$L^2(\mathcal{M})$ SPACE

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RÉSUMÉ. — Comme l'espace de fonctions sur un groupe localement compact et abélien $\ell^2(L^1)$, l'espace $\ell^2(\mathcal{M})$ est le plus grand espace solide de mesures admettant des mesures pour transformées de Fourier ; en fait pour qu'une mesure appartienne à $\ell^2(\mathcal{M})$, il suffit que pour chaque borélien la transformée de Fourier de la mesure induite soit une mesure.

ABSTRACT. — Similarly to the space $\ell^2(L^1)$ on a locally compact abelian group, the space $\ell^2(\mathcal{M})$ is the largest solid space of measures whose Fourier transforms are measures; in fact, in order for a measure to belong to $\ell^2(\mathcal{M})$ it suffices that the Fourier transforms of its restrictions to all Borel sets be measures.

1. $\mathcal{S}$ and $\mathcal{S}'$ on G.

A locally compact abelian group G can be represented as $G = G' \times \mathbb{R}^n$ where G' is locally compact and contains an open compact subgroup H-say. Then G'/H is discrete. Note that then $\widehat{G} = \widehat{G}' \times \mathbb{R}^n$ and that $\widehat{G}'$ has the same property as G', *i.e.* contains an open compact subgroup.

Define the space $\mathcal{S}(G)$ of test functions as follows

$$\mathcal{S}(G) = L^2(G'; \mathcal{S}(\mathbb{R}^n)) ,$$

where $\mathcal{S}(\mathbb{R}^n)$ is the usual space of C^∞ functions on $\mathbb{R}^n$ which are of rapid decrease together with all derivatives. The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined by the sequence of norms:

$$\|\varphi\|_{K_\ell} = \sup_x (1 + |x|^k) \sum_{|\alpha| \le \ell} |D^\alpha \varphi(x)| .$$

Accordingly, the topology on $\mathcal{S}(G)$ is defined by the sequence of norms

$$\|\Phi\|_{K_\ell} = \Big(\int_{G'} \|\Phi(g)\|^2_{K_\ell} \, dg \Big)^{1/2} .$$

The space $\mathcal{S}(\widehat{G})$ is defined similarly by symmetry.

The usefulness of these spaces manifests itself in the following observation concerning the Fourier transform $\widehat{\ }$:

$$\mathcal{S}(G) \xrightarrow{\widehat{\ }} \mathcal{S}(\widehat{G})$$

is a topological isomorphism.

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The Fourier transform on the topological dual $\mathcal{S}'$ of $\mathcal{S}$ can be defined by duality

$$\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle, \ \forall \varphi \in \mathcal{S} .$$

Again $\mathcal{S}'(G) \xrightarrow{\widehat{}} \mathcal{S}'(\widehat{G})$ is a topological isomorphism. A less obnoxious (but more restrictive) way of extending $\widehat{}$ is to use instead of $\mathcal{S}$ the space $\mathcal{K} = C_K(G)$ or the space $\ell^1(A)$. The definition $\langle \widehat{T}, \varphi \rangle = \langle T, \widehat{\varphi} \rangle$ for every φ with $\widehat{\varphi} \in \mathcal{K}$, *i.e.* for every $\varphi \in \widehat{\mathcal{K}}$ suffers of a lack of symmetry – this is not the case when $\ell^1(A)$ is used as a test space. $\mathcal{S}'$ definitions appears to be more general than either of the two possibilities mentioned above, $\mathcal{S}' \cap \mathcal{K}$ being dense in both $\mathcal{S}'$ and $\mathcal{K}$ and $\mathcal{S}' \cap \ell^1(A)$ being also dense in both $\mathcal{S}'$ and $\ell^1(A)$.

2. Definition of $\ell^p(L^q), \ell^1_p(\mathcal{M}), \ell^p(L^q)$, $(1 \le p, q \le +\infty)$.

We define $\ell^p(\mathcal{M})$ as the space of all Borel measures (or classes of functions) μ (or f) such that the function $t \to |\mu|(t+E) \in L^p(G)$, $(t \to \left(\int_{t+E} |f(s)|^q \, ds \right)^{1/q} \in L^p)$ with

$$(A) \qquad \|\mu\|_{\ell^p(\mathcal{M})} = \|\, |\mu|(\cdot + E)\|_{L^p(G)} \, ,$$

$$(A') \qquad \|f\|_{\ell^p(L^q)} = \|\|f\|_{L^q(t+E)}\|_{L^p(G)}$$

where E is a relatively compact neighborhood of the neutral element 0 in G. Different choices of E give rise to equivalent norms.

Recall that there are other equivalent ways of defining $\ell^2(\mathcal{M})$, $(\ell^p(L^q))$ notably:

$$(B) \qquad \|\mu\|_{\ell^p(\mathcal{M})} \sim \sup \left\{ \left\| \left(|\mu|(g+E) \right) \right\|_{\ell^p(\mathcal{G})} ; \mathcal{P} \right\} ,$$

$$(B') \qquad \|f\| \ell^p(L^q) \simeq \sup \left\{ \left\| \, \|f\|_{L^q(g+E)} \right\|_{\ell^p(\mathcal{G})} ; \mathcal{P} \right\} ,$$

where $\mathcal{P} = (g+E)_{g \in \mathcal{G}}$ is a maximal paving of G by disjoint translates of E (here we need to assume that E is symmetric. "Maximal" means that $\mathcal{P}$ cannot be enlarged to a paving of G by translates of E).

$$(C) \qquad \|\mu\|_{\ell^p(\mathcal{M})} = \left\| \left(\|\chi_\alpha \mu\|_{\mathcal{M}_b} \right) \right\|_{\ell^p} \, ,$$

$$(C') \qquad \|f\|_{\ell^p(L^q)} = \|\, \|\chi_\alpha f\|_{L^q}\|_{\ell^p}$$

where χ_α is a partition of unity of the form $\chi_\alpha(x) = \chi(x + g_\alpha)$, with $\chi \in \mathcal{K}$.

The least esthetically pleasing is the definition using the structural theorem and the definition of $\ell^p(M)$ on $\mathbb{R}^n$.

All properties of this space are developed in [BD1] and [BD3].

3. The maximality property of $L^2(\mathcal{M})$.

Let $\mu \in \mathcal{M} \cap \mathcal{S}'$. Denote by $L^{\infty 1}_{|\mu|}$ the closure of $L^1_{|\mu|} \cap L^\infty_{|\mu|}$ in $L^\infty_{|\mu|}$. The following statements are equivalent:

a) $\mu \in \ell^2(\mathcal{M})$,

b) $(f\,d\mu)^\wedge \in \mathcal{M}, \forall f \in L^{\infty 1}_{|\mu|}$,

c) $(f\,d\mu)^\wedge \in \mathcal{M}, \forall f \in C_0(G)$,

d) $(\chi_E d\mu)^\wedge \in \mathcal{M}, \forall E \in \mathcal{B}(G)$.

In the above statement $C_0(G)$ is the space of continuous functions vanishing at ∞ (the closure of $\mathcal{K}$ in L^∞) and $\mathcal{B}(G)$ is the algebra of Borel subsets of G.

We precede the proof with a number of remarks.

1. The implication $c) \Rightarrow a)$ includes the Bertrandias' version of Littlewood-Edwards' theorem (1984), also Demailly (1984) [B1], [De].

2. $b) \Rightarrow a)$ includes the result of Dupuis (1982) [D] (see also [BD2], [S3] and [S4]).

3. $a)$ implies the condition

b') $(f\,d\mu)^\wedge \in \ell^\infty(L^2) \subset \mathcal{M}$ for all $f \in L^\infty_{|\mu|}$

(see below) which *a priori* is stronger than $b)$.

4. One could also consider the condition $(f\,d\mu)^\wedge \in \mathcal{M}$ for all unimodular f. If only real $f-s$ are allowed then the condition is equivalent to $c)$. If all complex $f-s$ are allowed, then the condition is equivalent to *b')*

4. The local property of $(\widehat{\ell^2(\mathcal{M})})$.

We propose this generalisation of a result of J. J. Fournier [F]. Let $\sigma \in (\mathcal{S}')^+$, ($\sigma \in \mathcal{M}^+ \cap \mathcal{S}$) ; the following statements are equivalent:

a) there exists a neighborhood V of $0 \in \Gamma$ such that $\widehat{\sigma}_{|V} \in L^2(V)$;

b) $\widehat{\sigma} \in \ell^\infty(L^2)$;

c) $\sigma \in \ell^2(\mathcal{M})$.

A nonnegative distribution, by Schwartz is a nonnegative measure. By the fact $\widehat{\ell^2(\mathcal{M})} \subset \ell^\infty(L^2)$ it is sufficient to proof $a \Longrightarrow c)$. Let φ a test function with support in V, such that $\widehat{\varphi} > 0$. The condition *a)* implies $\varphi \cdot \widehat{\sigma} \in L^2$, *i.e.* $(\varphi \cdot \widehat{\sigma})^\wedge = \widehat{\varphi} * \sigma \geq 0$; the continuity of $\widehat{\varphi}$ assume the existence of an open set $\mathcal{O}$ and a scalar $\lambda > 0$ such that $\lambda(\widehat{\varphi} * \sigma) \geq \chi_\sigma * \sigma$ *i.e.* $\sigma \in \ell^2(\mathcal{M})$.

For the space $\widehat{\ell^2(\mathcal{M})}$, $1 \leq p \leq +\infty$, look the Bertrandias' result [1984] [B1].

5. Integral operators.

We recall a few facts about domains of integral operators which will be used in the proof of the statement in **3** (see [S1], [S2], [S3]).

We consider two measure spaces $(T, d\lambda)$ an $(S, d\nu)$ and denote by $L^0(T)$ (or respectively $L^0(S)$) the space of measurable, finite a.e. complex valued functions, equipped with the topology of convergence in measure on all sets of finite measure. An integral operator from $L^0(S)$ to $L^0(T)$ is given by the formula

$$Kf(t) = \int_s K(t,s)f(x)\ ds,\ \ f \in D_K\ ,$$

where $K \in L^0(T \times S)$ and D_K-the proper domain of K consists of all functions $f \in L^0(S)$ for which the integral in question exists as a Lebesgue integral for almost all $t \in T$.

Besides the proper domain, we associate with K in a canonical way its extended (or maximal) domain $\widetilde{D}_K$. It is a solid complete topological vector subspace of $L^0(S)$ with the following properties:

i) $D_K \subset \widetilde{D}_K$ with dense inclusion;

ii) $K : D_K \subset \widetilde{D}_K \to L^0$ is continuous and thus can be extended by continuity to an operator $\widetilde{K} : \widetilde{D}_K \to L^0$.

iii) If $L \subset L^0(S)$ is a solid topological vector space to which K can be extended by continuity from the dense subset $D_K \cap L$, then $L \subset \widetilde{D}_K$ with continuous inclusion and the extension of K to L is the restriction of $\widetilde{K}$ to L.

The extended domain of K is characterised by the following result:

$f \in \widetilde{D}_K$ if and only if for every sequence $g_n \in D_K$ of functions with disjoint supports satisfying $|g_n| \leq |f|$ a.e. we have $\sum |Kg_n(t)|^2 < \infty$ for a.e. $t \in T$.

The above result depends on properties of the Rademacker sequence, which appear in one form or another in various presentations of the results in **3**.

In the above condition it is sometime convenient to take g_n to be restrictions of the function f to disjoint subsets of S: this is possible if S and T are locally compact, ν, λ are Radon measures and K is continuous. In this case $\widetilde{D}_K \subset D_{K\,\mathrm{loc}}$ *i.e.* for every $f \in \widetilde{D}_K$ and E-a compact in S, $\chi_E f \in D_K$.

6. Application to the Fourier transform.

We apply the results of **5** to the case when $S = X$ is a locally compact abelian group, $T = \widehat{X} = \Xi$ is its group of characters.

We write the Fourier transform of the measure $f\,d\mu$ appearing in **3** in the form $\left(\frac{d\mu}{d|\mu|}(x)\right)$ denoting the Radon Nikodym derivative)

$$(f\,d\mu)^\wedge(\xi) = \int_X \xi(-x)f(x)\ d\mu(x) = \int_X \xi(-x)\frac{d\mu(x)}{d|\mu|}f(x)\ d|\mu|(x)\ .$$

Accordingly, we consider the integral operator K_μ with the kernel $K(\xi,x) = \xi(-x)\frac{d\mu(x)}{d|\mu|}$, on the measure spaces $(X, d|\mu|) \times (\Xi; d\xi)$ where $d\xi$ is the Haar measure.

Since $\left|\frac{d\mu}{d|\mu|}(X)\right| = 1$, $|\mu|$-almost everywhere, for the purpose of determining the domains of K_μ we may replace the factor $\frac{d\mu}{d|\mu|}$ by 1, and consider the operator, still denoted by K_μ:

$$K_\mu f(\xi) = \int_X \xi(-x) f(x) d|\mu|(x) .$$

It is obvious that $D_{K_\mu} = L^1(X, d|\mu|) = L^1_\mu$. To determine the extended domain $\widetilde{D}_{K_\mu}$ we use the remarks in **4**. Let E be a relatively compact neighborhood of 0 in X and $x_n + E$, $x_n \in X$ be disjoint. If $f \in \widetilde{D}_{K_\mu}$ then $\chi_{x_n+E} f \in D_K$ and $\sum |K_\mu(\chi_{x_n+E} f)|^2 = \sum |\int_{x_n+E} \xi(-x) f(x)\, d|\mu|(x)|^2 = \sum |\int_E \xi(-x) f(x_n + x)\, d|\mu|(x_n + x)|^2 < \infty$ for almost every ξ, in particular for almost every ξ in a neighborhood of 1 in Ξ , say F in which e.g. $|\xi(-x) - 1| < 1/2$ for all $x \in E$. Since $\widetilde{D}_{K_\mu}$ is solid, we may assume that $f \geq 0$. For $\xi \in F$ and $x \in E$ we have $\operatorname{Re} \xi(-x) > 3/4$ and

$$\frac{9}{16} \sum \left(\int_E f(x_n + x) d|\mu|(x_n + x)\right)^2 \leq \sum \left(\int_E \xi(-x) f(x_n + x) d|\mu|(x_n + x)\right)^2 < \infty .$$

It follows that $f \in \ell^2(L^1_\mu)$ and that $\widetilde{D}_{K_\mu} \subset \ell^2(L^1_\mu)$.

Since $\ell^2(L^1_\mu)$ is solid, the reverse inclusion also follows from remarks in **3**, once we observe that L^1_μ is dense in $\ell^2(L^1_\mu)$ and that the mapping $K_\mu : L^1_\mu \subset \ell^2(L^1_\mu) \to L^0(\Xi)$ is continuous. The first observation is obvious; as concerns the second observation we actually have continuity of $K_\mu : L^1_\mu \subset \ell^2(L^1_\mu) \to \ell^\infty(L^2)(\Xi)$.

Even though this this result is well known, we give a proof which seems to be of some interest.

We derive an estimate of the form

$$\int_\Xi \varphi(\xi) |K_\mu f(\xi)|^2\, d\xi \leq C \|f\|_{\ell^2(L^1_\mu)} ,$$

for $f \in L^1_\mu$ and $\varphi \in \mathcal{K}(\Xi)$, $\varphi \geq 0$ such that $\widehat{\varphi} \in \ell^1(L^\infty)(X)$; it is known that functions φ like this exist. We consider a partition of unity on G of the form $1 = \sum\limits_{\alpha \in A} \psi(x + x_\alpha)$ where $\psi \in \mathcal{K}_+$ and for every $x \# \{\alpha; \psi(x + x_\alpha) \neq 0\} \leq m$. If $f \in \ell^2(L^1_\mu)$ then $\left(\int_X \psi(x + x_\alpha) |f(x)|\, d|\mu|\right) \in \ell^2$, in particular the set $\left\{\alpha;\ \int_X \psi(x + x_\alpha) |f(x)|\, d|\mu| \neq 0\right\}$ is enumerable and $\|f\|_{\ell^2(L^1_\mu)} = \left(\sum\limits_\alpha \left(\int_X \psi(x + x_\alpha) |f(x)|\, d\mu\right)^2\right)^{1/2}$.

The integral $\int_\Xi \varphi(\xi) |K_\mu f(\xi)|^2\, d\xi$ can be written in the form

$$\begin{aligned}\int_{\Xi}\varphi(\xi)|K_\mu f(\xi)|^2\,d\xi &= \int_X\int_X \widehat{\varphi}(x-y)f(x)\overline{f(y)}d|\mu|(x)d|\mu|(y)\\ &\le \sum\int_X\int_X \widehat{\varphi}(x-y+x_\alpha-x_\beta)\psi(x+x_\alpha)\psi(y+x_\beta)\\ &\qquad\qquad\cdot f(x)\overline{f(y)}\,d|\mu|(x)d|\mu|(y)\\ &\le \sum a_{\alpha\beta}\int_X \psi(x+x_\alpha)|f(x)|\,d|\mu|(x)\int_X\psi(y+x_\beta)f(y)\,d|\mu|(y)\end{aligned}$$

where $a_{\alpha\beta} = \max\{|\widehat{\varphi}(z+x_\alpha-x_\beta)|,\ z\in U-U\}$, where $U = \operatorname{Supp}\psi$. Since for every α and z $\#\{\beta; z\in U-U+x_\alpha-x_\beta\} \le$ const, it follows that $\sum_\alpha a_{\alpha\beta}$, $\sum_\beta a_{\alpha\beta} \le$ const $\|\widehat{\varphi}\|_{\ell^1(L^\infty)}$ and the matrix $(a_{\alpha\beta})$ defines a bounded operator in $\ell^2(A)$. This readily implies the desired conclusion.

7. Proof of the maximality property of $\ell^2(\mathcal{M})$.

We proceed now to prove the statement made in **3**.

If $\mu\in\ell^2(\mathcal{M})$ then by **6** $(f\,d\mu)^\wedge\in\ell^\infty(L^2)$, in particular $a)\Rightarrow b)$, $a)\Rightarrow c)$ and $a)\Rightarrow d)$. We shall prove next that $b)\Rightarrow a)$ and then that $c)\Rightarrow b)$ and that $d)\Rightarrow b)$.

Supposed that μ satisfies $b)$ and $c)$, F be a compact subset of $\widehat{G}$. Then for $f\in L^{\infty 1}_\mu$, $(f\,d\mu)^\wedge\in\mathcal{M}(F)$ and by closed graph theorem $\|(f\,d\mu)^\wedge\|_{\mathcal{M}(F)}\le$ const $\|f\|_\infty$ for every $f\in L^{\infty 1}_\mu$. In particular, if $f\in L^1_\mu\cap L^\infty_\mu$, $(f\,d\mu)^\wedge\in C_0\subset L^1_{\text{loc}}$ and $(f\,d\mu)^\wedge)K_\mu f$. The last inequality implies that $K_\mu : L^\infty_\mu\cap L^1_\mu\subset L^{\infty 1}_\mu\to L^1_{\text{loc}}\subset L^0$ is continuous and since $L^{\infty 1}_\mu$ is solid, the discussion in **5** implies that $L^{\infty 1}_\mu\subset\ell^2(L^1_\mu)$ with continuous inclusion. In particular, if $\{x_\alpha+E\}$ is a paving in X, then every finite sum $\sum\chi_{x_\alpha+E}$ belongs to $L^1_\mu\cap L^\infty_\mu$ with $\|\sum\chi_{x_\alpha+E}\|_{L^\infty}\le 1$ and therefore $\|\sum\chi_{x_\alpha+E}\|^2_{\ell^2(L^1_\mu)} = \sum|\mu|(x_\alpha+E)^2\le$ const, *i.e.* $\mu\in\ell^2(\mathcal{M})$. We have shown that $b)\Rightarrow a)$.

To prove that $c)\Rightarrow b)$ we first note that $c)$ implies (using the closed graph theorem) that $\|(f\,d\nu)^\wedge\|_{\mathcal{M}(F)}\le C_F\|f\|_\infty$, $F\subset\subset\Xi$, for all $f\in C_0$. It suffices to show that the inequality persists for $f\in L^\infty_\mu\cap L^1_\mu$, in particular for $f\in L^\infty$ with compact support (the latter forming a dense subset in $L^{\infty 1}_\mu$). For such f, using Lusin's theorem and regularity of (the topological space) X, we find a sequence $f_n\in\mathcal{K}$ such that $f_n\to f$ in measure, that $\|f_n\|_\infty\le\|f\|_\infty$ and that supports of f_n remain in a fixed compact. Then $\|(f_n\,d\mu)^\wedge\|_{\mathcal{M}(F)} = \|(f_n\,d\mu)^\wedge\|_{L^1(F)}\le$ const $\|f_n\|_\infty\le$ const $\|f\|_\infty$ and one can pass to the limit in the left hand side using the dominated convergence theorem.

Proof that $d)\Rightarrow b)$.

The goal is to establish an inequality $\langle(f\ d\mu)^\wedge, g\rangle\le C_F\|f\|_\infty\,\|g\|_\infty$ for all $g\in C(F)$ where $F\subset\subset\Xi$ is fixed but arbitrary and $f\in L^{\infty 1}_\mu$. By a density argument, it suffices to obtain the estimate for $f\ge 0$ and finite valued. We begin with the case when $f=\chi_E$ is the characteristic function of $E\subset X$.

If $\mu = \mu_1 + i\mu_2$ satisfies $d)$ then also $\bar{\mu} = \mu_1 - i\mu_2$ satisfies $d)$ because of the formula $(\chi_E\, d\bar{\mu})^\wedge = \overline{(\chi_E\, d\mu)^\wedge}^\vee$ where $\varphi^\vee(\xi) = \varphi(-\xi)$; $\bar{\nu}^\vee$ of a measure ν is a measure.

It follows that $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ satisfy $d)$ and we may assume that μ is real. The Hahn decomposition allows us to assume that $\mu \geq 0$. In this case we write observe that is $\widehat{g} \in \widehat{C}(F)$ then $\widehat{g} \in \widehat{C}(-F)$ and $\widehat{g} = \widehat{g}_1 + ig_2$ where $g_j \in C(F \cup -F)$. Hence

$$\langle(\chi_E\, d\mu)^\wedge, g\rangle = \langle \chi_E\, d\mu, \widehat{g}\rangle = \langle \chi_E\, d\mu, \widehat{g}_1\rangle + i\langle \chi_E\, d\mu, \widehat{g}_2\rangle$$

and considering each term separately we may assume that $\widehat{g}$ is real we write then $\widehat{g} = \chi_{E_1}\widehat{g} + \chi_{E_2}\widehat{g}$ where $\chi_{E_1}\widehat{g} \geq 0$ and $\chi_{E_2}\widehat{g} \leq 0$.

$$\begin{aligned} |\langle \chi_E d\mu, \widehat{g}\rangle| &= |\langle \chi_E d\mu, \chi_{E_1}\widehat{g}\rangle + \xi_E\, d\mu, \chi_{E_2}\widehat{g}| \\ &\leq \langle d\mu, \chi_{E_1}\widehat{g}\rangle + \langle d\mu, -\chi_{E_2}\widehat{g}\rangle \\ &= \langle(\chi_{E_1} d\mu)^\wedge, g\rangle - \langle(\chi_{E_2}\, d\mu)^\wedge, g\rangle\ . \end{aligned}$$

It follows that $|\langle(\chi_E\, d\mu)^\wedge, g\rangle|$ is bounded uniformly in $E \in \mathcal{B}(X)$ by a constant depending on g. By Banach Steinhauss theorem $|\langle(\chi_E\, d\mu)^\wedge, g\rangle| \leq M\|g\|_\infty$ for all $E \in \mathcal{B}(X)$.

Suppose now that f is nonnegative and takes finitely many values. Such an f can be represented in the form

$$f(x) = \sum_{\ell=1}^{n} a_\ell \chi_{E_\ell}(x)\ ,$$

where $a_1, \ldots, a_n > 0$, $E_1 \supset E_2 \supset \cdots \supset E_n$, $|\mu|(E_\ell) > 0$ and $\overline{E}_1$ is compact. Clearly then $\|f\|_\infty = a_1 + \cdots + a_n$ and

$$\begin{aligned} |\langle(f\ d\mu)^\wedge, g\rangle| &= |a_1\langle(\chi_{E_1}\ d\mu)^\wedge, g\rangle + \cdots + a_n\langle(\chi_{E_n}\ d\mu)^\wedge, g\rangle| \\ &\leq a_1\,|\langle(\chi_1\ d\mu)^\wedge, g\rangle| + \cdots + a_n\,|\langle(\chi_n\ d\mu)^\wedge, g\rangle| \\ &\leq M(a_1 + \cdots a_n) = M\,\|f\|_\infty\,\|g\|_\infty\ . \end{aligned}$$

Since in this case $\langle f\ d\mu\rangle^\wedge$ is a function in L^1_{loc}, the inequality can be extended, as in the proof of $c) \Rightarrow b)$, to all $g \in L^\infty(F)$ and the proof of $d) \Rightarrow b)$ is complete.

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