta)=1, \\ \beta - \alpha_i \in \Delta_+ \cup \{0\}}} \mathfrak{g}_{-\beta}.$$

This proves that the dimension of $\hat{Y}_1$ is one half of the dimension of T_1, and that its tangent space at $X_{\alpha_1}v$ (hence at every point other than the origin) is isotropic: hence Y_1 is Legendrian. $\square$

The spaces T_1 and varieties Y_1 are given in the following table, where A^2, B^{m-4}, C^{12}, D^{m-4}, W^{56} are the minimal dimensional representations of $SL_2, SO_{m-4}, Sp_6, SL_6, E_7$ respectively, and superscripts denote their dimensions.

G	H	T_1	Y_1
SL_n	SL_{n-2}	$W_{\omega_2}\oplus W_{\omega_{n-2}}$	$\mathbb{P}W_{\omega_2} \sqcup \mathbb{P}W_{\omega_{n-2}}$
Sp_{2n}	SL_{n-1}	$S^2\mathbb{C}^{n-1}$	$v_2(\mathbb{P}^{n-2})$
SO_m	$SL_2 \times SO_{m-4}$	$A\otimes B$	$\mathbb{P}^1 \times \mathbb{Q}^{m-6}$
G_2	SL_2	S^3A	$v_3(\mathbb{P}^1)$
F_4	Sp_6	$\Lambda^{\langle 3\rangle}C$	$G_\omega(3,6)$
E_6	SL_6	$\Lambda^3 D$	$G(3,6)$
E_7	$Spin_{12}$	S_+	$\mathbb{S}_{12}$
E_8	E_7	W	$G_\omega(\mathbb{O}^3,\mathbb{O}^6)$

Remark 4.6. In the Sl_n case, and in that case only, Y_1 is disconnected.

Remark 4.7. $Y_1 \subset \mathbb{P}T_1$ is an irreducible Hermitian symmetric space in its fundamental embedding only for the exceptional groups F_4, E_6, E_7, E_8. We will have more to say about this in [22].

4.2 Their normal spaces

The following can be deduced from [15], proposition 6:

Proposition 4.8. *Let $X \subset \mathbb{P}(\mathfrak{g})$ be the adjoint variety of G. Then*

$$N_2 = \mathbb{C} \oplus \bigoplus_{\beta\in S} W_{-\beta} \oplus \mathfrak{h}, \quad \textit{and} \quad N_k = 0 \text{ for } k > 2.$$

In particular $\operatorname{Base}|\mathbb{FF}^2_{X,x}| = Y_1$.

Remark 4.9. Since the highest root is not short, the fact that $\operatorname{Base}|\mathbb{FF}^2_X| = Y_1$ is a special case of 2.19 if the adjoint representation is fundamental, and we will actually use it to prove 2.19 in §7. The two other cases are $\mathfrak{sl}_n$ where $T = W\oplus W^*\oplus \mathbb{C}$, and $\operatorname{Base}|\mathbb{FF}^2_X| = \mathbb{P}W \sqcup \mathbb{P}W^* \subset \mathbb{P}T$ and $\mathfrak{sp}_{2n}$, where $\operatorname{Base}|\mathbb{FF}^2_X| = \emptyset$ (here $\mathbb{FF}^2_X$ is the complete system of quadrics and the map is an isomorphism).

Remark 4.10. Adjoint varieties are examples of contact Fano manifolds, and it is conjectured that they are the only examples.

5 Classical homogeneous varieties

We have already examined the ordinary Grassmannians, the Lagrangian Grassmannians of maximal isotropic subspaces of a symplectic vector space (they are G-Hermitian symmetric spaces), and the orthogonal Grassmannians of isotropic two-planes with respect to a non-degenerate quadratic form (they are adjoint varieties). We now determine the higher normal spaces of the remaining classical homogeneous varieties. In these cases, the higher normal spaces are more difficult to determine. We give their decompositions into irreducible components.

5.1 Orthogonal Grassmannians

We begin with $G_o(k,n)$, the orthogonal Grassmannian of null k-planes in $V = \mathbb{C}^n$ where V is equipped with a nondegenerate quadratic form and $2k < n$. It is a subvariety of the ordinary Grassmannian, and its minimal embedding is the Plucker embedding in $\mathbb{P}V_{\omega_k} = \mathbb{P}(\Lambda^k V)$.

Proposition 5.1. *Let $E^k \subset E^\perp$ be the two tautological vector bundles on $G_o(k,n)$, and let $U^{n-2k} = E^\perp/E$. Then the tangent space and normal spaces of $G_o(k,n)$, as $H = P^{ss} = SL(E) \times SO(U)$ modules, are*

$$T_1 = E^*\otimes U, \qquad T_2 = \Lambda^2 E^*,$$

$$N_2 = (\Lambda^2 E^*\otimes\Lambda^2 U \oplus S^2E^*) \oplus (\Lambda^2E^*\otimes E^*\otimes U) \oplus (\Lambda^4 E^* \oplus S_{22}E^*).$$

$$N_p = \bigoplus_{a>0}(\Lambda^{p-a}E^*\otimes\Lambda^p E^*\otimes\Lambda^a U)\oplus \oplus(\Lambda^p E^*\otimes\Lambda^p E^*)_+ \oplus (\Lambda^{p-1}E^*\otimes\Lambda^{p-1}E^*)_-.$$

where $(\Lambda^p E^*\otimes\Lambda^p E^*)_+$ and $\Lambda^{p-1}E^*\otimes\Lambda^{p-1}E^*)_-$ are respectively the symmetric and skew-symmetric part of $\Lambda^p E^*\otimes\Lambda^p E^*$ when p is even (resp. odd). In particular, the length of the normal graduation is k when k is even, the last non zero term being $N_k \simeq \Lambda^k(U)$, and $k+1$ when k is odd, the last non zero term being $N_{k+1} \simeq \mathbb{C}$.

Remark 5.2. Note that the quadratic form on U induced by that on V is nondengenerate because E is isotropic. Also remark that the first term in N_2 is a component of S^2T_1, while the second term equals $T_1\otimes T_2$, and the last one, S^2T_2.

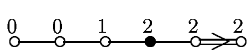
Remark 5.3. We may recover the statement that $N_3^* \neq N_2^{*(1)}$ proved in 2.20 by observing that $N_2^{*(1)}$ contains $S^3T_2^* = S^3(\Lambda^2 E)$, since N_2^* contains $S^2T_2^*$. In particular, it contains $S_{33}E$. But $S_{33}E \not\subset \Lambda^k V$, hence *a fortiori* $S_{33}E \not\subset N_3^*$.

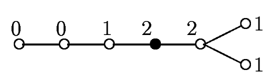
Corollary 5.4.

$$\text{Base}\,|\mathbb{FF}^2_{G_o(k,n),E}| = Y_1 = Seg(\mathbb{P}^{k-1}\times\mathbb{Q}^{n-2k-2}) \subset \mathbb{P}(T_1) \subset \mathbb{P}(T).$$

Proof of the corollary. This is follows from our formula for N_2. It can also be seen directly, recalling that Base $|\mathbb{FF}^2_{G_o(k,n),E}|$ is isomorphic to the set of lines in $G(k,n)$ through E contained in $G_o(k,n)$. As remarked above, the lines in $G(k,n)$ are defined by some $(k-1)$-plane $H \subset E$, and some $(k+1)$-plane K containing E. For a line to be included in the orthogonal Grassmannian, K must be isotropic, which means that the line K/E is isotropic in U, i.e., K/E defines a point of the quadric hypersurface $\mathbb{Q}^{n-2k-2} \subset \mathbb{P}(U)$. □

Proof of the proposition. The decomposition of the tangent space follows from Theorem 1.2, since the minimal roots with coefficient 2 on α_k are, for types B and D,



and


which in both cases are equal to ω_{k-1}.

We use the Maurer-Cartan form to compute N_2. We fix a basis $e_1,\dots,e_k$ of some isotropic k-plane $E \in G_o(k,n)$, and we complete it with a basis $e_{k+1},\dots,e_{n-k}$ of U, and a basis $e_{n-k+1},\dots,e_n$ of $E^* \simeq V/E^\perp$ Q-dual to the basis of E. A tangent vector w may be seen as a map from E to U, plus a skew-symmetric map from E to E^*. (And the using the induced inner product on U, the first map induces a dual map from U to E^*.) More explicitly, let J_k denote the order k antidiagonal matrix. The matrix of the quadratic form in our given basis is

$$Q = \begin{pmatrix} 0 & 0 & J_k \\ 0 & J_{n-2k} & 0 \\ J_k & 0 & 0 \end{pmatrix}, \quad \text{and} \quad w = \begin{pmatrix} 0 & 0 & 0 \\ \sigma & 0 & 0 \\ \tau & \sigma^t & 0 \end{pmatrix},$$

where τ is skew-symmetric and σ^t denotes the transpose matrix of σ. N_2 is generated by the vectors $w^2(e_1 \wedge \cdots \wedge e_k) \bmod w(e_1 \wedge \cdots \wedge e_k)$. There are four types of such tensors, corresponding to the following patterns:

$$E \xrightarrow{\sigma,\sigma} U \qquad E \xrightarrow{\sigma} U \xrightarrow{\sigma^t} E^* \qquad E \begin{smallmatrix} \nearrow^{\sigma} U \\ \searrow_{\tau} E^* \end{smallmatrix} \qquad E \xrightarrow{\tau,\tau} E^*$$

Tensors of the first kind are of the following form:

$$\sum_{i_1,i_2 \le k < j_1,j_2 \le n-2k} \sigma_{i_1}^{j_1} \sigma_{i_2}^{j_2} e_1 \wedge \cdots \wedge e_{j_1} \wedge \cdots \wedge e_{j_2} \wedge \cdots \wedge e_k,$$

where e_{j_1} (resp. e_{j_2}) appears in place of e_{i_1} (resp. e_{i_2}). This expression can be identified with

$$\sum_{i_1,i_2 \le k < j_1,j_2 \le n-2k} \sigma_{i_1}^{j_1} \sigma_{i_2}^{j_2} (e_{i_1} \wedge e_{i_2})^* \otimes e_{j_1} \wedge e_{j_2},$$

and such tensors generate $\Lambda^2 E^* \otimes \Lambda^2 U \subset S^2 T_1$. Tensors of the second kind are of the form

$$\sum_{i_1,i_2 \le k < j \le n-2k} \sigma_{i_1}^{j} \sigma_{i_2}^{n-k-j+1} e_1 \wedge \cdots \wedge e_{i_1-1} \wedge e_{2n-i_2+1} \wedge e_{i_1-1} \wedge \cdots \wedge e_k.$$

Again, such an expression can be identified with

$$\sum_{i_1,i_2 \le k < j \le n-2k} \sigma_{i_1}^{j} \sigma_{i_2}^{n-k-j+1} e_{i_1}^* \otimes e_{i_2}^*,$$

and is symmetric in i_1 and i_2. These tensors therefore generate $S^2E^* \subset S^2E^* \otimes S^2U \subset S^2T_1$.

Tensors of the third kind generate some subspace of $\Lambda^2 E^* \otimes E^* \otimes U$, this subspace being determined by the additional skew-symmetry in E. But this additional (skew-)symmetry is only apparent, and tensors possessing this symmetry actually generate $\Lambda^2 E^* \otimes E^* \otimes U$. To see this, we note that $\Lambda^2 E^* \otimes E^* \otimes U$ has only two irreducible components, and these components are not isomorphic. The simplest is $\Lambda^3 E^* \otimes U$, which is the image of $\Lambda^2 E^* \otimes E^* \otimes U$ by complete skew-symmetrization. But this skew-symmetrization does not cancel our tensors, so that they generate the component $\Lambda^3 E^* \otimes U$ in $\Lambda^2 E^* \otimes E^* \otimes U$. But since they are not completely skew-symmetric, they generate more, and therefore they must generate the other irreducible component as well.

The argument is similar for tensors of the fourth kind inside $S^2 T_2 = \Lambda^4 E^* \oplus S_{22} E^*$. Our claim is that they generate the whole of $S^2 T_2$.

To prove the claim, we check that their skew-symmetrization does not always give zero, and that they are not completely skew-symmetric.

To compute the higher normal spaces, we will need to separate the irreducible H-components of $\Lambda^k V$. The following lemma will be useful:

Lemma 5.5. *Let m, n, r be non negative integers, with $m \geq r$, and let A be a vector space of dimension at least $m+n+r$. Define a $GL(A)$-equivariant map*

$$\phi_{m,n,r} : \Lambda^m A \otimes \Lambda^{m+n} A \to (S^2 A)^{\otimes m-r} \otimes \Lambda^{n+2r} A$$

by

$$f_1 \wedge \cdots \wedge f_m \otimes g_1 \wedge \cdots \wedge g_{m+n} \mapsto \sum_{\sigma \in \mathcal{S}_m, \tau \in \mathcal{S}_{m+n}} \varepsilon(\sigma)\varepsilon(\tau) f_{\sigma(1)} g_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(m-r)} g_{\sigma(m-r)} \otimes \\ \otimes f_{\sigma(m-r+1)} \wedge \cdots \wedge f_{\sigma(m)} \wedge g_{\tau(m-r+1)} \wedge \cdots \wedge g_{\tau(m+n)}.$$

Then $\phi_{m,n,r}$ is non zero. Moreover, Image($\phi_{m,n,r}$) and coker($\phi_{m,n,r}$) are isomorphic to the unique irreducible component in its source and target which has highest weight $\omega_{m-r} + \omega_{m+n+r}$.

Proof of the lemma. Let $f_1, \ldots, f_{m+n+r}$ be independent vectors in A. Consider

$$\phi_{m,n,r}(f_1 \wedge \cdots \wedge f_m \otimes f_1 \wedge \cdots \wedge f_{m-r} \wedge f_{m+1} \wedge \cdots \wedge f_{m+n+r})$$

There is a term $f_1^2 \otimes \cdots \otimes f_{m-r}^2 \otimes f_{m-r+1} \wedge \cdots \wedge f_{m+n+r}$ with non-zero coefficient in the image tensor. Hence $\phi_{m,n,r}$ is not the zero map. But it is a consequence of Pieri's rules that $\Lambda^m A \otimes \Lambda^{m+n} A$ and $(S^2 A)^{\otimes m-r} \otimes \Lambda^{n+2r} A$ have a unique irreducible component in common, which has weight $\omega_{m-r} + \omega_{m+n+r}$ and multiplicity one. We conclude by Schur's lemma. □

We now determine all the normal spaces, as H-modules. Their sum must coincide with the whole representation $\Lambda^k V$, that is,

$$Res_H^G \Lambda^k V = \Lambda^k(E \oplus U \oplus E^*) = \bigoplus_{a,b} \Lambda^b E^* \otimes \Lambda^{a+b} E^* \otimes \Lambda^a U.$$

By Pieri's formula,

$$\Lambda^b E^* \otimes \Lambda^{a+b} E^* = \bigoplus_{c=0}^{b} S_{\underbrace{2 \ldots 2}_{b-c} \underbrace{1 \ldots 1}_{a+2c}} E^*.$$

For $a = 0$, we get the sum of two modules $(\Lambda^p E^* \otimes \Lambda^p E^*)_+$ and $(\Lambda^p E^* \otimes \Lambda^p E^*)_-$, given by the same formula but with $b - c$ even or odd, respectively. These modules are the symmetric (resp. skew-symmetric) part of $\Lambda^p E^* \otimes \Lambda^p E^*$ when p is even (resp. odd).

A pleasant feature of the decomposition of $\Lambda^k V$ into irreducible H-modules is that they all have multiplicity one, as follows from the above formulas. We thus need to find the minimal value of p for each component such that the component is contained in the p-th osculating space.

The p-th osculating space is generated by tensors of the form

$$T = \sum_{t,u,v,w} \Theta_{tu} \Omega_{vw} (e_s \wedge e_t \wedge e_v) \otimes (e_u \wedge e_w),$$

where s is a multi-index of length a (and $e_s = e_{s_1} \wedge \cdots \wedge e_{s_a}$), t and u have length b, v and w have length c. Moreover, Θ_{tu} and Ω_{vw} are respectively symmetric and skew-symmetric expressions that can be chosen arbitrarily.

We need to determine if such tensors generate the component of highest weight $\omega_{b+c-r} + \omega_{a+b+c+r}$ in $\Lambda^{a+b+c} E^* \otimes \Lambda^{b+c} E^*$. Moreover, we want to have $p = a + 2b + c$ as small as possible, that is b as small as possible, since $b + c$ and a are fixed. We will prove that it is possible already for $b = 0$, except when $a = 0$ and $b + c - r$ is odd, in which case we need at least $b = 1$.

We first suppose that $b = 0$. We want to prove that the map $\phi_{a,c,r}$ of lemma 5.3 is not identically zero on tensors of the form

$$T = \sum_{v,w} \Omega_{vw}(e_s \wedge e_v) \otimes e_w,$$

where Ω_{vw} is skew-symmetric. We need a lemma.

Lemma 5.6. *The map $\phi_{0,c,r}$ vanishes identically on tensors $T = \sum_{vw} \Omega_{vw} e_v \otimes e_w$, with Ω_{vw} skew-symmetric, if and only if $c - r$ is odd.*

Proof of the lemma. Each expression $\sum_{v_i,w_i} \Omega_{v_i w_i} e_{v_i} \otimes e_{w_i}$ can be considered as an element of $\Lambda^2 E^*$. Moreover, when we permute the indices $i = 1, ..., c$, the expression $\sum_{vw} \Omega_{vw} e_v \otimes e_w$ remains unchanged. We can therefore consider the tensor T as an element of $S^c(\Lambda^2 E^*)$. The map $\phi_{0,c,r}$ must vanish identically on T when $c - r$ is odd because $S^c(\Lambda^2 E^*)$ has an irreducible component of highest weight $\omega_{c-r} + \omega_{c+r}$ if and only if $c - r$ is even.

We now consider the case where $c - r$ is even. We begin with the case where $c = 2$ and $r = 0$. Here

$$\phi_{0,2,0}(T) = \sum_{v_1,v_2,w_1,w_2} \Omega_{v_1 w_1} \Omega_{v_2 w_2} e_{v_1} e_{w_2} \otimes e_{v_2} e_{w_1},$$

which is non-zero in general. The general case follows because the image of a general tensor will be a sum of products of such expressions plus terms of a different kind. $\square$

Suppose that $c - r$ is even. Then $\phi_{a,c,r}(T)$ contains a sum

$$\sum \Omega_{v_1 w_1} \cdots \Omega_{v_a w_a} e_{s_1} e_{w_1} \otimes \cdots \otimes e_{s_a} e_{w_a} \otimes \phi_{0,c,r}(T_{>a}) \wedge e_{v_1} \wedge \cdots \wedge e_{v_a}$$

plus other terms of a different kind, and the above lemma therefore implies that $\phi_{a,c,r}(T) \neq 0$ in general. For $c - r$ odd, these terms will cancel out, but if $a > 0$, then $\phi_{a,c,r}(T)$ also contains terms of type

$$\sum \Omega_{v_1 w_1} \cdots \Omega_{v_{a-1} w_{a-1}} e_{s_1} e_{w_1} \otimes \cdots \otimes e_{s_{a-1}} e_{w_{a-1}} \otimes \phi_{0,c,r}(T_{\geq a}) \wedge e_{v_1} \wedge \cdots \wedge e_{v_{a-1}} \wedge e_{s_a}$$

plus other terms of a different kind, and again we see that $\phi_{a,c,r}(T) \neq 0$ in general.

This proves two thirds of our claim, namely that for $b = 0$ we can generate the components we wish, except for $a = 0$ and $c - r$ odd. There just remains to check that we can generate the remaining components with $b = 1$. But this is clear, since $\phi_{a,c+1,r}(T)$ is equal to

$$\sum_{tu} \Theta_{tu} e_t e_u \otimes \phi_{a,c,r}(\sum_{vw} \Omega_{vw} e_v \otimes e_w)$$

plus other terms of a different kind, and is therefore non-zero in general. $\square$

5.2 Symplectic Grassmannians

The case of symplectic Grassmannians $G_\omega(k, 2n)$ is similar to that of orthogonal Grassmannians. One difference is that the minimal embedding of $G_w(k, 2n)$ is to a quotient of $\mathbb{P}(\Lambda^k \mathbb{C}^{2n})$, namely to $V_{\omega_k} = \Lambda^{\langle k \rangle} V = \Lambda^k V / \Omega \wedge \Lambda^{k-2} V$ is the k-th reduced exterior power of $V = \mathbb{C}^{2n}$.

A straightforward computation shows that V_{ω_k} has the following decomposition as an $H = SL_k \times Sp_{2n-2k} = P_k^{ss}$-module:

$$\Lambda^{\langle k \rangle} V = \bigoplus_{a,b} \Lambda^b E^* \otimes \Lambda^{a+b} E^* \otimes \Lambda^{\langle a \rangle} U.$$

Note that $U = E^{\perp}/E$ is endowed with a symplectic form induced by the symplectic form on $V = \mathbb{C}^{2n}$.

Proposition 5.7. *Let $E \subset E^{\perp}$ be the two tautological vector bundles on $G_\omega(k,n)$, and $U = E^{\perp}/E$. Then the tangent space and normal spaces of $G_\omega(k,n)$ are, as H-modules,*

$$T_1 = E^* \otimes U, \qquad T_2 = S^2 E^*,$$

$$N_2 = \Lambda^2 E^* \otimes \Lambda^{\langle 2 \rangle} U \oplus S_{21} E^* \otimes U \oplus S_{22} E^*,$$

$$\begin{aligned} N_p &= \bigoplus\nolimits_{a+b+c=p} \Lambda^{\langle a \rangle} U \otimes S_{\underbrace{2\dots2}_{b-c}\underbrace{1\dots1}_{a+2c}} E^* \\ &= \bigoplus\nolimits_{d+e=p} \Lambda^{d} U \otimes S_{\underbrace{2\dots2}_{e}\underbrace{1\dots1}_{d}} E^*. \end{aligned}$$

In particular, the length of the normal graduation is equal to k, the last non zero term being $N_k \simeq \Lambda^k(\mathbb{C} \oplus U)$.

The proof is similar to, but easier than, the orthogonal case.

Corollary 5.8.

$$\mathrm{Base}\,|\mathbb{FF}^2_{G_\omega(k,n),E}| = \overline{\mathbb{P}\{e \otimes u \oplus e^2 \mid e \in E^* \backslash \{0\},\ u \in U \backslash \{0\}\}}.$$

$\mathrm{Base}\,|\mathbb{FF}^2_{G_\omega(k,n),E}|$ *contains an open and dense H-orbit, the boundary of which is the union of the two (disjoint) closed H-orbits*

$$Y_1 \simeq \mathbb{P}^{k-1} \times \mathbb{P}^{2n-2k-1} \subset \mathbb{P}(T_1) \quad \text{and} \quad Y_2 \simeq v_2(\mathbb{P}^{k-1}) \subset \mathbb{P}(T_2).$$

We may also view $\mathrm{Base}\,|\mathbb{FF}^2_{G_\omega(k,n),E}|$ *as the total space of the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}E}(1) \otimes U \oplus \mathcal{O}_{\mathbb{P}E}(2))$ over $\mathbb{P}E$.*

Proof. This follows from our calculation of N_2 and also can be seen directly. A line in $G_\omega(k,n)$ through a point E is given by a $(k-1)$-plane $H \subset E$, and a $(k+1)$-plane $K \supset E$. K does not need to be isotropic, each point of the correponding line will be generated by H and a vector of K, and will be isotropic if and only if this vector is orthogonal to H (with respect to the symplectic form). The condition on K is thus $K \subset H^{\perp}$. K is therefore determined by a line in $H^{\perp}/E \simeq U \oplus H^{\perp}/E^{\perp}$.

If $e \in E^*$ is an equation of H, the line $H^{\perp}/E^{\perp} \simeq (E/H)^* \subset E^*$ is generated by E, so that a vector in $H^{\perp}/E$ can be written as $u \oplus \lambda e$, where $u \in E$. Our claim follows, the closed orbits Y_1 and Y_2 corresponding to the cases where u or λ is equal to zero. ☐

Remark 5.9. For C_n/P_l, the G-homogeneous linear spaces form a special class of linear spaces. The varieties parametrizing the unirulings by $\mathbb{P}^k$'s are also stratified by intermediate spaces in between these special classes and the generic. These intermediate spaces can be characterized by the extent they are integral for $\mathcal{Y}_1$. They may be integral for $\mathcal{Y}_1$ along a subspace, or only partially integral for $\mathcal{Y}_1$ along a subspace. Consider the case of $\mathbb{P}V = C_n/P_1$. There are $k+1$ types of $\mathbb{P}^k$'s, indexed by $\dim \mathbb{P}^k \cap \mathbb{P}^{k\perp}$, having various tangency properties with respect to $\mathcal{Y}_1$.

5.3 Odd spinor varieties

We call odd spinor varieties the homogeneous spaces B_n/P_n, i.e. the usual spinor varieties seen as B_n-homogeneous spaces.

Proposition 5.10. *The tangent space and normal spaces of the odd spinor varieties B_n/P_n, as $H = A_{n-1}$-modules, are* $(\dim E = n)$*:*

$$\begin{array}{rcl} T_1 & = & E^*, \qquad T_2 \ = \ \Lambda^2 E^*, \\ N_p & = & \Lambda^{2p-1}E^* \oplus \Lambda^{2p}E^*. \end{array}$$

Corollary 5.11.

$$\text{Base}\,|\mathbb{FF}^2_{B_n/P_n}| = \mathbb{P}\overline{\{e \oplus e \wedge f,\ e \in E^*\backslash\{0\},\ f \in E^*\backslash\mathbb{C}e\}}.$$

Base $|\mathbb{FF}^2_{B_n/P_n}|$ *contains an open and dense H-orbit, the boundary of which is the union of the two (disjoint) closed H-orbits*

$$Y_1 = \mathbb{P}E^* = \mathbb{P}(T_1) \quad \textit{and} \quad Y_2 \simeq G(2, E^*) = G(2, n-1) \subset \mathbb{P}(T_2).$$

This base locus is actually homogeneous, it is $G(2,n)$ in disguise.

Proof. Consider $V^{2n+1} \subset W^{2n+2}$, and $E^n \subset F^{n+1}$, where F is a null plane in W. let $L = E^\perp \subset F$ Then $\Lambda^\bullet E^* \simeq \Lambda^{even}F^*$. We may write $N_2 = \Lambda^3 E^* \otimes L \oplus \Lambda^4 E^* = \Lambda^4 F = I_2(G(2,F))$. The analogous identities hold for the higher normal spaces. After all, this is the same projective variety as D_n/P_n. □

Remark 5.12. The varieties parametrizing the unirulings of $B_n/P_n = D_{n+1}/P_{n+1}$ are stratified as B_n-spaces. For $\mathbb{P}^1$'s, they are D_{n+1}/P_{n-1}. There are two D_n spaces of $\mathbb{P}^2$'s, D_{n+1}/P_{n-2} and $D_{n+1}/P_{n-3,n}$.

For $k > 2$, there is a unique space $D_n/P_{n-k-1,n}$. In these cases, there is a preferred variety of $\mathbb{P}^k$'s that is a homogeneous variety of B_n, namely $B_n/P_{n-k-1,n}$ which is *not* B_n-homogeneous in the sense of Tits. To see this space, let $\tilde{V} = \mathbb{C}^{2n+2}$ have a nondegenerate quadratic form and let V be a hyperplane. A k-plane on D_{n+1}/P_{n+1} is given by a flag $F^{n-k-1} \subset H^{n+1}$ with $F \in D_n/P_{n-k-1} = G_o(n-k-1, \tilde{V})$ and $H \in D_{n+1}/P_n$, that is, the other spinor variety. The corresponding $\mathbb{P}^k$ is

$$\{E \in D_{n+1}/P_{n+1} \mid F \subset E \text{ and } \dim(E \cap H) = n\}.$$

To see this on B_n/P_n, we simply intersect these E with V. Our preferred family of $\mathbb{P}^k$'s correspond to the case where $F \subset V$, in which (F, H) descends to an element $(F, H \cap V)$ of $B_n/P_{n-k-1,n}$.

6 Exceptional short roots and the octonions

In this section we calculate Base $|\mathbb{FF}^2_{X,x}|$ for the exceptional spaces corresponding to short roots. (The cases of C_n/P_k and B_n/P_n are treated in §5.) We include interpretations of these spaces and their base loci in terms of the octonions. More detailed explanations and a fuller discussion of geometric interpretations of the exceptional spaces in terms of octonionic geometry will be given in [22].

6.1 $G_2/P_1 = \mathbb{P}(Im\mathbb{O})_0$

As an algebraic variety, G_2/P_1 is a familiar space, $G_2/P_1 = \mathbb{Q}^5 \subset \mathbb{P}^6$. Studying it from an octonionic perspective will help us to understand F_4/P_4 by analogy.

Identify $\mathbb{C}^7 \simeq Im\mathbb{O} = V_{\omega_1} = V$. Let $\phi \in \Lambda^3 V$ be a generic element and let $\rho : GL(V) \to GL(\Lambda^3 V^*)$ be the induced representation. Here are some descriptions of $G_2 \subset GL(V)$ (see [14] pp. 114, 116, 278):

$$\begin{aligned} G_2 &= \{g \in GL(V) \mid \rho(g)\phi = \phi\} \\ &= \mathrm{Aut}(\mathbb{O}) \\ &= \{g = (g_+, g_-, g_0) \in \mathrm{Spin}_8(V) \mid g_+ = g_- = g_0\}. \end{aligned}$$

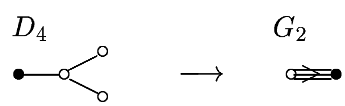
The third line should be understood as follows, let S_+, S_-, V_0 denote the vector and two spin representations of $Spin_8$ and choose appropriately an identification of the three spaces, so that they are acted on by $g \in Spin_8$ in three different ways, call them (g_+, g_-, g_0). In this case (see [14] p. 278), the *triality principle* of E. Cartan leads to the identification

$$Spin_8 = \{(g_+, g_-, g_0) \in SO(\mathbb{O}) \times SO(\mathbb{O}) \times SO(\mathbb{O}) \mid g_+(uv) = g_-(u) g_0(v)\},$$

where uv denotes octonionic multiplication. When the three coincide one obtains an automorphism of the octonions, showing the equivalence of the second and third definition. Harvey's description is explicit in bases. One may also use geometric folding to obtain this description, see [22]. The connection between the first two interpretations is that if one makes suitable identifications, for $u, v, w \in$ Im$\mathbb{O}$, we have $\phi(u, v, w) = Re[(uv)w]$.

The first definition is due to Bryant. It shows that G_2 is not really an exceptional group, because it is defined by a generic form. (Generic three forms on $\mathbb{C}^m$ for $m > 8$ are not preserved by a positive dimensional group. For $m = 6, 8$, the groups preserving such a form are classical.)

The third interpretation can be understood in terms of geometric folding:



Geometric folding indicates that G_2/P_1 should be be understandable in terms of $D_4/P_1 = \mathbb{Q}^6$ and in fact it is a generic hyperplane section. Im$\mathbb{O} \subset \mathbb{O}$ should be thought of as the *traceless elements*, where the trace of an element is its "real" part and we call the hyperplane section $\{\mathrm{tr} = 0\}$.

In what follows, uv etc... refers to octonionic multiplication.

Proposition 6.1. *Consider* $\mathbb{Q}^5 \simeq G_2/P_1 = \mathbb{P}(Im\mathbb{O})_0 \subset \mathbb{P}(Im\mathbb{O}) \simeq \mathbb{P}(V_{\omega_1})$.

Then $T_x G_2/P_1 = T_1 \oplus T_2 \oplus T_3$ *as a* $P_1^{ss} = SL_2$*-module. Let* $A = \mathbb{C}^2$*, the standard representation of* SL_2*. Then* $T_1 = A$*,* $T_2 = \mathbb{C}$ *(the trivial representation) and* $T_3 = A^*$*. Moreover, in a suitable normalization,*

$$\mathrm{Base}\, |\mathbb{FF}^2_{G_2/P_1}| = \mathbb{P}\{a \oplus t \oplus a^* \mid \langle a, a^* \rangle = t^2\}.$$

Proposition 6.2. *We have the following octonionic interpretations:*

$$\begin{aligned} G_2/P_1 &= \mathbb{P}(Im\mathbb{O})_0 = \{[u] \in \mathbb{P}(Im\mathbb{O}) \mid u^2 = 0\} \\ \hat{T}_{[u]}\mathbb{P}(Im\mathbb{O})_0 &= \{v \in Im\mathbb{O} \mid uv + vu = 0\} = \{v \in Im\mathbb{O} \mid Re(uv) = 0\} \\ \hat{T}_{[u]1}\mathbb{P}(Im\mathbb{O})_0 &= \{v \in Im\mathbb{O} \mid uv = 0\}. \end{aligned}$$

Proposition 6.3. *The space $\mathbb{G}(\mathbb{P}^1, \mathbb{P}(Im\mathbb{O})_0)$ of lines on G_2/P_1, has the following description:*

$$\mathbb{G}(\mathbb{P}^1, \mathbb{P}(Im\mathbb{O})_0) = \{\mathbb{P}\{u,v\} \mid [u],[v] \in G_2/P_1 \text{ such that } uv + vu = 0\}.$$

Note that $\mathbb{G}(\mathbb{P}^1, \mathbb{P}(Im\mathbb{O})_0) = G_o(2,7)$ and in particular is of dimension seven.

The space $G_2/P_2 = \mathbb{G}_0(\mathbb{P}^1, \mathbb{P}(Im\mathbb{O})_0)$ of G_2-homogeneous lines on G_2/P_1 has dimension 5, and admits the following descriptions:

i. $G_2/P_2 = \{\mathbb{P}E \in \mathbb{G}(\mathbb{P}^1, \mathbb{P}V_{\omega_1}) \mid E \lrcorner \phi = 0\}$

ii. $G_2/P_2 = \{\mathbb{P}E = \mathbb{P}\{u,v\} \mid [u],[v] \in G_2/P_1,\ uv = 0\}$.

Proofs are left to the reader. The arguments are similar to, but simpler than the arguments for the F_4/P_4 case below.

Remark 6.4. Recall that V_{ω_2} is the adjoint representation of G_2, so that G_2/P_2 is an adjoint variety. The relation with our description of G_2/P_2 as a space of special lines on the quadric G_2/P_1 is as follows. Let $[u],[v] \in G_2/P_1$ be such that $uv = vu = 0$. One can then check that the map

$$d_{u,v}(z) = u(vz) - v(uz), \quad z \in \mathbb{O},$$

defines a nilpotent derivation of $\mathbb{O}$,with $d_{u,v}^2 = 0$.

Remark 6.5. There is a distinguished family of $\mathbb{P}^2$'s on G_2/P_1 parametrized by G_2/P_1, namely, given $[u] \in G_2/P_1$ one takes the $\mathbb{P}^2_u$ such that $T_{[u]}\mathbb{P}^2_u = (T_1)_{[u]}$. Note that for all $[v] \in \mathbb{P}^2_u$, $T_{[v]}\mathbb{P}^2_u$ contains a line that is contained in $(T_1)_{[v]}$, but only at $[u]$ is the entire tangent space contained in T_1. This family is not deducible from Tits methods.

6.2 The Cayley plane $\mathbb{OP}^2$

Let $\mathcal{J}_3(\mathbb{O})$ be the space of 3×3 $\mathbb{O}$-Hermitian matrices

$$\mathcal{J}_3(\mathbb{O}) = \left\{ A = \begin{pmatrix} r_1 & \overline{x_3} & \overline{x_2} \\ x_3 & r_2 & \overline{x_1} \\ x_2 & x_1 & r_3 \end{pmatrix}, r_i \in \mathbb{C}, x_j \in \mathbb{O} \right\}.$$

$\mathcal{J}_3(\mathbb{O})$ can be equipped with the structure of a *Jordan algebra* for the commutative product $A \circ B = \frac{1}{2}(AB + BA)$, where AB is the usual matrix product. $\dim_{\mathbb{C}} \mathcal{J}_3(\mathbb{O}) = 27$ and it is a model for the E_6-module V_{ω_1}. There is a well-defined *determinant* on $\mathcal{J}_3(\mathbb{O})$, which is defined by same expression as the classical determinant in terms of traces:

$$\det A = \frac{1}{6}(\text{trace}\, A)^3 - \frac{1}{2}(\text{trace}\, A)(\text{trace}\, A^2) + \frac{1}{3}\text{trace}\, A^3.$$

E_6 is the subgroup of $GL(\mathcal{J}_3(\mathbb{O})) = GL(27, \mathbb{C})$ preserving *det*. The notion of rank one matrices is also well defined and the *Cayley plane*, $E_6/P_1 = \mathbb{OP}^2 \subset \mathbb{P}(\mathcal{J}_3(\mathbb{O}))$ is the projectivization of the rank one elements, with ideal the 2×2 minors (see [19]).

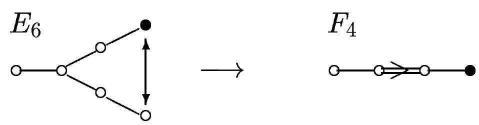
Since α_1 is not short, all linear spaces on $\mathbb{OP}^2$ are described by Tits geometries. In particular, E_6/P_3 is the space of lines on $\mathbb{OP}^1$ and E_6/P_2 is the space of $\mathbb{P}^5$'s on $\mathbb{OP}^2$.

6.3 $F_4/P_4 = \mathbb{OP}^2_0$

Here are some descriptions of $F_4 \subset GL(\mathcal{J}_3(\mathbb{O}))$:

$$\begin{aligned} F_4 &= \{g \in GL(\mathcal{J}_3(\mathbb{O})) \mid \mathrm{tr}((\rho(g)A)^i) = \mathrm{tr}A^i \text{ for } i = 1,2,3\} \\ &= \mathrm{Aut}(\mathcal{J}_3(\mathbb{O})) \\ &= \{g \in E_6 \mid g_+ = g_-\} \end{aligned}$$

The third description is motivated by geometric folding:



The equivalence of the second and third descriptions can be proved by using the quadratic form $\mathrm{tr}(A^2)$ to identify $\mathcal{J}_3(\mathbb{O})$ with $\mathcal{J}_3(\mathbb{O})^*$ and considering g_+ (resp. g_-) as the two resulting elements of $GL(\mathcal{J}_3(\mathbb{O}))$. We will give a general discussion of geometric folding in [22] and postphone proofs until then. (We do not use this description in the results that follow.) Harvey shows that the second definition implies the first [14] p. 296. For the first definition, one only needs two of the three forms to be preserved, as any group preserving two preserves the third.

Remark 6.6. Comparing the F_4 and G_2 cases, both groups preserve a trilinear form, but the symmetric cubic for F_4 is not generic. Both preserve a quadratic form. The G_2-quadratic form is deducible from the skew three-form. To deduce the F_4-quadratic form one needs the cubic form and the linear form.

Geometric folding indicates F_4/P_4 should be understood in terms of $\mathbb{OP}^2$, and, as with G_2/P_1 above, it is the hyperplane section $\{\mathrm{tr} = 0\}$. In what follows, AB denotes the usual matrix product of A and B. Note that $A^2 = A \circ A$.

Proposition 6.7. *Consider* $\mathbb{OP}^2_0 = F_4/P_4 \subset \mathbb{P}(\mathcal{J}_3(\mathbb{O})_0) \simeq \mathbb{P}(V_{\omega_4})$.

Then $T_x(F_4/P_4) = T_1 \oplus T_2$ *as a* $P_1^{ss} = Spin_7$*-module. Let* U *be the* 7*-dimensional vector representation of* $Spin_7$ *and* $\mathcal{S}(U)$ *the spin representation, then* $T_1 = \mathcal{S}(U)$ *and* $T_2 = U$*. The spinor variety* $Y_1 = \mathbb{S}(U)$ *is a six dimensional quadric, and* $Y_2 = \mathbb{Q}^5$ *is a five dimensional quadric. Moreover,*

$$\mathrm{Base}\,|\mathbb{FF}^2_{\mathbb{OP}^2_0}| \ = \mathbb{P}\overline{\{(x \wedge y \wedge z) \oplus z \in T_1 \oplus T_2 \mid [x \wedge y \wedge z] \in \mathbb{S}(U)\}} \ = \mathbb{S}_5 \cap H$$

where $\mathbb{S}_5 \cap H$ *is a generic hyperplane section of the spinor variety* $\mathbb{S}_5 = D_5/P_5$*. In particular,* $B = \mathrm{Base}\,|\mathbb{FF}^2_{\mathbb{OP}^2_0}|$ *is of dimension* 9*, and is the closure of a* $Spin_7$*-orbit, the boundary of which is the disjoint union of* Y_1 *and* Y_2*. It is not homogeneous for any group.*

Proof. The decomposition of the tangent space follows from §2.3. Moreover, B must be a hyperplane section of $\mathbb{S}_5 = \mathrm{Base}\,|\mathbb{FF}^2_{\mathbb{OP}^2}|$, it is generic because the trace is a generic hyperplane section. The explicit description follows because this is the only intersection of quadrics defined by an $Spin_7$-invariant subspace of S^2T^* that contains Y_1 and Y_2 and is of dimension nine. Base $|\mathbb{FF}^2_{\mathbb{OP}^2_0}|$ must intersect T_2 as the hyperplane section is generic, and therefore Base $|\mathbb{FF}^2_{\mathbb{OP}^2_0}|$ must contain Y_2. Finally, we check that there is no homogeneous space of dimension 9 homogeneously embedded in a $\mathbb{P}^{14}$. □

Proposition 6.8. *We have the following octonionic interpretations:*

$$\begin{aligned} \mathbb{OP}_0^2 = F_4/P_4 &= \{[A] \in \mathbb{P}\mathcal{J}_3(\mathbb{O})_0 \mid A^2 = 0\} \\ \hat{T}_{[A]}\mathbb{OP}_0^2 &= \{B \in \mathcal{J}_3(\mathbb{O})_0 \mid A \circ B = 0\} \\ \hat{T}_{1[A]}\mathbb{OP}_0^2 &= \{B \in \mathcal{J}_3(\mathbb{O})_0 \mid AB = 0\} \end{aligned}$$

Proof. A calculation shows that an element $A \in \mathcal{J}_3(\mathbb{O})$ is rank one and traceless if and only if $A^2 = 0$. Differentiation yields the second line.

To prove the third line, we first need to show that if $[A] \in \mathbb{OP}_0^2$ and $B \in \hat{T}_{[A]}\mathbb{OP}_0^2$, the equation $AB = 0$ is F_4 invariant (although the matrix product AB is *not* F_4 invariant). Note that F_4 is generated by SO_3 and $Spin_8$, where the action of $g \in SO_3$ is by $A \mapsto gA^tg$, and that of $(g_+, g_-, g_0) \in Spin_8$ by

$$\begin{pmatrix} r_1 & \overline{x_3} & \overline{x_2} \\ x_3 & r_2 & \overline{x_1} \\ x_2 & x_1 & r_3 \end{pmatrix} \mapsto \begin{pmatrix} r_1 & \overline{g_+(x_3)} & \overline{g_-(x_2)} \\ g_+(x_3) & r_2 & \overline{g_0(x_1)} \\ g_-(x_2) & g_0(x_1) & r_3 \end{pmatrix}.$$

(This defines an automorphism of the Jordan algebra $\mathcal{J}_3(\mathbb{O})_0$ because of the triality principle.) The SO_3 invariance is clear. Moreover, if we take

$$A = \begin{pmatrix} i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then

$$\hat{T}_{[A]}\mathbb{OP}_0^2 = \begin{pmatrix} i\mathrm{tr}(x_3) & \overline{x_3} & i\overline{x_1} \\ x_3 & -i\mathrm{tr}(x_3) & \overline{x_1} \\ ix_1 & x_1 & 0 \end{pmatrix}$$

where $\mathrm{tr}(u) = u - u_0 = \frac{1}{2}(u + \overline{u})$ is the "real" part of u. The $Spin_8$ invariance of the equation $AB = 0$ is a straightforward calculation, and follows again from the triality principle.

With this model, $\{B \in \hat{T}_{[A]}\mathbb{OP}_0^2 \mid AB = 0\} \simeq \{x_1, \mathrm{tr}(x_3)\}$ and we may consider $\{x_1\} \subset \hat{T}_{[A]}/\{A\} \simeq T$. Note that T is acted on by the subgroup of $Spin_8$ that preserves A, which means that $g_+(1) = 1$. By [14] p. 285,

$$\begin{aligned} Spin_7 &= \{(g_+, g_-, g_0) \in Spin_8 \mid g_- = g_0\} \\ &= \{(g_+, g_-, g_0) \in Spin_8 \mid g_+(1) = 1 \in \mathbb{O}\} \end{aligned}$$

(Note that this embedding of $Spin_7$ in $Spin_8$ is *not* the standard one). Thus we explicitly see the $Spin_7 = P_4^{ss}$ action on T and the decomposition of T into $T_1 \simeq \{x_1\}$ and $T_2 \simeq \{(x_3)_0\}$, respectively as the spin and vector representations. In particular, $\{T_1 + A\} = \{B \in \hat{T} \mid AB = 0\}$. □

Proposition 6.9. *The space $\mathbb{G}(\mathbb{P}^1, \mathbb{OP}_0^2)$ of lines on $\mathbb{OP}_0^2$ has dimension 23, and admits the following description:*

$$\mathbb{G}(\mathbb{P}^1, \mathbb{OP}_0^2) = \{\mathbb{P}\{A, B\} \mid [A], [B] \in \mathbb{OP}_0^2 \text{ such that } A \circ B = AB + BA = 0\}.$$

The space $F_4/P_3 = \mathbb{G}_0(\mathbb{P}^1, \mathbb{OP}_0^2)$ of F_4-homogeneous lines on $\mathbb{OP}_0^2$ has dimension 20, and admits the following description:

$$\mathbb{G}_0(\mathbb{P}^1, \mathbb{OP}_0^2) = \{\mathbb{P}\{A, B\} \mid [A], [B] \in \mathbb{OP}_0^2 \text{ such that } AB = 0\}.$$

Proof. The geometric descriptions of $\mathbb{G}(\mathbb{P}^1,\mathbb{OP}^2_0)$ and $\mathbb{G}_0(\mathbb{P}^1,\mathbb{OP}^2_0)$ follow immediately from corollary 2.5 and proposition 6.8, because, as will be proven later (theorem 7.8), F_4/P_3 is the space of lines on F_4/P_4 tangent to Y_1. Moreover,

$$\dim(\mathbb{G}(\mathbb{P}^1,\mathbb{OP}^2_0)) = \dim \mathbb{OP}^2_0 + \dim(\text{Base}\,|\mathbb{FF}^2_{\mathbb{OP}^2_0}|) - 1 = 15 + 9 - 1 = 23$$

verifies the dimension assertion. □

A straightforward consequence of 6.9 is the following

Proposition 6.10. *The space $\mathbb{G}(\mathbb{P}^2,\mathbb{OP}^2_0)$ of $\mathbb{P}^2$'s on $\mathbb{OP}^2_0$ has dimension 26, and admits the following description:*

$$\mathbb{G}(\mathbb{P}^2,\mathbb{OP}^2_0) = \{\mathbb{P}\{A,B,C\} \mid [A],[B],[C] \in \mathbb{OP}^2_0 \text{ such that } A\circ B = A\circ C = B\circ C = 0\}.$$

The space $F_4/P_2 = \mathbb{G}_0(\mathbb{P}^2,\mathbb{OP}^2_0)$ of F_4-homogeneous $\mathbb{P}^2$'s on $\mathbb{OP}^2_0$ has dimension 20, and admits the following description:

$$F_4/P_2 = \mathbb{G}_0(\mathbb{P}^2,\mathbb{OP}^2_0) = \{\mathbb{P}\{A,B,C\} \mid [A],[B],[C] \in \mathbb{OP}^2_0 \text{ such that } AB = AC = BC = 0\}.$$

Note that the general element of the first space of $\mathbb{P}^2$'s is nowhere tangent to the distribution $\mathcal{Y}_1$ defined by Y_1, where the second space is everywhere tangent to this distribution. There are two intermediate cases, $\{A,B,C \mid AB = AC = 0,\ B\circ C = 0\}$ and $\{A,B,C \mid AB = 0,\ A\circ C = B\circ C = 0\}$. The first has a distinguished point $[A] \in \mathbb{P}^2$ with the property that $\mathbb{P}T_{[A]}\mathbb{P}^2 \subset Y_{1[A]}$. It will appear in the next section as the space of lines on F_4/P_3. The second is nowhere contained in Y_1, but contains a distinguished line which is integral for $\mathcal{Y}_1$.

Remark 6.11. By 6.7 we calculate that the maximum dimension of a linear space on $\mathbb{OP}^2_0$ is 4. By Tits geometries, F_4/P_1 is a space of $\mathbb{P}^5$'s on F_4/P_4, which is correct, however these $\mathbb{P}^5$'s are embedded by the quadratic Veronese embedding. (See [22].)

Remark 6.12. The results of this section, appropriately modified, are valid for generic hyperplane sections of all four Severi varieties (see [19]) with $\mathbb{O}$ replaced by $\mathbb{A}$ (where $\mathbb{A}$ is one of the other composition algebras). This gives a unified presentation of the varieties on the first row of Freudenthal's magic chart (see [22]): $\mathbb{AP}^2_0$ is respectively $v_4(\mathbb{P}^1)$ (the quartic curve in $\mathbb{P}^4$), $\mathbb{F}_{1,3}$ (the adjoint variety of SL_3), $G_w(2,6)$, and $\mathbb{OP}^2_0$.

6.4 $F_4/P_3 = \mathbb{G}_0(\mathbb{P}^1,\mathbb{OP}^2_0)$

Proposition 6.13. *Consider $(F_4/P_3) \subset \mathbb{P}V_{\omega_3}$.*

Then $T = T_1 \oplus T_2 \oplus T_3$ as a $P_3^{ss} = SL_3 \times SL_2$-module. Let $\dim E = 3$ and $\dim U = 2$, then

$$T_1 = E^* \otimes U, \quad T_2 = E \otimes S^2U, \quad T_3 = U, \quad T_4 = E^*.$$

Moreover,

$$\text{Base}\,|\mathbb{FF}^2_{\mathbb{G}_0(\mathbb{P}^1,\mathbb{OP}^2_0)}| = \overline{\mathbb{P}\{e^*\otimes u + e\otimes u^2 \in T_1 \oplus T_2 \mid e^* \in E^*, e \in E, u \in U,\ \langle e^*, e\rangle = 0\}}.$$

$B = \text{Base}\,|\mathbb{FF}^2_{\mathbb{G}_0(\mathbb{P}^1,\mathbb{OP}^2_0)}|$ is a nontrivial $\mathbb{Q}^4$- bundle over $\mathbb{P}U = \mathbb{P}^1$. In particular, $\dim B = 5$ and it has a dense open $SL_3 \times SL_2$-orbit, the boundary of which is the union of the two closed orbits $Y_1 \subset \mathbb{P}T_1$ and $Y_2 \subset \mathbb{P}T_2$. It is not homogeneous for any Lie group.

Proposition 6.14. *The space $\mathbb{G}(\mathbb{P}^1, F_4/P_3)$ of $\mathbb{P}^1$'s on F_4/P_3 has dimension 24, and admits the following description:*

$$\mathbb{G}(\mathbb{P}^1, F_4/P_3) = \{\{A\} \subset \{A, B, C\} \mid [A], [B], [C] \in \mathbb{OP}^2_0 \text{ such that } AB = AC = 0, B \circ C = 0\}.$$

The space $F_4/P_{2,4} = \mathbb{G}_0(\mathbb{P}^1, \mathbb{OP}^2_0)$ of F_4-homogeneous $\mathbb{P}^1$'s on F_4/P_3 has dimension 22, and admits the following description:

$$F_4/P_{2,4} = \mathbb{G}_0(\mathbb{P}^1, \mathbb{OP}^2_0) = \{\{A\} \subset \{A, B, C\} \mid [A], [B], [C] \in \mathbb{OP}^2_0 \text{ with } AB = AC = BC = 0\}.$$

Proof of 6.14. We have $F_4/P_3 \subset G(2, 26)$ so a line on F_4/P_3 must be a line of the Grassmanian as well. Lines on $G(2, 26)$ are determined by the choice of a flag $\mathbb{P}^0 \subset \mathbb{P}^2$. Here we need $[A] = \mathbb{P}^0 \in F_4/P_4$ in both cases.

In the first case $AB = AC = 0$, $B \circ C = 0$ are necessary and sufficient conditions that the line be contained in F_4/P_3, as by 6.9, we need $A(sB + tC) = 0$ and $(sB + tC)^2 = 0$ for all $[s, t] \in \mathbb{P}^1$. Moreover, $\dim \mathbb{G}(\mathbb{P}^1, F_4/P_3) = 24$ because the choice of $[A]$ is 15 dimensions and then one needs an element of $G_o(2, 8)$, which is of dimension 9. (Here $\mathbb{C}^8 \simeq T_1$.)

In the second case, considering $F_4/P_{2,4}$ as a $\mathbb{P}^2$-bundle over F_4/P_2, the conditions $AB = AC = BC = 0$ follow from picking an element of F_4/P_2, and the choice of $[A]$ is a choice of an element in the fiber. □

Proof of 6.13. First, $\dim B = 5$ because

$$\dim (F_4/P_3) + \dim \mathrm{Base}\, |\mathbb{FF}^2_{F_4/P_3}| - 1 = \dim \mathbb{G}(\mathbb{P}^1, F_4/P_3).$$

Moreover, B must contain Y_1 and Y_2 by lemma 7.15, and is irreducible, as is seen from the explicit description of proposition 6.14.

Consider now $[y_1 + y_2] \in B$, with $y_j \in Y_j \subset \mathbb{P}T_j$. Write $y_1 = e \otimes u$ and $y_2 = e^* \otimes v^2$. Conditions for such a point to belong to B can only come from components of

$$T_1^* \otimes T_2^* = (E \otimes E^*) \otimes (U^* \otimes S^2 U^*) = (\mathbb{C} \oplus \mathfrak{sl}(E)) \otimes (U^* \oplus S^3 U^*).$$

Suppose that $\mathfrak{sl}(E) \otimes S^3 U^*$ were contained in N_2^*. Since $e \otimes e^*$ is not a homothety, this would force uv^2 to be zero in $S^3 U$, hence u or v to be zero. If this component were in N_2^*, then B would be included in $\mathbb{P}T_1 \sqcup \mathbb{P}T_2$, and would not be irreducible.

Suppose now that $\mathfrak{sl}(E) \otimes U^*$ were contained in N_2^*. This set of equations would force u and v to be parallel because under the contraction $U \otimes S^2 U \to U$, $u \otimes v^2$ maps to $\omega(u, v)v$, where $\omega \in \Lambda^2 U^*$. Similarly, the component $S^3 U^*$ would force $\langle e, e^* \rangle = 0$.

In conclusion, $\mathbb{P}\{e \otimes u \oplus e^* \otimes u^2 \mid \langle e, e^* \rangle = 0\} \subseteq B$. Since it is irreducible of dimension five, as is B, equality must hold.

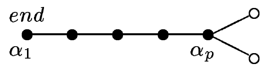
The quadric bundle structure is given by the application $B \to \mathbb{P}U$ defined by $[e^* \otimes u + e \otimes u^2] \mapsto [u]$, which is well defined. This is a nontrivial bundle structure. Finally, to see that B cannot be homogeneous, note that there are no homogeneous nontrivial quadric fibrations in dimension five. □

7 Linear spaces on homogeneous varieties

7.1 The role of Y_1

Among the homogeneous varieties G/P with P maximal, we have seen two preferred classes, the Hermitian symmetric spaces and the adjoint varieties. Another preferred class, which has considerable overlap with these two classes, is the class of homogeneous varieties corresponding to an extremity of a Dynkin diagram. We will call such varieties $X_{end} = G/P_{end}$ and will often relabel $\mathcal{D}(G)$ such that $P_{end} = P_{\alpha_1}$.

Fix an end of the Dynkin diagram of G, and label the end node α_1. Following [8], define the *branch* of α_1, $\mathcal{B}(\alpha_1)$, as the largest chain in $\mathcal{D}(G)$ containing α_1 that is isomorphic to $\mathcal{D}(A_p)$ or $\mathcal{D}(C_p)$, such that no node before in $\mathcal{B}(\alpha_1)$ before the last has valence three. We say such a branch has *length p*. We label the roots on $\mathcal{B}(\alpha_1)$ as $\alpha_1, ..., \alpha_p$ and denote the fundamental representation corresponding to ω_1 by $V = V_{end} = V_{\omega_1}$. Such an irreducible representation is called an *elementary representation* in [8].



By [29] the varieties G/P with P maximal can be understood as a certain family of $\mathbb{P}^k$'s on some (sometimes several varieties) X_{end}. Nevertheless, [29] does not address the following natural questions: are these $\mathbb{P}^k$'s always linearly embedded? Do these families comprise all linear spaces on X_{end}? If not, what are the other linear spaces and how can we describe them? How can the special families described by [29] be characterized? In this section, we answer most of these questions

We begin by announcing a corollary of 7.18, which answers these questions when α_{end} is not short:

Corollary 7.1. *Let G be a simple group, let $\alpha_{end} = \alpha_1$ be a root that is not short, corresponding to an end node on $\mathcal{D}(G)$ with branch of length p. Label the roots on the branch $\alpha_1, ..., \alpha_p$, and let $X_{end-k} = G/P_{\alpha_{k+1}}$ for $1 \leq k \leq p$. Then X_{end-k} is the space of linearly embedded k-planes on X_{end}.*

To obtain a more general result that includes the case where α is short, we first relate the fundamental representations corresponding to nodes on the branch of α_{end} to wedge powers of V. The following result is evidently due to Cartan, a proof can be found in [8] except for the 'moreover' assertions which may be verified on a case by case basis.

Proposition 7.2. *With the notations above, V_{ω_k} is an irreducible component of $\Lambda^k V$, with multiplicity one for $2 \leq k \leq p$. More precisely, ω_k is the unique extremal weight of $\Lambda^k V$.*

Moreover, $\Lambda^{p+1} V$ also has a unique extremal weight which is

1. *$2\omega_{p+1}$ for a double edge with arrow pointing away from ω_1.*
2. *$3\omega_{p+1}$ for a triple edge with arrow pointing away from ω_1.*
3. *$\omega_{p+1} + \omega_{p+2}$ if ω_p corresponds to a node of valence three.*

Idea of proof. One simply checks that among the weights of V, there is a maximal chain $\mu_1, \dots, \mu_{p+1}$ with $\mu_i = \omega_1 - (\alpha_1 + \cdots + \alpha_{i-1})$. In particular, $\mu_1 + \cdots + \mu_k$ is the unique maximal weight of $\Lambda^k V$ for $1 \leq k \leq p+1$, and it is straightforward to check that this weight is as announced in the proposition. □

A case by case analysis with LiE [25] leads to the following more precise result:

Proposition 7.3. *Notations as above.*

1. If X_{end} is Hermitian symmetric and V is not symplectic or $(D_n, \omega_n) \simeq (D_n, \omega_{n-1})$, then for $2 \leq k \leq p$, $V_{\omega_k} = \Lambda^k V$. This is the case for $(G, \omega) = (A_n, \omega_1)$, (B_n, ω_1), (D_n, ω_1) and (E_6, ω_1).

2. If X_{end} is Hermitian symmetric and V has a symplectic form Ω, set $\Lambda^{\langle k\rangle}V = \Lambda^k V/(\Omega \wedge \Lambda^{k-2}V)$. Then $V_{end-k} = \Lambda^{\langle k\rangle}V$. This is the case for $(G,\omega) = (C_n,\omega_1)$ and (E_7,ω_7).

3. If X_{end} is an adjoint variety, so $V = \mathfrak{g}$, let $\Lambda^{[2]}\mathfrak{g} = \ker[\ ,\ ]$, where we consider the Lie bracket as a map $[\ ,\ ] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$. The adjoint varietes corresponding to elementary representations are those of the exceptional groups: $(G,\omega) = (G_2,\omega_2)$, (F_4,ω_1), (E_6,ω_2), (E_7,ω_1) and (E_8,ω_8). In each of these cases, except for G_2, $V_{end-1} = \Lambda^{[2]}\mathfrak{g}$. (And afterwards there is always a double bond or node with triple valence.)

4. If $X_{end} = G_2/P_1$ then $V_{\omega_2} = \Lambda^2 V_{\omega_1}/(V^*_{\omega_1} \lrcorner \phi)$ where $\phi \in \Lambda^3 V_{\omega_1}$ is the defining three form.

Remark 7.4. A curious phenomenon occurs with the E_7/P_7 case. The E_7 module V_{ω_7} is symplectic, yet corollary 7.1 applies. This implies that the only linear spaces on E_7/P_7 are already isotropic for the symplectic form, in contrast to the case of C_n/P_1.

Let Z_k denote the space of $\mathbb{P}^k$'s in X_{end}. Z_k is naturally embedded in $\mathbb{P}(\Lambda^{k+1}V)$.

Lemma 7.5. *Let $\mathcal{C} \subset \mathcal{D}(G)$ be a connected chain isomorphic to $\mathcal{D}(A_{k+1})$ or $\mathcal{D}(C_{k+1})$ containing an end α_1 in $\mathcal{D}(G)$, and let $\beta_j = \alpha_1 + \cdots + \alpha_j \in \Delta_+$ for $1 \le j \le k$. Then the root vectors $v, X_{\beta_1}v, \ldots, X_{\beta_k}v$ generate a k-plane $\Lambda \in Z_k$.*

Proof of the lemma. In the case the chain is isomorphic to $\mathcal{D}(A_{k+1})$, let $\alpha_1 + \beta$ and $\alpha_1 + \gamma$ be roots of the form β_j. Since the roots with support on $\mathcal{D}(A_{k+1})$ form a root subsystem of type A_{k+1}, $2\alpha_1 + \gamma$ and $\alpha_1 + \beta + \gamma$ cannot be roots. Let us suppose that $X_{\alpha_1+\gamma} = [X_{\alpha_1}, X_\gamma]$. Then, using that $X^2_{\alpha_1}v = X_\gamma v = 0$, we get

$$\begin{aligned}
X_{\alpha_1}X_{\alpha_1+\gamma}v &= X_{\alpha_1}(X_{\alpha_1}X_\gamma - X_\gamma X_{\alpha_1})v \\
&= -X_{\alpha_1}X_\gamma X_{\alpha_1}v \\
&= -(X_{\alpha_1+\gamma} + X_\gamma X_{\alpha_1})X_{\alpha_1}v \\
&= -X_{\alpha_1+\gamma}X_{\alpha_1}v \\
&= -X_{\alpha_1}X_{\alpha_1+\gamma}v = 0.
\end{aligned}$$

This immediately implies that

$$\begin{aligned}
X_{\alpha_1+\beta}X_{\alpha_1+\gamma}v &= X_{\alpha_1}X_\beta X_{\alpha_1+\gamma}v \\
&= X_{\alpha_1}X_{\alpha_1+\gamma}X_\beta v = 0.
\end{aligned}$$

But these equalities allow one to write (note that the $X_{\alpha_1+\gamma}$ commute) that

$$\exp(\sum_{j=1}^k t_j X_{\beta_j})v = v + \sum_{j=1}^k t_j X_{\beta_j}v,$$

which implies our claim. The other case is similar, or one can simply verify the lemma for the two examples. □

Corollary 7.6. *If $\mathcal{B}(\alpha_1)$ has length p, then*

$$X_{end-k} \subseteq Z_k \qquad \forall k \le p,$$

and the induced projective embedding of X_{end-k} is its fundamental one.

We now study the relationship between X_{end-k} and Z_k more precisely. We first consider the case $k = 1$. Recall from corollary 2.5 that the set of lines in X_{end} passing through a given point is Base $|\mathbb{FF}^2_{X_{end},x}|$.

Proposition 7.7. *Let $X = G/P$ be a homogeneous variety and let X' be the space of G-homogeneous lines on X. Then X' is the space of lines tangent to the distribution defined by Y_1.*

To characterize X_{end-k} in terms of linear spaces on X_{end} for $k > 1$, we need a definition. We say that a $\mathbb{P}^k$ on X_{end} is *$\mathcal{Y}$-completely integral* if the following conditions hold. First, we require $\mathbb{P}^k$ to be an integral manifold of the distribution defined by Y_1. Next, for each $x \in \mathbb{P}^k$, the corresponding $Y_1 \subset \mathbb{P}T_{1x}X$ is a P^{ss}-homogeneous space, on which is defined a similar distribution Y_1'. Our $\mathbb{P}^k$ then defines a $\mathbb{P}^{k-1}$ on Y_1 which we require to be $\mathcal{Y}'$-completely integral, and so on. We denote by $Z_k^{\mathcal{Y}}$ the space of $\mathcal{Y}$-completely integral k-planes on X_{end}. $Z_k^{\mathcal{Y}}$ is a G-homogeneous subvariety of Z_k.

Theorem 7.8. *Let $X_{end} = G/P \subset \mathbb{P}V$ be a homogeneous variety in its fundamental embedding, where P is the maximal parabolic corresponding to an end in $\mathcal{D}(G)$. Let the branch of α_{end} have length p. Then for $k \leq p$,*

$$X_{end-k} = Z_k^{\mathcal{Y}}.$$

Proof. First note that the k-plane of highest weight, that is the k-plane generated by the root vectors $v, X_{\beta_1}v, \dots, X_{\beta_k}v$, is $\mathcal{Y}$-completely integral, so that $X_{end-k} \subset Z_k^{\mathcal{Y}}$, and we have a commutative diagram

$$\begin{array}{ccccc} X_{end-k} & & & & \\ \cap & & & & \\ Z_k^{\mathcal{Y}} & \subset & Z_k & \subset & G(k,V) \\ \cap & & & & \cap \\ \mathbb{P}V_{\omega_k} & & \subset & & \mathbb{P}(\Lambda^{k+1}V) \end{array}$$

We now estimate the dimension of $Z_k^{\mathcal{Y}}$. First consider the case $k = 1$. On the one hand, $\dim X_{end-1}$ is the number of positive roots with positive coefficient on α_2. Since the set of simple roots on which a given root has nonzero coefficient is always connected, we obtain

$$\dim X_{end-1} = \#\Delta_+ - 1 - \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = m_2(\alpha) = 0\}.$$

On the other hand, it follows from the definition of $Z_1^{\mathcal{Y}}$ that $\dim Z_1^{\mathcal{Y}} \leq \dim X + \dim Y_1 - 1$, that is,

$$\begin{aligned} \dim Z_1^{\mathcal{Y}} &\leq \#\Delta_+ - \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = 0\} + \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = 0,\ \alpha + \alpha_1 \in \Delta_+\} - 1 \\ &= \#\Delta_+ - \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = 0\} + \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = 0, m_2(\alpha) > 0\} - 1. \end{aligned}$$

For $k > 1$, a similar count shows that $\dim X_{end-k} \geq \dim Z_k^{\mathcal{Y}}$. Indeed, on the one hand,

$$\begin{aligned} \dim X_{end-k} &= \#\{\alpha \in \Delta_+ \mid m_k(\alpha) > 0\} \\ &= \#\Delta_+ - \tfrac{k(k+1)}{2} - \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = \cdots = m_k(\alpha) = 0\}, \end{aligned}$$

because the roots with support on the chain $\mathcal{C}$ form a root system of type A_k, so there are $k(k+1)$ of them. On the other hand, if $Y_1^{(1)} = Y_1^{end}$, and if $Y_1^{(j+1)}$ is defined similarly as the preferred homogeneous subvariety of the tangent space at a point of $Y_1^{(j)}$, then each complete flag of subspaces of a $\mathcal{Y}$-integral $\mathbb{P}^k$ defines a sequence of points in the $Y_1^{(j)}$ for $1 \leq j \leq k$. Hence

$$\dim Z_k^{\mathcal{Y}} \leq \dim X + \sum_{j=1}^{k} \dim Y_1^{(j)} - \frac{k(k+1)}{2},$$

and the conclusion follows from the fact, which is checked inductively, that

$$\dim Y_1^{(j)} = \#\{\alpha \in \Delta_+ \mid m_1(\alpha) = \cdots = m_{j-1}(\alpha) = 0,\ m_j(\alpha) > 0\}.$$

Now, since $X_{end-k} \subset Z_k^{\mathcal{Y}}$ and $\dim X_{end-k} \geq \dim Z_k^{\mathcal{Y}}$, we conclude that X_{end-k} is a connected component of $Z_k^{\mathcal{Y}}$. But a consequence of our dimension estimation for $Z_k^{\mathcal{Y}}$ is also that, to test if a $\mathbb{P}^k$ belongs to $Z_k^{\mathcal{Y}}$, we only need to check that it is tangent to Y_1 at one point, then that some line is tangent to the similar distribution on Y_1, and so on. This implies in particular that $Z_k^{\mathcal{Y}}$ is homogeneous, and the proof is complete. □

Remark 7.9. Note that a line on $Z_k \subset \mathbb{P}(\Lambda^{k+1}V)$ corresponds to a $\mathbb{P}^{k+1}$ on X. In particular, we see that if a component of Z_p is maximal, it cannot contain any lines. Thus if it is homogeneous (and not a point), it cannot be in its minimal embedding.

After crossing an end, we have the following results:

Proposition 7.10. *If $\mathcal{D}(G)$ is simply laced, then the space of $\mathbb{P}^{p+1}$'s on X_{end} is $G/P_{\omega_{p+1}+\omega_{p+2}}$ in its minimal homogeneous embedding.*

Proposition 7.11. *If α_{end} is not short with branch of length p and there is a double (resp. triple) bond adjacent to α_p, then the space of $\mathbb{P}^{p+1}$'s on X_{end} is $v_2(G/P_{p+1})$ (resp. $v_3(G/P_{p+1})$).*

Remark 7.12. Consider the case of A_n with the node α_0 of its completed Dynkin diagram. The roots α_1 and α_n are connected to α_0 in $\mathcal{D}^+(A_n)$, and the homogeneous variety $X = G/P_{1,n}$ is the adjoint variety $\mathbb{F}\mathbb{F}_{1,n}$ for $\mathfrak{sl}_{n+1}$. The space of lines in X is disconnected: it is the disjoint union of $\mathbb{F}\mathbb{F}_{2,n}$ and $\mathbb{F}\mathbb{F}_{1,n-1}$, the corresponding embeddings of which are not their minimal ones, since

$$\Lambda^{[2]}\mathfrak{sl}_{n+1} = V_{2\omega_1+\omega_{n-1}} \oplus V_{\omega_2+2\omega_n}.$$

When α_{end} is a short root, our dimension count gives the following statement.

Corollary 7.13. *Let G be a simple group, let $\alpha_{end} = \alpha_1$ be a short root corresponding to an end node on $\mathcal{D}(G)$. Let d be the codimension of Y_1 in* Base $|\mathbb{F}\mathbb{F}^2_{X_{end}}|$.

Then X_{end-1} also has codimension d in the space of lines on X_{end}. In particular, the space of lines on X_{end} is not G-homogeneous.

Example 7.14. For $X_{end} = C_n/P_1$, Y_1 has codimension one in $\mathbb{P}T$, which is coherent with the fact that $G_\omega(2, V)$, for a symplectic vector space V of dimension $2n$, is a hyperplane section of $G(2, V)$.

Also note that a k-plane $\mathbb{P}H$ is tangent to Y_1 at a point of X_{end} corresponding to a line L, if and only if $H \subset L^\perp$, so that $\mathbb{P}H$ is tangent to Y_1 along $\mathbb{P}(H \cap H^\perp)$. This gives in particular the classification of k-planes on C_n/P_1, by the dimension of $H \cap H^\perp$, or equivalently, the rank of the restriction to H of the symplectic form.

7.2 Proof of the main theorem on the base locus

As promised in section 2, we now come to the proof of Theorem 2.19, following which the base locus of the second fundamental form of $X = G/P_\alpha$ is minimal, if and only if α is not a short root. Our proof will have three steps:

Lemma 7.15. *If α is short, then $Y_2 \subset$* Base $|\mathbb{F}\mathbb{F}^2_X|$.

Proof. We may suppose that $G \neq G_2$, since the case of G_2/P_2 was examined in §6.1. Then, except for (F_4, α_4), for which the lemma holds by 6.7, there is a subdiagram $\mathcal{D} \subseteq \mathcal{D}(G)$, isomorphic to $\mathcal{D}(C_p)$, with α at one end. Let β be the minimal root with coefficient 2 on α. Then β is in the root subsystem consisting of roots with support on $\mathcal{D}$. But we know that for C_p/P_1 the base locus of the second fundamental form is not minimal, and contains the closed orbit in $\mathbb{P}(T_2)$. This implies that the relation $X_\beta^2 v = 0$ must also hold in V_{ω_α} if v is a highest root vector, so $Y_2 \subset \mathrm{Base}\,|\mathbb{FF}^2_X|$. □

We now suppose that α is not a short root, and we want to prove that $\mathrm{Base}\,|\mathbb{FF}^2_X| = Y_1$.

Lemma 7.16. *Suppose that $X = X_{end}$ corresponds to an extremity of the Dynkin diagram of G. Then* $\mathrm{Base}\,|\mathbb{FF}^2_X| = Y_1$.

Proof. We know that $\mathrm{Base}\,|\mathbb{FF}^2_X| = Y_1$ if X is Hermitian symmetric (corollary 3.7) or adjoint (corollary 4.8). This implies our lemma, except for E_7/P_1, E_8/P_1 and E_8/P_2, for which we give ad hoc arguments.

We begin with the case of E_7/P_2. The minimal root β with coefficient 2 on α_2 has coefficient zero on α_7. Thus if the relation $X_\beta^2 v = 0$ were to hold, because of Lemma 7.2.5 in [7], it would also hold in the representation of E_6 with highest weight ω_2. But we know this in not the case, since it is the adjoint representation of E_6. Hence $\mathrm{Base}\,|\mathbb{FF}^2_{E_7/P_2}|$ does not contain Y_2; since it is closed and H-stable, and since the tangent space T only has two irreducible components, it must be contained in $\mathbb{P}T_1$. And Y_1 is the unique H-orbit in $\mathbb{P}T_1$ defined by quadratic equations.

The case of E_8/P_1 is similar, and is left to the reader. We treat the remaining case, E_8/P_2. The semi-simple part of P_2 is a copy of SL_8, and the tangent space has three irreducible components

$$T_1 = \Lambda^3 V, \quad T_2 = \Lambda^6 V, \quad T_3 = V,$$

where V is eight-dimensional. We want to identify the second normal space N_2, which is a sum of irreducible components of S^2T. So we compute its symmetric square, and we check that each of its components has multiplicity one, except for $\Lambda^4 V$, which is both inside $T_1 \otimes T_3$ (as the image of the map $a \otimes c \to a \wedge c$) and $S^2 T_2$ (as the image of $b^* \lrcorner \omega \otimes b^* \lrcorner \omega \to (b^* \wedge b^*) \lrcorner \omega$, where ω is a non zero element of $\Lambda^8 V = \mathbb{C}$ and $b^* \in \Lambda^2 V^*$). If N_2^* were to contain a copy of $\Lambda^4 V$, its base locus would be of the form

$$B = \mathbb{P}\{(a, b^* \lrcorner \omega, c) \in T, \quad \lambda a \wedge c = \mu (b^* \wedge b^*) \lrcorner \omega\}$$

for some complex coefficients λ and μ. In particular, it would contain Y_2. But Y_2 is not in $\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$, since otherwise this would also be true for E_7/P_2. The only component of N_2^* which does not vanish identically on Y_2 can therefore only be the Cartan product of T_2^* with itself. But the base locus of this Cartan product being empty, this implies that each element of $\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$ has a zero component on T_2, hence also on T_3, since the linear span of $\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$ must be P_2-stable. Thus $\mathrm{Base}\,|\mathbb{FF}^2_{X,x}|$ must actually be contained in $\mathbb{P}(T_1)$, and being a proper subvariety, it must be Y_1. □

Lemma 7.17. *If* $\mathrm{Base}\,|\mathbb{FF}^2_X| = Y_1$ *for each homogeneous space corresponding to a non-short root at an extremity of a Dynkin diagram, it is also true for any non-short root.*

Proof. Our X corresponds to a node of $\mathcal{D}(G)$, which is on the branch of an end which is also a non-short root, hence by the above lemma has a minimal base locus for its second fundamental form.

Moreover, by proposition 7.8 we know that $X = X_{end-k}$ and $X_{end-k-1}$, are the spaces of k-planes and $(k+1)$-planes on X_{end}, respectively. But a line on X is given by a $(k-1)$-plane contained in a $(k+1)$-plane on X_{end}, which is a point of $X_{end-k-1}$, so that

$$\dim X + \dim \text{Base}\,|\mathbb{FF}^2_X| = \dim X_{end-k-1} + 2(k-1).$$

We can therefore perform the same dimension count as in the proof of proposition 7.8 to check that the base locus of the second fundamental form must be minimal. □

7.3 Unirulings of X

If α is not a short root, then Base $|\mathbb{FF}^2_{X,x}|$ is Hermitian symmetric, and for any homogeneously embedded Hermitian symmetric space the base locus of its second fundamental form is again Hermitian symmetric.

To choose a k-plane on X, one must choose a point $x \in X$ and a $k-1$ plane in Base $|\mathbb{FF}^2_{X,x}|$, or equivalently a point $x \in X$, a point in Base $|\mathbb{FF}^2_{X,x}|$, a point in Base $|\mathbb{FF}^2_{\text{Base}\,|\mathbb{FF}^2_{X,x}|}|$ etc... Since all these choices are acted on transitively by G, we see that if α is not short root, the space of $\mathbb{P}^k$'s on X is the (disjoint union of) G-homogeneous varieties and we may apply Tits' methods to describe them. For short roots where $G/P_\alpha = \tilde{G}/P_{\alpha'}$ in the notation of the introduction, then we are reduced to the previous case. For other short roots we have seen that Base $|\mathbb{FF}^2_{X,x}|$, although it is always smooth, is not homogeneous for any group.

In summary:

Theorem 7.18. *Let G be a simple group, let α be a simple root, let $P = P_\alpha$ be the corresponding parabolic and let $X = G/P \subset \mathbb{P}V$ be the corresponding homogeneous variety in its minimal homogeneous embedding.*

1. If α is not short short (i.e. if no arrow in $\mathcal{D}(G)$ points towards α), then for all k, the variety parametrizing the $\mathbb{P}^k$'s on X is the disjoint union of homogeneous varieties $G/P_{\Sigma\beta_j}$ where $\{\beta_j\} \subset \Delta_+$ is a reduced set such that the component of $\mathcal{D}(G)\backslash\{\beta_j\}$ containing α is isomorphic to $\mathcal{D}(A_k)$ or $\mathcal{D}(B_k)$ and α is an extremal node of this component.

2. If $G/P_\alpha = B_n/P_n, G_2/P_1$ or C_n/P_1, then the space of $\mathbb{P}^k$'s on X is not G-homogeneous, but $\tilde{G}$-homogeneous, where $\mathcal{D}(G)$ is the fold of $\mathcal{D}(\tilde{G})$.

3. If $G/P_\alpha = C_n/P_k$ with $1 < k < n$, or F_4/P_4 or F_4/P_3, then the space of $\mathbb{P}^1$'s on X is not homogeneous.

Similarly, using Base $|\mathbb{FF}^2_{X,x}|$ or the Dynkin diagram, we can quickly compute the maximum dimension of a linear space on X. The maximum dimension of a linear space on X is one plus the maximum dimension of a linear space on Base $|\mathbb{FF}^2_{X,x}|$, which is two plus the maximum dimension of a linear space on the base locus of the second fundamental form of Base $|\mathbb{FF}^2_{X,x}|$, etc... If one does not have the Dynkin diagram handy, one can compute in this manner. However, combined with the observation above, we have

Corollary 7.19. *Let G be a simple Lie group and let P_α be a maximal parabolic with α not short. Let $X = G/P$ be the corresponding homogeneous variety in its fundamental embedding. Suppose that the longest of the A or B chains in $\mathcal{D}(G)$ beginning at α is isomorphic to $\mathcal{D}(A_n)$ or $\mathcal{D}(B_n)$. Then the largest linear space on X is a $\mathbb{P}^n$.*

Note that already from [28] one knows that the longest such chain gives a lower bound, whether or not P corresponds to a short root. In fact one can either use Tits geometries or the discussion above to conclude that for each maximal chain of length p, there is a $\mathbb{P}^p \subset X$ and a corresponding uniruling by $\mathbb{P}^p$'s.

For example, the largest linear space on E_n/P_1 is a $\mathbb{P}^{n-1}$, via the chain terminating with α_n, so E_n/P_1 is maximally uniruled by $\mathbb{P}^{n-1}$'s and there is a second chain terminating with α_2, so E_n/P_1 is also maximally uniruled by $\mathbb{P}^4$'s. (The unirulings by the $\mathbb{P}^4$'s are maximal in the sense that none of the $\mathbb{P}^4$'s of the uniruling are contained in any $\mathbb{P}^5$ on X.) The varieties parametrizing these rulings are respectively E_n/P_2 and E_n/P_5.

Similarly, E_n/P_2 has maximal unirulings by $\mathbb{P}^{n-2}$'s and $\mathbb{P}^4$'s.

The largest linear space contained in $C_n/P_n = G_{Lag}(n, 2n)$ is a $\mathbb{P}^1$.

Note also that the maximum dimension of a linear space on $C_n/P_1 = \mathbb{P}^{2n-1}$ is $2n-1$, not n.

7.4 Dynkin diagrams via second fundamental forms

As a final remark, we describe how to recover $\mathcal{D}^*(G)$, the marked Dynkin diagram of G, from the second fundamental form at a point of any $X = G/P$ where P is maximal and not short (i.e. no arrow points towards P). We use the notations of section 3.

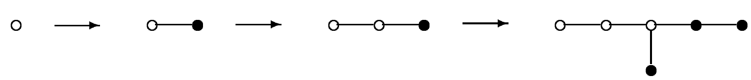
Fix $x \in X$ and start with a marked node, which will correspond to P. Say Y_1 is the Segre product of k irreducible Hermitian symmetric spaces. Then attach k edges to the node, with nodes at the end of each edge. For each factor in the Segre that is minimally embedded, the edge is simple. If there is a factor that is a quadratic (resp. cubic) Veronese, then make the corresponding edge a double (resp. triple) bond. Now compute Base $|\mathbb{FF}^2_{X,x}|$ of each factor and repeat the process starting with the node corresponding to the factor. Continue until arriving at the empty set. The resulting diagram is $\mathcal{D}(G)$.

A shortcut: if at any point one obtains an H/Q as a factor where the marked Dynkin diagram associated to H/Q is known, one can simply attach the diagram. In particular, if one arrives at a $\mathbb{P}^l$, just attach a copy of $\mathcal{D}(A_l)$.

Example 7.20. Beginning with $X = E_n/P_1$, one has

$$\begin{array}{ll} \text{Base}\,|\mathbb{FF}^2_X| = \mathbb{S}_{n-1}, & \text{Base}\,|\mathbb{FF}^2_{\mathbb{S}_{n-1}}| = G(2, n-1), \\ \text{Base}\,|\mathbb{FF}^2_{G(2,n-1)}| = Seg(\mathbb{P}^1 \times \mathbb{P}^{n-2}), & \text{Base}\,|\mathbb{FF}^2_{Seg(\mathbb{P}^1 \times \mathbb{P}^{n-2})}| = \mathbb{P}^0 \sqcup \mathbb{P}^{n-3}. \end{array}$$

So the construction is:



Other aspects of the interplay between the geometry of homogeneous spaces and fundamental algebraic data such as Dynkin diagrams will be studied in [23].

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