General Curvature Estimates for Stable H-Surfaces Immersed into a Space Form

Pierre Bérard Laurent Hauswirth

bh98l.tex (98sep13)

Abstract

In this paper, we give general curvature estimates for constant mean curvature surfaces immersed into a simply-connected 3-dimensional space form. We obtain bounds on the norm of the traceless second fundamental form and on the Gaussian curvature at the center of a relatively compact stable geodesic ball (and, more generally, of a relatively compact geodesic ball with stability operator bounded from below). As a by-product, we show that the notions of weak and strong Morse indices coincide for complete non-compact constant mean curvature surfaces. We also derive a geometric proof of the fact that a complete stable surface with constant mean curvature 1 in the usual hyperbolic space must be a horosphere.

Mathematics Subject Classification (1991): 53C42, 53C21, 35J60

Key words: constant mean curvature, curvature estimates, stability, Morse index

1 Introduction

In 1983, R. Schoen [16] proved a curvature estimate for stable minimal surfaces in $\mathbb{R}^3$. The Gauss curvature K of a stable minimal surface M, with boundary ∂M, immersed in $\mathbb{R}^3$, satisfies the estimate

$$|K(x_0)| \leq C d(x_0, \partial M)^{-2}, \tag{1}$$

where C is a universal constant and $d(x_0, \partial M)$ the distance of the point x_0 to the boundary. This estimate is very useful to study minimal surfaces. For instance, when M is a complete stable minimal surface immersed in $\mathbb{R}^3$, letting R tend to infinity, estimate (1) implies that M is a plane (a result proved independently by do Carmo – Peng and Fischer-Colbrie – Schoen).
Previously, Heinz [10], Osserman [13] had proved similar estimates in some particular cases and Schoen, Simon, Yau [18] other curvature type estimates in higher dimensions.

The purpose of the present paper is to prove similar estimates for stable surfaces M with constant mean curvature H immersed in a 3-manifold $\overline{M}(c)$ with constant curvature c. The methods are very much inspired by those of [16]. Denoting by A^0 the traceless second fundamental form of the immersion, we shall prove estimates of the form

$$|A^0|^2(x_0) \leq C(\Lambda)R^{-2} \text{ and } |K_g(x_0)| \leq C(\Lambda)R^{-2}.$$

provided that the ball $B(x_0, R) \subset M$ is relatively compact and that R satisfies one of the following conditions

$$(A) \qquad c + H^2 \leq 0 \text{ and } 4R^2(c + H^2)_- \leq \Lambda,$$

or

$$(B) \qquad c + H^2 > 0, \text{ and } 4(c + H^2)R^2 \leq \pi^2,$$

where Λ is a free parameter.

Note that F. Sauvigny [15] obtained an estimate of the form $|K(x_0)| \leq CR^{-2}$, with a constant which depends on the product HR for surfaces immersed in $\mathbb{R}^3$. Let us also point out that the estimate of Heinz and Osserman has been generalized to the constant mean curvature case by Spruck [21] and that Ecker and Huisken [6] obtained similar curvature estimates for graphs with prescribed mean curvature in the Euclidean n-space.

When $c + H^2 = 0$, there are no restrictions on the size of R in our estimate. This is not very surprising in view of Schoen's result [16] and of the Lawson correspondance between minimal surfaces in $\mathbb{R}^3$ and surfaces with constant mean curvature 1 in $\mathbb{H}^3$ (see [2, 12] and Section 6.3). We are then able to give a different proof of Silveira's result [19] which states that a complete stable surface with constant mean curvature 1 in $\mathbb{H}^3$ is a horosphere. We refer to Section 4 for more details.

When $c + H^2 > 0$, one can show (see Section 6.5), following an argument of H. Rosenberg and A.Ros in the Euclidean case and of Ronaldo Freire de Lima in the general case that the limitation on the radius R is necessary.

We shall in fact give stronger results and consider the case in which the immersion is only assumed to have finite index (see Theorem 4.2 for a precise statement).

As is well-known, there are two different notions of stability for complete constant mean curvature surfaces. Both involve the stability operator L of the immersion (see Section 6.2.2). For *strong stability*, one considers the operator L acting on all smooth functions with compact support in M, while for *weak stability*, one considers the operator L acting on smooth functions with compact support having mean-value equal to zero on M. Using our curvature estimates and [1], one can show that these notions coincide for complete non-compact surfaces.

Notations: Let $i : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion of an oriented Riemann surface into a simply-connected 3-manifold with constant curvature c. We choose a unit normal field ν along the immersion. Let $A : T_pM \to T_pM$ be the shape operator associated to the second fundamental form and let k_1, k_2 be the eigenvalues of A. The mean curvature H of the immersion is given by $2H = k_1 + k_2$. We assume $H = Ct$ and we note $A^0 = A - HId$ the operator associated with the traceless second fundamental form. Both tensors A, A^0 satisfy the Codazzi equation. The stability operator L_g is given by

$$L_g = \Delta_g + \{|A^0|^2 + 2(c + H^2)\}$$

where Δ_g is the non-positive Laplacian.

We assume furthermore that the immersion i is (strongly) stable *i.e.* that the second variation of the area is non-negative for all deformations with compact support:

$$-\int_M \phi L_g \phi \, dv_g \geq 0$$

for all smooth functions ϕ with compact support in M, with ϕ vanishing on ∂M if M has a boundary. Here dv_g is the Riemannian measure associated with the metric g.

The stability assumption implies that the inequality

$$\int_M \zeta^2 \phi L_g \phi \, dv_g \leq \int_M \phi^2 |d\zeta|_g^2 \, dv_g \tag{2}$$

holds for any C^∞ function ϕ and for any Lipschitz function with compact support ζ on M (Lemma 6.2 in the Appendix). We have denoted by $|d\zeta|_g$ the norm of the differential of the function ζ in the metric g.

As in [16], the proof of our curvature estimates consists in applying (2) to different well chosen functions. The paper is organized as follows.

In Section 2, we recall the well-known iteration method of de Giorgi, Moser and Nash; it will be used repeatedly in the paper.

Section 3 is devoted to studying conformal isometric immersions of the unit disk. Similar results, in the stable case, were obtained in [16] (Theorem 1) and in [5] (in a more general setting). Our result (Theorem 3.2) is more precise and applies in the finite index case as well.

In Section 4, we state our curvature estimates and we give some applications (in particular to the equivalence between weak and strong stability in the complete case).

Section 5 is devoted to the proof of Theorem 4.2.

Several results, which we use throughout the text are gathered in the Appendices.

The authors are grateful to Manfredo do Carmo, Pascal Collin, Geraldo de Oliveira Filho, Harold Rosenberg and Walcy Santos for stimulating discussions during the preparation of this paper.

2 The de Giorgi–Moser–Nash iteration method

In this paper, we will apply the de Giorgi–Moser–Nash iteration method repeatedly, with slight variations, in order to obtain our curvature estimates. The purpose of this section is to recall the main lines of this method for the convenience of the reader. The iteration method is based on Sobolev inequalities.

2.1 Sobolev inequalities

Let (M,g) be a Riemannian surface. The main assumption we need is that (M,g) satisfies a Sobolev inequality of the form

$$(3) \qquad \Big(\int_M f^2\, dv_g\Big)^{1/2} \le A_M \Big(\int_M |df|_g\, dv_g + \int_M B_M |f|\, dv_g\Big)$$

for all real valued, C^1-functions with compact support in M, $f \in C_0^1(M,\mathbb{R})$. Here $|df|_g$ denotes the pointwise norm of the differential of f (or equivalently of its gradient) with respect to the Riemannian metric g and dv_g denotes the Riemannian measure. This Sobolev inequality involves a constant A_M and a non-negative function B_M which a priori depend on the geometry of (M,g).

Such an inequality, with $A_M = A(n)$, a constant which only depends on the dimension n, and with $B_M = 0$, holds when (M,g) is the Euclidean space $(\mathbb{R}^2, e)$ or when M is a minimal surface immersed in Euclidean 3-space.
A similar inequality, with $A_M = A(n)$ and $B_M = |H|$, holds when (M,g) is isometrically immersed with mean curvature H into a simply-connected Riemannian manifold $(\overline{M},\overline{g})$ with non-positive sectional curvatures (see [11], where a more general situation is described and [4], where it is shown that one can choose $A_M = A(n,h)$ and $B_M = 0$ when $\overline{M}$ has constant sectional curvatures equal to -1 and $|H| \le h < 1$). Note that no completeness assumption is made on (M,g).
Given $p \ge 1$ and $u \in C_0^1(M,\mathbb{R})$ we apply inequality (3) to $f = |u|^p$ and we obtain

$$\Big(\int_M |u|^{2p}\, dv_g\Big)^{1/2} \le pA_M\Big(\int_M |u|^{p-1}\, |du|_g\, dv_g + \int_M B_M |u|^p\, dv_g\Big).$$

Hölder's inequality with $\frac{p+1}{2p} + \frac{p-1}{2p} = 1$ gives

$$\Big(\int_M |u|^{2p}\, dv_g\Big)^{\frac{1}{2p}} \le pA_M\Big\{\Big(\int_M |du|_g^{\frac{2p}{p+1}}\, dv_g\Big)^{\frac{p+1}{2p}} + \Big(\int_M (B_M|u|)^{\frac{2p}{p+1}}\, dv_g\Big)^{\frac{p+1}{2p}}\Big\}.$$

Hölder's inequality with $\frac{p}{p+1} + \frac{1}{p+1} = 1$ gives

$$\Big(\int_M |u|^{2p}\, dv_g\Big)^{\frac{1}{2p}} \le pA_M\Big(\int_{\mathrm{Supp}\,(u)} dv_g\Big)^{\frac{1}{2p}} \Big\{\Big(\int_M |du|_g^2\, dv_g\Big)^{\frac12} + \Big(\int_M B_M^2 u^2\, dv_g\Big)^{\frac12}\Big\}$$

and

$$(4) \Big(\int_M |u|^{2p}\, dv_g\Big)^{\frac{1}{p}} \le 2p^2 A_M^2\Big(\int_{\mathrm{Supp}\,(u)} dv_g\Big)^{\frac{1}{p}} \Big\{\int_M |du|_g^2\, dv_g + \int_M B_M^2 u^2\, dv_g\Big\},$$

for any $u \in C_0^1(M,\mathbb{R})$ and any $p \ge 1$.

2.2 The de Giorgi–Moser-Nash lemma

Lemma 2.1 *Assume that the Riemannian manifold (M, g) satisfies the Sobolev inequality (3). Let $B(R)$ be some relatively compact geodesic ball in (M, g), centered at some point x_0 and assume that it satisfies the volume estimate*
(a) there exists some constant C_1 such that $\int_{B(R)} dv_g \leq C_1 R^2$.
Let f, h be real valued C^2 functions on $B(R)$ such that $h \geq 0$ and $\Delta_g h + fh \geq 0$ pointwise in $B(R)$, where Δ_g is the non-positive Laplacian on (M, g). Assume furthermore that
(b) there exist some number $q \geq 6$ and some constant C_2 such that

$$\Big(\int_{B(3R/4)} h^{2q}\, dv_g\Big)^{1/q} \leq C_2 R^{-2+2/q},$$

(c) there exists some constant C_3 such that for all $\alpha \in [0, 1/2]$,

$$\int_{B(3R/4)} (f + B_M^2)_+^{1+\alpha}\, dv_g \leq C_3 R^{-2\alpha}.$$

Then there exists a constant $C := C(A_M, C_1, C_2, C_3)$ such that

$$\sup_{B(R/2)} h^2 \leq q^2 C R^{-2}.$$

Proof. The proof of this lemma uses Sobolev inequality (4) and the de Giorgi–Nash–Moser iteration method. In the proof, we will denote by c_i constants which only depend on A_M, C_1, C_2, C_3.
• Step 1 : integration by parts.
Let ζ be a non-negative Lipschitz function with compact support in $B(R)$. Let $k \in \mathbb{R}$, with $k \geq 1$. Then

$$|d(\zeta h^k)|_g^2 = h^{2k}|d\zeta|_g^2 + k^2\zeta^2 h^{2k-2}|dh|_g^2 + 2k\zeta h^{2k-1}\langle dh, d\zeta\rangle_g$$

and

$$\langle d(\zeta^2 h^{2k-1}), dh\rangle_g = 2\zeta h^{2k-1}\langle dh, d\zeta\rangle_g + (2k-1)\zeta^2 h^{2k-2}|dh|_g^2.$$

Since $k \geq 1$, we obtain

$$|d(\zeta h^k)|_g^2 \leq h^{2k}|d\zeta|_g^2 + k\langle d(\zeta^2 h^{2k-1}), dh\rangle_g.$$

Multiplying the inequality $(\Delta_g + f)h \geq 0$ by $\zeta^2 h^{2k-1}$ and integrating by parts, we find

$$-\int_{B(R)} \zeta^2 h^{2k-1}\Delta_g h\, dv_g = \int_{B(R)} \langle d(\zeta^2 h^{2k-1}), dh\rangle_g\, dv_g \leq \int_{B(R)} f\zeta^2 h^{2k}\, dv_g$$

and finally

$$(5) \qquad \int_{B(R)} |d(\zeta h^k)|_g^2\, dv_g \leq \int_{B(R)} h^{2k}|d\zeta|_g^2\, dv_g + k\int_{B(R)} f\zeta^2 h^{2k}\, dv_g.$$

Now given $a \in [1/2, 3/4[$ and $r \in [0, 3/4 - a]$, we define

$$B_a := B(aR) \subset B_{a+r} := B((a+r)R) \subset B(3R/4),$$

and we choose a family of Lipschitz functions $\zeta = \theta \circ \rho$ depending on a, r, R, where ρ is the geodesic distance to the given point x_0 in (M, g) and where θ is a smooth function such that $0 \leq \theta \leq 1, \theta = 1$ on $[0, aR], \theta = 0$ on $[(a+r)R, \infty[$ and $|\theta'| \leq 2/(rR)$.

• Step 2 : using the Sobolev inequality.

Plugging $u := \zeta h^k$ into the Sobolev inequality (4), with ζ as above and $p = q$, we obtain

$$\begin{aligned}(\int_{B(R)} (\zeta h^k)^{2q}\, dv_g)^{\frac{1}{q}} \;&\leq\; q^2 c_1 (\int_{B(R)} dv_g)^{\frac{1}{q}} \{ \int_{B(R)} |d(\zeta h^k)|_g^2\, dv_g \\ &\quad + \int_{B(R)} B_M^2 (\zeta h^k)^2\, dv_g \}\end{aligned}$$

which gives, using formula (5)

$$\begin{aligned}(\int_{B_a} h^{2kq}\, dv_g)^{\frac{1}{q}} \;&\leq\; k q^2 c_1 (\int_{B(R)} dv_g)^{\frac{1}{q}} \{ \int_{B_{a+r}} h^{2k} |d\zeta|_g^2\, dv_g \\ &\quad + \int_{B_{a+r}} (f + B_M^2)_+ h^{2k}\, dv_g \}.\end{aligned} \tag{6}$$

• Step 3 : applying Hölder's inequality.

We now apply Hölder's inequality with $\frac{2}{q} + \frac{q-2}{q} = 1$. Since $q \geq 6$, we have $q/(q-2) \leq 3/2$ and

$$\begin{aligned}(\int_{B_a} h^{2kq}\, dv_g)^{\frac{1}{q}} &\leq k q^2 c_1 (\int_{B(R)} dv_g)^{\frac{1}{q}} \Big\{ \big(\int_{B_{a+r}} |d\zeta|_g^{\frac{2q}{q-2}}\, dv_g \big)^{\frac{q-2}{q}} \\ &+ \big(\int_{B_{a+r}} (f + B_M^2)_+^{\frac{q}{q-2}}\, dv_g \big)^{\frac{q-2}{q}} \Big\} \big(\int_{B_{a+r}} h^{qk}\, dv_g \big)^{\frac{2}{q}}.\end{aligned} \tag{7}$$

Applying assumption (a) and the fact that $|\theta'| \leq 2/(rR)$ we get

$$\big(\int_{B_{a+r}} |d\zeta|_g^{\frac{2q}{q-2}}\, dv_g \big)^{\frac{q-2}{q}} \leq c_2 r^{-2} R^{-2} \big(\int_{B(R)} dv_g \big)^{\frac{q-2}{q}} \leq c_3 r^{-2} R^{-\frac{4}{q}}. \tag{8}$$

Using assumption (c) we obtain

$$\big(\int_{B_{a+r}} (f + B_M^2)_+^{\frac{q}{q-2}}\, dv_g \big)^{\frac{q-2}{q}} \leq c_4 R^{-\frac{4}{q}}. \tag{9}$$

We can now plug inequalities (8) and (9) into (7) to obtain

$$\big(\int_{B_a} h^{2qk}\, dv_g \big)^{\frac{1}{q}} \leq c_5 q^2 k R^{-\frac{2}{q}} (r^{-2} + 1) \big(\int_{B_{a+r}} h^{qk}\, dv_g \big)^{\frac{2}{q}} \tag{10}$$

for all $k \geq 1$.

• Step 4 : the iteration.

We now define $k_i = 2^i, r_i = 2^{-i-3}, a_0 = 3/4, a_{i+1} = a_i - r_i$, for $i \geq 0$, i.e. $a_i = \frac{1}{2} + \frac{1}{2^{i+2}}$, and

$$I(i) = \Big(\int_{B_{a_i}} h^{2q2^i}\, dv_g\Big)^{\frac{1}{q2^i}}.$$

Rewriting the formula (10) with k_{i+1}, a_{i+1} and r_i, we obtain

$$I(i+1)^{2^{i+1}} \leq q^2 c_6 2^{i+1} R^{-2/q}(2^{2i+6}+1) I(i)^{2^{i+1}}.$$

Then:

$$I(i+1) \leq \big(c_6 2^6 q^2\big)^{1/2^{i+1}} (2^{3(i+1)})^{1/2^{i+1}} R^{-2/q2^{i+1}} I(i). \tag{11}$$

Iterating (11), we obtain

$$I(i+1) \leq C(i+1) R^{-2d_{i+1}/q} I(0)$$

where $d_{i+1} = \sum_{j=1}^{i+1} \frac{1}{2^j}$ and $C(i+1) = (q^2 c_6 2^6)^{d_{i+1}} \prod_{j=1}^{i+1} \big(8^j\big)^{1/2^j}$.

Assumption (b) gives the initial estimate for $I(0)$:

$$I(0) = \Big(\int_{B(3R/4)} h^{2q}\, dv_g\Big)^{1/q} \leq C_2 R^{-2+2/q}.$$

Thus we get:

$$I(i+1) \leq C(i+1) C_2 R^{-2+2/q-2d_{i+1}/q} \tag{12}$$

Letting i tend to infinity in (12), we obtain

$$\lim_{i\to\infty} I(i+1) = \sup_{B(R/2)} h^2 \leq q^2 C R^{-2} \tag{13}$$

where the constant C only depends on A_M, C_1, C_2, C_3. ∎

3 On conformal disks

Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into a 3-manifold with constant sectional curvatures c (we do not need $\overline{M}$ to be simply-connected in this section). Assume furthermore that the immersion has constant mean curvature H. Write the metric g as

$$g = i^*\overline{g} = \lambda^2 e = h^{-2} e$$

where $e = |dz|^2$ is the Euclidean metric in D_r and $2\lambda^2 = |di|^2_g$ (where $|di|^2_g := |di(\frac{\partial}{\partial x}|^2_g + |di(\frac{\partial}{\partial y}|^2_g)$. The purpose of this section is to give a lower bound on the function λ (or equivalently an upper bound on the function h) under a stability assumption on the immersion i. Theorem 3.1 below generalizes Theorem 1 in [16]. The method of proof is similar.

Theorem 3.1 *Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into $\overline{M}^3(c)$. Assume that i has constant mean curvature H. Let $B(R)$ denote the geodesic g-ball of radius R with center at 0 and assume that $B(R)$ is relatively compact in D_r. Assume finally that the*

immersion i is stable on $B(R)$, i.e. that the stability operator $-L_g$ is non-negative on the space $C_0^\infty(B(R))$.
Then there exists a universal constant $C_0 > 0$ such that

$$\inf_{B(R/2)} |di|_g^2 \geq C_0 r^{-2} R^2 \big(1 + R^2 (c+H^2)_-\big)^{-1}.$$

We shall in fact prove the following stronger result.

Theorem 3.2 *Let $i : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion of the disk of radius r in $\mathbb{R}^2$ into $\overline{M}^3(c)$. Assume that i has constant mean curvature H. Let $B(R)$ denote the geodesic g-ball of radius R with center at 0 and assume that $B(R)$ is relatively compact in D_r. Assume finally that the the stability operator L_g of the immersion i is bounded from above by some non-negative number 2ℓ on the space $C_0^\infty(B(R))$.*
Then there exists a universal constant $C_0 > 0$ such that

$$\inf_{B(R/2)} |di|_g^2 \geq C_0 r^{-2} R^2 \big\{1 + R^2 \big((c+H^2)_- + \ell\big)\big\}^{-1}.$$

Remarks. The assumption that L_g is bounded from above on $C_0^\infty(B(R))$ is equivalent to saying that the least eigenvalue of L_g on $B(R)$, with Dirichlet boundary condition, is bounded from below by -2ℓ. Such an assumption is verified for a complete immersion with finite Morse index in the sense of [7].

Proof of Theorem 3.2. In the following, $c_i > 0, i = 1, 2...$, will denote universal constants.

Step 0: Rescaling.
Let $i_r : (D_r, g) \to (\overline{M}^3(c), \overline{g})$ as above, let $m_r(D, \tilde{g}) \to (D_r, g)$ be the r-dilation and let $i = i_r \circ m_r$. The equality $i^*\overline{g} = m_r^* g = \tilde{g}$ implies that $g = \lambda^2 e$ and $\tilde{g} = \tilde{\lambda}^2 e$ with $\tilde{\lambda}(z) = r\lambda(rz)$.
By rescaling, we may therefore assume that $r = 1$ and we will denote by D the unit disk D_1.
Recall the general formula which relates the curvatures K_g, K_{g_0} of two conformal metrics $g = e^{2u} g_0$ in dimension 2:

$$K_g = e^{-2u}\{K_{g_0} - \Delta_{g_0} u\} = e^{-2u} K_{g_0} - \Delta_g$$

(with non-positive Laplacians).
Since the metric g is conformal to the Euclidean metric $e = |dz|^2$, the (intrinsic) Gauss curvature of the metric g is given by $K_g = -\Delta_g \ln \lambda = \Delta_g \ln h$, where $\Delta_g = \lambda^{-2}\big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\big)$ is the Laplacian for the metric g. Since $\Delta_g \ln h = h^{-1}\Delta_g h - h^{-2}|dh|_g^2$, it follows that

$$\Delta_g h = K_g h + h^{-1}|dh|_g^2 \geq K_g h. \tag{14}$$

The Gauss equation of the immersion can be written as

$$2K_g = -|A^0|^2 + 2(c+H^2) \tag{15}$$

(see (80) in the Appendix and recall that A^0 is the traceless second fundamental form of the immersion) so that inequality (14) gives

$$\Delta_g h + \{\frac{1}{2}|A^0|^2 - (c+H^2)\}h \geq 0. \tag{16}$$

Let $a := (c+H^2)_- = \max\{0, -(c+H^2)\}$ denote the negative part of $(c+H^2)$ and let f denote the function $f := \frac{1}{2}|A^0|^2 + a$. With these notations, inequality (16) implies that

$$\Delta_g h + fh \geq 0. \tag{17}$$

We will apply a variant of the de Giorgi-Moser-Nash lemma to this inequality in order to bound h from above. For this purpose, we need some initial estimates (compare with Lemma 2.1). As in [16], they will be given by the stability assumption (more precisely, by the following lemma applied to suitable functions ζ and ϕ). We will repeatedly use the fact that the metric g is conformal to the Euclidean metric e.
In the sequel, we denote by $|d\varphi|_g$ and $|d\varphi|_e$ the norm of the differential of a function φ respectively in the metrics g and e. Recall that $g = h^{-2}e$ and observe that $|d\varphi|_g^2 = h^2|d\varphi|_e^2$ and that the Riemannian measures are related by $dv_g = h^{-2}dv_e$ (notice that $|d\varphi|_e^2 = \varphi_x^2 + \varphi_y^2$ on D).

Lemma 3.3 *Under the assumptions of Theorem 3.2, the stability inequality*

$$\int_D \zeta^2 \phi L_g \phi \, dv_g \leq \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2 \zeta^2 \, dv_g \tag{18}$$

holds for all $\phi \in C^\infty(D)$ and all $\zeta \in C_0^\infty(B(R))$.

This lemma follows from Lemma 6.2 in the Appendix, applied to the operator $L = L_g - 2\ell$. ∎

Step 1: Initial estimates.
As in Lemma 2.1, we need estimates of $\int_{B(3R/4)} h^{2p}\, dv_e$, for some $p \in [6, +\infty[$, and of $\int_{B(3R/4)} f^\alpha \, dv_e$, for all $1 \leq \alpha \leq 3/2$. Using the expression of the stability operator $L_g = \Delta_g + \{|A^0|^2 + 2(c+H^2)\}$ (see formula (83) in the Appendix), equation (14) and the stability condition (18) with $\phi = h$, we obtain

$$\int_D \zeta^2 \{|dh|_g^2 + \big(K_g + 2(c+H^2) + |A^0|^2\big)h^2\} \, dv_g \leq \int_D h^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D h^2 \zeta^2 \, dv_g$$

for any function $\zeta \in C_0^\infty(B(R), \mathbb{R})$ and, more generally, for any Lipschitz function with compact support in $B(R)$. Using (15), taking into account the relations between the metrics g and e and using the conformal invariance of the Dirichlet integral, we obtain

$$\int_D \zeta^2 \{|dh|_e^2 + \big(\frac{1}{2}|A^0|^2 + 3(c+H^2) - 2\ell\big)\} \, dv_e \leq \int_D h^2 |d\zeta|_e^2 \, dv_e. \tag{19}$$

• Estimate of $\displaystyle\int_{B(3R/4)} h^{2p} \, dv_e$:

Let us denote by a' the number $a' := (c+H^2)_- + \ell = a+\ell$. Using equation (19) and the relationship between the metrics g and e, we obtain the inequalities

$$
\begin{aligned}
\int_D |d(\zeta h)|_e^2 \, dv_e &\leq 2\int_D \zeta^2 |dh|_e^2 \, dv_e + 2\int_D h^2 |d\zeta|_e^2 \, dv_e \\
&\leq 4\int_D h^2 |d\zeta|_e^2 \, dv_e + 6a' \int_D \zeta^2 \, dv_e \qquad (20)\\
&= 4\int_D |d\zeta|_g^2 \, dv_e + 6a' \int_D \zeta^2 \, dv_e .
\end{aligned}
$$

Let ζ be a cut-off function of the geodesic distance in the g-metric, with $\zeta = 1$ on $B(3R/4), \zeta = 0$ outside $B(7R/8)$, and $|d\zeta|_g^2 \leq c_1 R^{-2}$ for some universal constant $c_1 > 0$ independent of R. The Euclidean Sobolev inequality, *i.e.* inequality (4) with $B_M = 0$, applied to the function $u = \zeta h$ gives

$$\Big(\int_D (\zeta h)^{2p} \, dv_e\Big)^{1/p} \leq p^2 c_2 \Big(\int_D dv_e\Big)^{1/p} \int_D |d(\zeta h)|_e^2 \, dv_e .$$

Taking (20) into account, with the above choice of ζ, we obtain our first initial estimate (to be compared with assumption (b) in Lemma 2.1)

$$
\begin{aligned}
\Big(\int_{B(3R/4)} h^{2p} \, dv_e\Big)^{\frac{1}{p}} &\leq 4p^2 c_2 \Big(4\int_D |d\zeta|_g^2 \, dv_e + 6a' \int_D \zeta^2 \, dv_e\Big) \qquad (21)\\
&\leq p^2 c_3 (R^{-2} + a')
\end{aligned}
$$

for any $p \in [1, +\infty[$, where c_3 is a universal constant.

• Estimate of $\int_{B(3R/4)} f^\alpha \, dv_e$:

From the definitions of a and f, we have $2f = |A^0|^2 + 2a$. Choosing a suitable cut-off function ζ, with compact support in $B(R)$ and such that $\zeta = 1$ on $B(7R/8)$, we obtain from (19)

$$(22)\int_{B(7R/8)} (|A^0|^2 + 2a) \, dv_e \leq 2\int_D |d\zeta|_g^2 \, dv_e + 8a' \int_D \zeta^2 \, dv_e \leq c_4(R^{-2} + a').$$

Lemma 3.4 *Let $\phi := h(|A^0|^2 + 2a)^\alpha$, for $0 < \alpha \leq 1$. Then*

$$\phi \Delta_g \phi \geq \phi^2 \Delta_g \ln \phi \geq -(\frac{1}{2} + 2\alpha) h^2 (|A^0|^2 + 2a)^{1+2\alpha}.$$

Proof of the lemma. Using moving frame techniques as in [22] or J. Simons' equation (Proposition 6.7), we have $\Delta_g \ln |A^0|^2 = 4K_g$, which implies (using Lemma 6.1 in the Appendix) that

$$\Delta_g \ln(|A^0|^2 + 2a) \geq \frac{4K_g |A^0|^2}{|A^0|^2 + 2a}.$$

Since $K_g \leq 0$ when $a \neq 0$, if follows that $\Delta_g \ln(|A^0|^2 + 2a) \geq 4K_g$. One can now write

$$\Delta_g \ln \phi \geq (1+4\alpha)K_g \geq -(\frac{1}{2}+2\alpha)(|A^0|^2+2a),$$

and the lemma follows. ∎

By applying the stability condition (18), with $\phi = h(|A^0|^2+2a)^\alpha$ and with a suitable cut-off function ζ, we obtain

$$(23)\qquad \begin{aligned}(\frac{1}{2}-2\alpha)\int_{B(3R/4)} h^2(|A^0|^2 \quad &+ \quad 2a)^{1+2\alpha}\, dv_g \leq \int_D h^2(|A^0|^2+2a)^{2\alpha}|d\zeta|_g^2\, dv_g \\ &+ \quad (4a+2\ell)\int_D h^2\zeta^2(|A^0|^2+2a)^{2\alpha}\, dv_g.\end{aligned}$$

Choosing α small enough, for example $\alpha = 1/8$, and taking the relationship between the metrics g and e into account, we obtain

$$(24)\qquad \begin{aligned}\frac{1}{4}\int_{B(3R/4)}(|A^0|^2 \quad &+ \quad 2a)^{5/4}\, dv_e \leq \int_D (|A^0|^2+2a)^{1/4}|d\zeta|_g^2\, dv_e \\ &+ \quad 4a'\int_D \zeta^2(|A^0|^2+2a)^{1/4}\, dv_e\end{aligned}$$

since $a' = a+\ell$. Using (22), Hölder's inequality and the fact that the Euclidean volume is controled we have

$$(25)\qquad \begin{aligned}\int_{B(7R/8)}(|A^0|^2+2a)^{1/4}\, dv_e \quad &\leq \quad c_5\big(\int_{B(7R/8)}(|A^0|^2+2a)\, dv_e\big)^{1/4} \\ &\leq \quad c_6(R^{-2}+a')^{1/4}\end{aligned}$$

for some universal constants c_5, c_6. With a suitable choice of ζ in (24), we get

$$\begin{aligned}\int_{B(3R/4)}(|A^0|^2+2a)^{5/4}\, dv_e \quad &\leq \quad c_7(R^{-2}+a')\int_{B(7R/8)}(|A^0|^2+2a)^{1/4}\, dv_e \\ &\leq \quad c_8(R^{-2}+a')^{5/4}.\end{aligned}$$

Finally, we obtain our second initial estimate

$$(26)\qquad \big(\int_{B(3R/4)}(|A^0|^2+2a)^{5/4}\, dv_e\big)^{4/5} \leq c_9(R^{-2}+a').$$

Step 2: The iteration.

With estimates (21) and (26) at hand, we can apply the de Giorgi–Moser–Nash iteration method to (17), $\Delta_g h + fh \geq 0$. Formula (5) is still valid

$$\int_{B(R)}|d(\zeta h^k)|_g^2\, dv_g \leq \int_{B(R)} h^{2k}|d\zeta|_g^2\, dv_g + k\int_{B(R)} f\zeta^2 h^{2k}\, dv_g$$

and gives, by conformal invariance and using the relation $h^2 dv_g = dv_e$,

$$(27)\quad \int_{B(R)} |d(\zeta h^k)|_e^2\, dv_e \le \int_{B(R)} h^{2k-2}|d\zeta|_g^2\, dv_e + k\int_{B(R)} f\zeta^2 h^{2k-2}\, dv_e.$$

Now given $t \in [1/2, 3/4[$ and $r \in [0, 3/4 - t]$, we define

$$B_t := B(tR) \subset B_{t+r} := B((t+r)R) \subset B(3R/4)$$

and we choose a family of Lipschitz functions $\zeta = \theta \circ \rho$ depending on t, r, R, where ρ is the geodesic distance to the point 0 in (D, g) and where θ is a smooth function such that $0 \le \theta \le 1, \theta = 1$ on $[0, tR], \theta = 0$ on $[(t+r)R, \infty[$ and $|\theta'| \le 2/(rR)$.
We apply the Euclidean Sobolev inequality to the function ζh^k and, using (27), we get

$$(28)\quad \Big(\int_{B_t} h^{2kp}\, dv_e\Big)^{\frac{1}{p}} \le p^2 k c_{10} \int_{B_{t+r}} \{h^{2k-2}|d\zeta|_g^2 + f\zeta^2 h^{2k-2}\}\, dv_e.$$

Let $p_1 := \dfrac{kp}{2(k-1)}$ and $p_2 := \dfrac{kp}{k(p-2)+2}$ and choose $p \ge 10$ so that $p_2 \le 5/4$.
We can apply Hölder's inequality, with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, to inequality (28) and we obtain the analog of inequality (6)

$$(29)\quad \begin{aligned} \Big(\int_{B_t} h^{2kp}\, dv_e\Big)^{\frac{1}{p}} &\le p^2 k c_{11}\Big\{\Big(\int_{B_{t+r}} |d\zeta|_g^{2p_2}\, dv_e\Big)^{1/p_2} \\ &+ \Big(\int_{B_{t+r}} f^{p_2}\, dv_e\Big)^{1/p_2}\Big\}\Big(\int_{B_{t+r}} h^{pk}\, dv_e\Big)^{2(k-1)/pk}. \end{aligned}$$

Since $B_{t+r} \subset D$, we have

$$\Big(\int_{B_{t+r}} |d\zeta|_g^{2p_2}\, dv_e\Big)^{1/p_2} \le c_{12} r^{-2} R^{-2}$$

and, applying Hölder's inequality and using (26),

$$\Big(\int_{B_{t+r}} f^{p_2}\, dv_e\Big)^{1/p_2} \le c_{13}\Big(\int_{B_{t+r}} f^{5/4}\, dv_e\Big)^{4/5} \le c_{14}(R^{-2} + a')$$

since $p_2 \le 5/4$. Plugging these inequalities into (29) we obtain

$$(30)\Big(\int_{B_t} h^{2pk}\, dv_e\Big)^{1/p} \le p^2 k c_{15}(R^{-2} + a')(r^{-2} + 1)\Big(\int_{B_{t+r}} h^{pk}\, dv_e\Big)^{2(k-1)/pk}.$$

We now perform the iteration (as in the proof of Lemma 2.1, Step 4). We choose $q \ge 1$ and we define $k_i = q2^i, r_i = 2^{-i-3}, t_0 = 3/4$ and $t_{i+1} = t_i - r_i$, for $i \ge 0$, and

$$I(i, q) := \Big(\int_{B_{t_i}} h^{2pk_i}\, dv_e\Big)^{1/pk_i}.$$

We can rewrite (30) with k_{i+1}, t_{i+1} and r_i as

$$I(i+1,q)^{k_{i+1}} \le p^2 c_{15} k_{i+1} (R^{-2}+a')(r_i^{-2}+1) I(i,q)^{k_{i+1}-1}.$$

From which we obtain

$$(31) \quad I(i+1,q) \le \left(p^2 c_{15} k_{i+1} 2^{2i+8}\right)^{1/k_{i+1}} (R^{-2}+a')^{1/k_{i+1}} I(i,q)^{1-1/k_{i+1}}.$$

Define the sequence

$$s_{i,j} = \begin{cases} (1-\frac{1}{k_i})(1-\frac{1}{k_{i-1}})\dots(1-\frac{1}{k_{j+1}})\frac{1}{k_j} & \text{if } j<i \\ \frac{1}{k_i} & \text{if } j=i \end{cases}$$

Iterating inequality (31), we obtain

$$(32) \qquad I(i+1,q) \le C(i+1,q)(R^{-2}+a')^{\alpha(i+1,q)} I(0,q)^{\beta(i+1,q)}$$

with

$$\begin{aligned} \alpha(i,q) &= \sum_{j=1}^{i} s_{i,j} \\ \beta(i,q) &= (1-\frac{1}{k_i})(1-\frac{1}{k_{i-1}})\dots(1-\frac{1}{k_1}) \\ C(i,q) &= (p^2 c_{15} 2^6)^{\alpha(i,q)} \prod_{j=1}^{i} (4^j k_j)^{s_{i,j}}. \end{aligned}$$

Applying inequality (21), we have

$$I(0,q) = \Big(\int_{B(3R/4)} h^{2pq}\, dv_e\Big)^{1/pq} \le p^2 q^2 c_3 (R^{-2}+a').$$

Since $\beta(i+1,q) = 1-\alpha(i+1,q)$, we have

$$I(i+1,q) \le C(i+1,q)(p^2q^2c_3)^{\beta(i+1,q)}(R^{-2}+a').$$

Let us define $\beta(q) = \lim_{i\to\infty} \beta(i+1,q)$ and $\alpha(q) = \lim_{i\to\infty}\alpha(i+1,q) = 1-\beta(q)$. A straightforward computation gives

$$\frac{1}{q} \le -\ln\beta(q) \le \frac{1}{q} + \frac{4}{3q^2}$$

which implies that $\beta(q) = 1-\frac{1}{q}+O(\frac{1}{q^2})$ and $\alpha(q) = \frac{1}{q}+O(\frac{1}{q^2})$ when q goes to infinity. We also have, $\lim\limits_{i\to\infty}(p^2c_{15}2^6)^{\alpha(i+1,q)} = (p^2c_{15}2^6)^{\alpha(q)}$.
Moreover

$$\lim_{i\to\infty}\prod_{j=1}^{i+1}(4^j k_j)^{s_{i+1,j}} = \lim_{i\to\infty}\prod_{j=1}^{i+1}(q8^j)^{s_{i+1,j}} = \lim_{i\to\infty} e^{\alpha(i+1,q)\ln q} e^{\gamma(i+1,q)\ln 8}$$

where $\gamma(i+1,q) = \sum_{j=1}^{i+1} j s_{i+1,j}$. Then $j s_{i+1,j} \leq \frac{j}{q 2^j}$ implies that $\gamma(i+1,q) \leq \delta/q$ where $\delta = \sum_{j=1}^{\infty} \frac{j}{2^j}$.

Thus

$$1 \leq \lim_{i\to\infty} \prod_{j=1}^{i+1} (4^j k_j)^{s_{i+1,j}} \leq q^{\alpha(q)} 8^{\delta/q}$$

This gives

$$\sup_{B(R/2)} h^2 \leq c_{17} p^2 q^{1+\beta(q)} (R^{-2} + a').$$

and

$$\inf_{B(R/2)} \lambda^2 \geq C_0 (R^{-2} + a')^{-1} = C_0 R^2 (1 + R^2 a')^{-1}.$$

once we have fixed some $p \geq 10$ and some $q \geq 1$. This is the estimate we wanted to prove. ∎

4 Curvature Estimates

The purpose of this section is to prove the following theorem which generalizes Theorem 3 in [16].

Theorem 4.1 *Let (M,g) be an oriented Riemannian surface. Let $i : (M,g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion with constant mean curvature H of (M,g) into a simply-connected 3-manifold with constant curvature c. Assume there are positive numbers $R' > R$ such that the geodesic g-ball $B(R')$ centered at some x_0 in M, with radius R', is relatively compact in (M,g) and that the stability operator L_g, with Dirichlet boundary conditions, is non-positive on $B(R')$. Let A^0 and K_g denote respectively the traceless second fundamental form and the Gaussian curvature of the immersion i.*
Given $\Lambda > 0$, there exists a constant $C(\Lambda)$, which only depends on Λ, such that

$$|A^0|^2(x_0) \leq C(\Lambda) R^{-2} \quad \text{and} \quad |K_g(x_0)| \leq C(\Lambda) R^{-2}.$$

under one of the following conditions

$$(A) \qquad c + H^2 \leq 0 \quad \text{and} \quad 4R^2(c+H^2)_- \leq \Lambda,$$

or

$$(B) \qquad c + H^2 > 0, \quad \text{and} \quad 4(c+H^2)R^2 \leq \pi^2.$$

We shall in fact prove the following, more general theorem.

Theorem 4.2 *Let (M,g) be an oriented Riemannian surface. Let $i:(M,g)\to(\overline{M}^3(c),\overline{g})$ be an isometric immersion with constant mean curvature H of (M,g) into a simply-connected 3-manifold with constant curvature c. Assume there are positive numbers $R' > R$ such that the geodesic g-ball $B(R')$ centered at some x_0 in M, with radius R', is relatively compact in (M,g) and that the stability operator L_g, with Dirichlet boundary conditions, is bounded from above by some number $2\ell \geq 0$ on $B(R')$. Let A^0 and K_g denote respectively the traceless second fundamental form and the Gaussian curvature of the immersion i.*
Given $\Lambda > 0$, there exist positive constants $C(\Lambda), c(\Lambda)$, which only depends on Λ, such that

$$|A^0|^2(x) \leq C(\Lambda)R^{-2} \quad \textit{and} \quad |K_g(x| \leq C(\Lambda)R^{-2}.$$

for all $x \in B(c(\Lambda)R)$, provided one of the following conditions holds,

$$(A) \qquad c+H^2 \leq 0 \quad \textit{and} \quad 4R^2\big((c+H^2)_- + \ell\big) \leq \Lambda,$$

or

$$(B) \begin{cases} c+H^2 > 0, \quad 4(c+H^2)R^2 \leq \pi^2, \\ \quad \textit{and} \quad 4\ell R^2 \leq \Lambda. \end{cases}$$

Remarks. Notice that when $c + H^2 = 0$, the condition $4R^2(c+H^2)_- \leq \Lambda$ in Theorem 4.1 is empty. Theorem 4.2 implies Theorem 4.1 by taking $\ell = 0$. The assumption that L_g is bounded from above by 2ℓ in Theorem 4.2 means that the least eigenvalue of the operator L_g in $B(R')$, with Dirichlet boundary condition, is bounded from below by -2ℓ. Such an assumption is verified when i is an immersion of a complete surface (M,g), with finite Morse index in the sense of [7].

We postpone the proof of Theorem 4.2 to Section 5 and we now give two applications.
Let $(M,g)\to(\overline{M}^3(c),\overline{g})$ be a complete immersion with constant mean curvature H. Once c, H are fixed, choosing some $\Lambda > 0$, we obtain from Theorem 4.2 uniform estimates for $|A^0|$ and for K_g over M. Recall that one can introduce two indices for the stability operator L_g, namely the *strong index* $\mathrm{Ind}_s(L_g)$, resp. the *weak index* $\mathrm{Ind}_w(L_g)$, defined as the maximal dimension of a subspace E of $C_0^\infty(M)$, resp. as the maximal dimension of a subspace E of $C_0^\infty(M)\cap\{f \mid \int_M f\,dv_g = 0\}$, such that $\int_M \varphi L_g\varphi\,dv_g \geq 0$ for all $\varphi \in E$ (see [7]). These indices satisfy the inequality

$$(33) \qquad \mathrm{Ind}_w(L_g) \leq \mathrm{Ind}_s(L_g) \leq \mathrm{Ind}_w(L_g) + 1.$$

Application 1. It follows from inequality (33) that weak stability implies that the strong index is at most 1. On the other-hand, Theorem 4.1 implies a uniform estimate for the Gauss curvature of a weakly stable surface M and hence, according to [1], that the surface is strongly stable provided it is complete and non-compact.

Application 2. Let M be a weakly stable complete immersion with constant mean curvature 1 in $\mathbb{H}^3$. It follows from the preceding application that the immersion is in fact strongly stable. We can then apply Theorem 4.1 with no restriction on R because $c + H^2 = 0$. This implies that $A^0 \equiv 0$ and hence that the immersion is totally umbilic. We have therefore obtained a new proof of Silveira's theorem [19].

5 Proof of Theorem 4.2

Proof of Theorem 4.2. The proof will take the remainder of this section. For the sake of clarity, and although some arguments will be repeated, we will give two separate proofs, one for each condition (A) and (B). Notice that we may slightly restrict R in the arguments if necessary.
In the course of the proof, we will use the following notations :

$$\left\{ \begin{array}{l} S := c + H^2 \text{ and } a := S_- = (c + H^2)_-, \\ S' := S - \ell \text{ and } a' := a + \ell. \end{array} \right. \tag{34}$$

Recall that the Gauss equation of the immersion i can be written as

$$K_g = -\frac{1}{2}|A^0|^2 + S. \tag{35}$$

5.1 Proof of Theorem 4.2 under Condition (A)

• **Step 1.** Since $S \leq 0$, we have $K_g \leq 0$ and the exponential map $\exp_{x_0}$ is a local diffeomorphism. Let us consider the ball $B_0(R') \subset (T_{x_0}M, \exp^*_{x_0} g)$.

We need the following lemma which appears in [8].

Lemma 5.1 *Let (M, g) be a Riemannian manifold. Let $q : M \to I\!R$ be a continuous function and let L be the operator $L := \Delta_g + q$.*
(i) If $\Omega \subset\subset M$ is a smooth relatively compact domain in M, then $L \leq 0$ on $C_0^\infty(\Omega)$ if and only if there exists a function $u : \Omega \to I\!R$, with $u \geq 0$ and $u \not\equiv 0$ on Ω, such that $Lu \leq 0$ in Ω;
(ii) If M is complete non-compact, $L \leq 0$ on $C_0^\infty(M)$ if and only if there exists a function $u : M \to I\!R$, with $u \geq 0$ and $u \not\equiv 0$ in M, such that $Lu = 0$ in M.

Proof. We shall in fact only use Assertion (i) which follows by applying Green's formula and the fact that the first eigenfunction of L does not vanish in the interior of Ω. We refer to [8] for Assertion (ii). ∎

Applying Assertion (i) of the above Lemma to $\Omega = B(R')$, gives us a non-negative function u on Ω, such that $(L_g - 2\ell)u \leq 0$. We now consider the ball $\widetilde{\Omega} := B_0(R') \subset (T_{x_0}M, \tilde{g})$, where $\tilde{g} = \exp^*_{x_0} g$. The function $\tilde{u} = u \circ \exp_{x_0}$ is non-negative in $\widetilde{\Omega}$. Since $\exp_{x_0}$ is a local isometry, we have $(L_{\tilde{g}} - 2\ell)\tilde{u} \leq 0$. Assertion (i) of the Lemma implies that the operator $L_{\tilde{g}} - 2\ell$ is non-positive on $B_0(R')$.

Step 0. Since the immersion $i \circ \exp_{x_0} : (B_0(R'), \exp^*_{x_0} g) \to (\overline{M}^3(c), \overline{g})$ is also an isometric immersion with constant mean curvature H, it follows from the preceding argument that we can from now on assume that M is diffeomorphic to a disk.

Since the ball $B_0(R')$ is simply-connected, there exists some R'', with $R' \geq R'' > R$, such that $(B_0(R''), \exp_{x_0}^* g)$ is conformally equivalent to the unit disk (D, e), i.e. there exists a diffeomorphism $\varphi : D \to B_0(R'')$ such that $\varphi(0) = 0_{T_{x_0}M}$ and $\varphi^*(\exp_{x_0}^* g) = \lambda^2 e$ for some function λ. We may also assume that $R'' \leq 2R$.
Since $\exp_{x_0}$ is a local isometry, curvature estimates in $B_0(R')$ imply curvature estimates in $B(R)$. We are then reduced to proving the following result.

Proposition 5.2 *Let D be the unit disk in $\mathbb{C}$, equiped with a Riemannian metric g. Let $i : (D, g) \to (\overline{M}^3(c), \overline{g})$ be a conformal isometric immersion (i.e. $g = i^*\overline{g} = \lambda^2 e$, where e is the Euclidean metric), with constant mean curvature H, such that the geodesic g-ball $B(R) := B^g(0, R)$ is relatively compact in D. Assume furthermore that $c + H^2 \leq 0$ and $4R^2\big((c+H^2)_- + \ell\big) \leq \Lambda$ for some $\Lambda > 0$ and that the stability operator L_g of the immersion, with Dirichlet boundary condition, is bounded from above by 2ℓ on $B(R)$. Then, there exists a constant $C(\Lambda)$ such that*

$$|A^0|^2(x_0) \leq C(\Lambda) R^{-2} \quad \textit{and} \quad |K_g(x_0)| \leq C(\Lambda) R^{-2}.$$

• **Step 2.** We shall now continue with the proof of Proposition 5.2.
Let ρ denote the Riemannian distance to the point $0 \in D$ with respect to the metric g. Since $K_g \leq 0$ by (35), Bishop's theorem gives

$$\Delta_g \rho^2 \geq 2 \tag{36}$$

where Δ_g is the non-positive Laplacian for the metric g. It follows from Lemma 3.3 that the following stability inequality holds

$$\int_D \zeta^2 \phi L_g \phi \, dv_g \leq \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2 \zeta^2 \, dv_g \tag{37}$$

for any C^∞ function ϕ and any Lipschitz function ζ with compact support in $B(R)$. Recall that the stability operator L_g is given by

$$L_g = \Delta_g + |A^0|^2 + 2S. \tag{38}$$

A recurrent idea in the proof is to get estimates by plugging well chosen functions ϕ and ζ into (37). In the sequel, we shall denote by c_i universal constants, by $c_i(\Lambda)$ constants which only depend on Λ, *etc.* . We shall also denote by $D(1/2)$ the Euclidean disk of radius $1/2$.

We begin by choosing $\phi = e^{\Lambda \rho^2 / R^2}$, where Λ is the positive number given in the assumptions of Theorem 4.2. Since we may assume $R'' \leq 2R$ in the above construction, it follows that

$$1 \leq \phi \leq e^{4\Lambda} \quad \text{in} \quad D.$$

Using (36) a direct computation gives

$$\begin{aligned} \phi L_g \phi &= \big(\Lambda R^{-2} \Delta_g \rho^2 + \Lambda^2 R^{-4} |d\rho^2|_g^2 + |A^0|^2 + 2S\big)\phi^2 \\ &\geq \big(2\Lambda R^{-2} + 2S + |A^0|^2\big)\phi^2. \end{aligned}$$

Using this inequality, the stability condition (37) and the conformal invariance of the Dirichlet integral, we obtain

$$\begin{aligned}\int_D \left(2\Lambda R^{-2} - 2a + |A^0|^2\right)\phi^2\zeta^2 \, dv_g &\leq \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2\zeta^2 \, dv_g \\ &= \int_D \phi^2 |d\zeta|_e^2 \, dv_e + 2\ell \int_D \phi^2\zeta^2 \, dv_g\end{aligned}$$

where $a := (c + H^2)_-$, see (34). Using a suitable function ζ of the Euclidean distance to $0 \in D$, we obtain the following important estimate

$$\int_{D(1/2)} \left(2\Lambda R^{-2} - 2a' + |A^0|^2\right) dv_g \leq c_1(\Lambda) \tag{39}$$

where $a' = a + \ell$, see (34). This estimate is meaningfull only when the integrand in the left-hand side is positive. This is why we have to assume that $4a'R^2 \leq \Lambda$ (unless $a' = 0$ i.e. $c + H^2 = 0$ and $\ell = 0$), see Condition (A) in the statement of Theorem 4.2.

• **Step 3.** The above estimate involves the Euclidean disk $D(1/2)$; in order to be able to take the metric $g = \lambda^2 e$ into account, we make use of Theorem 3.2 which gives the estimate

$$\lambda^2 \geq C_0 R^2 (1 + a'R^2)^{-1} \quad \text{on} \quad B(R/2). \tag{40}$$

We then have the following lemma.

Lemma 5.3 *Under the assumptions of Proposition 5.2, define*

$$b(\Lambda) := \min\{\frac{1}{2}, (1+\Lambda)^{-\frac{1}{2}} \frac{\sqrt{C_0}}{2}\} \quad \textit{and} \quad R_\Lambda := b(\Lambda) R, \tag{41}$$

where C_0 is the universal constant given by Theorem 3.2. We then have,

$$B(R_\Lambda) := B^g(R_\Lambda) \subset D(1/2) \tag{42}$$

and

$$\int_{B(R_\Lambda)} \left(\Lambda R^{-2} + |A^0|^2\right) dv_g \leq c_1(\Lambda). \tag{43}$$

In particular

$$\begin{cases} \int_{B(R_\Lambda)} |A^0|^2 \, dv_g \leq c_1(\Lambda), \\ \mathrm{Vol}(B(R_\Lambda)) \leq c_2(\Lambda) R_\Lambda^2. \end{cases} \tag{44}$$

Proof. We have $g = \lambda^2 e$, with $\lambda^2 \geq C_0 R^2 (1 + a'R^2)^{-1}$ on the ball $B(R/2)$. Let c be a geodesic issued from 0 and parametrized by arc-length in the metric g. The inequality $\int_0^{R_\Lambda} |\dot{c}(t)|_e dt \leq 1/2$ impies that $c(R_\Lambda) \in D(1/2)$. To achieve the inequality, it suffices to have $\int_0^{R_\Lambda} \lambda^{-1} |\dot{c}(t)|_g dt \leq 1/2$ i.e. , if $R_\Lambda \leq R/2$, $R_\Lambda C_0^{-1} R^{-1} (1 + a'R^2)^{1/2} \leq 1/2$ and $4a'R^2 \leq \Lambda$. These conditions are satisfied if $2R_\Lambda / R \leq C_0^{1/2} (1 + \Lambda)^{-1/2}$. ∎

• **Step 4.**
It follows from Lemma 6.1 [for $m > 0, \Delta_g \ln u = f \Longrightarrow \Delta_g(u+m)^\beta \geq \beta f u(u+m)^{\beta-1}$] and Proposition 6.7 [$\Delta_g \ln |A^0|^2 = 4K_g$] that the function $f := (|A^0|^2 + m)^{1/2}$, with $m \geq 0$, satisfies the inequality

$$f^{1/2}\Delta_g f^{1/2} \geq f K_g |A^0|^2 f^{-2}$$

where K_g is the Gaussian curvature of the metric g. It follows that

$$f^{1/2}(L_g - 2\ell)f^{1/2} = f^{1/2}\Delta_g f^{1/2} + |A^0|^2 f + 2S'f \geq Ff \tag{45}$$

where $F := |A^0|^2 + 2S' + K_g|A^0|^2 f^{-2}$ (recall formula (38) and the notation $S' := c + H^2 - \ell$).

We now define the function

$$h := (\Lambda R^{-2} + 2S' + |A^0|^2)^{1/2} = (\Lambda R^{-2} - 2a' + |A^0|^2)^{1/2} \tag{46}$$

where $S' := S - \ell, a' := a + \ell$, see (34). Using the assumption $4a'R^2 \leq \Lambda$, we see that h actually exists and that

$$\frac{1}{2}(\Lambda R^{-2} + |A^0|^2)^{1/2} \leq h \leq (\Lambda R^{-2} + |A^0|^2)^{1/2}. \tag{47}$$

We can deduce from Lemma 5.3 the estimate

$$\int_{B(R_\Lambda)} h^2 \, dv_g \leq c_3(\Lambda). \tag{48}$$

Choose $0 < t < t + r < 1$ and let θ be a C^∞ function with compact support on $\mathbb{R}$ such that $0 \leq \theta \leq 1$, $\theta\big|[0, tR_\Lambda] = 1$, $\theta\big|[(t+r)R_\Lambda, \infty[= 0$ and $|\theta'| \leq c_4(rR_\Lambda)^{-1}$. Applying the stability inequality (37) with $\phi = h^{1/2}$ and $\zeta = \theta \circ \rho$ and inequality (45) with $f = h$, we obtain

$$\begin{aligned}
\int_{B(tR_\Lambda)} Fh \, dv_g &\leq \int_{B(R_\Lambda)} \zeta^2 h^{1/2}(L_g - 2\ell)h^{1/2} \, dv_g \\
&\leq \int_{B(R_\Lambda)} h|d\zeta|_g^2 \, dv_g \leq c_6(rR_\Lambda)^{-2} \int_{B(R_\Lambda)} h \, dv_g \\
&\leq c_6(rR_\Lambda)^{-2} \Big(\int_{B(R_\Lambda)} dv_g\Big)^{1/2} \Big(\int_{B(R_\Lambda)} h^2 \, dv_g\Big)^{1/2}
\end{aligned}$$

where $F := K_g|A^0|^2 h^{-2} + |A^0|^2 + 2S'$, and it follows from Lemma 5.3 and (48) that

$$\int_{B(tR_\Lambda)} Fh \, dv_g \leq c_7(\Lambda) r^{-2} R_\Lambda^{-1}. \tag{49}$$

Since $K_g \leq 0$, we have

$$F := K_g|A^0|^2 h^{-2} + |A^0|^2 + 2S - 2\ell > K_g + |A^0|^2 + 2S - 2\ell = \frac{1}{2}|A^0|^2 + 3S - 2\ell$$

and finally (recall that $S' := S - \ell, \ell \geq 0$)

$$F \geq \frac{1}{2}|A^0|^2 + 3S'. \tag{50}$$

One can then write

$$\begin{aligned}
\int_{B(tR_\Lambda)} h^3\, dv_g &= \int_{B(tR_\Lambda)} h\big(\Lambda R^{-2} + 2S' + |A^0|^2\big)\, dv_g \\
&= \int_{B(tR_\Lambda)} \big(6S' + |A^0|^2\big) h\, dv_g + \big(\Lambda R^{-2} - 4S'\big) \int_{B(tR_\Lambda)} h\, dv_g \\
&\leq 2\int_{B(tR_\Lambda)} Fh\, dv_g + 2\Lambda R^{-2} \int_{B(tR_\Lambda)} h\, dv_g
\end{aligned}$$

where we have used the inequality (50) and the assumption $4a'R^2 \leq \Lambda$. Finally, using Cauchy-Schwarz, Lemma 5.3 and inequalities (48) and (49), we obtain

$$\int_{B(tR_\Lambda)} h^3\, dv_g \leq 2c_7(\Lambda) r^{-2} R_\Lambda^{-1} + c_8(\Lambda) R_\Lambda^{-1}. \tag{51}$$

Choosing $t = 7/8$ and $r = 1/16$, we obtain

Lemma 5.4 *Recall the notations (34). Define* $h := \big(\Lambda R^{-2} + 2S' + |A^0|^2\big)^{1/2}$ *and* $f := -2K_g|A^0|^2 h^{-2}$. *Under the assumptions* $S := c + H^2 \leq 0$ *and* $4a'R^2 \leq \Lambda$, *we have*

$$\Delta_g h + fh \geq 0. \tag{i}$$

There exists a constant $c_{10}(\Lambda)$*, which only depend on* Λ*, such that:*

$$\int_{B(7R_\Lambda/8)} h^2\, dv_g \leq c_{10}(\Lambda), \tag{ii}$$

$$\int_{B(7R_\Lambda/8)} h^3\, dv_g \leq c_{10}(\Lambda) R_\Lambda^{-1}, \tag{iii}$$

and, for all $\alpha \in [0, 1/2]$,

$$\int_{B(7R_\Lambda/8)} h^{2(\alpha+1)}\, dv_g \leq c_{10}(\Lambda) R_\Lambda^{-2\alpha}. \tag{iv}$$

Furthermore,

$$f \leq h^2 \quad \text{and hence} \quad \int_{B(7R_\Lambda/8)} f_+^{1+\alpha}\, dv_g \leq c_{10}(\Lambda) R_\Lambda^{-2\alpha}. \tag{v}$$

Proof. Assertion (i) follows from Lemma 6.1. We have already proved Assertions (ii) and (iii) and Assertion (iv) follows by interpolation. Furthermore, since $K_g \le 0$, $f = f_+ \le -2K_g = |A^0|^2 - 2S = h^2 - 4S + 2\ell - \Lambda R^{-2} \le h^2 + 4a' - \Lambda R^{-2}$. The assumption $4a'R^2 \le \Lambda$ implies that $f \le h^2$ which proves Assertion (v). ∎

This lemma says that Assumptions (a) and (c) of Lemma 2.1 are satisfied (indeed, (a) follows from Lemma 5.3, control of the volume of the ball, and (c) follows from Lemma 5.4 if we can prove that $B_M = 0$). In order to be able to apply Lemma 2.1 to the present situation, it therefore remains to show that Assumption (b) in that lemma is satisfies as well.

• **Step 5.** Let us prove

Lemma 5.5 *Given $q \ge 1$, there exists a constant $c_{11}(q, \Lambda)$ such that*

$$\Big(\int_{B(3R_\Lambda/4)} h^{2q}\, dv_g\Big)^{1/q} \le c_{11}(q,\Lambda) R^{-2+2/q}$$

provided that $4a'R^2 \le \Lambda$, see notation (34).

In order to prove this result, we need another lemma.

Lemma 5.6 *Under the assumption of Proposition 5.2, the surface (D, g) satisfies a Sobolev inequality of Euclidean type (i.e. an inequality of the form (3), with $B_M = 0$)*

$$\forall \phi \in C_0^\infty(D), \quad \Big(\int_D \phi^2\, dv_g\Big)^{1/2} \le A_D \int_D |d\phi|_g\, dv_g.$$

Proof. To prove this lemma, we use the Lawson correspondence. Recall that $i : (D, g) \to (\overline{M}^3(c), \overline{g})$ is an isometric immersion with constant mean curvature H and that $c + H^2 \le 0$. Let W be the shape operator associated with i. This operator satisfies

$$\begin{cases} \operatorname{Det} W + c = K_g \quad \text{(Gauss equation)},\\ (D^g_X W)(Y) = (D^g_Y W)(X) \quad \text{(Codazzi equation)}, \end{cases}$$

for all vector fields X, Y. The operator $W^0 := W - HId$ satisfies the equations

$$\begin{cases} \operatorname{Det} W^0 + c + H^2 = K_g,\\ (D^g_X W^0)(Y) = (D^g_Y W^0)(X), \end{cases}$$

for all vector fields X, Y, because H is constant. Since D is simply-connected, it follows from [17] (Volume IV, Theorem 19, p. 71 ff) that there exists an isometric immersion $j : (D, g) \to (\overline{M}^3(c + H^2), \overline{g})$ whose shape operator is precisely W^0. Since $\operatorname{Trace} W^0 = 0$, this immersion is a minimal immersion of (D, g) into a simply-connected space form with non-positive curvature $c + H^2$. We may therefore apply the Sobolev inequality given by Hoffman-Spruck [11]: there exists a universal constant A_D such that

$$\forall \phi \in C_0^\infty(D), \quad \Big(\int_D \phi^2\, dv_g\Big)^{1/2} \le A_D \int_D |d\phi|_g\, dv_g. \tag{52}$$ ∎

From inequality (52), we deduce that

$$(53) \qquad \Big(\int_D |\phi|^{2p}\, dv_g\Big)^{1/p} \leq p^2 A_D^2 \Big(\int_{\mathrm{Supp}\,\phi} dv_g\Big)^{1/p} \int_D |d\phi|_g^2\, dv_g,$$

for all $\phi \in C_0^\infty(D)$ and for all $p \geq 1$.

In order to prove Lemma 5.5, we apply inequality (53) to the function $\phi = \zeta h$, with h as in Lemma 5.4 and $\zeta \in C_0^\infty(B(R_\Lambda))$ (recall that $B(R)$ is relatively compact in D). Assuming that $\zeta \big| B(3R_\Lambda/4) = 1$, we get

$$(54) \quad \Big(\int_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Big)^{1/p} \leq c_{15}(p) \Big(\int_{B(R_\Lambda)} dv_g\Big)^{1/p} \int_{B(R_\Lambda)} |d(\zeta h)|_g^2\, dv_g.$$

Using

$$|d(\zeta h)|_g^2 = h^2 |d\zeta|_g^2 + \zeta^2 |dh|_g^2 + 2\zeta h \langle d\zeta, dh\rangle_g$$

and

$$\langle d(\zeta^2 h), dh\rangle_g = 2\zeta h \langle d\zeta, dh\rangle_g + \zeta^2 |dh|_g^2$$

integration by parts and the inequality $\Delta_g h + fh \geq 0$, we obtain

$$\begin{aligned}
\int_{B(R_\Lambda)} |d(\zeta h)|_g^2\, dv_g &= \int_{B(R_\Lambda)} h^2 |d\zeta|_g^2\, dv_g + \int_{B(R_\Lambda)} \langle d(\zeta^2 h), dh\rangle_g\, dv_g \\
&\leq \int_{B(R_\Lambda)} h^2 |d\zeta|_g^2\, dv_g + \int_{B(R_\Lambda)} f \zeta^2 h^2\, dv_g
\end{aligned}$$

Using Lemma 5.3, one can rewrite inequality (54) as

$$\begin{aligned}
\Big(\int_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Big)^{1/p} &\leq c_{16}(p,\Lambda) R_\Lambda^{2/p} \Big\{ \int_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g \\
&+ \int_{B(7R_\Lambda/8)} f h^2\, dv_g \Big\}
\end{aligned}$$

provided that $\mathrm{Supp}\,\zeta \subset B(7R_\Lambda/8)$. Using the inequality $f \leq h^2$ of Lemma 5.4, one finally deduces that

$$(55) \quad \begin{aligned}
\Big(\int_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Big)^{1/p} &\leq c_{16}(p,\Lambda) R_\Lambda^{2/p} \Big\{ \int_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g \\
&+ \int_{B(7R_\Lambda/8)} h^4\, dv_g \Big\}.
\end{aligned}$$

Choosing ζ such that $\zeta \big| B(3R_\Lambda/4) = 1, \mathrm{Supp}\,\zeta \subset B(7R_\Lambda/8)$ and $|d\zeta|_g^2 \leq c_{17} R_\Lambda^{-2}$ and applying Lemma 5.3, we obtain

$$\int_{B(7R_\Lambda/8)} h^2 |d\zeta|_g^2\, dv_g \leq c_{18}(p,\Lambda) R_\Lambda^{-2} \int_{B(7R_\Lambda/8)} h^2\, dv_g \leq c_{19}(p,\Lambda) R_\Lambda^{-2}.$$

We now need an analogous control of $\int_{B(7R_\Lambda/8)} h^4\, dv_g$. For this purpose, we apply the Sobolev inequality (52) to the function ζh^2 and we choose ζ such that $\zeta \big| B(7R_\Lambda/8) = 1, \operatorname{Supp}\zeta \subset B(15R_\Lambda/16)$ and we obtain

$$\Big(\int_{B(7R_\Lambda/8)} h^4\, dv_g\Big)^{1/2} \le c_{20} \int_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g.$$

We also have

$$\int_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g \le \int_{B(R_\Lambda)} h^2 |d\zeta|_g\, dv_g + \int_{B(R_\Lambda)} \zeta |dh^2|_g\, dv_g$$

$$\int_{B(R_\Lambda)} |d(\zeta h^2)|_g\, dv_g \le c_{21}(\Lambda) R_\Lambda^{-1} + \int_{B(15R_\Lambda/16)} |dh^2|_g\, dv_g,$$

and,

$$\begin{aligned}
\int_{B(tR_\Lambda)} |dh^2|_g\, dv_g &= \int_{B(tR_\Lambda)} |d(h^{1/2})^4|_g\, dv_g = 4\int_{B(tR_\Lambda)} h^{3/2}|dh^{1/2}|_g\, dv_g \\
&\le 4\Big(\int_{B(tR_\Lambda)} h^3\, dv_g\Big)^{1/2}\Big(\int_{B(tR_\Lambda)} |dh^{1/2}|_g^2\, dv_g\Big)^{1/2} \\
&\le c_{22}(\Lambda) R_\Lambda^{-1/2}\Big(\int_{B(tR_\Lambda)} |dh^{1/2}|_g^2\, dv_g\Big)^{1/2}
\end{aligned}$$

where we have used Lemma 5.4 in the last inequality.
From the definition of the function $h := (\Lambda R^{-2} + 2S' + |A^0|^2)^{1/2}$ and from the equation $\Delta_g \ln |A^0|^2 = 4K_g$ (Proposition 6.7), we deduce the equality

$$\Delta_g \ln h^{1/2} = K_g |A^0|^2 h^{-2} + \frac{1}{4}\big|d|A^0|^2\big|^2 (\Lambda R^{-2} + 2S')|A^0|^{-2} h^{-4}$$

and, since $\Lambda R^{-2} + 2S' \ge 0$,

$$\Delta_g \ln h^{1/2} \ge K_g |A^0|^2 h^{-2}$$

and

$$h^{1/2}\Delta_g h^{1/2} \ge \big|dh^{1/2}\big|_g^2 + hK_g|A^0|^2 h^{-2}.$$

According to (45) with $f = h$, one also has

$$h^{1/2}(L_g - 2\ell) h^{1/2} \ge \big|dh^{1/2}\big|_g^2 + h\big(K_g|A^0|^2 h^{-2} + 2S' + |A^0|^2\big).$$

With an appropriate choice of ζ, with $\operatorname{Supp}\zeta \subset B((t+r)R_\Lambda), \zeta|B(tR_\Lambda) = 1$, we can write

$$\begin{aligned}
\int_{B(R_\Lambda)} \zeta^2 h^{1/2}(L_g &- 2\ell) h^{1/2} dv_g \ge \int_{B(tR_\Lambda)} h^{1/2}(L_g - 2\ell)h^{1/2}\, dv_g \\
&\ge \int_{B(tR_\Lambda)} \{|dh^{1/2}|_g^2 + h(K_g|A^0|^2 h^{-2} + 2S' + |A^0|^2)\}\, dv_g.
\end{aligned}$$

On the other-hand, using the stability inequality (37), we have

$$\begin{aligned}\int_{B(R_\Lambda)} \zeta^2 h^{1/2}(L_g &- 2\ell)h^{1/2}\, dv_g = \int_{B((t+r)R_\Lambda)} \zeta^2 h^{1/2}(L_g - 2\ell)h^{1/2}\, dv_g \\ &\leq \int_{B((t+r)R_\Lambda)} h|d\zeta|_g^2\, dv_g \leq c_{24}(\Lambda, r)R_\Lambda^{-2} \int_{B((t+r)R_\Lambda)} h\, dv_g \\ &\leq c_{25}(\Lambda, r)R_\Lambda^{-1}\end{aligned}$$

using Cauchy-Schwarz, Lemma 5.3 and Lemma 5.4. On the other-hand, since $K_g \leq 0$, we have $K_g|A^0|^2 h^{-2} + 2S' + |A^0|^2 \geq K_g + 2S' + |A^0|^2 \geq \frac{1}{2}|A^0|^2 + 3S'$, which implies that

$$\int_{B(tR_\Lambda)} h\big(K_g|A^0|^2h^{-2} + 2S' + |A^0|^2\big)\, dv_g \geq -3a' \int_{B(tR_\Lambda)} h\, dv_g$$

and

$$-\int_{B(tR_\Lambda)} h\big(K_g|A^0|^2h^{-2} + 2S' + |A^0|^2\big)\, dv_g \leq 3a' \int_{B(tR_\Lambda)} h\, dv_g$$

and hence, using Lemma 5.3, (47) and the assumption $4a'R^2 \leq \Lambda$,

$$-\int_{B(tR_\Lambda)} h\big(K_g|A^0|^2h^{-2} + 2S' + |A^0|^2\big)\, dv_g \leq 3a'R_\Lambda \leq c_{26}(\Lambda)R_\Lambda^{-1}.$$

Finally,

$$\int_{B(tR_\Lambda)} |dh^{1/2}|_g^2\, dv_g \leq c_{27}(\Lambda)R_\Lambda^{-1}$$

which implies

$$\int_{B(tR_\Lambda)} |dh^2|_g\, dv_g \leq c_{28}(\Lambda)R_\Lambda^{-1}$$

and, choosing t, r appropriately,

$$\int_{B(7R_\Lambda/8)} h^4\, dv_g \leq c_{29}(\Lambda)R_\Lambda^{-2}$$

Plugging this inequality into (55), we conclude that

$$\forall p \geq 1, \quad \Big(\int_{B(3R_\Lambda/4)} h^{2p}\, dv_g\Big)^{1/p} \leq c_{30}(p, \Lambda)R_\Lambda^{-2+2/p}.$$

We can now apply Lemma 2.1 to obtain the estimate

$$\sup_{B(R_\Lambda/2)} h^2 \leq c_{31}(\Lambda)R_\Lambda^{-2}. \tag{56}$$

Recalling that $h := \big(\Lambda R^{-2} + 2S' + |A^0|^2\big)^{1/2}$, that $|K_g| \leq \frac{1}{2}|A^0|^2 + a$ and the assumption $4a'R^2 \leq \Lambda$, we obtain the estimates

$$|A^0|^2 \le c_{32}(\Lambda) R_\Lambda^{-2} \quad \text{on} \quad B(R_\Lambda/2),$$

$$|K_g| \le c_{33}(\Lambda) R_\Lambda^{-2} \quad \text{on} \quad B(R_\Lambda/2).$$

This finishes the proof of Theorem 4.2 under Condition (A).

5.2 Proof of Theorem 4.2 under Condition (B)

We assume that there exist two positive numbers R_1', R_1, with $2R_1 \ge R_1' > R_1$, such that $B(R_1') := B(x_0, R_1') \subset\subset (M, g)$, i.e. $B(R_1')$ is relatively compact in M. Recall that $S := c + H^2 > 0$.
We also assume that the stability operator $L_g := \Delta_g + |A^0|^2 + 2S$ is bounded from above by some number $2\ell \ge 0$ on $C_0^\infty(B(R_1))$. Then, according to Lemma 3.3, we have the *stability condition*

$$\int_M \zeta^2 \phi L_g \phi \, dv_g \le \int_M \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_M \phi^2 \zeta^2 \, dv_g \tag{57}$$

for all $\phi \in C^\infty(B(R_1))$ and for all $\zeta \in C_0^\infty(B(R_1))$ (or more generally for any Lipschitz function ζ with compact support in $B(R_1)$).
Note. We shall in fact work in smaller balls, namely in balls $B(R)$ with $R \le \min\{R_1, \pi/2\sqrt{c+H^2}, \sqrt{\Lambda/4\ell}\}$. Since we reduce the size of the domain, (57) is still valid in such balls.
According to (35), $K_g \le S$ and hence the exponential map $\exp_{x_0}$ is a local diffeomorphism on $B(0, R_2') \subset T_{x_0}M$ where $R_2' := \min\{R_1', \pi/\sqrt{S}\}$.

First reduction.
Applying Assertion (i) of Lemma 5.1 to $\Omega = B(R_2')$, gives us a non-negative function u on Ω, such that $(L_g - 2\ell)u \le 0$. We now consider the ball $\widetilde{\Omega} := B(0, R_2') \subset (T_{x_0}M, \tilde{g})$, where $\tilde{g} = \exp_{x_0}^* g$. The function $\tilde{u} = u \circ \exp_{x_0}$ is non-negative in $\widetilde{\Omega}$. Since $\exp_{x_0}$ is a local isometry, we have $(L_{\tilde{g}} - 2\ell)\tilde{u} \le 0$. Assertion (i) of the same lemma implies that the operator $L_{\tilde{g}} - 2\ell$ is non-positive on $B(0, R_2')$.
By reducing R_2' if necessary, we may assume that $2R_2 \ge R_2' > R_2$, where $R_2 := \min\{R_1, \pi/\sqrt{S}\}$, and that $B(0, R_2')$ is conformally equivalent to the unit disk $D \subset \mathbb{C}$, i.e. that there exists a diffeomorphism $\Phi : D \to B_0(R_2')$ such that $\Phi(0) = 0_{T_{x_0}M}$ and $\Phi^*(exp_{x_0}^* g) = \lambda^2 e$, where e is the Euclidean metric in D.
This is our first reduction: we shall now work in a ball $B(R)$ such that $2R \ge R' > R$, with $B(R')$ conformally equivalent to the unit disk D. We shall also assume that

$$\begin{cases} (i) & 4SR^2 \le \pi^2, \\ (ii) & 4\ell R^2 \le \Lambda. \end{cases} \tag{58}$$

Condition (i) in (58) is a strong restriction. Condition (ii) depends on the free parameter Λ, a positive constant which can be chosen appropriately. Note that condition (ii) is empty if $\ell = 0$.

Second reduction. In order to make the second reduction, we make use of the Lawson correspondence back and forth.

• Assume we are given an isometric immersion $i_1 : (D,g) \to (\overline{M}(c_1), \overline{g})$ with constant mean curvature H_1 (with $H_1 = H, c_1 = c$) such that $c_1 < 0$ and $0 < c_1 + H_1^2 < -c_1$. Let W_1 denote its shape operator.
Consider the operator $W_2 := W_1 - (H_1 - \sqrt{c_1 + H_1^2})\,\mathrm{Id}$. The Lawson correspondence gives us an isometric immersion $i_2 : (D,g) \to (\mathbb{R}^3, e)$, with constant mean curvature $H_2 = \sqrt{c_1 + H_1^2}$ and with the same metric g on D, hence with the same Gauss curvature K_g. It follows that the corresponding stability operators L_1 and L_2 coincide.

• Define $\mu := H_2/\sqrt{-c_1} \in]0,1[$ and make a dilation in $\mathbb{R}^3$ to obtain an isometric immersion $i_3 : (D, g_3) \to (\mathbb{R}^3, e)$ with constant mean curvature $H_3 = \mu^{-1} H_2 = \sqrt{-c_1}$, metric $g_3 = \mu g$ and shape operator W_3. It follows that the Gauss curvature is $K_3 = \mu^{-2} K_g$.
Define $W_4 := W_3 + (-H_3 + \sqrt{-2c_1})\,\mathrm{Id}$. The Lawson correspondence gives us an immersion $i_4 : (D, g_4) \to (\overline{M}(c_4), \overline{g})$ with metric $g_4 = g_3$, constant mean curvature $H_4 = \sqrt{-2c_1}, c_4 = c_1, c_4 + H_4^2 = -c_4$.
Since $g_4 = \mu^2 g$, we have $K_4 = \mu^{-2} K_g, |A_4^0|^2 = \mu^{-2} |A^0|^2, B^{g_4}(\mu R) = B^g(R) \subset \subset (D, g_4)$. The corresponding stability operator satisfies $L_4 = \mu^{-2} L_g$ which impliese that $L_4 \leq 2\ell \mu^{-2}$ on $C_0^\infty(B^{g_4}(\mu R))$ and we have: $S_4 := c_4 + H_4^2 = -c_4, (2\ell\mu^{-2})(\mu R)^2 \leq \Lambda, 4 S_4 (\mu R)^2 = -4 c_1 H_2^2/(-c_1) R^2 = 4 S_1 R^2 \leq \pi^2$.
We are therefore reduced to studying an immersion satisfying

$$\text{(59)} \qquad \begin{cases} \text{either} \;\; c \geq 0, \\ \text{or} \;\; c < 0 \;\; \text{and} \;\; S = c + H^2 \geq -c > 0. \end{cases}$$

Assuming the estimates are proved under the assumptions (59), we obtain for the immersion i_4 the estimates $|K_4|, |A_4^0|^2 \leq C(\Lambda)(\mu R)^{-2}$ in the ball $B^{g_4}(c(\Lambda)\mu R)$. Using the relations between the invariants in the g_4 and in the g metric, we obtain the desired estimates for the immersion i_1

$$|K_1|, |A_1^0|^2 \leq C(\Lambda) R^{-2} \;\; \text{in} \;\; B^{g_4}(c(\Lambda)R).$$

In order to prove Assertion B of Theorem 4.1, we are now reduced to proving the following proposition.

Proposition 5.7 *Let $D \subset \mathbb{C}$ be the unit disk and let g be a Riemannian metric on D. Fix some positive constant Λ. Make the following assumptions:*

1. *There exists a conformal isometric immersion $i : (D,g) \to (\overline{M}^3(c), \overline{g})$, $i^*\bar{g} = g = \lambda^2 e$, with constant mean curvature H, with $S := c + H^2 > 0$ if $c \geq 0$ and $S := c + H^2 \geq -c$ if $c < 0$;*

2. *There exists $R > 0$ such that the ball $B(R)$ is relatively compact in (D,g), where $B(R) := B^g(0,R) \subset D$, and such that $D \subset B(2R)$;*

3. *The stability operator L_g of the immersion, $L_g := \Delta_g + |A^0|^2 + 2S$, is bounded from above by 2ℓ on $C_0^\infty(B(R))$, for some $\ell \geq 0$;*

4. *The number R satisfies (58),*

$$\begin{cases} (i) & 4SR^2 \;\; \leq \;\; \pi^2, \\ (ii) & 4\ell R^2 \;\; \leq \;\; \Lambda. \end{cases}$$

Then, there exist positive constants $C(\Lambda), c(\Lambda)$, which only depend on Λ, such that

$$\left\{\begin{array}{lll} |A^0|^2 & \leq & C(\Lambda)R^{-2} \\ |K_g| & \leq & C(\Lambda)R^{-2} \end{array}\right. \quad on \;\; B(c(\Lambda)R).$$

Proof. The remainder of this section will be devoted to the proof of Proposition 5.7. In the course of the proof, we will denote by $c_i, c_i(\Lambda), \dots$ constants which depend on the indicated arguments.

• **Step 1.**
Let ρ denote the Riemannian distance to the point $0 \in D$ with respect to the metric g. Since $K_g \leq S$ by (35), using (58) (i), Bishop's theorem gives

$$\Delta_g \rho^2 \geq 2 \tag{60}$$

where Δ_g is the non-positive Laplacian for the metric g.
We define $\phi = e^{\Lambda\rho^2/R^2} \in C^\infty(D)$, where Λ is the positive number given in the assumptions of Theorem 4.2. Then, we have $1 \leq \phi \leq e^{4\Lambda}$ because $D \subset B(2R)$ and

$$\begin{array}{rcl} \phi L_g \phi & = & \big(\Lambda R^{-2}\Delta_g\rho^2 + \Lambda^2 R^{-4}|d\rho^2|_g^2 + |A^0|^2 + 2S\big)\phi^2 \\ & \geq & \big(2\Lambda R^{-2} + 2S + |A^0|^2\big)\phi^2. \end{array}$$

It follows from (57) that

$$\int_D \big(2\Lambda R^{-2} + 2S + |A^0|^2\big)\phi^2\zeta^2 \, dv_g \leq \int_D \phi^2 |d\zeta|_g^2 \, dv_g + 2\ell \int_D \phi^2\zeta^2 \, dv_g.$$

Using the conformal invariance of the Dirichlet integral and (58ii), we obtain

$$\int_D \big(\Lambda R^{-2} + 2S + |A^0|^2\big)\phi^2\zeta^2 \, dv_g \leq \int_D \phi^2 |d\zeta|_e^2 \, dv_e.$$

Using a suitable function ζ of the Euclidean distance to $0 \in D$ and the inequality $1 \leq \phi \leq e^{4\Lambda}$, we obtain the following important estimate

$$\int_{D(1/2)} \big(\Lambda R^{-2} + 2S + |A^0|^2\big) \, dv_g \leq c_1(\Lambda). \tag{61}$$

• **Step 2.** The above estimate involves the Euclidean disk $D(1/2)$; in order to be able to take the metric g into account, we make use of Theorem 3.2 which gives the estimate

$$\lambda^2 \geq C_0 R^2 (1 + \ell R^2)^{-1} \;\; \text{on} \;\; B(R/2). \tag{62}$$

We obtain the analogue of Lemma 5.3.

Lemma 5.8 *Under the assumptions of Proposition 5.7, define*

$$A = A(\Lambda) := \min\{1, \sqrt{C_0(1+\Lambda)^{-1}}\}/2 \;\; and \;\; R_A := AR \tag{63}$$

where C_0 is given by Theorem 3.2. Then

$$B(R_A) \subset D(1/2), \tag{64}$$

$$\int_{B(R_A)} \left(\Lambda R^{-2} + 2S + |A^0|^2\right) dv_g \leq c_1(\Lambda). \tag{65}$$

In particular,

$$\mathrm{Vol}(B(R_A)) \leq c_2(\Lambda) R_A^2. \tag{66}$$

Remarks. Note that $A(\Lambda)$ does not depend on Λ when $\ell = 0$. By Gauss equation (35), and since $S > 0$, we have

$$|K_g| \leq \max\{S, \frac{1}{2}|A^0|^2\}.$$

In order to control K_g it therefore suffices to control $|A^0|^2$ or any function of the form $(m + |A^0|^2)$, for some $m > 0$. From now on, let

$$h := \left(\Lambda R^{-2} + 2S + |A^0|^2\right)^{1/2}. \tag{67}$$

As in Step 4 of the proof of Theorem 4.2 under Condition (A), we have

$$h^{1/2} L_g h^{1/2} \geq h\left(|A^0|^2 + 2S + K_g |A^0|^2 h^{-2}\right)$$

and it follows easily, looking at the cases $K_g \leq 0$ and $K_g \geq 0$, that

$$h^{1/2} L_g h^{1/2} \geq h\left(\frac{1}{2}|A^0|^2 + 2S\right). \tag{68}$$

We then have the following lemma.

Lemma 5.9 *Define* $h := \left(\Lambda R^{-2} + 2S + |A^0|^2\right)^{1/2}$ *and* $f := -2K_g|A^0|^2 h^{-2}$ *and let*

$$f_1 := \begin{cases} (f + H^2)_+ & \text{if } c \leq 0, \\ (f + c + H^2)_+ & \text{if } c \geq 0. \end{cases}$$

Under the assumptions of Proposition 5.7, we have
(i) $\Delta_g h + fh \geq 0$.
There exists a constant $c_{10}(\Lambda)$*, which only depends on* Λ*, such that*
(ii) $\int_{B(7R_A/8)} h^2 \, dv_g \leq c_{10}(\Lambda)$,
(iii) $\int_{B(7R_A/8)} h^3 \, dv_g \leq c_{10}(\Lambda) R_A^{-1}$,
(iv) $\forall \alpha \in [0, 1/2]$, $\int_{B(7R_A/8)} h^{2(1+\alpha)} \, dv_g \leq c_{10}(\Lambda) R_A^{-2\alpha}$,
(v) $f_1 \leq h^2$ *and hence,* $\forall \alpha \in [0, 1/2]$, $\int_{B(7R_A/8)} f_1^{1+\alpha} \, dv_g \leq c_{10}(\Lambda) R_A^{-2\alpha}$.

Proof. Assertion (i) follows from Lemma 6.1 and Proposition 6.7. Assertion (ii) directly from the definition of h and from inequality (65). Take $0 < a < a + r < 1$ (to be chosen later) and choose a smooth function θ such that $\theta = 1$ on $[0, aR_A]$, $\theta = 0$ on $[(a+r)R_A, R_A]$ and $|\theta'| \leq c_4 (rR_A)^{-1}$. Plugging $\phi = h^{1/2}$ and $\zeta = \theta \circ \rho$ into (57), using the inequality $h^{1/2} L_g h^{1/2} \geq hF$, with $F := \left(|A^0|^2 + 2S + K_g|A^0|^2 h^{-2}\right)$, we obtain

$$\begin{aligned}\int_{B(aR_A)} Fh\,dv_g &\le \int_{B(R_A)} \zeta^2 h^{1/2} L_g h^{1/2}\,dv_g\\ &\le \int_{B(R_A)} h|d\zeta|_g^2\,dv_g + 2\ell\int_{B(R_A)}\zeta^2 h\,dv_g.\end{aligned}$$

Using (65) and (66), we have

$$\int_{B(R_A)} h\,dv_g \le \Big(\int_{B(R_A)} h^2\,dv_g\Big)^{1/2}\Big(\int_{B(R_A)} dv_g\Big)^{1/2} \le c_7(\Lambda)R_A \tag{69}$$

and hence, using (58ii),

$$\int_{B(aR_A)} hF\,dv_g \le c_8(\Lambda)r^{-2}R_A^{-1}. \tag{70}$$

Since

$$F \ge \frac{1}{2}|A^0|^2 + 2S \tag{71}$$

by (68), we can finally write

$$\begin{aligned}\textstyle\int_{B(aR_A)} h^3\,dv_g &= \textstyle\int_{B(aR_A)} h\big(\Lambda R^{-2} + 2S + |A^0|^2\big)\,dv_g\\ &\le \Lambda R^{-2}\textstyle\int_{B(aR_A)} h\,dv_g + 2\int_{B(aR_A)} hF\,dv_g.\end{aligned}$$

Using (69) and (70) this leads to

$$\int_{B(aR_A)} h^3\,dv_g \le c_9(\Lambda)r^{-2}R_A^{-1}. \tag{72}$$

Finally, using interpolation, the values $a := 7/8$ and $r := 1/16$, (ii) and (72) give Assertion (iv).

In order to prove Assertion (v), it suffices to show that $f_1 \le h^2$.

Claim: $f_1 \le h^2$. Indeed, using the definition of f, h, f_1 and (58), we have:

• If $c \ge 0$ and $K_g \ge 0$, then $f \le 0$ and $f + c + H^2 \le c + H^2 = S \le h^2$.

• If $c \ge 0$ and $K_g \le 0$, then $f \le -2K_g$ and $f + c + H^2 \le -2K_g + c + H^2 \le |A^0|^2 - S \le h^2$.

• If $c < 0$ and $K_g \le 0$, then $f \le -2K_g$, since $H^2 \le 2(H^2 + c)$ under the assumptions of Proposition 5.7. We obtain $0 \le f + H^2 \le |A^0|^2 \le h^2$.

• If $c < 0$ and $K_g \ge 0$, then $f + H^2 \le H^2 \le 2(c + H^2) \le h^2$.

This finishes the proof of Lemma 5.9. ∎

Note. With the above notations (in partcular under the additionnal condition $-cR^2 \le \Lambda$), we claim that $f_1 \le h^2$ provided that $4(c + H^2)R^2 \le \pi^2$ and $-cR^2 \le \Lambda$, under the sole assumption $c + H^2 > 0$ (this avoids using the Lawson correspondence to reduce to the case $c + H^2 > 0$ if $c \ge 0$, or $c + H^2 \ge -c$ if $c < 0$). Indeed,

• If $c \ge 0$ and $K_g \ge 0$, as above.

• If $c \ge 0$ and $K_g \le 0$, as above.

• If $c < 0$ and $K_g \le 0$, then $f \le -2K_g = |A^0|^2 - 2S$. Furthermore, $0 \le f + H^2 \le |A^0|^2 - (c + H^2) - c \le |A^0|^2 - c \le |A^0|^2 + \Lambda R^{-2} \le h^2$.

• If $c < 0$ and $K_g \geq 0$, then $f \leq 0$ and hence $f + H^2 \leq H^2 = c + H^2 - c \leq S + \Lambda R^{-2} \leq h^2$.

• **Step 3.** Let us prove

Lemma 5.10 *Under the assumption of Proposition 5.7, given $q \geq 1$, there exists a constant $c_{11}(q, \Lambda)$ such that*

$$\Big(\int_{B(3R_A/4)} h^{2q}\, dv_g\Big)^{1/q} \leq c_{11}(q,\Lambda) R_{R_A}^{-2+2/q}.$$

In order to prove this result, we need another lemma.

Lemma 5.11 *Under the assumptions of Proposition 5.7 (in particular $c + H^2 > 0$), the surface (D, g) satisfies the Sobolev inequality*

$$\Big(\int_D \phi^2\, dv_g\Big)^{1/2} \leq A_D \Big\{ \int_D |d\phi|_g\, dv_g + B_D \int_D |\phi|\, dv_g \Big\}$$

for all $\phi \in C_0^\infty(D)$, where A_D is a universal constant and where B_D is defined by

$$B_D = \begin{cases} H & \text{if } c \leq 0, \\ \sqrt{c + H^2} & \text{if } c \geq 0. \end{cases}$$

Proof. To prove this lemma, we use the fact that the immersion $i : (D, g) \to (\overline{M}^3(c), \overline{g})$ is an isometric immersion with constant mean curvature H. When $c \leq 0$ we can directly apply the Sobolev inequality in [11]. When $c > 0$, we compose i with the isometric immersion of the 3-sphere of curvature c into $\mathbb{R}^4$. This gives an isometric immersion j whose mean curvature vector has norm $\sqrt{c + H^2}$ and we apply [11] again. ∎

From this inequality, we deduce that

$$(73) \quad \Big(\int_D |\phi|^{2p}\, dv_g\Big)^{1/p} \leq c_{15}(p) \Big(\int_{\mathrm{Supp}\,\phi} dv_g\Big)^{1/p} \Big\{ \int_D |d\phi|_g^2\, dv_g + B_D^2 \int_D \phi^2\, dv_g \Big\},$$

for all $\phi \in C_0^\infty(D)$ and for all $p \geq 1$.

In order to prove Lemma 5.10, we apply inequality (73) to the function $\phi = \zeta h$, with h as in Lemma 5.9 and $\zeta \in C_0^\infty(B(R_A))$ with $\zeta\big|B(3R_A/4) = 1$. We get

$$(74) \quad \begin{aligned} \Big(\int_{B(3R_A/4)} h^{2p}\, dv_g\Big)^{1/p} &\leq c_{15}(p) \Big(\int_{B(R_A)} dv_g\Big)^{1/p} \Big\{ \int_{B(R_A)} |d(\zeta h)|_g^2\, dv_g \\ &+ B_D^2 \int_{B(R_A)} h^2 \zeta^2\, dv_g \Big\}. \end{aligned}$$

Using

$$|d(\zeta h)|_g^2 = h^2 |d\zeta|_g^2 + \zeta^2 |dh|_g^2 + 2\zeta h \langle d\zeta, dh \rangle_g$$

and

$$\langle d(\zeta^2 h), dh\rangle_g = 2\zeta h\langle d\zeta, dh\rangle_g + \zeta^2|dh|_g^2$$

integration by parts and the inequality $\Delta_g h + fh \geq 0$, we obtain

$$\begin{aligned}\int_{B(R_A)} |d(\zeta h)|_g^2\, dv_g &= \int_{B(R_A)} h^2|d\zeta|_g^2\, dv_g + \int_{B(R_A)} \langle d(\zeta^2 h), dh\rangle_g\, dv_g \\ &\leq \int_{B(R_A)} h^2|d\zeta|_g^2\, dv_g + \int_{B(R_A)} f\zeta^2 h^2\, dv_g\end{aligned}$$

Using Lemma 5.8, one can rewrite inequality (74) as

$$\begin{aligned}\Big(\int_{B(3R_A/4)} h^{2p}\, dv_g\Big)^{1/p} &\leq c_{16}(p,\Lambda)R_A^{2/p}\Big\{\int_{B(7R_A/8)} h^2|d\zeta|_g^2\, dv_g \\ &+ \int_{B(7R_A/8)} (f+B_D^2)_+ h^2\, dv_g\Big\}\end{aligned}$$

provided that $\operatorname{Supp}\zeta \subset B(7R_A/8)$ and $0 \leq \zeta \leq 1$. Using the inequality $f_1 \leq h^2$ of Lemma 5.9 and a suitable function ζ, we obtain

$$\begin{aligned}\Big(\int_{B(3R_A/4)} h^{2p}\, dv_g\Big)^{1/p} &\leq c_{16}(p,\Lambda)R_A^{2/p}\Big\{R_A^{-2}\int_{B(7R_A/8)} h^2\, dv_g \\ &+ \int_{B(7R_A/8)} h^4\, dv_g\Big\}.\end{aligned} \tag{75}$$

We now need to control $\int_{B(7R_A/8)} h^4\, dv_g$. For this purpose, we apply Lemma 5.11 to the function ζh^2 and we choose a suitable function ζ such that

$$\zeta\big|B(7R_A/8) = 1, \operatorname{Supp}\zeta \subset B(15R_A/16).$$

We obtain

$$\Big(\int_{B(7R_A/8)} h^4\, dv_g\Big)^{1/2} \leq c_{20}\Big\{\int_{B(R_A)} |d(\zeta h^2)|_g\, dv_g + B_D\int_{B(R_A)} \zeta h^2\, dv_g\Big\}.$$

We also have

$$\int_{B(R_A)} |d(\zeta h^2)|_g\, dv_g \leq \int_{B(R_A)} h^2|d\zeta|_g\, dv_g + \int_{B(R_A)} \zeta|dh^2|_g\, dv_g$$

$$\int_{B(R_A)} |d(\zeta h^2)|_g\, dv_g \leq c_{21}(\Lambda)R_A^{-1} + \int_{B(15R_A/16)} |dh^2|_g\, dv_g,$$

and,

$$\begin{aligned}\int_{B(aR_A)} |dh^2|_g \, dv_g &= \int_{B(aR_A)} |d(h^{1/2})^4|_g \, dv_g = 4\int_{B(aR_A)} h^{3/2}|dh^{1/2}|_g \, dv_g \\ &\le 4\Big(\int_{B(aR_A)} h^3 \, dv_g\Big)^{1/2}\Big(\int_{B(aR_A)} |dh^{1/2}|_g^2 \, dv_g\Big)^{1/2} \\ &\le c_{22}(\Lambda)R_A^{-1/2}\Big(\int_{B(aR_A)} |dh^{1/2}|_g^2 \, dv_g\Big)^{1/2}\end{aligned}$$

where we have used Lemma 5.9 in the last inequality.
From the definition of the function $h := (\Lambda R^{-2} + 2S + |A^0|^2)^{1/2}$ and from the equation $\Delta_g \ln |A^0|^2 = 4K_g$ (Proposition 6.7), we deduce the equality

$$\Delta_g \ln h^{1/2} = K_g|A^0|^2h^{-2} + \frac{1}{4}\big|d|A^0|^2\big|^2(\Lambda R^{-2} + 2S)|A^0|^{-2}h^{-4}$$

and, since $\Lambda R^{-2} + 2S \ge 0$,

$$\Delta_g \ln h^{1/2} \ge K_g|A^0|^2h^{-2}$$

and

$$h^{1/2}\Delta_g h^{1/2} \ge \big|dh^{1/2}\big|_g^2 + hK_g|A^0|^2h^{-2}.$$

According to (45) with $f = h$, one also has

$$h^{1/2}L_g h^{1/2} \ge \big|dh^{1/2}\big|_g^2 + h\big(K_g|A^0|^2h^{-2} + 2S + |A^0|^2\big).$$

If $K_g \ge 0$, we have $|dh^{1/2}|_g^2 \le h^{1/2}L_g h^{1/2}$. If $K_g \le 0$ we have $K_g|A^0|^2h^{-2} + 2S + |A^0|^2 \ge K_g + 2S + |A^0|^2 \ge \frac{1}{2}|A^0|^2 + 3S > 0$ and again $|dh^{1/2}|_g^2 \le h^{1/2}L_g h^{1/2}$. With an appropriate choice of ζ, this implies

$$\begin{aligned}\int_{B(aR_A)} |dh^{1/2}|_g^2 &\le \int_{B(aR_A)} h^{1/2}L_g h^{1/2} \, dv_g \\ &\le \int_{B(aR_A)} \zeta^2 h^{1/2}L_g h^{1/2} \, dv_g \\ &\le \int_{B(R_A)} h|d\zeta|_g^2 \, dv_g + 2\ell\int_{B(R_A)} h\zeta^2 \, dv_g \\ &\le c_{24}(\Lambda)R_A^{-2}\int_{B(R_A)} h \, dv_g\end{aligned}$$

Where we have used (58ii). Using (69), Lemma 5.8, Lemma 5.9 and (58ii) we get

$$\int_{B(aR_A)} |dh^{1/2}|_g^2 \, dv_g \le c_{25}(\Lambda)R_A^{-1},$$

$$\int_{B(aR_A)} |dh^2|_g \, dv_g \le c_{28}(\Lambda)R_A^{-1},$$

$$\int_{B(aR_A)} |d(\zeta h^2)|_g \, dv_g \le c_{29}(\Lambda)R_A^{-1},$$

$$\Big(\int_{B(7R_A/8)} h^4\,dv_g\Big)^{1/2} \le c_{30}(\Lambda)\big\{R_A^{-1} + B_D \int_{B(R_A)} h^2\,dv_g\big\}.$$

Using the definition of B_D,

$$B_D h^2 = \begin{cases} Hh^2 & \text{if } \; c \le 0, \\ \sqrt{c+H^2}\,h^2 & \text{if } \; c > 0. \end{cases}$$

In the case $c \le 0$ we have, using $H^2 \le 2(H^2+c)$, $h = \big(|A^0|^2 + 2(c+H^2) + \Lambda R^2\big)^{1/2} \ge H$. In the case $c > 0$, we have $h \ge \sqrt{c+H^2}$. These inequalities imply that $B_D h^2 \le h^3$. Using Lemma 5.9, Assertion (iii), we obtain

$$\int_{B(7R_A/8)} h^4\,dv_g \le c_{31}(\Lambda) R_A^{-2}.$$

Finally, we have obtained the estimate

$$\Big(\int_{B(3R_A/4)} h^{2p}\,dv_g\Big)^{1/p} \le c_{32}(p,\Lambda) R_A^{-2+2/p} \tag{76}$$

and we can now apply Lemma 2.1 to obtain the estimate

$$\sup_{B(R_A/2)} h^2 \le c_{33} R_A^{-2}$$

from which Assertion B follows in view of the Gauss equation (35). ∎

6 Appendices

6.1 Some Analytic Formulas

Let (M,g) be a Riemannian manifold. We denote by $|du|_g$ the norm of the differential of the function u on M (for the induced metric on T^*M) and by dv_g the Riemannian measure on M. We denote by Δ_g the non-positive Laplace operator acting on functions.

Lemma 6.1 *Given any $u \in C^\infty(M,\mathbb{R})$ on M with $u \ge 0$ and any $f \in C^\infty(M,\mathbb{R})$ such as $\Delta_g \ln u(x) = f(x)$ when $u(x) > 0$, we have*

(i) $$\Delta_g \ln u(x) = \frac{\Delta_g u(x)}{u(x)} - \frac{|du(x)|_g^2}{u^2(x)} \;\; \textit{where } u(x) > 0,$$

(ii) $$u(x)\Delta_g u(x) - |du(x)|_g^2 = f(x)u^2(x) \;\; \textit{where } u(x) \ge 0,$$

(iii) $$\Delta_g u(x) \ge f(x)u(x) \;\; \textit{where } u(x) \ge 0.$$

If a is a positive constant ($a > 0$) and if $\beta \in \mathbb{R}$,

(iv) $$\Delta_g (u+a)^\beta \ge \frac{\beta f u}{u+a}(u+a)^\beta.$$

Proof. Clearly (i), (ii), (iii) hold. By (i),

$$\Delta_g (u+a)^\beta \geq (u+a)^\beta \Delta_g \ln(u+a)^\beta$$

and

$$\Delta_g \ln(u+a)^\beta = \beta \Delta_g \ln(u+a) = \frac{\beta}{u+a}\big(\Delta_g u - \frac{|du|_g^2}{u+a}\big) \geq \frac{\beta f u}{u+a}$$

which implies (iv). ∎

Lemma 6.2 *Let q be a continuous function on M and assume that the operator $L := \Delta_g + q$ is non-positive on $C_0^\infty(M)$, i.e.*

$$\int_M \varphi L_g \varphi \, dv_g = -\int_M \left(|d\varphi|_g^2 - q\varphi^2\right) dv_g \leq 0,$$

for all $\varphi \in C_0^\infty(M, I\!R)$.
Then, for all $\varphi \in C^\infty(M, I\!R)$, for all $\zeta \in C_0^\infty(M, I\!R)$,

$$\int_M \zeta^2 \varphi L \varphi \, dv_g \leq \int_M \varphi^2 |d\zeta|_g^2 \, dv_g.$$

Proof. Write $\int_M |d(\varphi\zeta)|_g^2 \, dv_g = \int_M \left(\varphi^2 |d\zeta|_g^2 + \zeta^2 |d\varphi|_g^2 + \frac{1}{2}\langle d\varphi^2, d\zeta^2\rangle\right) dv_g$. Integration by parts and the formula $\Delta_g(\varphi^2) = 2\varphi\Delta_g\varphi + 2|d\varphi|_g^2$ give

$$\int_M |d(\varphi\zeta)|_g^2 \, dv_g = \int_M \varphi^2 |d\zeta|_g^2 \, dv_g - \int_M \zeta^2 \varphi \Delta_g \varphi \, dv_g.$$

Then

$$\begin{aligned}
\int_M \zeta^2 \varphi L \varphi \, dv_g &= \int_M \zeta^2 \varphi \big(\Delta_g \varphi + q\varphi\big) \, dv_g \\
&= -\Big\{ \int_M |d(\zeta\varphi)|_g^2 \, dv_g - \int_M q\zeta^2\varphi^2 \, dv_g \Big\} + \int_M \varphi^2 |d\zeta|_g^2 \, dv_g
\end{aligned}$$

which gives the result. ∎

6.2 Some Geometric Formulas

6.2.1 The Gauss Equation

In this section, we consider an isometric immersion $f : (M, g) \to (\overline{M}^3(c), \overline{g})$. We denote by ν a (local) unit normal vector field along $f(M)$ and by $A : T_pM \to T_pM$ the shape operator of f. We also denote by $2H = k_1 + k_2$ the mean curvature of f and by $|A|^2 = k_1^2 + k_2^2$ the square of the norm of A. Here, k_1, k_2 are the eigenvalues of A, i.e. the principal curvatures of the immersion f. We assume that $H = Ct$ and we define $A^0 := A - H\mathrm{Id}$. This operator is traceless and satisfies Codazzi equation since H is constant. Notice that the norm of A^0 is given by $|A^0|^2 = \frac{1}{2}(k_1 - k_2)^2$ and measures how much M deviates from being umbilic at a given point.

The following formulas hold:

$$\text{(77)}\qquad \begin{cases} 2H = k_1 + k_2 \;\text{ and }\; |A|^2 = k_1^2 + k_2^2, \\ |A|^2 = |A^0|^2 + 2H^2, \\ k_1 k_2 = 2H^2 - \frac{1}{2}|A|^2 = H^2 - \frac{1}{2}|A^0|^2. \end{cases}$$

We choose a local adapted frame $\{e_1, e_2, e_3 = \nu\}$, where the first two vectors fields are tangent to M. The Gauss equation of the immersion reads (with obvious notations, see [20])

$$\text{(78)}\qquad K_g = \mathrm{Sect}_M\{e_1, e_2\} = \mathrm{Sect}_{\overline{M}}\{e_1, e_2\} + k_1 k_2.$$

This formula can also be written as

$$\text{(79)}\qquad \begin{cases} K_g = \mathrm{Scal}_{\overline{M}} - \mathrm{Ric}_{\overline{M}}(\nu) + 2H^2 - \frac{1}{2}|A|^2, \\ K_g = \mathrm{Scal}_{\overline{M}} - \mathrm{Ric}_{\overline{M}}(\nu) + H^2 - \frac{1}{2}|A^0|^2. \end{cases}$$

In the particular case in which the ambient manifold $\overline{M}$ has constant curvature c, the Gauss equation can be rewritten as

$$\text{(80)}\qquad K_g = c + H^2 - \frac{1}{2}|A^0|^2.$$

6.2.2 The Stability Operator

The stablity operator of the immersion is the operator

$$\text{(81)}\qquad L_g = \Delta_g + \{\mathrm{Ric}_{\overline{M}}(\nu) + |A|^2\}$$

where Δ_g is the non-positive Laplace operator on (M, g) (i.e. $\int_M u\Delta_g u\, dv_g = -\int_M |du|_g^2\, dv_g$). The following obvious formulas for L_g will be usefull.

$$\text{(82)}\qquad \begin{cases} L_g = \Delta_g + \{\mathrm{Scal}_{\overline{M}} + 2H^2 + \frac{1}{2}|A|^2 - K_g\}, \\ L_g = \Delta_g + \{\mathrm{Scal}_{\overline{M}} + 3H^2 + \frac{1}{2}|A^0|^2 - K_g\}. \end{cases}$$

and, in the case $\overline{M}$ has constant curvature c,

$$\text{(83)}\qquad \begin{cases} L_g = \Delta_g + \{3(c + H^2) + \frac{1}{2}|A^0|^2 - K_g\}, \\ L_g = \Delta_g + \{2(c + H^2) + |A^0|^2\}. \end{cases}$$

6.3 The Lawson Correspondance

In this Appendix we recall a correspondance introduced by B. Lawson [12] (see also [17]).

Lemma 6.3 *Let M be a simply connected Riemannian surface and let $i_1 : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an isometric immersion with constant mean curvature $H_1 \geq \sqrt{-c} > 0$ (here $c < 0$). Then the metric g can be induced by an isometric immersion $i_2 : (M, g) \to \mathbb{R}^3$, with constant mean curvature $H_2 = \sqrt{c + H_1^2}$. Conversely, any isometric immersion $i_2 : (M, g) \to \mathbb{R}^3$, with constant mean curvature $H_2 \geq 0$ induces, for every $c < 0$, an isometric immersion $i_3 : (M, g) \to (\overline{M}^3(c), \overline{g})$ with constant mean curvature $H_3 = \sqrt{H_2^2 - c} \geq \sqrt{-c} > 0$.*

Proof. We use the fundamental existence theorem for surfaces (see [17]). This theorem is local in general and global on simply-connected surfaces. Let $i_1 : (M, g) \to (\overline{M}^3(c), \overline{g})$ be the isometric immersion and let be W_1 be its shape operator. The tensor W_1 satisfies the Gauss and Codazzi equations in $(\overline{M}^3(c), \overline{g})$. In particular $\mathrm{Det}\, W_1 + c = K$. We consider $W_2 = W_1 - \big(H_1 - \sqrt{(c + H_1^2)}\big)\mathrm{Id}$. Since H_1 is constant, $DW_1 = DW_2$ and $\mathrm{Det}\, W_2 = \mathrm{Det}\, W_1 + c$ which implies that W_2 satisfies the Gauss and Codazzi equations in $\mathbb{R}^3$. Furthermore, $\mathrm{Trace}\, W_2 = 2H_2 = 2\sqrt{(c + H_1^2)}$. Conversely, we consider $W_3 = W_2 - \big(H_2 - \sqrt{(H_2^2 - c)}\big)\mathrm{Id}$ to find the isometric immersion $i_3 : (M, g) \to (\overline{M}^3(c), \overline{g})$. ∎

6.4 On Simons' Equation

In this Appendix, we consider an isometric immersion $(M^n, g) \to (\overline{M}^{\bar{n}}, \bar{g})$. Given a tensor field $U : NM \to SM$ from the normal bundle NM of M into the bundle of symmetric endomorphisms of the tangent bundle TM of M, we introduce two tensor fields $\underset{\sim}{U}$ and $\widetilde{U}$, defined as follows. Using the usual extensions of the Riemannian metric g to tensor fields, we define the transpose map ${}^{\dagger}U : SM \to NM$ by the formula

$$\bar{g}({}^{\dagger}U(s), w) = g(s, U^w) := Trace_g(U^w \circ s) := \sum_{i=1}^{n} U^w(e_i)s(e_i),$$

where $\{e_i\}$ is a local orthonormal basis in M. We then define $\widetilde{U}$ by

$$\widetilde{U}: NM \to NM, \quad \widetilde{U} := {}^{\dagger}U \circ U$$

i.e $\bar{g}(\widetilde{U}\,(w), v) = \bar{g}({}^{\dagger}U(U^w), v) = g(U^w, U^v) = Trace_g(U^v \circ U^w)$, for all $v, w \in NM$.
In particular,

$$\bar{g}(\widetilde{U}\,(w), w) = Trace_g(U^w \circ U^w) = |U^w|^2 \tag{84}$$

for all $w \in NM$.
It is straightforward that for all $s \in SM$ and $v, w \in NM$, $[U^v, [U^w, s]] \in SM$. We define $\underset{\sim}{U}: SM \to SM$ as

$$\underset{\sim}{U}\,(s) := \sum_{\alpha=1}^{\bar{n}-n} [U^{v_\alpha}, [U^{v_\alpha}, s]]$$

where $\{v_\alpha\}$is an orthonormal basis of the normal spaces to M, i.e. as the trace of the bilinear map $NM \times NM \ni (v, w) \to Ad_{U^v} \circ Ad_{U^w} \in End(SM)$.
The case in which M is a hypersurface in $\overline{M}$ is of particular interest. In this case, one has:

Proposition 6.4 *In the case $\bar{n} = n + 1$ (ie M has codimension 1 in $\overline{M}$), for any tensor field $U : NM \to SM$ as above, and for all $w \in NM$, one has*

(i) $U \circ \widetilde{U}\,(w) = |U^w|^2 U^w$;
(ii) $\underset{\sim}{U} \circ U = 0$.

Proof. From formula (84) and from the fact that the codimension is 1, one can deduce that $\widetilde{U}(w) = |U^w|^2 w$ and hence that $U \circ \widetilde{U}(w) = |U^w|^2 U^w$.
In codimension 1, for a unit normal vector w, one can write $\underset{\sim}{U}(s) = [U^w, [U^w, s]]$ and hence, $\underset{\sim}{U} \circ U^w = \underset{\sim}{U}(U^w) = [U^w, [U^w, U^w]] = 0$. ∎

Given an isometric immersion $(M^n, g) \to (\overline{M}^{\bar{n}}, \bar{g})$, the second fundamental form can be viewed as the tensor field $A : NM \to SM$ defined (with the obvious notations, see [20]) by the formula

$$A^w(x) := -(\overline{\nabla}_x W)^\top.$$

A symmetric tensor field $B : TM \times TM \to NM$ can also be defined by

$$B(x, y) := (\overline{\nabla}_x Y)^\perp.$$

These tensor fields are related by the formula:

$$\forall m \in M, x, y \in T_m M, w \in N_m M,\ g_m(A^w(x), y) = \bar{g}_m(B(x, y), w).$$

Finally, the mean curvature vector is defined by $H := \frac{1}{n} Trace_g\, B$.

When the codimension $(\bar{n} - n)$ is 1, we can choose a unit normal vector field ν along M (at least locally) and we define A as A^ν and the associated mean curvature by $H := \frac{1}{n} Trace_g\, A$. In this case, we also introduce the *traceless second fundamental form* of the immersion $A^0 := A - H\,\mathrm{Id}$.

The only assumption we have used on the tensor U in Proposition 6.4 is that U is symmetric. We can in particular apply it to the second fundamental form A or to the traceless second fundamental form A^0.

Associated with the second fundamental form $A : NM \to SM$ of the isometric immersion $M \to \overline{M}$, J. Simons defined the tensor $\overline{R}(A)^w(x)$ as follows ([20], formula 4.2.2 page 81).

$$\begin{aligned}
\langle \overline{R}(A)^w(x), y\rangle &= \sum_{i=1}^{n} \big\{ 2\langle \overline{R}_{e_i,y}\big(B(x,e_i)\big), w\rangle + 2\langle \overline{R}_{e_i,x}\big(B(y,e_i)\big), w\rangle \\
&- \langle A^w(x), \overline{R}_{e_i,y} e_i\rangle - \langle A^w(y), \overline{R}_{e_i,x} e_i\rangle \\
&+ \langle \overline{R}_{e_i,B(x,y)} e_i, w\rangle - 2\langle A^w(e_i), \overline{R}_{e_i,x} y\rangle \big\}
\end{aligned}$$

where $x, y \in T_m M,\ w \in N_m M$.
When the codimension is 1, we can take $w = \nu$ (at least locally). The first two terms in the above formula vanish since $B(\cdot,\cdot)$ is parallel to ν and the fifth term can be written as $\langle A(x), y\rangle \langle \overline{R}_{e_i,\nu} e_i, \nu\rangle$ (here we have written A instead of A^ν). It is then clear that the preceding formula can be applied to any symmetric tensor U and we therefore define $\overline{R}(U)$, in codimension 1, by the formula

$$\begin{aligned}
\langle \overline{R}(U)(x), y\rangle &= \sum_{i=1}^{n} \big\{ -\langle U(x), \overline{R}_{e_i,y} e_i\rangle - \langle U(y), \overline{R}_{e_i,x} e_i\rangle \\
&+ \langle U(x), y\rangle \langle \overline{R}_{e_i,\nu} e_i, \nu\rangle - 2\langle U(e_i), \overline{R}_{e_i,x} y\rangle \big\}.
\end{aligned}$$

A straightforward computation then yields

$$\langle \overline{R}(U + \lambda \,\mathrm{Id})(x), y\rangle = \langle \overline{R}(U)(x), y\rangle - \lambda \overline{Ric}(\nu)\langle x, y\rangle \tag{85}$$

where $\overline{Ric}$ is the Ricci curvature of $\overline{M}$. We shall in particular apply this formula to $A = A^0 + H\,\mathrm{Id}$.

Another straightforward computation ([3], Corollary 2) yields the following formula when $\overline{M}$ has constant curvature c (ie when $\overline{R}_{t_1,t_2}t_3 = -c\langle t_1, t_3\rangle t_3 + c\langle t_2, t_3\rangle t_1$)

$$\langle \overline{R}(A)(x), y\rangle = nc(\langle A(x), y\rangle - 2H\langle x, y\rangle). \tag{86}$$

For an isometric immersion $(M^n, g) \to (\overline{M}^{\,n+1}, \bar{g})$, with codimension 1 and constant mean curvature H, J. Simons' generalized equation reads

$$\overline{\Delta} A(x) = |A|^2 A(x) - \overline{R}(A)(x) - nH\langle \overline{R}_{\nu,x}\cdot, \nu\rangle - nHA \circ A(x) - \overline{R}\,'(x). \tag{87}$$

This formula follows from [3], Theorem 2 and from Proposition 6.4 (we used A instead of A^ν). We also used $\overline{\Delta}$ to denote the (positive) Bochner Laplacian ($\overline{\Delta} = -\nabla^2$, see [20]) and $\overline{R}\,'$ to denote the tensor field

$$\langle \overline{R}\,'(x), y\rangle = \sum_{i=1}^{n} \{\langle (\overline{\nabla}_x \overline{R})_{e_i,y} e_i, \nu\rangle + \langle (\overline{\nabla}_{e_i}\overline{R})_{e_i,x} y, \nu\rangle\}$$

([20], formula 4.2.1, page 80).
Since the mean curvature is constant, one has $\overline{\Delta} A^0 = \overline{\Delta} A$. Substituting $A^0 + H\,\mathrm{Id}$ to A in formula (87) we obtain:

Proposition 6.5 *Let $(M^n, g) \to (\overline{M}^{\,n+1}, \bar{g})$, be an isometric immersion, with codimension 1 and constant mean curvature H. The traceless second fundamental form A^0 of M satisfies J. Simons' generalized equations*

$$\begin{aligned}
\overline{\Delta} A^0(x) &= |A^0|^2 A^0(x) - nHA^0 \circ A^0(x) + H|A^0|^2 x - nH^2 A^0(x) \\
&- \overline{R}(A^0)(x) + H\overline{Ric}(\nu)x - nH\langle \overline{R}_{\nu,x}\cdot, \nu\rangle - \overline{R}\,'(x),
\end{aligned}$$

and

$$\begin{aligned}
\langle \overline{\Delta} A^0, A^0\rangle &= |A^0|^4 - nH\, Trace(A^0)^3 - nH^2|A^0|^2 - \langle \overline{R}(A^0), A^0\rangle \\
&- nH\sum_{i=1}^{n} \langle \overline{R}_{\nu,e_i} A^0(e_i), \nu\rangle - \langle \overline{R}\,', A^0\rangle.
\end{aligned}$$

The second formula in Proposition 6.5 is the formula we are really interested in. It is valid for the traceless tensor field A^0, in codimension 1. We now consider several cases.

Case 1: codimension 1, no further assumption.

In this general case, one can estimate the right hand side of the second formula in Proposition 6.5 as follows

$$
\begin{aligned}
\langle\overline{\Delta}A^0, A^0\rangle \leq |A^0|^4 \quad &+ \quad (C(\overline{R}) - nH^2)|A^0|^2 \\
&+ \quad nH\frac{n-2}{\sqrt{n(n-1)}}|A^0|^3 + C(\overline{R}\,')|A^0|
\end{aligned}
\tag{88}
$$

where $C(\overline{R})$ (resp. $C(\overline{R}\,')$) is a constant which only depends on the dimension n and on bounds on the curvature tensor of $\overline{M}$ (resp. on the dimension n and on bounds on the curvature tensor and its first order covariant derivatives). The estimate of the term $Trace_g(A^0)^3$ is due to H. Alencar and M. do Carmo (see [3] §5) ; this term vanishes in dimension 2.

Case 2: codimension 1, $\overline{M}$ has constant curvature c.
In this case, the formula reads, using (86),

$$
\langle\overline{\Delta}A^0, A^0\rangle = |A^0|^4 - nH\,Trace(A^0)^3 - n(c+H^2)|A^0|^2 \tag{89}
$$

and, if necessary, the second term can be estimated by $nH\frac{n-2}{\sqrt{n(n-1)}}|A^0|^3$ in absolute value.

Case 3: codimension 1, $n = 2$.
The above estimate gives

$$
\langle\overline{\Delta}A^0, A^0\rangle \leq |A^0|^4 + (C(\overline{R}) - 2H^2)|A^0|^2 + C(\overline{R}\,')|A^0|. \tag{90}
$$

where the constants are as above.

Case 4: codimension 1, $n = 2$ and $\overline{M}$ has constant curvature c.
In this particular case, the above formula reads

$$
\langle\overline{\Delta}A^0, A^0\rangle = |A^0|^4 - 2(c+H^2)|A^0|^2 \tag{91}
$$

We have the following general formulas for any symmetric tensor field U

$$
\Delta_g|U|^2 = 2|U|\Delta_g|U| - 2|d|U||^2,
$$

where Δ_g is the non-negative Laplacian, and

$$
\Delta_g|U|^2 = 2\langle U, \overline{\Delta}U\rangle - 2|\nabla^g U|^2.
$$

In general, Cauchy-Schwarz inequality gives $|\nabla u|^2 \geq ||d|U||^2$. In the particular case $Trace\,U = 0$, $\overline{M}$ has constant sectional curvature and M has codimension 1, this inequality can be improved to $|\nabla U|^2 \geq (1+\frac{2}{n})|d|U||^2$ (see [3], §5 or [18] for a more general estimate). This gives

$$
|U|\Delta_g|U| + \frac{2}{n}|d|U||^2 \leq \langle\overline{\Delta}U, U\rangle
$$

when $Trace\,U = 0$ and when M has codimension 1. In the particular case in which $n = 2$, this estimate can still be improved.

Lemma 6.6 *When $\overline{M}$ has dimension 3 and constant curvature and when M has dimension 2, one has*

$$|\nabla A^0|^2 = 2|d|A^0||^2.$$

Proof. Let $\{e_i\}$ be a local orthonormal basis, and let $U := A^0$. The matrix $U_{ij} := U(e_i, e_j)$ is symmetric. We also define $U_{ijk} := (\nabla_{e_k} U)(e_i, e_j)$. Due to Codazzi equations ([3], Theorem 1), U_{ijk} is symmetric with respect to the three indices. Since the dimension is 2 and since U is traceless, we have $U_{11} + U_{22} = 0$ and similarly $U_{11k} + U_{22k} = 0$ ([3], Theorem 1). We set $U_{11} = -U_{22} = a$ and $U_{11k} = -U_{22k} = a_k$. We can also assume that the orthonormal basis diagonalizes U at the point we consider. This gives $|U|^2|\nabla U|^2 = 8a^2(a_1^2 + a_2^2)$ if we take into account the symmetries of U_{ijk}. On the other hand, we can write $|U|^2|d|U||^2 = \sum_k (\sum_i U_{iik} U_{ii})^2 = 4a^2 \sum_k a_k^2$. These equalities give the result. ∎

Proposition 6.7 *Let $(M^2, g) \to (\overline{M}^3(c), \bar{g})$ be an isometric immersion of a surface in a 3-space of constant curvature c, with constant mean curvature H and traceless second fundamental form A^0.*
(i) The Gauss equation can be written as

$$2K_g = 2(c + H^2) - |A^0|^2,$$

(ii) Simons' equation gives

$$-\langle \overline{\Delta} A^0, A^0 \rangle = |A^0| \Delta_g |A^0| - |d|A^0||^2 = -|A^0|^4 + 2(c + H^2)|A^0|^2,$$

$$\Delta_g \ln |A^0| = -|A^0|^2 + 2(c + H^2) = 2K_g,$$

where K_g is the Gauss curvature of (M, g) and where Δ_g is the Laplacian for the metric g on M such that $\int_M u \Delta_g u \, dv_g = -\int_M |du|_g^2 \, dv_g$.

Proof. Straightforward using (91), Lemma 6.6 and formula (80). ∎

The last formula in Proposition 6.7 gives the classical formula $\Delta \ln |A| = 2K_g$ for minimal surfaces in $\mathbb{R}^3$ (see also [22]).

6.5 Bounds on the radius of stable balls

In this section, we follow an argument of A. Ros – H. Rosenberg ([14]) and R. Freire de Lima ([9]) to give a bound on the radius of stable balls. More precisely, we have

Proposition 6.8 *Let $i : (M, g) \to (\overline{M}^3(c), \overline{g})$ be an immersion with constant mean curvature H such that $c + H^2 > 0$. Then, any (strongly) stable ball contained in M must have radius R less than $\sqrt{2\pi^2/3(c + H^2)}$.*

Proof. Assume that there is a number $R' > 0$ such that the ball $B(R') := B^g(x_0, R')$ is relatively compact in M. Assume furthermore that $B(R')$ is stable, i.e. the stability operator L_g is non-positive on $C_0^\infty(B(R'))$. Then according to Lemma 5.1, there exists a non-negative function u on $B(R')$ such that $L_g u \leq 0$. Given some $R < R'$, since $u > 0$ on $B(R)$ by the maximum principle, we can consider the metric $\tilde{g} := u^2 g$.

• **Fact.** Let y_0 be a point on $\partial B(R)$ such that $d_{\tilde{g}}(x_0, y_0) = \inf\{d_{\tilde{g}}(x_0, y) \mid y \in \partial B(R)\}$. Then, there exists a $\tilde{g}$-minimizing geodesic γ from x_0 to y_0.
Assume that γ is parametrized by arc-length in the $\tilde{g}$-metric and that it has length $\widetilde{R}$. Let $s(t)$ denote the arc-length for the g-metric along γ,

$$s(t) = \int_0^t |\dot{\gamma}(\tau)|_g \, d\tau. \tag{92}$$

Define $v(t) := u \circ \gamma(t)$. Since $\tilde{g} := u^2 g$, we have

$$s'(t)v(t) = 1. \tag{93}$$

It follows that

$$\int_0^{\widetilde{R}} |\dot{\gamma}(\tau)|_g \, d\tau = g - \text{length}(\gamma)\big|_0^{\widetilde{R}} = \overline{R} \geq R$$

since $d_g(x_0, y_0) = R$.

Since γ is $\tilde{g}$-minimizing, the second variation formula give the inequality

$$\int_0^{\widetilde{R}} \left\{(\phi')^2(t) - \tilde{k}(t)\phi^2(t)\right\} dt \geq 0 \tag{94}$$

for any smooth function ϕ such that $\phi(0) = \phi(\widetilde{R}) = 0$, where $\tilde{k}(t) = \widetilde{K} \circ \gamma(t)$ is the curvature along the curve γ for the metric $\tilde{g}$.
We change from parameter t to parameter s along γ and we write

$$\phi(t) =: \varphi(s(t)), \quad \gamma(t) =: c(s(t)), \quad v(t) =: w(s(t)).$$

Equation (94) can be re-written as

$$\int_0^{\widetilde{R}} \left\{(\varphi')^2(s(t))(s'(t))^2 - \tilde{k}(t)\varphi^2(s(t))\right\} dt \geq 0 \tag{95}$$

and, changing variables in the integral,

$$\int_0^{\overline{R}} \left\{(\varphi')^2(s)w^{-1}(s) - \widetilde{K} \circ c(s)\varphi^2(s)w(s)\right\} ds \geq 0. \tag{96}$$

On the other-hand, the relationship $\tilde{g} := u^2 g$ gives, for the Gauss curvature,

$$\begin{aligned} \widetilde{K} &= u^{-2}\big(K - \Delta_g \ln u\big) \\ &= u^{-4}\big(|du|_g^2 - u\Delta_g u + K_g u^2\big) \\ &\geq u^{-4}\big(|du|_g^2 + 3(c + H^2)u^2\big) \end{aligned} \tag{97}$$

where the inequality follows from the inequality $L_g u \leq 0$ (indeed, $L_g u = \Delta_g u - K_g u + 3(c+H^2)u + \frac{1}{2}|A^0|^2 u$, according to Section 6.2.2).
Since $w(s) := u \circ c(s)$, we have $w'(s) = du(c'(s))$ and $(w')^2(s) \leq |du|_g^2 |c'(s)|_g^2$. It follows that

$$(w')^2(s) \leq |du|_g^2 \tag{98}$$

because $|c'(s)|_g = 1$.

Using (96) – (98), we obtain

$$\begin{aligned}
\int_0^{\overline{R}} (\varphi')^2 w^{-1}\, ds &\geq \int_0^{\overline{R}} \widetilde{K} \circ c \varphi^2 w\, ds \\
&\geq \int_0^{\overline{R}} w^{-4}\{|du|_g^2 + 3(c+H^2)u^2\} \circ c\varphi^2 w\, ds \\
&\geq \int_0^{\overline{R}} \{(w')^2 w^{-3} + 3(c+H^2)w^{-1}\}\varphi^2\, ds
\end{aligned}$$

i.e.

$$\int_0^{\overline{R}} (\varphi')^2 w^{-1} ds \geq \int_0^{\overline{R}} \varphi^2\{(w')^2 w^{-3} + 3(c+H^2)w^{-1}\} ds. \tag{99}$$

Let us now write φ as $\varphi(s) = w(s)\psi(s)$. A straightforward computation transforms (99) to (100)

$$\int_0^{\overline{R}} \{2\psi(s)\psi''(s) + (\psi')^2(s) + 3(c+H^2)\psi^2(s)\} w(s)\, ds \leq 0. \tag{100}$$

Choosing $\psi(s) = \sin \frac{\pi s}{\overline{R}}$, equation (100) gives

$$\int_0^{\overline{R}} \{(3(c+H^2) - \frac{2\pi^2}{\overline{R}^2}) \sin^2 \frac{\pi s}{\overline{R}} + \frac{\pi^2}{\overline{R}^2} \cos^2 \frac{\pi s}{\overline{R}}\}\, ds \leq 0. \tag{101}$$

When $3\overline{R}^2(c+H^2) \geq 2\pi^2$, the inequality (101) is not possible. It follows that we necesarily have $3\overline{R}^2(c+H^2) < 2\pi^2$ and hence $3R^2(c+H^2) < 2\pi^2$ since $R \leq \overline{R}$.
∎

References

[1] Barbosa, J.M. Lucas – Bérard, Pierre. — On a twisted eigenvalue problem, preprint 1998

[2] Bryant, Robert. — Surfaces of mean curvature one in hyperbolic space, Astérisque 154-155, Soc. Math. de France, 1987, p. 321–347

[3] Bérard, Pierre. — Simons' equation revisited, An. Acad. bras. Ci. 66(1994), 397–403

[4] Castillon, Philippe. — Inégalité isopérimétrique pour certaines sous-variétés, Preprint Institut Fourier (1995), 12 p.

[5] Cheung, Leung–Feung. — Geometric properties of stable noncompact constant mean curvature surfaces. Dissertation, Rheinische Friedrich-Wilhems-Universität Bonn, 1990. Bonner Math. Schriften, 220. Universität Bonn, Math. Institut, Bonn, 1991. iv+65 pp.

[6] Ecker, Klaus – Huisken, Gerhard. — Interior curvature estimates for hypersurfaces of prescrided mean curvature, Ann. Inst. Henri PoincarÚ, Anal. Non LinÚaire, 6 (1989), 251–260

[7] Fischer Colbrie, Doris. — On complete minimal surfaces with finite Morse index in three manifolds, Inventiones Math. 82 (1987), 121–132

[8] Fischer Colbrie, Doris – Schoen, Richard. — The structure of complete minimal surfaces in 3-manifolds of non-negative scalar curvature, Comm. Pure Appl. Math. 33 (1980), 199–211

[9] Freire de Lima, Ronaldo. — Thesis in progress, IMPA 1998

[10] Heinz, Ernst. — Über die Lösungen der Minimalflächengleichung, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl II (1952), 51-56

[11] Hoffman, David – Spruck, Joel. — Sobolev and isoperimetric inequalities for Riemannian submanifolds, Communications in Pure and Applied Math. 27 (1974), 715-727

[12] Lawson, H. Blaine. — Complete minimal surfaces in S^3, Annals of Math. 92 (1970), 335–374

[13] Osserman, Robert. — An extension of certain results in function theory to a class of surfaces, Proc. Nat Acad. Sci 45 (1959), 1031–1035

[14] Rosenberg, Harold. — Private communication, 1998

[15] Sauvigny, Friedrich. — A priori estimates of the principal curvatures for immersions of prescribed mean curvature and a theorem of Bernstein type, Math. Zeit. 205 (1990), 567–582

[16] Schoen, Richard. — Estimates for stable minimal surfaces in three dimensional manifolds, Seminar on minimal submanifolds, Ed. by E. Bombieri, Annals of Math. Studies, Princeton University Press, 1983.

[17] Spivak, Michael. — A comprehensive introduction to differential geometry, Volume IV (Second edition), Publish or Perish Inc., 1975

[18] Schoen, Richard – Simon, Leon – Yau, Shing-Tung. — Curvature estimates for minimal hypersurfaces, Acta Math. 134 (1975), 275–288

[19] da Silveira, Alexandre. — Stability of complete noncompact surfaces with constant mean curvature, Math. Ann. 277 (1987), 629–638

[20] Simons, James. — Minimal varieties in Riemannian manifolds, Annals of Math. 88 (1968), 62–105

[21] Spruck, Joel. — Gauss curvature estimates for surfaces of constant mean curvature, Comm. Pure Applied Math., 27 (1974), 547–557

[22] Traizet, Martin. — On stable surfaces of constant Gauss curvature in spaces forms, Annals Global Anal. Geom. 13 (1995), 141–148

Pierre Bérard
Institut Fourier
UMR 5582 UJF–CNRS
Université Joseph Fourier
B.P. 74
38402 St Martin d'Hères Cedex
France
Pierre.Berard@ujf-grenoble.fr

Laurent Hauswirth
Departamento de Matemática
Universidade Federal do Ceará
Campus do Pici
60455-760 Fortaleza
Ceará, Brazil
laurent@mat.ufc.br