On the general faces of the moment polytope

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1. Introduction

Let X be a projective algebraic variety endowed with an action of a connected reductive group G, both defined over $\mathbb{C}$. Let $\mathcal{L}$ be an ample line bundle over X. Assume that $\mathcal{L}$ is G-linearized, that is, G acts on the total space of $\mathcal{L}$ compatibly with its action on X, and linearly on fibers. Then the space $\Gamma(X,\mathcal{L})$ of global sections of $\mathcal{L}$ is a rational G-module. Its decomposition into simple modules is described by its highest weight vectors, that is, the set $\Gamma(X,\mathcal{L})^{(B)}$ of eigenvectors of a fixed Borel subgroup B of G.

To study simultaneously the G-modules $\Gamma(X,\mathcal{L}^{\otimes n})$ for all positive integers n, let us introduce a set $P_G(X,\mathcal{L})$ as follows. Let $\mathcal{X}$ be the character group of B and set $\mathcal{X}_{\mathbb{Q}} := \mathcal{X} \otimes_{\mathbb{Z}} \mathbb{Q}$. For each dominant weight $\chi \in \mathcal{X}$, let $V(\chi)$ be a simple G-module with highest weight χ. Set

$$P_G(X,\mathcal{L}) := \{p \in \mathcal{X}_{\mathbb{Q}} \mid V(np) \text{ occurs in } \Gamma(X,\mathcal{L}^{\otimes n}) \text{ for some positive integer } n\}.$$

In other words, $P_G(X,\mathcal{L})$ is the set of all $p \in \mathcal{X}_{\mathbb{Q}}$ which can be written as χ/n for some positive integer n and some weight χ occuring in $\Gamma(X,\mathcal{L}^{\otimes n})^{(B)}$. Then $P_G(X,\mathcal{L})$ is a convex polytope in $\mathcal{X}_{\mathbb{Q}}$, see [B]. By definition, $P_G(X,\mathcal{L})$ is contained in the Weyl chamber $\mathcal{C}$ of dominant rational weights.

We shal call $P_G(X,\mathcal{L})$ the *moment polytope*. It is indeed closely related to the image of the moment map of symplectic geometers: Assume that X is a smooth closed G-stable subvariety of projective space $\mathbb{P}(V)$ where V is a rational G-module. Assume furthermore that $\mathcal{L}$ is the restriction to X of the line bundle $\mathcal{O}(1)$ with its natural G-linearization. Let K be a maximal compact subgroup of G and let $(\ ,\ )$ be a K-invariant Hermitian inner product on V. These data define a moment map

$$\mu : \mathbb{P}(V) \to \mathrm{Lie}(K)^*$$

(the dual over $\mathbb{R}$ of the Lie algebra of K) by the formula

$$\mu([v])(\xi) = \frac{1}{2\pi i}\frac{(v,\xi v)}{(v,v)}$$

where $v \in V$ and $\xi \in \mathrm{Lie}(K)$. The intersection of $\mu(X)$ with a positive Weyl chamber turns out to be a convex polytope with rational vertices, which identifies to the closure of $P_G(X,\mathcal{L})$. See Mumford's appendix to [N], and also [B].

Back to the algebraic setting, let $\mathcal{C}_X$ be the smallest face of the Weyl chamber which contains $P_G(X,\mathcal{L})$ (it turns out that $\mathcal{C}_X$ is independent of $\mathcal{L}$). A face of $P_G(X,\mathcal{L})$ which meets the relative interior of $\mathcal{C}_X$ will be called *general*.

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In this article, we describe the general faces of the moment polytope in terms of fixed points of certain subtori of G. When X is normal, we describe the corresponding sets of highest weight vectors as well. As applications, we determine the general vertices of polytopes associated with decompositions of tensor products of simple modules, and with their Schur powers. In the former case, we obtain an inductive description of the facets of the polytope.

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2. General results

If G fixes a point x of X, then G acts on the fiber $\mathcal{L}_x$ by a character χ, and $P_G(X,\mathcal{L})$ contains $-\chi$ (indeed, there exists a positive integer n and a section $\sigma \in \Gamma(X,\mathcal{L}^{\otimes n})$ such that $\sigma(x) \neq 0$; because G is reductive and fixes x, we may assume that σ is an eigenvector of G, then its weight is $-n\chi$). We shall see that fixed points of subtori of G contribute to $P_G(X,\mathcal{L})$ as well. For this, we need more notation.

Fix a maximal torus T of B. Let T' be a subtorus of T, with fixed point subset $X^{T'}$ in X. Let G' be the centralizer of T' in G. Then G' is a connected reductive group; it acts on $X^{T'}$ and stabilizes each irreducible component X' of that space. Set $B' := B \cap G'$; then B' is a Borel subgroup of G' containing T. The character group of B' identifies with $\mathcal{X}$, and the corresponding Weyl chamber $\mathcal{C}'$ contains $\mathcal{C}$. Furthermore, $\mathcal{L}$ restricts to an ample G'-linearized line bundle over X', which defines a convex polytope $P_{G'}(X',\mathcal{L})$ contained in $\mathcal{C}'$.

Recall that the subtori of T correspond bijectively to the linear subspaces of $\mathcal{X}_{\mathbb{Q}}$, by assigning to a subtorus T' the span of the characters of T which restrict trivially to T'. For a face F of $P_G(X,\mathcal{L})$, with direction $\mathrm{lin}(F)$, let $T_F \subset T$ be the subtorus associated with $\mathrm{lin}(F)$.

Theorem 1. *(i) For any subtorus T' of T with centralizer G' and for any irreducible component X' of $X^{T'}$, the set $P_{G'}(X',\mathcal{L}) \cap \mathcal{C}$ is contained in $P_G(X,\mathcal{L})$.*

(ii) Any general face F of $P_G(X,\mathcal{L})$ is the intersection $P_{G_F}(X_F,\mathcal{L}) \cap \mathcal{C}$ for a unique irreducible component X_F of X^{T_F}.

(iii) In particular, if $x \in X$ is a T-fixed point such that T acts on the fiber $\mathcal{L}_x$ via the opposite of a dominant weight, then this weight is in $P_G(X,\mathcal{L})$. Furthermore, all general vertices of $P_G(X,\mathcal{L})$ are obtained in this way; in particular, all such vertices are integral.

Remark. Statements (ii) and (iii) do not extend to arbitrary faces of $P_G(X,\mathcal{L})$. Consider indeed the G-variety $X = \mathbb{P}(V)$ where V is a rational G-module, and the line bundle $\mathcal{L} = \mathcal{O}(1)$ with its natural G-linearization. Then the G-module $\Gamma(X,\mathcal{L}^{\otimes n})$ is the dual of the n-th symmetric power S^nV. Thus, $P_G(X,\mathcal{L})$ contains 0 iff there exists a non constant G-invariant polynomial function on V. If moreover G is semisimple, then 0 is a vertex of $P_G(X,\mathcal{L})$; the corresponding subtorus is T. The irreducible components X' of X^T are the projective spaces of the T-weight subspaces of V; the corresponding polytopes $P_{G'}(X',\mathcal{L})$

consist of the opposites of the weights. Thus, if 0 is not a T-weight in V, but V admits non constant G-invariant polynomial functions, then (ii) does not hold for X.

There are many examples of pairs (G, V) satisfying these assumptions, e.g. $(\mathrm{SL}_r, S^2\mathbb{C}^r)$ for $r \geq 3$. In this case, using the explicit description of the SL_r-modules $S^n(S^2\mathbb{C}^r)$, one sees that $P_G(X, \mathcal{L})$ is the convex hull of 0 and the points $2\pi_j/(r-j)$ for $1 \leq j \leq r-1$, where $\pi_1, \ldots, \pi_{r-1}$ are the fundamental weights of SL_r. In particular, $P_G(X, \mathcal{L})$ has non integral vertices as well if $r \geq 4$.

In the setting of symplectic geometry, statement (iii) is known, and (i), (ii) can be deduced from the "local cross-section theorem" [G-S]. But our algebraic approach gives a characterization of all general faces, as follows.

For a face F of $P_G(X, \mathcal{L})$, choose a point $f \in F^0$ (the relative interior of F) and define the *tangent cone to* $P_G(X, \mathcal{L})$ *at* F as the convex cone generated by $-f + P_G(X, L)$. This cone depends only on F, and describes the shape of $P_G(X, \mathcal{L})$ along that face. The direction of F is the largest linear subspace contained in its tangent cone.

Theorem 2. *For a subtorus $T' \subset T$ and an irreducible component $X' \subset X^{T'}$ such that $P_{G'}(X', \mathcal{L})$ meets $\mathcal{C}_X^0$, the following conditions are equivalent:*

(i) $P_{G'}(X', \mathcal{L}) \cap \mathcal{C}$ is a face F of $P_G(X, \mathcal{L})$.

(ii) For generic $x \in X'$, all nonzero weights of T' in $T_x(X)/T_x(Ux)$ are contained in an open half space.

(Here $T_x(X)$ denotes the Zariski tangent space to X at x; U is the unipotent part of B, and Ux is the U-orbit through x). Then the convex cone generated by these weights identifies with the opposite of the tangent cone to $P_G(X, \mathcal{L})$ at F.

If $\mathcal{C}_X = \mathcal{C}$, we shall see in the proof that the isotropy group U_x is trivial for x as in (ii), so that Ux is isomorphic to U. If moreover X is smooth, then condition (ii) can be formulated in more combinatorial terms. Indeed, as a component of the fixed point set of a torus, X' is smooth as well. It follows that the structure of the T'-module $T_x(X)$ is independent of $x \in X'$. In other words, the multiset of weights of T' in $T_x(X)$ (with their multiplicity) is independent of x. Thus, (ii) is equivalent to:

(ii)' For some $x \in X'^T$, the multiset of nonzero weights of T' in $T_x(X)$, minus the multiset of restrictions to T' of roots of (B, T), consists of weights in an open half space.

The polytope $P_G(X, \mathcal{L})$ only entails information on the asymptotic behaviour of the sets $\Gamma(X, \mathcal{L}^{\otimes n})^{(B)}_{np}$ of eigenvectors of B of weight np, for rational weights p. Our next result gives a more precise description of these sets.

Theorem 3. *Assume that X is normal. Let F be a general face of $P_G(X, \mathcal{L})$; let T_F and X_F be as above. Then, for any positive integer n and for any $p \in F \cap \mathcal{C}_X^0$, the restriction map*

$$\Gamma(X, \mathcal{L}^{\otimes n})^{(B)}_{np} \to \Gamma(X_F, \mathcal{L}^{\otimes n})^{(B_F)}_{np}$$

is bijective. In particular, for any general vertex p of $P_G(X,\mathcal{L})$, with corresponding irreducible component X_p of X^T, the restriction map

$$\Gamma(X,\mathcal{L}^{\otimes n})^{(B)}_{np} \to \Gamma(X_p,\mathcal{L}^{\otimes n})$$

is bijective.

As a consequence of Theorem 3, we see that for normal X such that X^T is finite, all general vertices of $P_G(X,\mathcal{L})$ are weights of elements of $\Gamma(X,\mathcal{L})^{(B)}$.

3. The polytope associated with decomposition of tensor products

Let B^- be the Borel subgroup of G such that $B^- \cap B = T$. Consider the projective variety

$$X = G/B^- \times G/B^-$$

with diagonal action of G. Any character χ of T defines a homogeneous line bundle $\mathcal{L}_\chi$ on G/B^-. Recall that $\mathcal{L}_\chi$ is generated by its global sections (resp. ample) if and only if $-\chi$ is dominant (resp. regular dominant); the G-module $\Gamma(G/B^-,\mathcal{L}_\chi)$ is isomorphic to $V(-\chi)$.

Let λ, μ be regular dominant weights. Then

$$\mathcal{L} := \mathcal{L}_{-\lambda} \times \mathcal{L}_{-\mu}$$

is an ample G-linearized line bundle on X, and

$$\Gamma(X,\mathcal{L}) = V(\lambda) \otimes V(\mu).$$

Thus, $P_G(X,\mathcal{L})$ is the rational polytope

$$P_G(\lambda,\mu) := \{p \in \mathcal{X}_{\mathbb{Q}} \mid V(np) \text{ occurs in } V(n\lambda)\otimes V(n\mu) \text{ for some positive integer } n\}.$$

Remark. The real points of $P_G(\lambda,\mu)$ are the intersection in $Lie(K)^*$ of the subset $K\lambda + K\mu$ with the positive Weyl chamber; here K is a maximal compact subgroup of G such that $K \cap T$ is the maximal compact subgroup of T, and we identify λ, μ to elements of $Lie(T)^* \subset Lie(K)^*$.

Indeed, the K-variety X identifies to $K\lambda \times K\mu \subset Lie(K)^* \times Lie(K)^*$, and the moment map associated with $\mathcal{L}$ is inclusion followed by projection to the diagonal $Lie(K)^*$.

In the case where $G = \mathrm{GL}_n$ with its standard torus T of diagonal invertible matrices, we take K to be the unitary group. Then $Lie(K)^*$ identifies with the space of Hermitian matrices, and the positive Weyl chamber consists of diagonal matrices with real entries in decreasing order. Thus, the K-orbits in $Lie(K)^*$ are the sets of Hermitian matrices with a prescribed spectrum; and $P_G(\lambda,\mu)$ consists of all spectra of sums of two Hermitian matrices with spectra λ, μ.

Clearly, $P_G(\lambda,\mu)$ contains $\lambda+\mu$, a regular dominant weight. It follows that $\mathcal{C}_X = \mathcal{C}$: the general faces of $P_G(\lambda,\mu)$ are those which contain regular (rational) weights.

In order to describe the general vertices of $P_G(\lambda,\mu)$, we introduce the following notation. Let W be the Weyl group of (G,T) and let $\Phi \subset \mathcal{X}$ be the corresponding root system with the subset Φ^+ of positive roots associated with B, and the subset Δ of simple roots. For $w \in W$, let

$$\mathrm{Inv}(w) := w(\Phi^+) \cap \Phi^-$$

be the corresponding "inversion set".

Definition. A pair $(u,v) \in W \times W$ will be called *admissible* if there exists $w \in W$ such that $\mathrm{Inv}(w)$ is the disjoint union of $\mathrm{Inv}(u)$ and $\mathrm{Inv}(v)$. Then w is uniquely determined by (u,v); we shall call (u,v,w) an *admissible triple.*

For a subset I of Δ, let W_I be the subgroup of W generated by the s_α for $\alpha \in I$, and let w_I be the longest element of W_I. Then examples of admissible triples are $(u, w_I u, w_I)$ and $(u, w_I w_\Delta, u)$ where u is an arbitrary element of W_I.

Theorem 4. *With notation as above, the general vertices of $P_G(\lambda,\mu)$ are the regular dominant weights of the form $u(\lambda)+v(\mu)$ where (u,v,w) is an admissible triple. Under this assumption, $V(u(\lambda)+v(\mu))$ occurs in $V(\lambda)\otimes V(\mu)$ with multiplicity one, and the tangent cone to $P_G(\lambda,\mu)$ at $u(\lambda)+v(\mu)$ is generated by $-w(\Delta)$.*

In particular, the direction of the affine space generated by $P_G(\lambda,\mu)$ is the span of Φ, so that the dimension of that polytope is the semisimple rank ℓ of G. And all general vertices of $P_G(\lambda,\mu)$ are contained in precisely ℓ edges.

The fact that $V(u(\lambda)+v(\mu))$ occurs in $V(\lambda)\otimes V(\mu)$ is a special case of the former PRV conjecture, proved by Kumar and Mathieu [Ku], [Mat].

We now turn to the description of the general facets of $P_G(\lambda,\mu)$. Let Z be the connected center of G. Because G is the almost direct product of Z with a connected semisimple group, any $p \in \mathcal{X}_{\mathbb{Q}}$ decomposes uniquely as

$$p = p_Z + \sum_{\alpha\in\Delta} p_\alpha \alpha$$

where p_Z is a rational weight of Z, and the p_α's are in $\mathbb{Q}$. For a subset S of $\mathcal{X}_{\mathbb{Q}}$, we denote by S_α the multiset consisting of all coordinates p_α ($p \in S$) with their multiplicities.

Definition. For $\alpha \in \Delta$, a triple (u,v,w) in $W \times W \times W$ will be called *α-compatible* if

$$u(\Phi^+)_\alpha \cup v(\Phi^+)_\alpha = \Phi^+_\alpha \cup w(\Phi^+)_\alpha \text{ or } u(\Phi^+)_\alpha \cup v(\Phi^+)_\alpha = \Phi^-_\alpha \cup w(\Phi^+)_\alpha$$

where $\cup$ denotes union as multisets. We then set $\varepsilon_\alpha(u,v,w) = +1$ in the former case, and $\varepsilon_\alpha(u,v,w) = -1$ in the latter case.

The relation between α-compatible and admissible triples is discussed briefly in Remark (ii) below. Finally, we denote by Φ^α the sub-root system of Φ with basis $\Delta \setminus \{\alpha\}$, and by G^α the corrresponding connected reductive subgroup of G containing T. For a weight ω of T, we denote by $V^\alpha(\omega)$ the simple G^α-module with highest weight ω.

Theorem 5. *With notation as above, $P_G(\lambda,\mu)$ is the set of all dominant rational weights ν satisfying the inequalities*

$$\varepsilon_\alpha(u,v,w)\, w(\nu)_\alpha \le u(\lambda)_\alpha + v(\mu)_\alpha \qquad (*)$$

for all $\alpha \in \Delta$ and all α-compatible triples (u,v,w) such that $P_{G^\alpha}(u(\lambda),v(\mu))$ meets $w(\mathcal{C}^0)$. Equality holds in $(*)$ if and only if $w(\nu) \in P_{G^\alpha}(u(\lambda),v(\mu))$, and this defines a general facet F of $P_G(\lambda,\mu)$. For $\nu \in F$ and a positive integer n, the multiplicity of $V(n\nu)$ in $V(n\lambda)\otimes V(n\mu)$ is equal to the multiplicity of $V^\alpha(nw(\nu))$ in $V^\alpha(nu(\lambda))\otimes V^\alpha(nv(\mu))$. Finally, all general facets of $P_G(\lambda,\mu)$ are obtained in this way.

This inductive description of $P_G(\lambda,\mu)$ is in the spirit of a theorem of Klyachko, but both results are not equivalent. Specifically, let C_G be the cone of all triples (λ,μ,ν) of rational dominant weights such that $V(n\nu)$ occurs in $V(n\lambda)\otimes V(n\mu)$ for some positive integer n. In the case where G is the special linear group, Klyachko obtained an inductive description of C_G by linear inequalities (see [Kl] and also [F], [KT] for relevant recent work). Because $P_G(\lambda,\mu)$ is the intersection of C_G with the affine space $(\lambda,\mu)\times\mathcal{X}_{\mathbf{Q}}$, any facet F of $P_G(\lambda,\mu)$ is the intersection of this affine space with some facet of C_G. But only those facets of C_G which intersect transversally the affine space contribute to facets of $P_G(\lambda,\mu)$.

Examples. (i) In the case where $G = \mathrm{GL}_2$, there is a unique simple root α, and W is equal to $\{1,-1\}$. The α-compatible triples are all triples except for $(1,1,-1)$ and $(-1,-1,1)$. The set of inequalities deduced from Theorem 5 is equivalent to

$$|\lambda_\alpha - \mu_\alpha| \le \nu_\alpha \le \lambda_\alpha + \mu_\alpha.$$

Of course, this is also a direct consequence of the Clebsch-Gordan decomposition.

(ii) More generally, consider $G = \mathrm{GL}_n$ with the maximal torus T of diagonal invertible matrices, and the Borel subgroup B of upper triangular invertible matrices. Then W is the symmetric group S_n, and we can see Φ as the set of all pairs (j,k) such that $1 \le j,k \le n$ and $j \ne k$. The simple roots are the pairs $(r,r+1) =: \alpha_r$ for $\le r \le n-1$; the dominant weights are the decreasing sequences of n integers. For two such sequences λ, μ, we denote by $P_n(\lambda,\mu)$ the corresponding polytope.

For $u \in W$ and $1 \le r \le n-1$, denote by $l_r^+(u)$ (resp. $l_r^-(u)$) the numbers of pairs (j,k) of integers such $1 \le j < k \le n$ and $u(k) \le r < u(j)$ (resp. $u(j) \le r < u(k)$). Then the multiset $u(\Phi^+)_{\alpha_r}$ consists of: 1 with multiplicity $l_r^-(u)$, -1 with multiplicity $l_r^+(u)$, and 0 with multiplicity $\frac{n(n-1)}{2} - r(n-r)$. It follows that a triple (u,v,w) is α_r-admissible with sign $\pm$ if and only if:

$$l_r^\pm(u) + l_r^\pm(v) = l_r^\pm(w).$$

With obvious notation, the inequality $(*)$ translates into:

$$\pm \sum_{k=1}^{r} \nu_{w^{-1}(k)} \le \sum_{i=1}^{r} \lambda_{u^{-1}(i)} + \sum_{j=1}^{r} \mu_{v^{-1}(j)}$$

which matches with Klyachko's result.

Observe that this inequality and α_r-admissibility are unchanged when (u,v,w) is replaced by $(u'u, v'v, w'w)$ for u', v' and w' in the subgroup $S_r \times S_{n-r}$ of S_n. Thus, we may assume that $u^{-1}(1) > \cdots > u^{-1}(r)$ and $u^{-1}(r+1) > \cdots > u^{-1}(n)$, and similarly for v, w.

Finally, $G^{\alpha_r} = \mathrm{GL}_r \times \mathrm{GL}_{n-r}$ and

$$P_{G^{\alpha_r}}(u(\lambda), v(\mu)) = P_r(u(\lambda)_{\leq r}, v(\mu)_{\leq r}) \times P_{n-r}(u(\lambda)_{>r}, v(\mu)_{>r})$$

where $u(\lambda)_{\leq r} = (\lambda_{u^{-1}(1)}, \ldots, \lambda_{u^{-1}(r)})$ and $u(\lambda)_{>r} = (\lambda_{u^{-1}(r+1)}, \ldots, \lambda_{u^{-1}(n)})$. It would be interesting to obtain a more tractable form of the condition: $P_{G^{\alpha_r}}(u(\lambda), v(\mu))$ meets $w(\mathcal{C}^0)$. The latter consists of all $(t_1, \ldots, t_n) \in \mathbb{Q}^n$ such that $t_{w(1)} > \cdots > t_{w(n)}$.

Remarks. (i) For arbitrary connected reductive G, the facet of $P_G(\lambda,\mu)$ associated with an α-compatible triple (u,v,w) depends only on the left cosets of u, v and w modulo $W_{\Delta\setminus\{\alpha\}}$. This follows from the formula

$$w(\nu)_\alpha = \langle \nu, w^{-1}(\pi_\alpha)\rangle$$

where π_α is the fundamental weight corresponding to α, and from the fact that $W_{\Delta\setminus\{\alpha\}}$ is the isotropy group of π_α in W.

(ii) Let (u,v,w) be an admissible triple such that $u(\lambda)+v(\mu)$ is regular dominant. Then, by Theorem 4, the cone generated by the $w^{-1}u(\lambda) + w^{-1}v(\mu) - w^{-1}(\nu)$ $(\nu \in P_G(\lambda,\mu))$ is generated by Δ as well. Thus, $(w^{-1}u, w^{-1}v, w^{-1})$ is α-compatible for all $\alpha \in \Delta$, and the corresponding sign is $+1$ (of course, this can be checked directly). By Theorem 4 again, all facets of $P_G(\lambda,\mu)$ which contain a general vertex arise in this way; but there may be other general faces.

(iii) If (u,v,w) is α-compatible, then $(uw_\Delta, vw_\Delta, ww_\Delta)$ is α-compatible as well, with opposite sign. This reflects the fact that

$$\nu \in P_G(\lambda,\mu) \Leftrightarrow -w_\Delta(\nu) \in P_G(-w_\Delta(\lambda), -w_\Delta(\mu))$$

because the dual of $V(\nu)$ is $V(-w_\Delta(\nu))$.

4. The general vertices of a polytope associated with plethysm

Let π be a partition, that is, a finite decreasing sequence of positive integers. Denote by S^π the Schur functor associated with π (a construction of this functor is recalled below). Let λ be a dominant weight, then $S^\pi V(\lambda)$ is a rational G-module. To study its decomposition into simple modules, a generalization of the operation of plethysm, consider the set

$$P_G(\pi,\lambda) := \{p \in \mathcal{X}_{\mathbb{Q}} \mid V(np) \text{ occurs in } S^{n\pi}V(\lambda) \text{ for some positive integer } n\}.$$

This set was introduced by Manivel [Man]; he interpreted it as a moment polytope, as follows. Write

$$\pi = (\pi_1^{a_1}, \ldots, \pi_r^{a_r})$$

where $(\pi_1, \ldots, \pi_r)$ is a strictly decreasing sequence, and each a_j is the multiplicity of π_j in π. For a finite dimensional vector space V, denote by $\mathcal{F}_\pi(V)$ the variety of partial flags

$$f = (V = V^0 \supset V^1 \supset \cdots \supset V^r)$$

where $\dim(V^{j-1}/V^j) = a_j$ for all indices j. We have quotient bundles Q_j on $\mathcal{F}_\pi(V)$, whose fiber at f is V^{j-1}/V^j, and we have an ample line bundle

$$\mathcal{L}_\pi := \bigotimes_{j=1}^{r} (\bigwedge^{a_j} Q_j)^{\otimes \pi_j}$$

which is homogeneous for the transitive action of $\mathrm{GL}(V)$ on $\mathcal{F}_\pi(V)$. Furthermore, we have

$$\Gamma(\mathcal{F}_\pi(V), \mathcal{L}_\pi) = S^\pi V.$$

It follows that

$$P_G(\pi, \lambda) = P_G(X, \mathcal{L}_\pi)$$

where $X = \mathcal{F}_\pi(V(\lambda))$ and G acts on X via its linear action on $V(\lambda)$.

Some vertices of $P_G(\pi, \lambda)$ were constructed by Manivel, as components of $S^\pi V(\lambda)$ associated with fixed points of B in $\mathcal{F}_\pi(V(\lambda))$ which satisfy a certain condition [Man; 2.3]; for vertices which are regular dominant weights, it is equivalent to condition (ii)' in Theorem 2. In the case where G is the general linear group GL_d and $V(\lambda)$ is a symmetric power of $\mathbb{C}^d$, Manivel conjectured that his construction gives all general vertices, see Conjecture 2 in [Man]. The following result answers partially this hope, in a more general setting.

Theorem 6. *If p is a general vertex of $P_G(\pi, \lambda)$, then $V(p)$ occurs in $S^\pi V(\lambda)$. If moreover all weights of T in $V(\lambda)$ have multiplicity one, then the multiplicity of $V(p)$ in $S^\pi V(\lambda)$ is one, too.*

5. Proof of Theorem 1.

Let $p \in \mathcal{C}$. Let n_0 be the smallest positive integer such that $n_0 p$ is in $\mathcal{X}$. Let P be the largest subgroup of G containing B such that $n_0 p$ extends to a character of P. For any positive multiple n of n_0, we have an ample homogeneous line bundle $\mathcal{L}_{np}$ on G/P, and

$$\Gamma(G/P, \mathcal{L}_{np}) = V(np)^*.$$

By definition, p is in $P_G(X, \mathcal{L})$ if and only if the G-module $\Gamma(X, \mathcal{L}^{\otimes n}) \otimes \Gamma(G/P, \mathcal{L}_{np})$ contains nonzero G-invariants for some positive multiple n of n_0; that is, $X \times G/P$ contains semistable points for the ample G-linearized line bundle $\mathcal{L}^{\otimes n_0} \times \mathcal{L}_{n_0 p}$. Equivalently, there exists $x \in X$ such that (x, e_p) is semistable for this line bundle, where e_p is the base point of G/P.

Let now $p \in P_{G'}(X', \mathcal{L}) \cap \mathcal{C}$. Observe that $G'/(G' \cap P)$ embeds into $(G/P)^{T'}$ as the G'-orbit of e_p. By assumption, we can find $x' \in X'$ and a positive multiple n of n_0 such that (x', e_p) is a semistable point of $X' \times G'e_p$ for the action of G' and the line bundle $\mathcal{L}^{\otimes n} \times \mathcal{L}_{np}$. Then it follows from [L] Corollaire 2 that (x', e_p) is semistable for G as well. Thus, $p \in P_G(X, \mathcal{L})$, and (i) is proved.

Let F be a general face of $P_G(X, \mathcal{L})$ and let $p \in F^0$; in particular, $p \in \mathcal{C}_X^0$, so that the parabolic subgroup P depends only on $\mathcal{C}_X$. Let n_0 be as above. Then there exists a positive multiple n of n_0, and a section $\sigma \in \Gamma(X, \mathcal{L}^{\otimes n})_{np}^{(B)}$. This section generates a G-submodule of $\Gamma(X, \mathcal{L}^{\otimes n})$ isomorphic to $V(np)$; it defines a G-equivariant rational map

$$\varphi : X - \to \mathbb{P}(V(np)^*).$$

Let X_σ be the complement in X of the zero set of σ; then X_σ is the preimage under φ of the open subset of $\mathbb{P}(V(np)^*)$ where the coordinate of weight np is nonzero. Because $\mathcal{L}$ is ample, X_σ is affine; let $\mathbb{C}[X_\sigma]$ be its coordinate ring. Then, for any non negative integer m and for any $s \in \Gamma(X, \mathcal{L}^{\otimes nm})$, the quotient s/σ^m is in $\mathbb{C}[X_\sigma]$. Moreover, because $\mathcal{L}$ is ample, $\mathbb{C}[X_\sigma]$ is the increasing union of its subspaces

$$\Gamma(X, \mathcal{L}^{\otimes nm})/\sigma^m$$

where $\Gamma(X, \mathcal{L}^{\otimes nm})$ is mapped to $\Gamma(X, \mathcal{L}^{\otimes n(m+1)})$ by multiplication by $\sigma \in \Gamma(X, \mathcal{L}^{\otimes n})$. Each of these subspaces is P-stable.

Let L be the Levi subgroup of P which contains T; then the roots of L are those roots which are orthogonal to p. We have the Levi decomposition $P = P^u L$ where P^u denotes the unipotent radical of P. By the local structure theorem [BLV;1.2], there exists a closed L-stable subset Z_σ of X_σ such that the map

$$\begin{array}{ccc} P^u \times Z_\sigma & \to & X_\sigma \\ (g, z) & \to & gz \end{array}$$

is an isomorphism. Thus, Z_σ is affine, and there is a L-equivariant isomorphism of $\mathbb{C}[Z_\sigma]$ onto the subalgebra $\mathbb{C}[X_\sigma]^{P^u}$ of regular functions on X_σ, invariant under P^u. It follows that $\mathbb{C}[Z_\sigma]$ is isomorphic to the increasing union of the spaces

$$\Gamma(X, \mathcal{L}^{\otimes nm})^{P^u}/\sigma^m.$$

Write $B = P^u(B \cap L)$, then $B \cap L$ is a Borel subgroup of L. Let C be the convex cone of $\mathcal{X}_{\mathbb{Q}}$ generated by all weights of $\mathbb{C}[Z_\sigma]^{(B \cap L)}$. The latter being the increasing union of the spaces $\Gamma(X, \mathcal{L}^{\otimes nm})^{(B)}/\sigma^m$, the cone C is generated by $-p + P_G(X, \mathcal{L})$. But $P_G(X, \mathcal{L})$ is contained in $\mathcal{C}_X$, the smallest face of $\mathcal{C}$ containing p. It follows that any simple root orthogonal to p is orthogonal to C as well. In other words, any root of L is orthogonal to all weights of $\mathbb{C}[Z_\sigma]^{(B \cap L)}$. So the derived subgroup $[L, L]$ acts trivially on $\mathbf{C}[Z_\sigma]$, and on Z_σ as well (because that variety is affine). Thus, $\mathbb{C}[Z_\sigma]^{(B \cap L)} = \mathbb{C}[Z_\sigma]^{(T)}$.

Because $p \in F^0$, the largest linear subspace contained in C is equal to $\mathrm{lin}(F)$. Let T_F be the corresponding subtorus of T. By Lemma 1 below, the set $Z_\sigma^{T_F}$ is non empty and irreducible. Thus,

$$X_\sigma^{T_F} \cong (P^u)^{T_F} \times Z_\sigma^{T_F} = (G_F \cap P^u) \times Z_\sigma^{T_F}$$

is non empty and irreducible as well, where G_F denotes the centralizer of T_F in G. Let X_F be the closure of $X_\sigma^{T_F}$ in X, then X_F is an irreducible component of X^{T_F}. Moreover, restriction of σ to X_F is a section of $\mathcal{L}^{\otimes n}$, eigenvector of weight np of the group $B_F = G_F \cap B$. Thus, p is in $P_{G_F}(X_F, \mathcal{L})$.

We now check that X_F is independent of the choice of p in F^0. Indeed, if p' is another point in F^0, we can construct as above a global section σ' of a positive power $\mathcal{L}^{\otimes n'}$, which is an eigenvector of B of weight $n'p'$. Then $\sigma\sigma'$ is a section of $\mathcal{L}^{\otimes(n+n')}$ with weight $np+n'p'$, and

$$\frac{np + n'p'}{n + n'} \in F^0.$$

Furthermore, arguing as above with σ replaced by $\sigma\sigma'$, we see that $X_{\sigma\sigma'}^{T_F}$ is not empy, that, is, $X_\sigma^{T_F}$ meets $X_{\sigma'}^{T_F}$. Thus, the irreducible component X_F of X^{T_F} is uniquely determined by F. Furthermore, $p \in P_{G_F}(X_F, \mathcal{L})$ so that F is contained in $P_{G_F}(X_F, \mathcal{L}) \cap \mathcal{C}$.

Conversely, let $q \in P_{G_F}(X_F, \mathcal{L}) \cap \mathcal{C}_X^0$. Then the G_F-variety $X_F \times (G_F/G_F \cap P)$ contains semistable points for $\mathcal{L}^{\otimes m} \times \mathcal{L}_{mq}$ where m is some positive integer. On the other hand, because T_F acts trivially on the irreducible variety X_F, it acts on the fiber of $\mathcal{L}$ at any point of X_F via a character χ. Then the character $m\chi + mq|_{T_F}$ must be trivial (otherwise, there are no semistable points). This holds for q replaced with p; thus, $q - p \in \mathrm{lin}(F)$, and q is in the affine space $\mathrm{aff}(F)$ spanned by F. So we have

$$F^0 \subset P_{G_F}(X_F, \mathcal{L}) \cap \mathcal{C}^0 \subset P_G(X, \mathcal{L}) \cap \mathrm{aff}(F).$$

Because F is a face of $P_G(X, \mathcal{L})$, we conclude that

$$F = P_{G_F}(X_F, \mathcal{L}) \cap \mathcal{C}_X = P_{G_F}(X_F, \mathcal{L}) \cap \mathcal{C}.$$

This proves (ii).

If F consists of a vertex, then $T_F = T$. Let χ be as above; then $P_{G_F}(X_F, \mathcal{L})$ consists of the point $-\chi$. Conversely, if $P_{G_F}(X_F, \mathcal{L})$ is a general vertex of $P_G(X, \mathcal{L})$, then $T_F = T$ by (ii). Together with (i), this proves (iii).

Remark. Let $p \in P_G(X, \mathcal{L})^0$. Then we saw that X contains a nonempty open subset, stable by P and isomorphic to $P^u \times Z$ where Z is stable by L and fixed pointwise by $[L, L]$. It follows that P is the stabilizer of a general B-orbit in X [Kn;§2]. In particular, P, and hence $\mathcal{C}_X$, is independent of $\mathcal{L}$.

Lemma 1. *Let Z be an affine variety with an action of a torus T. Let $\mathcal{X}$ be the character group of T, let C be the convex cone of $\mathcal{X}_\mathbb{Q}$ generated by all weights of T in $\mathbb{C}[Z]$, let $\mathrm{lin}(C)$ be the largest linear subspace contained in C, and let T' be the subtorus of T associated with $\mathrm{lin}(C)$. Then $Z^{T'}$ is non empty, irreducible and contains all closed T-orbits in Z. Moreover, for all $z \in Z^{T'}$, the nonzero weights of T' in the space $T_z(Z)$ are contained in an open half space, and the convex cone generated by these weights identifies with the image of $-C$ in the quotient $\mathcal{X}_\mathbb{Q}/\mathrm{lin}(C)$.*

6. Proofs of Lemma 1, Theorem 2 and Theorem 3

Proof of Lemma 1. We can choose a one parameter subgroup λ of T such that λ is positive on all points of $C\setminus\mathrm{lin}(C)$. Then λ vanishes on $\mathrm{lin}(C)$ and thus, the image of λ is contained in T'. Moreover, for each $z\in Z$, the limit of $\lambda(t)z$ as $t\to\infty$ exists and defines a surjective morphism $Z\to Z^\lambda$. In particular, Z^λ is irreducible and contains all closed T-orbits in Z.

Because the linear part of C is the space of characters of T which vanish on T', we can find λ as above, such that $Z^\lambda = Z^{T'}$. So $Z^{T'}$ is non empty and irreducible. Let I be the ideal of z in $\mathbb{C}[Z]$. Then I is T'-stable and the T'-module I/I^2 is the dual $T_z(Z)^*$. Thus, all weights of T' in $T_z(Z)^*$ are in $C/\mathrm{lin}(C)$. Conversely, let χ be the weight of an eigenvector f of T' in $\mathbb{C}[Z]$. If $\chi\notin\mathrm{lin}(C)$ then $f(z)=0$ (because z is fixed by T'), that is, $f\in I$. Let n be the integer such that $f\in I^n\setminus I^{n+1}$. Then χ is a weight of T' in I^n/I^{n+1}. Thus, χ is a sum of n weights of T' in I/I^2. It follows that the convex cone $C/\mathrm{lin}(C)$ is generated by the weights of T' in $T_z(Z)^*$.

Proof of Theorem 2. Set $F:=P_{G'}(X',\mathcal{L})\cap\mathcal{C}$. Choose $p\in F^0$ and define n_0, P, σ, X_σ, Z_σ and C as in the proof of Theorem 1. Then, as observed in this proof, the cone C is generated by $-p+P_G(X,\mathcal{L})$. Furthermore, X_σ is isomorphic to $P^u\times Z_\sigma$ and Z_σ is fixed pointwise by $[L,L]$. Thus

$$T_x(X)/T_x(Ux) = T_x(X)/T_x(P^u x) = T_x(Z_\sigma)$$

for all $x\in Z_\sigma$. Together with Lemma 1, this shows that (i) implies (ii), and that C is generated by the opposite of the weights of T' in $T_x(X)/T_x(Ux)$.

Conversely, assume (ii). Then, as in the beginning of the proof of Theorem 1, we can choose σ such that X'_σ is not empty. Thus, X' contains contains $(G'\cap P^u)\times Z_\sigma^{T'}$ as a dense open subset, and we conclude by Lemma 1 again.

Proof of Theorem 3. Set

$$R:=\bigoplus_{n=0}^{\infty}\Gamma(X,\mathcal{L}^{\otimes n}).$$

Because $\mathcal{L}$ is ample and G-linearized, R is a finitely generated graded algebra, where G acts by automorphisms; because X is normal, R is normal as well.

By [Kr], it follows that the graded subalgebra R^U is finitely generated and normal. Set

$$A:=\bigoplus_{n=0}^{\infty}\Gamma(X,\mathcal{L}^{\otimes n})^{(B)}_{np}=\bigoplus_{n=0}^{\infty}(R^U)^{(T)}_{n,np}.$$

Then A is a graded, finitely generated normal algebra as well (to see this, consider the algebra $R^U[t]$ graded by the group $\mathcal{X}_{\mathbb{Q}}\times\mathbb{Z}$, where t is an indeterminate of degree $(-p,-1)$; then A is the subalgebra of $R^U[t]$ consisting of all elements of degree 0).

Similarly, the space

$$A_F:=\bigoplus_{n=0}^{\infty}\Gamma(X_F,\mathcal{L}^{\otimes n})^{(B_F)}_{np}$$

is a finitely generated graded domain. Restriction to X_F defines an algebra homomorphism

$$r : A \to A_F.$$

For any homogeneous element σ of positive degree in A, we check that r is an isomorphism after inverting σ. By a classical argument (see Lemma 2 below), it will follow that r is an isomorphism.

Let $\sigma \in \Gamma(X, \mathcal{L}^{\otimes n})^{(B)}_{np}$ be a homogeneous element of A, with $n > 0$. Then

$$R[1/\sigma] = \bigoplus_{m=-\infty}^{\infty} \Gamma(X_\sigma, \mathcal{L}^{\otimes m})$$

whence

$$A[1/\sigma] = \bigoplus_{m=-\infty}^{\infty} \Gamma(X_\sigma, \mathcal{L}^{\otimes m})^{(B)}_{mp}.$$

Let Z_σ be as in the proof of Theorem 1. Then we have

$$A[1/\sigma] = \bigoplus_{m=-\infty}^{\infty} \Gamma(Z_\sigma, \mathcal{L}^{\otimes m})^{(T)}_{mp}$$

and

$$A_F[1/\sigma] = \bigoplus_{m=-\infty}^{\infty} \Gamma(Z_\sigma^{T_F}, \mathcal{L}^{\otimes m})^{(T)}_{mp}$$

because $X_\sigma^{T_F} = (G_F \cap P^u) \times Z_\sigma^{T_F}$, and $Z_\sigma^{T_F}$ is fixed by $[G_F \cap L, G_F \cap L]$. We claim that the restriction map

$$r_m : \Gamma(Z_\sigma, \mathcal{L}^{\otimes m})^{(T)}_{mp} \to \Gamma(Z_\sigma^{T_F}, \mathcal{L}^{\otimes m})^{(T)}_{mp}$$

is an isomorphism for all m.

Indeed, this map is surjective, because $Z_\sigma^{T_F}$ is a closed T-stable subvariety of the affine T-variety Z_σ. For injectivity, consider first the case where $p \in F^0$. Let $s \in \Gamma(Z_\sigma, \mathcal{L}^{\otimes m})^{(T)}_{mp}$ such that $r_m(s)$ vanishes everywhere on $Z_\sigma^{T'}$. Then s^n/σ^m is a regular T-invariant function on Z_σ; by Lemma 1, this function vanishes on all closed T-orbits. Thus, $s = 0$.

In the general case where $p \in F \cap \mathcal{C}_X^0$, let $F(p)$ be the smallest face of $P_G(X, \mathcal{L})$ which contains p. Then $p \in F(p)^0 \subset F$, whence $T_{F(p)} \supset T_F$ and $X_\sigma^{T_{F(p)}} \subset X_\sigma^{T_F}$. By the argument above, restriction to $X_\sigma^{T_{F(p)}}$ is injective; thus, restriction to $X_\sigma^{T_F}$ is injective as well.

If moreover p is a vertex of $P_G(X, \mathcal{L})$, then $T_F = T = G_F = B_F$. Let $x \in X_F$, then T acts on the fiber $\mathcal{L}_x$ via a character which depends only on X_F, and which must be equal to $-p$. It follows that T acts on $\Gamma(X_F, \mathcal{L}^{\otimes n})$ via the character np.

Lemma 2. *Let $r : A \to A'$ be a homomorphism between finitely generated graded domains. Assume that A is normal and that the localization $r : A[1/\sigma] \to A'[1/\sigma]$ is an isomorphism for any homogeneous $\sigma \in A$ of positive degree. Then r is an isomorphism.*

Proof. The assumptions imply that r is injective; we shall treat it as an inclusion, and identify the fraction fields of A and A'. If $\dim(A) = 1$ then A is a polynomial ring in a homogeneous variable and the assertion is clear. If $\dim(A) > 1$, let P be a prime ideal of A of height one. Then $A \setminus P$ contains a homogeneous element of positive degree; thus, A' is contained in the localization A_P. Because A is normal, it follows that $A = A'$.

7. Proofs of Theorems 4, 5 and 6

Proof of Theorem 4. Let e_{B^-} be the base point of G/B^-. Then the T-fixed points in X are the

$$x(u,v) := (ue_{B^-}, ve_{B^-})$$

where $(u,v) \in W \times W$. Observe that T acts on $\mathcal{L}_{x(u,v)}$ via the character $-u(\lambda) - v(\mu)$. Moreover, the multiset of weights of T in $T_{x(u,v)}(X)$ is the union (with multiplicities) of $u(\Phi^+)$ and $v(\Phi^+)$.

Let $(u,v) \in W \times W$ such that $u(\lambda) + v(\mu)$ is dominant regular. Then, for $\alpha \in \Phi^+$, we must have $\langle u(\lambda), \check{\alpha}\rangle > 0$ or $\langle v(\beta), \check{\alpha}\rangle > 0$. It follows that

$$\Phi^+ \subset u(\Phi^+) \cup v(\Phi^+).$$

Taking complements in Φ and then opposites, we obtain

$$u(\Phi^+) \cap v(\Phi^+) \subset \Phi^+.$$

Thus, all weights of T in $T_{x(u,v)}(X)/T_{x(u,v)}(Ux(u,v))$ have multiplicity one, and there are as many such weights as positive roots. So condition (ii)' in Theorem 2 is equivalent to: there exists $w \in W$ such that

$$u(\Phi^+) \cup v(\Phi^+) = \Phi^+ \cup w(\Phi^+) \text{ and } u(\Phi^+) \cap v(\Phi^+) = \Phi^+ \cap w(\Phi^+).$$

This implies in turn that (u,v) is admissible. Conversely, for admissible (u,v), we have $u(\Phi^+) \cap v(\Phi^+) \subset \Phi^+$ and the multiset of weights of T in $T_{x(u,v)}(X)/T_{x(u,v)}(Ux(u,v))$ is $w(\Phi^+)$ (with all multiplicities equal to one). Thus, condition (ii)' holds, and $u(\lambda)+v(\mu)$ is a general vertex of $P_G(\lambda,\mu)$. By Theorem 2, the tangent cone at that vertex is generated by $-w(\Phi^+)$. Moreover, by Theorem 3 and the fact that $x(u,v)$ is an isolated fixed point, the space $(V(\lambda)\otimes V(\mu))^{(B)}_{u(\lambda)+v(\mu)}$ is one dimensional.

Proof of Theorem 5. Because $P_G(\lambda,\mu)$ is contained in $\mathcal{C}$ and has the same dimension, this polytope is the set of all $\nu \in \mathcal{C}$ which satisfy the inequalities corresponding to all general facets. Let F be such a facet, with associated subtorus $T_F \subset T$. Let G_F be the centralizer of T_F in G, then the irreducible components of X^{T_F} are the spaces

$$(G_F \times G_F)x(u,v)$$

for $(u,v) \in W \times W$; each of them is isomorphic to $G_F/B_F^- \times G_F/B_F^-$.

Choose (u,v) such that $X_F = (G_F \times G_F)x(u,v)$. Because $F = P_{G_F}(X_F,\mathcal{L}) \cap \mathcal{C}$ is of dimension $\ell - 1$, the same holds for $P_{G_F}(X_F,\mathcal{L})$. It follows that the semisimple rank of G_F is $\ell - 1$. Thus, T_F is the center of G_F, and is the image of a one parameter subgroup θ of T. Furthermore, there exists $w \in W$ and $\alpha \in \Delta$ such that the root system of (G_F, T) has a basis consisting of the $w(\beta)$ where $\beta \in \Delta$ and $\beta \neq \alpha$. Then θ is a positive multiple of $w(\pi_\alpha)$ where π_α is the fundamental weight associated with α.

By condition (ii)' of Theorem 2, there exists a multiset E consisting of integers of the same sign together with 0, such that

$$\theta(u(\Phi^+)) \cup \theta(v(\Phi^+)) = \theta(\Phi^+) \cup E$$

(union as multisets). Taking opposites and then unions, we obtain

$$\theta(\Phi) = E \cup (-E),$$

that is, $E = \Phi_\alpha^+$ or $E = \Phi_\alpha^-$ (because $\theta(\Phi) = \Phi_\alpha$). In the former (resp. latter) case, we obtain

$$w^{-1}u(\Phi^+)_\alpha \cup w^{-1}v(\Phi^+)_\alpha = w^{-1}(\Phi^+)_\alpha \cup \Phi_\alpha^+ \text{ (resp. } w^{-1}(\Phi^+)_\alpha \cup \Phi_\alpha^-),$$

that is, $(w^{-1}u, w^{-1}v, w^{-1})$ is α-compatible with sign $+1$ (resp. -1). Furthermore, by Theorem 2, $P_{G_F}(X_F,\mathcal{L})$ meets $\mathcal{C}^0$; it follows that $P_{G^\alpha}(w^{-1}u(\lambda), w^{-1}v(\lambda))$ meets $w^{-1}(\mathcal{C}^0)$. Thus, the triple $(w^{-1}u, w^{-1}v, w^{-1})$ satisfies the conditions of Theorem 5. Set $\varepsilon = +1$ in the former case, and -1 in the latter case. As $u(\lambda) + v(\mu) \in P_{G_F}(X_F,\mathcal{L})$, we have by Theorem 2:

$$\varepsilon \langle \theta, \nu \rangle \geq \langle \theta, u(\lambda) + v(\mu) \rangle$$

for all $\nu \in P_G(\lambda,\mu)$, with equality on F. Equivalently,

$$\varepsilon\, w^{-1}(\nu)_\alpha \geq w^{-1}u(\lambda)_\alpha + w^{-1}v(\lambda)_\alpha$$

with equality on F. Finally, the assertion on multiplicities of tensor products follows from Theorem 3.

Proof of Theorem 6. Set $X := \mathcal{F}_\pi(V(\lambda))$ and $\mathcal{L} := \mathcal{L}_\pi$. Let p be a general vertex of $P_G(\pi,\lambda)$ and let X_p be the corresponding irreducible component of X^T. Then X_p is an orbit of the centralizer of T in $\mathrm{GL}(V(\lambda))$. As a consequence, X_p is a product of partial flag varieties.

By Theorem 3, the restriction map

$$\bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{L}^{\otimes n})_{np}^{(B)} \to \bigoplus_{n=0}^{\infty} \Gamma(X_p, \mathcal{L}^{\otimes n})$$

is an isomorphism. Because $\mathcal{L}$ is an ample line bundle on a product of partial flag varieties, the graded algebra in the right hand side is generated by its elements of degree one. It follows that $\Gamma(X,\mathcal{L})$ contains eigenvectors of B of weight p.

If moreover all weights of T in $V(\lambda)$ have multiplicity one, then T fixes only finitely many linear subspaces of $V(\lambda)$. It follows that X^T is finite, and that X_p is a point. Thus, the multiplicity of $V(p)$ in $S^\pi V(\lambda)$ is one.

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