On open 3-manifolds proper homotopy equivalent to geometrically simply-connected polyhedra *

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Abstract

We prove that an open 3-manifold proper homotopy equivalent to a geometrically simply connected polyhedron is simply connected at infinity thereby generalizing the theorem proved by Poénaru in [6].

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1 Introduction

The immediate antecedent to this paper is [6], the principal theorem of which is the following.

Theorem 1.1 *(V.Poénaru) If U is an open contractible 3-manifold and, for some n, $U \times D^n$ has a handlebody decomposition without 1-handles then U is simply connected at infinity.*

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(**Note**: D^n denotes the n-ball; see [6] for the definition of "handlebody decomposition without 1-handles "; an open contractible 3-manifold U^3 is *simply connected at infinity* (s.c.i.), and we write also $\pi_1^\infty(U^3) = 0$, if given a compactum $X \subset U$ there exists another compactum Y with $X \subset Y \subset U^3$, such that any loop in $U - Y$ is null-homotopic in $U - X$.) All 3-manifolds we consider in the sequel will be orientable unless the contrary is explicitly stated.

It is easily seen that any non-compact polyhedral manifold U which has a handlebody decomposition with no 1-handles is "geometrically simply-connected" (a non-compact polyhedron P is *geometrically simply connected (g.s.c.)* if it is exhausted by compact 1-connected polyhedra). In addition the projection map $p : U \times D^n \to U$ is a proper simple-homotopy equivalence (defined in [7]). In [6] Poénaru hinted at the conjecture which results when the hypothesis of the above-stated theorem is replaced by the (therefore) weaker hypothesis that U be proper simple-homotopy equivalent to a g.s.c. polyhedron. This conjecture was subsequently established in [3] using the techniques of [6]. The following theorem is an immediate corollary of the principal result of this paper (which is proven using only basic 3-manifold theory). It further generalizes the theorem stated above.

Theorem 1.2 *(L.Funar-T.L.Thickstun) Any open 3-manifold which is proper homotopy equivalent to a g.s.c. polyhedron is simply-connected at infinity.*

Remark 1.1 *If M^n $(n > 3)$ is a compact, contractible n-manifold with non-simply-connected boundary (e.g. those constructed in [4] and [5]) then int(M^n) is easily seen to be g.s.c. but not simply connected at infinity. This demonstrates that the above theorem cannot be extended to include open contractible manifolds of dimension greater than three.*

Provisos: We remain in the polyhedral category throughout and all homology groups are with **Z** coefficients.

2 Statement of results

We first require the following definitions.

Definition 2.1 A proper map $f : X \to Y$ is H_3-*nontrivial* if given non-null compacta $L \subset Y$ and $K \subset X$ such that $f(X - K) \subset Y - L$ then $f_* : H_3(X, X - K) \to H_3(Y, Y - L)$ is nontrivial (i.e. its image is not a singleton).

Definition 2.2 Given noncompact polyhedra X and Y we say Y *is H_3-semi-dominated by* X if there exists an H_3-nontrivial proper map $f : X \to Y$.

Definition 2.3 An open connected 3-manifold U^3 is *simple-ended* if it has an exhaustion $\{M_i\}_{i=1}^{\infty}$ by compact 3-submanifolds where, for all i, the genus of ∂M_i is zero.

Remark 2.1 *1. If U is an open orientable 3-manifold and $Y \subset U$ a non-null compactum then $H_3(U, U - Y)$ is nontrivial.*

2. If $f, g : X \to Y$ are properly homotopic maps and f is H_3-nontrivial the g is H_3-nontrivial.

Our main result is the following.

Theorem 2.1 *An open, connected, orientable 3-manifold U is H_3-semi-dominated by a g.s.c. polyhedron if and only if U is the connect-sum of a 1-connected, simply connected at infinity open 3-manifold and a closed, orientable 3-manifold with finite fundamental group.*

Remark 2.2 *1. Observe that the theorem stated in the introduction is an immediate corollary of the above theorem.*

2. The proof of the "if" part of the theorem is very brief. Just observe that if U satisfies the hypothesis then the universal covering of U is 1-connected and simply connected at infinity (hence g.s.c. -see (3) below) and the covering projection is proper and H_3-nontrivial. (In the sequel when we refer to the hypothesis or conclusion of the theorem we will mean the "only if" part.)

3. The class of 1-connected, simply connected at infinity open 3-manifolds is well-known to be equal to each of the following two classes of open 3-manifolds. Those which can be constructed as follows: delete a tame, 0-dimensional, compact subspace from S^3, denote the result by U and replace each element of a pairwise disjoint, proper family of 3-balls in U by a homotopy 3-ball. Those each of which has an exhaustion $\{M_i\}_{i=1}^{\infty}$ by compact 3-submanifolds where, for each i, M_i is 1-connected (and hence the genus of ∂M_i is 0) and each component of $M_{i+1} - M_i$ is homeomorphic to a space obtained by taking finitely many pairwise disjoint 3-balls in S^3, replacing one by a homotopy 3-ball and deleting the interiors of the rest.

To establish the theorem we will demonstrate the following three propositions.

Proposition 2.1 *If U is as in the hypothesis of the theorem then U is simple-ended.*

Proposition 2.2 *If U is as in the hypothesis of the theorem then $\pi_1(U)$ is a torsion group.*

Proposition 2.3 *If U is an open, connected simple-ended 3-manifold such that $\pi_1(U)$ is a torsion group then U is as in the conclusion of the theorem.*

3 Proof of Proposition 2.1

Lemma 3.1 *Suppose the following: U is an open, orientable 3-manifold; K is a compact, connected 3-submanifold of U such that each component of $cl(U-K)$ is noncompact and has connected boundary; and $f : (M, \partial M) \to (U, U-K)$ is a map of a compact 3-manifold such that ∂M has genus zero and $f_* : H_3(M, \partial M) \to H_3(U, U-K)$ is nontrivial. Then there exists a compact, connected 3-submanifold N of U such that K is in N and the genus of ∂N is zero.*

Proof: It will suffice to find, for each component V of $cl(U-K)$ a compact, connected 3-submanifold $N(V)$ of V such that ∂V is in $N(V)$ and the genus of $\partial N(V) - \partial V$ is zero. So let V be such a component. We assume f is transverse to ∂V and denote the intersection of $f^{-1}(V)$ and ∂M by C. From the hypothesis on f_* we conclude that $f|_{f^{-1}(\partial V)} : f^{-1}(\partial V) \to \partial V$ has nonzero degree (where the orientations are induced from the orientations of the ambient spaces). Note that $f|_C : C \to V$ and $f\ |_{f^{-1}(\partial V)}: f^{-1}(\partial V) \to V$ are homologous and hence $f|_C$ is homologous (in V) to a nonzero multiple of ∂V.

Applying the prime factorization theorem for compact 3-manifolds to a regular neighborhood of $f(C)$ in $int(V)$ we conclude the existence of an embedding R in $int(V)$ where R is a closed, oriented surface of genus zero such that R is homologous to $f|_C$ in $int(V)$ (recall that the prime factorization theorem implies that π_2 of a compact 3-manifold is generated, as a π_1-module, by a finite family of pairwise disjoint embedded 2-spheres). Now let W be a regular neighborhood of R in $int(V)$ and denote by $N(V)$ that component of $cl(V-W)$ containing ∂V. It remains only to show that $N(V)$ is compact. Suppose otherwise. Then there exists a proper ray in V extending from ∂V to infinity and avoiding R. Such a ray has intersection number one with ∂V but intersection number zero with R. This contradicts the fact that R is homologous in V to a multiple of ∂V. □

Definition 3.1 An *admissible pair* is a map $f : (X, Y) \to (M, M-L)$ and a subspace $K \subset L$ satisfying the following conditions:

1. (X, Y) is a pair of compact simplicial complexes and X is simply connected.
2. M is a compact orientable 3-manifold and K and L are compact 3-submanifolds with $K \subset int(L)$, $L \subset int(M)$ and L connected.
3. The map f is simplicial and M is an (abstract) regular neighborhood of $f(X)$.
4. Only one component of $X - f^{-1}(\partial L)$ has image under f meeting K.

5. The map $f_* : H_3(X,Y) \to H_3(M, M-L)$ is non-trivial.

We will refer to the M above as the *target* of the admissible pair.

Notation: If $X \subset P$ then ∂X denotes the frontier of X in P.

Lemma 3.2 *If $f : P \to U$ is the map of the theorem and K is a compact 3-submanifold of U then we can choose $X \subset P$ and a compact 3-submanifold L of U such that the pair $\{f|_X : (X, \partial X) \to (M, M-L), K\}$ is admissible where M is a regular neighborhood of $f(X)$ in U.*

Proof: Consider L_0 be a regular neighborhood of K in U. The hypothesis implies the existence of some X with $p^{-1}(L_0) \subset X_0 \subset P$, such that $f_* : H_3(P, P-X) \to H_3(U, U-L_0)$ is non-trivial. Moreover once such an X_0 is chosen then larger X with $int(X) \supset X_0$ are also convenient for this purpose. Moreover f is proper implies the existence of an $X_1 \supset X_0$ such that $f(\partial X_1) \cap L_0 = \emptyset$. Using excision on both sides above we find that the map $(f|_{X_1})_* : H_3(X_1, \partial X_1) \to H_3(M, M-L_0)$ is non-trivial. Denote the set X_1 with these properties by $X(L_0)$. Remark that $X(L_0)$ is defined for any compact L_0 engulfing K. Assume now that $X - f^{-1}(\partial L_0)$ has at least two components whose image meets K, or equivalently, that $f^{-1}(K) \cap (X(L_0) - f^{-1}(\partial L_0))$ is not connected. Since f is simplicial the latter is a simplicial subcomplex of P and so it cannot be path connected. In particular there exist points $x, y \in f^{-1}(K)$ such that any path connecting them in $X(L_0)$ should meet $f^{-1}(\partial L_0)$. But $f^{-1}(K)$ is a compact (thus finite) simplicial complex because f is proper hence it has a finite number of components. Consider some arcs joining these components in P. Their union is contained in some compact subset $K' \subset P$. Consider now L large enough such that $f^{-1}(\partial L) \cap K' = \emptyset$, and $K' \subset int(f^{-1}(L))$. Then $X(L) \supset f^{-1}(L) \supset K'$, and we claim that this X fulfills all conditions needed. If x, y are two points in $f^{-1}(K)$ then there exist path connecting them inside K' and so there exists a path inside $X(L) - f^{-1}(\partial L)$. □

Lemma 3.3 *Given an open, connected 3-manifold U and compactum X in U there exists a compact 3-submanifold K of U containing X and such that each component of $cl(U-K)$ is noncompact and has connected boundary.*

Proof: Let M be a compact 3-submanifold of U containing X. Let N be the union of M and all compact components of $cl(U-M)$. To obtain K from N add 1-handles to N (in $cl(U-N)$) which connect different components of ∂N which are in the same component of $cl(U-N)$. □

The proof of Proposition 2.1 will proceed by applying the tower construction to the admissible pair of Lemma 3.2 (where K is also chosen to satisfy the conclusion of Lemma 3.3) to obtain a map satisfying the hypothesis of Lemma 3.1. It is convenient to state first the following definition.

Definition 3.2 A *reduction of the admissible pair* $\{f_0 : (X,Y) \to (M_0, M_0 - L_0), K_0\}$ is a second admissible pair $\{f_1 : (X,Y) \to (M_1, M_1 - L_1), K_1\}$ such that there exists a map $p : M_1 \to M_0$, the "projection map" of the reduction, satisfying the following conditions:

1. $p \circ f_1 = f_0$.

2. $p(K_1) = K_0$, $p(L_1) = L_0$, and the maps $p|_{L_1} : L_1 \to L_0$, $p|_{K_1} : K_1 \to K_0$ are boundary preserving.

3. $p|_{L_1} : L_1 \to L_0$ has non-zero degree.

4. The complexity of f_1 is strictly less than the complexity of f_0 (where the complexity of a simplicial map g with compact domain is the number of simplices s in the domain for which $g^{-1}(g(s)) \neq s$).

5. The image under p of only one component of $M_1 - \partial L_1$ mets K_0. Observe that, by (3), the image of L_1 must meet K_0.

Lemma 3.4 *If f_1 is a reduction of f_0 and f_2 is a reduction of f_1 then f_2 is a reduction of f_0.*

Proof: This is obvious.□

Lemma 3.5 *Any admissible pair with non-simply connected target has a reduction with simply-connected target.*

Proof: It will suffice to show that any admissible pair with non-simply connected target has a reduction. Because then, by iteration (which could occur at most finitely many times by condition 4) and applying Lemma 3.4 we obtain a reduction with simply connected target.

Let $\{f|_X : (X,Y) \to (M, M - L), K\}$ be the admissible pair. Let $\widetilde{M}$ be the universal covering of M and p be the covering projection. Since X is simply connected there exists a lift $\widetilde{f} : X \to \widetilde{M}$ of f. We will show that $\{f_1 : (X,Y) \to (M_1, M_1 - L_1), K_1\}$ is a reduction, where

1. $f_1 = \widetilde{f}$.

2. M_1 is a regular neighborhood of $\widetilde{f}(X)$.

3. L_1 is the only component of $p^{-1}(L)$ which is contained in $f_1(X) = \widetilde{f}(X)$.

4. $K_1 = p^{-1}(K_0) \cap L_1$.

5. The covering map restricted to M_1 is the map p from the definition 3.2.

Let us show that f_1 is well-defined and admissible.

f_1 is well-defined: First we show that $f_1(Y) \subset M_1 - L_1$. We know that $f(Y) \subset M - L$ since f is admissible, and so $f_1(Y) = M_1 \cap p^{-1}(f(Y)) \subset M_1 \cap p^{-1}(M - L) = M_1 - p^{-1}L \subset M_1 - L_1$. Observe also that $p(M_1) \subset p(\widetilde{M}) \subset M$ hence p is well-defined.

In order to have a consistent definition of L_1 we must show that there exists one and only one component of $p^{-1}(L)$ contained in $f_1(X)$.

First we prove the existence. Since f is H_3-nontrivial on H_3 (the condition (5) for f) we derive that $\widetilde{f}_* : H_3(X, Y) \to H_3(\widetilde{M}, \widetilde{M} - p^{-1}(L))$ is non-trivial. In particular the Abelian group $H_3(\widetilde{M}, \widetilde{M} - p^{-1}(L))$ is non-zero. Since this group is freely generated by the compact components of $p^{-1}(L)$, there exists at least one such (the deck transformations act transitively on the components of $p^{-1}(L)$ and so every component is compact). Since L is connected $H_3(M, M - L)$ is generated by the orientation class of L and f_* nontrivial implies that L is in $f(X)$. In fact a cycle c representing the orientation class can be a regular neighborhood of L in M.

Observe now that $p(M_1 \cap p^{-1}(L)) = L$ since $p(M_1) = M$, and also $p(M_1 - M_1 \cap p^{-1}(L)) = M - L$. Then since f is H_3-nontrivial the map $\widetilde{f}_* : H_3(X, Y) \to H_3(M_1, M_1 - M_1 \cap p^{-1}(L))$ should be non-trivial. The same argument used above shows that $\widetilde{f}(X) \supset M_1 \cap p^{-1}(L)$. Let L_1 be a component of $p^{-1}(L)$ which meets $\widetilde{f}(X)$. Suppose that L_1 is not completely contained in $\widetilde{f}(X)$. Since L_1 is connected then $L_1 \cap (M_1 - \widetilde{f}(X)) \neq \emptyset$, or in other words the regular neighborhood of $\widetilde{f}(X)$ meets a larger subset of L_1 than the image $\widetilde{f}(X)$. This contradicts the fact that $M_1 \cap L_1 \subset \widetilde{f}(X)$. Thus any component of $p^{-1}(L)$ meeting $\widetilde{f}(X)$ is entirely contained in $\widetilde{f}(X)$. At least one component has non-null intersection with the image because $p(\widetilde{f}(X) \cap p^{-1}(L)) = p(M_1 \cap p^{-1}(L)) = L$.

Suppose now that there are two components, L_1 and L_1', meeting $\widetilde{f}(X)$. The two components are then disjoint and contained in $\widetilde{f}(X)$. Furthermore there are two components of $M_1 - p^{-1}(\partial L_1)$, namely $int(L_1)$ and $int(L_1')$ whose images under p meet K. However there is only one component, say ξ, of $X - f^{-1}(\partial L)$ whose image by f meets K. Then $\widetilde{f}(\xi) \subset M_1 - \partial L_1 \cup \partial L_1'$ since $\widetilde{f}(\xi)$ avoids $p^{-1}(\partial L)$ and $\widetilde{f}(\xi) \supset int(L_1) \cup L_1' \supset K_1 \cup K_1'$ because it meets K. This is a contradiction as $\widetilde{f}(\xi)$ must be connected.

f_1 is admissible: Conditions (1-3) from definition 3.1 are immediate. The condition (5) for f_1 is satisfied since $f = p \circ f_1$ is H_3-nontrivial. Finally (4) is implied by $f_1^{-1}(\partial L_1) = f^{-1}(\partial L)$.

f_1 is a reduction of f: With the exception of (3) and (5) we leave them to the reader. Condition (3) follows from the fact that $p|_{L_1} : L_1 \to L$ is a covering map, and any covering map from

one compact orientable 3-manifold to another has non-zero degree. Otherwise, the map $p_* : H_3(M_1, M_1 - L_1) \to H_3(M, M - L)$ is multiplication by $deg_{L_1}p$. In fact $L_1 = p^{-1}(L) \cap M_1$, because we have seen above that $p^{-1}(L) \cap M_1 = p^{-1}(L) \cap \widetilde{f}(X) = L_1$ and then we apply the result from [1], p.267. If $deg_{L_1}p$ is vanishing then f cannot be H_3-nontrivial, which is a contradiction. But the degree of $p|_{L_1}$ is the same of $deg_{L_1}p$ by standard results and thus the former is non-zero, as claimed. We already saw before that $int(L_1)$ is the only component of $M_1 - p^{-1}(\partial L)$ whose image meets K hence establishing (5). □

Proof of Proposition 2.1 from the Lemmas: Begin with the admissible pair of Lemma 3.2 where K also satisfies the conclusion of Lemma 3.3. Apply Lemma 3.5 to that admissible pair. Note that the projection map of that reduction satisfies the hypothesis of Lemma 3.1 whose application then completes the proof. □

4 Proof of Proposition 2.2

Lemma 4.1 *Suppose the diagram*

$$\begin{array}{ccccc} & & X & & \\ & \varphi \nearrow & & \searrow \psi & \\ M & \xrightarrow{h} & N & \subset & U \end{array}$$

is commutative and satisfies the following conditions:

1. *M and N are compact, orientable 3-manifolds with $h(\partial M) \subset \partial N$.*

2. *h has non-zero degree.*

3. *X is simply-connected.*

Then $e_\sharp(\pi_1(N))$ is a torsion subgroup of $\pi_1(U)$ (where e denotes the inclusion $N \subset U$).

Proof: It will suffice to show that given $\alpha : S^1 \to N$, some "multiple" of α is null-homotopic in U. We can assume that α is an embedding and that h is transverse to $\alpha(S^1)$. The preimage of $\alpha(S^1)$ under h is is then a disjoint union of circles $C_1, C_2, ..., C_k$. Since h has non-zero degree we can choose homeomorphisms $\alpha_i : S^1 \to C_i$, $1 \leq i \leq k$ such that the element $[h \circ \alpha_1] + [h \circ \alpha_2] + ... + [h \circ \alpha_k]$ in $H_1(\alpha(S^1))$ is not zero (where the brackets indicate "homology class of"). But since h factors through X, each map $h \circ \alpha_i$ must be null-homotopic in U. The rest is easy. □

Now to prove Proposition 2.2 it will suffice to show that if N is a compact, connected 3-submanifold of U then $e_\sharp(\pi_1(N))$ is a torsion subgroup of $\pi_1(U)$. Let $f : X \to U$ be as in the definition of "H_3-semi-dominated" (where X is g.s.c.). By excision we have $f_* : H_3(Y, \partial Y) \to H_3(N, \partial N)$ is nontrivial, where $Y = f^{-1}(N)$. We can "realize" any element of $H_3(Y, \partial Y)$ by a map $g : (M, \partial M) \to (Y, \partial Y)$, where M is a compact orientable 3-manifold (i.e. $g_* : H_3(M, \partial M) \to H_3(Y, \partial Y)$ sends the orientation class of M to the preassigned element of $H_3(Y, \partial Y)$ -see [8]). We choose the preassigned element to be one not sent to zero by f_*. Now apply the lemma with $h = f \circ g$. □

5 Proof of Proposition 2.3

Lemma 5.1 *If A and B are groups and $a \in A$, $b \in B$ are neither the identity then $a * b$ is not a torsion element in $A * B$ (the free product of A and B).*

Proof: Let X and Y be spaces such that $\pi_1(X) = A$ and $\pi_1(Y) = B$. Suppose $(a*b)^n$ is the identity for some n and choose a loop representing it in the wedge of X and Y. Consider the preimage of the wedge point under the null-homotopy of the loop. We leave the details to the reader. □

Corollary 5.1 *The prime factorization of a closed 3-manifold whose fundamental group is a torsion group can have at most one non-simply connected factor.*

Now to prove Proposition 2.3 we assume that U is not simply-connected (otherwise we are done) and let M be a non-simply connected 3-submanifold of U such that ∂M has genus zero. We can express M as a connect-sum of a punctured homotopy 3-ball and a closed orientable 3-manifold N where $\pi_1(N)$ is a torsion group. By the above corollary we can assume N is irreducible. By [2] any orientable, irreducible closed 3-manifold with torsion must have finite fundamental group. It remains only to show that U can have no other non-simply connected factor but this also follows from the Corollary. □

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