LOCALLY CONSTANT ALMOST EVERYWHERE FOURIER TRANSFORM

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1. Introduction.

Let $\mathbb{T}$ be the torus $\mathbb{R}/2\pi\mathbb{Z}$, and $E_0(\mathbb{T})$ the space of these continuous functions v on $\mathbb{T}$ such that there exists $K \subset \mathbb{T}$, compact, of zero Lebesgue measure, with v restricted to $\mathbb{T} \setminus K$ locally constant. More concisely :

$$E_0(\mathbb{T}) = \big\{ v \in C(\mathbb{T})\,;\ v \text{ locally constant almost everywhere} \big\}\,.$$

In this paper we are interested in the functions in $E_0(\mathbb{T})$ whose Fourier series are absolutely convergent, *i.e.* by the space

$$A_0(\mathbb{T}) = E_0(\mathbb{T}) \cap A(\mathbb{T})\,,$$

where

$$A(\mathbb{T}) = \Big\{ v \in C(\mathbb{T})\,;\ \sum_{-\infty}^{+\infty} |\hat{v}(n)| < +\infty \Big\}\,.$$

$A(\mathbb{T})$ is a Banach space for the norm $\|v\|_{A(\mathbb{T})} = \sum_{-\infty}^{+\infty} |\hat{v}(n)|$. $A_0(\mathbb{T})$ is a dense subspace of $A(\mathbb{T})$ (see § 2). We will prove in § 3 that $A_0(\mathbb{T})$, equipped with the norm of $A(\mathbb{T})$, is an ultrabornological space.

Let $U(\mathbb{T})$ be the space of uniformly convergent trigonometric series :

$$U(\mathbb{T}) = \Big\{ u \in C(\mathbb{T})\,;\ \sum_{-N}^{+N} \hat{u}(n) e^{int} \underset{N\to\infty}{\longrightarrow} u\,,\ \text{uniformly on } \mathbb{T} \Big\}\,.$$

It is a well known result of R. Salem that $U(\mathbb{T})$ is not an algebra and more precisely that there exists $u \in U(\mathbb{T})$ and $v \in A(\mathbb{T})$ with the product $uv \notin U(\mathbb{T})$ (see J. P. Kahane [2], page 6). The ultrabornological property allows us to deduce immediately from this result the fact there exists $u \in U(\mathbb{T})$ and $v \in A_0(\mathbb{T})$ with $uv \notin U(\mathbb{T})$: we do it in § 4.

We give some remarks in § 5.

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2. Density.

PROPOSITION 1. — *$A_0(\mathbb{T})$ is dense in $A(\mathbb{T})$.*

$A_0(\mathbb{T})$ is obviously a subspace of $A(\mathbb{T})$, closed under multiplication. Moreover $A_0(\mathbb{T})$ is translations invariant, so its closure in $A(\mathbb{T})$ is an ideal of $A(\mathbb{T})$ for convolution. So the density announced is an easy consequence of the following lemma:

LEMMA 1. — *$\forall n_0 \in Z, \exists v \in A_0(\mathbb{T})$ s.t. $\hat{v}(n_0) \neq 0$.*

Proof. — As a consequence of the theorem of Ivasev-Musatov (see T.W. Körner [3]) there exists a compact $K \subset \mathbb{T}$, of Lebesgue measure 0, and a measure $\mu \neq 0$, supported by K, such that

$$|\hat{\mu}(n)| = \mathcal{O}(|n|^{-\frac{1}{2}}) \text{ when } |n| \to +\infty .$$

Multiplying μ by e^{int} if necessary, for a well chosen n, one can assume $\hat{\mu}(n_0) \neq 0$. Let h be the indicator function of the image in $\mathbb{T}$ of the interval $[-1, +1]$ and define $v = \mu * h$, the convolution of μ with the L^1 function h. One has $\hat{x}(n) = \hat{\mu}(n) \cdot \hat{h}(n) = \hat{\mu}(n) \times \frac{\sin(n)}{n}$ so $|\hat{v}(n)| = \mathcal{O}(|n|^{-\frac{3}{2}})$ which implies $v \in A(\mathbb{T})$, and $\hat{v}(n_0) \neq 0$. But v is locally constant on the complement of $(K-1) \cup (K+1)$. So v has all the desired properties.

3. Ultrabornology.

Define $\mathcal{J} = \{K \subset \mathbb{T}; K \text{ compact of zero Lebesgue measure}\}$, and for each $K \in \mathcal{J}$, $A_K(\mathbb{T}) = \{v \in A(\mathbb{T}); v \text{ is locally constant on the complement of } K\}$. One has $A_0(\mathbb{T}) = \cup\{A_K(\mathbb{T});\ K \in \mathcal{J}\}$. $A_K(\mathbb{T})$ is a closed subspace of $A(\mathbb{T})$, so is Banach for the $A(\mathbb{T})$ norm. The following proposition is the announced result of ultrabornology:

PROPOSITION 2. — *Let B be a normed space and $\ell : A_0(\mathbb{T}) \to B$ a linear mapping. If for each $K \in \mathcal{J}$, ℓ restricted to $A_K(\mathbb{T})$ is continuous, then ℓ is continuous (all spaces being equipped with the $A(\mathbb{T})$ norm).*

The proof will use the following lemma:

LEMMA 2. — *$\forall \varepsilon > 0, \exists w \in A_0(\mathbb{T})$, $w(0) \neq 0$ and support $(w) \subset]-\varepsilon, +\varepsilon[$.*

Proof of lemma 2. — By lemma 1 there exists $v \in A_0(\mathbb{T})$, non constant. Let $V = \{t \in \mathbb{T};\ v \text{ is locally constant on a neighborhood of } t\}$ and $K = \mathbb{T} \setminus V$. K is a perfect compact subset of $\mathbb{T}$, and so there exists $t \in K$ such that for each $\varepsilon > 0$, $]t-\varepsilon, t[\cap K \neq \emptyset$ and $]t, t+\varepsilon[\cap K \neq \emptyset$. Translating the function v one can assume $t = 0$. Now for each $\varepsilon > 0$ there exists a_ε and b_ε in $\mathbb{T}$, with $a_\varepsilon \in]-\varepsilon, 0[$ and $b_\varepsilon \in]0, +\varepsilon[$, and $\eta > 0$, such that v is constant on $]a_\varepsilon - \eta, a_\varepsilon + \eta[$ and on $[b_\varepsilon - \eta, b_\varepsilon + \eta[$, and such that $v(a_\varepsilon) \neq v(0)$ and $v(b_\varepsilon) \neq v(0)$. Take $h \in A(\mathbb{T})$, for instance $h \in C^\infty(\mathbb{T})$, such that $h = 1$ on $[a_\varepsilon + \eta, b_\varepsilon - \eta]$

and $h = 0$ outside $]a_\varepsilon - \eta, b_\varepsilon + \eta[$. Then w defined by $w = h\cdot(v - v(a_\varepsilon))\cdot(v - v(b_\varepsilon))$ has the desired properties.

Proof of proposition 2. — By standard arguments (using the fact that $A_0(\mathbb{T})$ is a self adjoint function algebra, translations invariant, in which every element without zero is invertible) you get from lemma 2 the existence of partitions of unity. Precisely:

For each finite open covering $V_1 \cdots V_n$ of $\mathbb{T}$, the mapping $S : A_0(V_1) \times \cdots \times A_0(V_n) \to A_0(\mathbb{T})$ defined by $S(v_1, \ldots, v_n) = v_1 + \cdots + v_n$ is open (where $\forall V \subset \mathbb{T}$, $A_0(V) = \{v \in A_0(\mathbb{T})\,;\ \text{support } v \subset V\}$, equipped with the $A(\mathbb{T})$ norm).

Now, if our linear mapping ℓ were not continuous, there would then exist $t_0 \in \mathbb{T}$ such that for each neighborhood V to t_0, ℓ restricted to $A_0(V)$ would not be continuous (standard Borel-Lebesgue argument). Then one could construct a sequence $(u_n,\ n \in \mathbb{N})$ in $A_0(\mathbb{T})$ with $\|u_n\|_{A(\mathbb{T})} \le \frac{1}{n}$, support $(u_n) \subset]t_0 - \frac{1}{n}, t_0 + \frac{1}{n}[$ and $\|\ell(u_n)\|_B \ge 1$. But for each n, $u_n \in A_{K_n}(\mathbb{T})$ for some $K_n \in \mathcal{J}$, which care be choosen included in $]t_0 - \frac{1}{n}, t_0 + \frac{1}{n}[$. Put $K = \big(\bigcup_{n\in\mathbb{N}} K_n\big) \cup \{t_0\}$. K is still in $\mathcal{J}$ and for each n, $u_n \in A_K$. So ℓ restricted to $A_K(\mathbb{T})$ is not continuous. Contradiction.

4. Multipliers of $U(\mathbb{T})$.

$U(\mathbb{T})$, as defined in the introduction, is a Banach space for the norm $\|u\|_{U(\mathbb{T})} =$ support $\Big\{\Big\|\sum_{-N}^{+N} \hat{u}(n)e^{int}\Big\|_\infty;\ \ N = 0, 1, \ldots\Big\}$: see J. P. Kahane [2] where it is proven that if $(\varepsilon_n,\ n \in \mathbb{N})$ is a sequence decreasing to zero, then $\sum_1^\infty \varepsilon_n \frac{\sin nt}{n}$ defines an element u_0 of $U(\mathbb{T})$ such that there exists $v \in A(\mathbb{T})$ with the product $uv \notin U(\mathbb{T})$. Due to the ultra-bornological property of $A_0(\mathbb{T})$ we have the following:

Proposition 3. — *Let $u \in U(\mathbb{T})$ be such that for each $v \in A_0(\mathbb{T})$, $uv \in U(\mathbb{T})$. Then for each $v \in A(\mathbb{T})$, $uv \in U(\mathbb{T})$.*

Proof. — $\ell : v \rightsquigarrow uv$ defines a linear mapping of $A_0(\mathbb{T})$ into $U(\mathbb{T})$. For each K, compact subset of $\mathbb{T}$ of Lebesgue measure zero the restriction of ℓ to $A_K(\mathbb{T})$ has a closed graph (use the uniform convergence implied by the $A(\mathbb{T})$ convergence). $A_K(\mathbb{T})$ being Banach, this restriction is continuous, by the closed graph theorem. So ℓ is continuous by proposition 2. Now take $v \in A(\mathbb{T})$ and $\forall n$, $v_n \in A_0(\mathbb{T})$, with $v_n \xrightarrow[n\to\infty]{} v$ in the $A(\mathbb{T})$ norm (proposition 1): $(v_n,\ n \in \mathbb{N})$ is Cauchy in $A_0(\mathbb{T})$, so $(uv_n,\ n \in \mathbb{N})$ is Cauchy in $U(\mathbb{T})$, so $(uv_n,\ n \in \mathbb{N})$ converges in $U(\mathbb{T})$. But the limit has to be uv (by uniform convergence). So $uv \in U(\mathbb{T})$.

5. Remarks.

1. — Spaces of continuous functions on a compact X locally constant on a dense subset of X appear in several papers: see [1] for a bibliography. In [1] the ultrabornological property of such spaces, in case X is metrisable, is proven. Our paper is inspired by [1].

2. — It is easy to construct an $A(\mathbb{T})$ function locally constant on a dense subset of $\mathbb{T}$, and not constant: take K a nowhere dense compact subset of $\mathbb{T}$, of positive Lebesgue measure, denote by k its indicator function, and take $v = k * h$, where h is the indicator function of the interval $[-1, +1]$. The difficulties appear when one ask K to be of Lebesgue measure zero.

3. — It is an exercise to prove that the Cantor "middle third" function, suitably periodicized and defining then an $E_0(\mathbb{T})$ function, does not define an $A_0(\mathbb{T})$ function.

4. — There exist perfect subcompacts K of $\mathbb{T}$ such that the only elements of $A_K(\mathbb{T})$ are the constants! See J. P. Kahane [2], page 21.

5. — To produce a non constant $A_0(\mathbb{T})$ function we used a non zero measure μ, supported by a compact subset K of $\mathbb{T}$ of Lebesgue measure zero, for which $\hat{\mu}(n) \underset{n\to\infty}{\longrightarrow} 0$ in a controlled way. The theorem of Ivasev-Musatov gives such a measure with $\hat{\mu} \in \mathcal{O}(|n|^{-\frac{1}{2}})$. In fact the existence of such a function (and such a measure, but only with $\hat{\mu} \in \mathcal{O}(|n|^{-\alpha})$ for some unspecified $\alpha > 0$) can be taken out from a 1936 paper of J. E. Littlewood [4].

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