ON A Λ-ADIC ANDRIANOV L-FUNCTION FOR $GSp(4)$

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Introduction

The present paper is based on a joint lecture of I.I. Piatetski-Shapiro and A.A. Panchishkin of June 5, 1998 in the Seminar in Honour of I.I. Piatetski-Shapiro held in the Institut Fourier, Grenoble (June 4-5, 1998) during his visit to Grenoble in May-June 1998.

Let G be a reductive group over a number field F, and p be a prime number. The arithmetic of L-functions attached to automorphic forms on $G(\mathbb{A}_F)$, in particular the study of their special values, is closely related to the theory of Eisenstein series via Rankin's method [Ran39], [Ran52]. This method uses Eisenstein series in an integral representation for certain rather general complex automorphic L-functions [PSh-R], [Ge-PSh]. In order to construct p-adic automorphic L-functions out of their complex special values one can successfully use p-adic integration along a (many variable) Eisenstein measure which was introduced by N.Katz [Ka76, Ka77, Ka78] and used by H.Hida [Hi86, Hi91, Hi93] in the case of $G = \mathrm{GL}_2$ over a totally real field F (i.e. for the elliptic modular forms and Hilbert modular forms). The application of such a measure to a given p-adic family of modular forms provides a general construction of p-adic L-functions of several variables. On the other hand, the evaluation of this measure at certain points gives another important source of p-adic L-functions [Ka78]. In the Siegel modular case the Eisenstein measure was studied in [PaSE]. The purpose of this paper is to construct a Λ-adic version of the Andrianov L-function for the symplectic group

$$\mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}_4 \mid {}^t g J_4 g = \nu(g) J_4, \nu(\alpha) \in \mathrm{GL}_1 \right\},$$

over a totally real field F where

$$J_4 = \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}$$

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using the Eisenstein measure and a p-adic analogue of the Petersson product for Λ-adic automorphic forms on GL_2 over a totally real field F, see [Hi90]. Main Theorem is given in Section 4.

1. Complex analytic L-functions for $GSp(4)$

Let F be a global field of characteristic $\neq 2$, and V a four dimensional vector space over F endowed with a non-degenerate skew-symmetric form $\rho : V \times V \to F$,

$$G_\rho = \mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}(V) \mid \rho(gu, gv) = \nu_g \rho(u,v),\ \nu_g \in F^\times \right\},$$

the algebraic group of symplectic similitudes of ρ over F. Let $\pi = \otimes_v \pi_v$ be an irrreducible cuspidal automorphic representation of $G_\rho(\mathbf{A}_F)$ where v run over all places of F, then according to Langlands' classification of irreducible supercuspidal representations π_v of $G_\rho(F_v)$ for almost all v π_v correspond to a semisimple conjugacy class of a diagonal matrix

$$h_v = \mathrm{diag}\{\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2\} \in {}^L G_\rho(\mathbf{C}) \overset{r}{\tilde{\to}} GSP_4(\mathbf{C})$$
$$(\alpha_j = \alpha_j(v), v \notin S, |S| < \infty).$$

The Andrianov L-function (or the spinor L-function) of π is then the following Euler product

$$L(s, \pi, r) = \prod_{v \notin S} \det\left(1_4 - h_v \cdot Nv^{-s}\right)^{-1} \times \begin{pmatrix} \text{a finite Euler product} \\ \text{over } v \in S \end{pmatrix} \qquad (1.1)$$

This L function plays an impotant role in the arithmetic, in particular it is related to the l-adic Galois representation on H^3 of the corresponding Siegel threefold [Tay].

This L function was introduced by Andrianov [AndBud], [And74] in the classical fashion, for $F = \mathbf{Q}$, and for $\pi = \pi_f$ coming from a holomorphic Siegel cusp eigenform $f = \sum_\xi a_\xi q^\xi$ for the Siegel modular group $\Gamma_2 = Sp_4(\mathbf{Z})$ over the Sigel half plane

$$H_2 = \{z = {}^t z \in M_2(\mathbf{C}), \mathrm{Im}\,(z) > 0\},$$

where ξ run over the semigroup A_4 of positive definite half integral symmetric matrices ξ, $a_\xi \in \mathbf{C}$, $q^\xi = \exp(2\pi i \mathrm{Tr}(\xi z))$. Consider the Hecke algebra $\mathcal{H} = \langle(\Gamma_2 g \Gamma_2)\rangle = \otimes_p \mathcal{H}_p$ generated by all double coset classes $(\Gamma_2 g \Gamma_2)$ with $g \in GSp_4(\mathbf{Q})$. Then we have that $\mathcal{H}_p = \mathbf{Q}[x_0^\pm, x_1^\pm, x_2^\pm]^{W_2}$ (W_2 the Weyl group) and one has a $\mathbf{Q}$-algebras homomorphism $\lambda_f : \mathcal{H} \to \mathbf{C}$ given by $f|X = \lambda_f(X) f$, $X \in \mathcal{H}$, and α_j are defined as $\lambda_f(x_j)$, $j = 0, 1, 2$. In the notation of Andrianov,

$$Z_f(s) = \prod_p \det\left(1_4 - h_p p^{-s}\right)^{-1} \qquad (1.2)$$

is called the spinor L function of f, and he proved that it coincides essentially with the Dirichlet series $L(s,f,\xi_0)=\sum_{m=1}^{\infty}\frac{a_{m\xi_0}}{m^s}$ where $\xi_0>0$ in a fixed positive definit matrix. Starting from this identification, he obtained an integral representation for $Z_f(s)$ using the group $GL_2(K)$ where $K=\mathbf{Q}(\sqrt{-\det\xi_0})$ an imaginary quadratic field. This integral representation implied an analytic continuation of $Z_f(s)$ to the whole complex plane and the functional equation of the type

$$\Psi_f(s)=\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s-k+2)=(-1)^k\Psi_f(2k-2-s). \tag{1.3}$$

where $\Gamma_{\mathbf{C}}=(2\pi)^{-s}\Gamma(s)$ is the standard Γ-factor. Its analytic properties were studied by A. N. Andrianov [And74] but stil little is known about algebraic and arithmetic properties of the special values of this function; however, from the general Deligne conjecture on critical values of L-functions it follows that algebraicity properties could exist only for $s=k-1$.

This work was extended by I.I.Piatetski-Shapiro [PShBud], [PshPac] to arbitrary F using an arbitrary quadratic extension K/F and the folloing construction. Put

$$V=K^2=\left\{x=\begin{pmatrix}x_1\\x_2\end{pmatrix}, x_j\in K, j=1,2\right\}$$

then V may be viewed as a four dimensional F vector space, $\dim_F V=4$, and define $\rho(x,y)=\mathrm{Tr}_{K/F}(x_1y_2-x_2y_1)$. Consider the following F-algebraic group

$$G=\{g\in GL_2(K)\mid \det g\in F^\times\},\ \ G(F)\subset GL_2(K) \tag{1.4}$$

then there is an imbedding of F-algebraic groups $i: G\hookrightarrow G_\rho$ because $x_1y_2-x_2y_1=\det(x,y)$ and $\det(gx,gy)=\det g\cdot\det(x,y)$, $\rho(gx,gy)=\det g\cdot\rho(x,y)$. Note that $G(\mathbf{A}_F)\subset GL_2(\mathbf{A}_K)$ and $G(\mathbf{A}_F)\hookrightarrow G_\rho(\mathbf{A}_F)=GSp_4(\mathbf{A}_F)$. It turns out that there is an integral representation for $L(s,\pi,r)$ of the following type:

$$L(s,\pi,r)=\int_{G(F)C_F\backslash G(\mathbf{A}_F)}\varphi(i(g))E(g,s,\mu)dg \tag{1.5}$$

where φ is an automorphic form on $G_\rho(\mathbf{A}_F)=GSp_4(\mathbf{A}_F)$ from the representation space of π, C_F the center of $G(F)\subset GL_2(K)$, $E(g,s,\mu)$ is an Eisenstein series on $G(\mathbf{A}_F)\subset GL_2(\mathbf{A}_K)$ attached to a quasicharacter $\mu: K^\times\backslash\mathbf{A}_K^\times\to\mathbf{C}^\times$ ([PshPac], § 5).

2. A p-adic construction

Let $p\geq 5$ be a prime number. We consider the case of two totally real fields $K\supset F$ and a representation π_f attached to a holomorphic Siegel-Hilbert cusp form $f(z)=\tilde\varphi$ of scalar weight $k=(k,\dots,k)$ on the Siegel-Hilbert half plane

$$H_{2,F}=H_2\times\cdots\times H_2\ \ (n\text{ copies}); \tag{2.1}$$

in this case there is also a critical value $s = k-1$ for L-functions of the type $L(s, \pi_f, \otimes\chi, r)$ where χ is a character of finite order of $F^\times\backslash\mathbf{A}_F^\times$. According to general conjectures on motivic L-functions there should exist p-adic L-functions which interpolate p-adically their critical values, see [Co], [Co-PeRi], [PaIF]. However in our present construction instead of p-adic interpolation of their special values of the type $L(k-1, \pi_f \otimes \chi, r)$ we use directly a p-adic version of (1.5) using techniques of Λ-adic modular forms (see Section 3). We hope that the resulting p-adic L-function provide also the above p-adic interpolation.

3. Λ-adic modular forms

Recall that the Iwasawa algebra [Iw] $\Lambda = \mathbf{Z}_p[[T]] \cong \mathbf{Z}_p[[\Gamma]]$ is the completed group ring of the profinite group $\Gamma = 1 + p\mathbf{Z}_p = \langle 1+p \rangle \subset \mathbf{Z}_p^\times$. According to the theorem of Kubota-Leopoldt [Ku-Le], there exists a unique element $g(T) \in \Lambda$ such that for all $k \geq 1$, $k \equiv 1 \bmod (p-1)$

$$g((1+p)^k - 1) = \zeta^*(1-k)$$

where $\zeta^*(1-k)$ denotes the special value at $s = 1-k$ of the Riemann zeta-function with a modified Euler p-factor: $\zeta^*(s) = (1-(1+p)^{-s+1})(1-p^{-s})\zeta(s)$.

Definition 3.1. *The Serre ring* $\Lambda[[q]]$ is the ring of all formal q-expansions with coefficients in Λ:

$$\Lambda[[q]] = \{f = \sum_{n=0}^{\infty} a_n(T)q^n \mid a_n(T) \in \Lambda\};$$

Definition 3.2. The Λ-module $M(\Lambda) \subset \Lambda[[q]]$ of Λ-*adic modular forms* (of some fixed level N, $(N,p) = 1$, consists of all $f = \sum_{n=0}^{\infty} a_n(T)q^n \in \Lambda[[q]]$ such that for each $k \geq 5$, $k \equiv 1 \bmod (p-1)$ the specialisation

$$f_k = f|_{T=(1+p)^k-1} \in \mathbf{Z}_p[[q]]$$

is a classical modular form of weight k and level Np. In other terms f is given by a p-adic measure μ_f on $\mathbf{Z}_p^\times$ with values in $\mathbf{Z}_p[[q]]$ such that the integrals

$$\int_{\mathbf{Z}_p^\times} x_p^k \mu_f = f_k \tag{3.1}$$

are classical modular forms.

Example 3.3. *The* Λ*-adic Eisenstein series* $f \in M(\Lambda)$ (of level $N = 1$) is defined by

$$f_k = \frac{\zeta^*(1-k)}{2} + \sum_{n\geq 1} \sigma_{k-1}^*(n)q^n, \quad \sigma_{k-1}^*(n) = (1-(1+p)^k) \sum_{d|n, p\nmid d} d^{k-1}. \tag{3.2}$$

Example 3.3. *Hida's families* f are elements of

$$S^{\text{ord}}(\Lambda) = eS(\Lambda), \ \ e = \lim_{n\to\infty} U_p^{n!}$$

$(U_p(\sum_{n\geq 0} a_n q^n) = \sum_{n\geq 0} a_{pn} q^n$ is the Atkin U-operator), $S(\Lambda)$ is the Λ-submodule of Λ-adic cusp forms.

The Hilbert modular case. According to the classical theorem of Klingen [Kli], for a totally real field F and for $k \geq 1$ the special values $\zeta_F(1-k)$ are rational numbers where $\zeta_F(s)$ is the Dedekind zeta function of F.

The Deligne-Ribet p-adic zeta function [De-Ri] interpolates p-adically these special values as an element g_F of a version of the Iwasawa algebra over F, $\Lambda_F = \mathbf{Z}_p[[G_{p,F}]]$, where $G_{p,F} = \mathrm{Gal}(F^{\text{ab}}_{p,\infty}/F)$ is the Galois group of the maximal abelian extension unramified outside of prime divisors over p and ∞. For $F = \mathbf{Q}$ we have that $G_{p,\mathbf{Q}} \xrightarrow{\sim} \mathbf{Z}_p^\times$, and there is the natural restriction homomorphism (or the norm homomorphism) $\mathcal{N} : G_{p,F} \to \mathbf{Z}_p^\times$, so that for an integer k the group homomorphism $\mathcal{N}^k : G_{p,F} \to \mathbf{Z}_p^\times$ induces the ring homomorphism $\mathcal{N}^k : \Lambda_F = \mathbf{Z}_p[[G_{p,F}]] \to \mathbf{Z}_p$ and the numbers $\mathcal{N}^k(g_F)$ interpolate $\zeta_F(1-k)$. Also $g_\mathbf{Q}$ coincides essentially with the Kubota-Leopoldt zeta-function $g(T)$.

A Λ-adic Hilbert modular form could be defined as a formal Fourier expansion

$$f = \sum_{\eta \in L_F} a_\eta q^\eta \in \Lambda_F[[q^{L_F}]] \ \ (L_F \subset F \text{ a lattice})$$

(η runs over totally positive elements or 0) whose appropriate specialisations are classical Hilbert modular form. More precisely, for an integer k there is a homomorphism $\mathcal{N}^k : \Lambda_F[[q^{L_F}]] \to \mathbf{Z}_p[[q]]$ and it is required that for all appropriate sufficiently large k the specialization $f_k = \mathcal{N}^k(f)$ be the Fourier expansion of a classical Hilbert modular form. As over $\mathbf{Q}$, the first example of a Λ-adic Hilbert modular form is given by an Eisenstein series (more precisely, this seris is given by the Katz-Hilbert-Eisenstein measure, see [Ka78]). Also, Hida's theory could be extended to the Hiilbert modular case [Hi91].

The Siegel-Hilbert modular case. A Λ-adic Siegel-Hilbert modular form could be defined as a formal Fourier expansion

$$f = \sum_{\xi \in B_{2,F}} a_\xi q^\xi \in \Lambda_F[[q^{B_{2,F}}]] \ \ (B_{2,F} \subset M_{2,F})$$

($B_{2,F}$ is the semi-group of all symmetric totally non-negative matrices ξ in a sublattice of $M_{2,F}$) whose appropriate specialisations are classical Siegel-Hilbert modular form. More precisely, for an integer k there is a homomorphism $\mathcal{N}^k : \Lambda_F[[q^{L_F}]] \to \mathbf{Z}_p[[q]]$ and it is required that for all appropriate sufficiently large k the specialization $f_k = \mathcal{N}^k(f)$ be the Fourier expansion of a classical Siegel-Hilbert modular form. The first example of a Λ-adic Siegel-Hilbert modular form is given by an Eisenstein series (for $F = \mathbf{Q}$ these series are described in [PaSE]. It seems that Hida's theory also could be extended to the Siegel-Hilbert modular case [Hi98], [Til-U],[Til].

4. Λ-adic L-functions

Recall that we consider the case of two totally real fields $K \supset F$ and a representation π_f attached to a holomorphic Siegel-Hilbert cusp form $f(z) = \tilde{\varphi}$ of scalar weight $k = (k, \dots, k)$ on the Siegel-Hilbert half plane

$$H_{2,F} = H_2 \times \cdots \times H_2 \ \ (n \text{ copies});$$

Then we rewrite the integral representation (1.5) in the form of the Petersson scalar product over K:

$$L(s, \pi, r) = \langle \tilde{i}^* \tilde{\varphi}, \tilde{E}(s, \mu) \rangle_K \tag{4.1}$$

where i denotes both the imbedding $i : G \hookrightarrow G_\rho$ and the corresponding modular imbedding

$$i : H_F \times H_F \to H_{2,F}, \ \ H_F = H \times \cdots \times H\,; H_{2,F} = H_2 \times \cdots \times H_2 \ \ (n \text{ copies}); \tag{4.2}$$

(which looks like

$$(z_1, z_2) \mapsto \begin{pmatrix} z_1 & \alpha(z_1 - z_2) \\ \alpha(z_1 - z_2) & z_2 \end{pmatrix} \ \text{ with } \ \alpha \in F,$$

(see [Ham])), $i^* \tilde{\varphi} = \varphi \circ i$ is a rapidly decreasing (but not cuspidal) holomorphic form. For the Λ-adic construction take a Λ-adic Siegel-Hilbert cusp form $\tilde{\varphi}$ on $GSp_{4,F}$ then $i^* \tilde{\varphi}$ is a Λ-adic Hilbert modular form over K explicitely described by its Fourier expansion. Now take G to be the Λ-adic Katz-Hilbert-Eisenstein measure for $GL_{2,K}$. In order to define the Petersson product

$$\langle \tilde{i}^* \varphi, G \rangle_K \tag{4.3}$$

we put $\mathcal{L} = \mathrm{Quot}(\Lambda)$ then it suffices to define

$$\langle 1_{\mathrm{Eis}}(i^* \tilde{\varphi}), G \rangle_K$$

where $1_{\mathrm{Eis}}(i^* \tilde{\varphi})$ denotes the projection in the $\mathcal{L}$-vector space $M(\mathcal{L})$ to the $\mathcal{L}$-subspace $Eis_K(\mathcal{L})$ of Hilbert-Eisenstein series. The projection $1_{\mathrm{Eis}}(i^* \tilde{\varphi})$ could be explicitely computed using higher terms of the Fourier expansions of $i^* \tilde{\varphi}$ and of the Fourier expansions of a $\mathcal{L}$-basis of $Eis_K(\mathcal{L})$. Then we are reduced to the case of $\langle G_1, G_2 \rangle_K$, where G_1 and G_2 are two Hilbert-Eisenstein series, and in order to define their Petersson product we use the method of Rankin. If G_1, G_2 were two cusp forms of weight k their Petersson product would coincide with a normalized residue of the Rankin zeta function $L_{G_1,G_2}(s)$ at $s = k$. In the case of normalised Eisenstein series the Rankin zeta function $L_{G_1,G_2}(s)$ is explicitely evaluated via Rankin's lemma as a product of abelian Dirichlet L-functions, and we define the

$\langle G_1, G_2\rangle_K$ in a similar fashion as in [Ko-Za] as the normalised residue of $L_{G_1,G_2}(s)$ in terms of the corresponding Deligne-Ribet p-adic zeta functions.

Main theorem. *Let $\tilde{\varphi}$ be a Λ-adic Siegel-Hilbert cusp form then*

1) there exists a canonically defined element

$$\mathcal{L}_\varphi = \langle 1_{\mathrm{Eis}}(i^*\tilde{\varphi}), G\rangle_K \in \mathcal{L}_F$$

where G is the Katz-Hilbert-Eisenstein series, $i^*\tilde{\varphi}$ the Λ-adic pullback of φ, $1_{\mathrm{Eis}}(i^*\tilde{\varphi})$ its Eisenstein projection and $i^*\tilde{\varphi}$ is a Λ-adic Hilbert modular form over K explicitely described by its Fourier expansion.

2) the element $\mathcal{L}_\varphi$ gives the p-adic interpolation of the residue of the normalized Rankin L function $L^*_{1_{\mathrm{Eis}}(i^*\tilde{\varphi}_k),G_k}(s)$ (at $s = k$, the scalar weight of a specialisation $\tilde{\varphi}_k$):

$$\mathcal{N}^k(\mathcal{L}_\varphi) = \mathrm{Res}_{s=k} L^*_{1_{\mathrm{Eis}}(i^*\tilde{\varphi}_k),G_k}(s)$$

In order to explain some details of the proof we let S be a finite set of primes containing p. In the rest of this section we consider properties of the Rankin convolutions of Hilbert modular forms; they correspond to certain automorphic forms on the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field F and have the form of the following Dirichlet series

$$L(s, \mathfrak{f}, \mathfrak{g}) = \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathfrak{f})\, C(\mathfrak{n}, \mathfrak{g})\mathcal{N}(\mathfrak{n})^{-s}, \tag{4.4}$$

where $\mathfrak{f}$, $\mathfrak{g}$ are Hilbert automorphic forms of "holomorphic type" over F , and $C(\mathfrak{n}, \mathfrak{f})$, $C(\mathfrak{n}, \mathfrak{g})$ are their normalized Fourier coefficients (indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F \subset F$). We view $\mathfrak{f}$, $\mathfrak{g}$ as functions on the adelic group $G_{\mathbf{A}} = \mathrm{GL}_2(\mathbf{A}_F)$, where $\mathbf{A}_F$ is the ring of adeles of F and we suppose that $\mathfrak{f}$ is a primitive cusp form of scalar weight $k \geq 2$, conductor $\mathfrak{c}(\mathfrak{f}) \subset \mathcal{O}_F$, and charachter ψ and $\mathfrak{g}$ a primitive cusp form of weight $l < k$, conductor $\mathfrak{c}(\mathfrak{g})$, and character ω (here $\psi, \omega : \mathbf{A}_F^\times \to \mathbf{C}^\times$ are Hecke characters of finite order).

Let ψ^*, ω^* be the characters of the ideal group of F which are associated with ψ ,ω and let

$$L_{\mathfrak{c}}(s, \psi\omega) = \sum_{\mathfrak{n}+\mathfrak{c}=\mathcal{O}_F} \psi^*(\mathfrak{n})\omega^*(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}+\mathfrak{c}=\mathcal{O}_F} (1-\psi^*(\mathfrak{p})\omega^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1} \tag{4.5}$$

be the correspoding Hecke L-function (here $\mathfrak{c} = \mathfrak{c}(\mathfrak{f})\mathfrak{c}(\mathfrak{g})$). We now define the normalized zeta function by setting

$$\Psi(s, \mathfrak{f}, \mathfrak{g}) = \gamma_n(s) L_{\mathfrak{c}}(2s + 2 - k - l, \psi\omega) L(s, \mathfrak{f}, \mathfrak{g})$$

where $n = [F : \mathbf{Q}]$ is the degree of F,

$$\gamma_n(s) = (2\pi)^{-2ns}\Gamma(s)^n\Gamma(s + 1 - l)^n$$

is the gamma-factor. Then the function $\Psi(s,\mathfrak{f},\mathfrak{g})$ admits an analytic continuation onto the entire comlex plane, and it satisfies a certain functional equation [Ja], [Shi78]. For the non-Archimedean construction we consider the S-adic completion

$$\mathcal{O}_S = \prod_{q\in S}(\mathcal{O}_F \otimes \mathbf{Z}_q) = \prod_{\mathfrak{p}|q\in S} \mathcal{O}_{\mathfrak{p}}$$

of the ring $\mathcal{O}_F$.

We set

$$S_F = \{\mathfrak{p} \mid \mathfrak{p} \text{ divides } q \in S\},\ \ \mathfrak{m}_0 = \prod \mathfrak{p}\ (\text{over all}\ \mathfrak{p} \in S_F),$$

and let $\mathrm{Gal}_S = \mathrm{Gal}(F(S)/F)$ denote the Galois group of the maximal abelian extension of F unramified outside S and ∞.

The domain of definition of the non-Archimedean L-functions is the p-adic analytic Lie group

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}_S, \mathbf{C}_p^\times)$$

of all continuouos p-adic characters of the Galois group Gal_S ($\mathbf{C}_p$ is the Tate field). Elements of finite order $\chi \in \mathcal{X}_S$ can be identified with those Hecke characters of finite order whose conductors contain only prime divisors in S_F; this identification uses the map

$$\chi : \mathbf{A}_F^\times \overset{\mathrm{CFT}}{\longrightarrow} \mathrm{Gal}_S \to \overline{\mathbf{Q}}^\times \overset{i_p}{\to} \mathbf{C}_p^\times,$$

where CTF is the homomorphism of class field theory. Recall that the essential property of the convolution

$$L(s,\mathfrak{f},\mathfrak{g}(\chi)) = \sum_{\mathfrak{n}} \chi^*(\mathfrak{n}) C(\mathfrak{n},\mathfrak{f})\, C(\mathfrak{n},\mathfrak{g}) \mathcal{N}(\mathfrak{n})^{-s}$$

is the following Euler product decomposition

$$\begin{aligned} &L_{\mathfrak{c}}(2s+2-k-l,\psi\omega\chi^2)L(s,\mathfrak{f},\mathfrak{g}(\chi)) = \\ &\prod_{\mathfrak{q}}(1-\chi^*(\mathfrak{q})\alpha(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi^*(\mathfrak{q})\alpha(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})\times \\ &\times (1-\chi^*(\mathfrak{q})\alpha'(\mathfrak{q})\beta(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s})(1-\chi^*(\mathfrak{q})\alpha'(\mathfrak{q})\beta'(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{-s}), \end{aligned} \tag{4.6}$$

where the numbers $\alpha(\mathfrak{q})$, $\alpha'(\mathfrak{q})$, $\beta(\mathfrak{q})$, and $\beta'(\mathfrak{q})$ are roots of the Hecke polynomials

$$X^2 - C(\mathfrak{q},\mathfrak{f})X + \psi^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{k-1} = (X-\alpha(\mathfrak{q}))(X-\alpha'(\mathfrak{q})),$$

and

$$X^2 - C(\mathfrak{q},\mathfrak{g})X + \omega^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{l-1} = (X-\beta(\mathfrak{q}))(X-\beta'(\mathfrak{q})).$$

The decomposition (4.6) is not difficult to deduce from the following elementary lemma on rational functions, applied to each of the Euler $\mathfrak{q}$-factors: if

$$\sum_{i=0}^{\infty} A_i X^i = \frac{1}{(1-\alpha X)(1-\alpha' X)},\quad \sum_{i=0}^{\infty} B_i X^i = \frac{1}{(1-\beta X)(1-\beta' X)},$$

then

$$\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1-\alpha\alpha'\beta\beta' X^2}{(1-\alpha\beta X)(1-\alpha\beta' X)(1-\alpha'\beta' X)(1-\alpha'\beta' X)}. \tag{4.7}$$

5.1. Hilbert modular forms. Hilbert-Eisenstein series. — Let the symbols

$$\mathcal{O}_F,\ I,\ F_{\mathbf{A}},\ F_{\mathbf{A}}^{\times},\ \mathfrak{d}\subset\mathcal{O}_F,\ D_F=\mathcal{N}(\mathfrak{d})$$

denote, respectively, the maximal order, the group of fractional ideals, the ring of adeles, the group of ideles, the different and the discriminant of a totally real field F of degree n over $\mathbf{Q}$. Let $\Sigma=\Sigma_\infty\cup\Sigma_0$ denote the set of places (i.e. eqivalence classes of valuations) of F where $\Sigma_\infty=\{\infty_1,\cdots,\infty_n\}$ are the Archimedean places, $\Sigma_0=\{\mathfrak{p}=\mathfrak{p}_v\subset\mathcal{O}_F\}$ finite (non-Archimedean) places. The Archimedean places are induced by the real embeddings of $F: x\mapsto x^{(v)}\in\mathbf{R}\ \ (v=1,\cdots,n)$. An element $x\in F^\times$ is called totally positive ($x\gg 0$) if one has $x^{(v)}>0$ for all v and let $F_+^\times$ denote the multiplicative group of all totally positive elements of F. We put also $F_\infty=F\otimes_{\mathbf{Q}}\mathbf{R}\cong\mathbf{R}^n\subset F_{\mathbf{A}}$, and let $\hat F\cong\hat{\mathcal{O}}_F\otimes_{\mathbf{Z}}\mathbf{Q}\subset F_{\mathbf{A}}$ be the subring of finite adeles where $\hat{\mathcal{O}}_F$ is the profinite completion of the ring $\mathcal{O}_F$ (with respect to all its ideals). Then $F_{\mathbf{A}}=F_\infty\oplus\hat F$, and for an adele $x=(x_v)_{v\in\Sigma}$ we write $x=x_\infty+x_0$ where $x_\infty\in F_\infty,\ x_0\in\hat F$. On the other hand there is the decomposition $F_{\mathbf{A}}^\times=F_\infty^\times\times\hat F^\times$ and we shall allow ourselves convenient abuse of notation by writing $y=y_\infty\cdot y_0$ with $y_\infty\in F_\infty^\times,\ y_0\in\hat F^\times$. For the idele $y\in F_{\mathbf{A}}^\times$ let the symbol $\tilde y\in I$ denote the fractional ideal associated with y (so that $\tilde y\widehat{\mathcal{O}}_F=y_0\widehat{\mathcal{O}}_F$).

We view the group $\mathrm{GL}_2(F)$ as the group $G_{\mathbf{Q}}$ of all $\mathbf{Q}$-rational points of a certain $\mathbf{Q}$ -subgroup $G\subset\mathrm{GL}_{2n}$. Then the adelization $G_{\mathbf{A}}=G(\mathbf{A})$ can be identified with the product

$$\mathrm{GL}_2(F_{\mathbf{A}})\cong G_\infty\times G_{\widehat{\mathbf{Q}}},$$

where

$$G_\infty=\mathrm{GL}_2(F_\infty)\cong\mathrm{GL}_2(\mathbf{R})^n,\ \ G_{\widehat{\mathbf{Q}}}=\mathrm{GL}_2(\widehat F).$$

The subgroup

$$G_\infty^+\cong\mathrm{GL}_2^+(\mathbf{R})^n\subset G_\infty$$

consists of all elements

$$\alpha=(\alpha_1,\cdots,\alpha_n),\quad \alpha_v=\begin{pmatrix}\alpha_v & \beta_v\\ \gamma_v & \delta_v\end{pmatrix},$$

such that $\det\alpha_v>0$, $v=1,\cdots,n$. Every element $\alpha\in G_\infty^+$ acts on the product $\mathfrak{H}^n$ of the n copies of the upper half planes according to the formula

$$\alpha(z_1,\cdots,z_n)=(\alpha_1(z_1),\cdots,\alpha_n(z_n)),$$

where

$$\alpha_v(z_v)\ =\ (a_vz_v+b_v)/(c_vz_v+d_v).$$

For $z=(z_1,\cdots,z_n)$ we put $\{z\}=z_1+\cdots+z_n$ and $e_F(z)=e(\{z\})$, with $e(x)=\exp(2\pi ix)$. Let $\mathbf{i}=(i,\cdots,i)\in\mathfrak{H}^n$, then

$$(\{\alpha\in G_\infty^+\mid\alpha(\mathbf{i})=\mathbf{i}\})/\mathbf{R}_+^\times$$

is a maximal compact subgroup in $G_\infty^+/\mathbf{R}_+^\times$. For $\alpha \in G_\infty^+$, an integer k and an arbitrary function $f : \mathfrak{H}^n \to \mathbf{C}$ we use the notation

$$(f|_k\alpha)(z) = \mathcal{N}(cz+d)^{-k} f(\alpha(z))\mathcal{N}\det(\alpha)^{k/2},$$

with $\mathcal{N}(z)^k = z_1^k \cdots z_n^k$. Let $\mathfrak{c} \subset \mathcal{O}_F$ be an integral ideal, $\mathfrak{c}_\mathfrak{p} = \mathfrak{c}\mathcal{O}_\mathfrak{p}$ its $\mathfrak{p}$-part,$\mathfrak{d}_\mathfrak{p} = \mathfrak{d}\mathcal{O}_\mathfrak{p}$ the local different. We shall need the open subgroups $W = W_\mathfrak{c} \subset G_\mathbf{A}$ defined by

$$\begin{aligned} &W = G_\infty^+ \times \prod_\mathfrak{p} W(\mathfrak{p}),\\ &W(\mathfrak{p}) = \\ &\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p}) | b \in \mathfrak{d}_\mathfrak{p}^{-1}, c \in \mathfrak{d}_\mathfrak{p}\mathfrak{c}_\mathfrak{p},\ a, d \in \mathcal{O}_\mathfrak{p}, ad-bc \in \mathcal{O}_\mathfrak{p}^\times \right\}. \end{aligned} \tag{5.1}$$

Let $h = |\widetilde{Cl_F}|$ be the number of ideal classes of F (in the narrow sense),

$$\widetilde{Cl_F} = I/\{(x)|x \in F_+^\times\},$$

and let us choose the ideles $t_1, \cdots, t_h$ so that $\tilde{t}_\lambda \subset \mathcal{O}_F$ form a complete system of representatives for $\widetilde{Cl_F}$, $(t_\lambda)_\infty = 1$ and $\tilde{t}_\lambda + \mathfrak{m}_0 = \mathcal{O}_F$ ($\lambda = 1, \cdots, h$, $\mathfrak{m}_0 = \prod_{\mathfrak{q}\in S_F} \mathfrak{q}$). If we put $x_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & t_\lambda \end{pmatrix}$ then there is the following decomposition into a disjoint union ("the approximation theorem"):

$$G_\mathbf{A} = \cup_\lambda G_\mathbf{Q} x_\lambda W = \cup_\lambda G_\mathbf{Q} x_\lambda^{-\iota} W, \tag{5.2}$$

where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, ι denotes the involution given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(see [Shi78], p.647).

5.2. — Definition of Hilbert automorphic forms of weight k and level $\mathfrak{c} \subset \mathcal{O}_F$ with a Hecke character ψ of finite order. By a *Hilbert automorphic form* of weight k, level $\mathfrak{c} \subset \mathcal{O}_F$, and Hecke character ψ we mean a function $\mathfrak{f} : G_\mathbf{A} \to \mathbf{C}$ satisfying the following conditions (5.3) - (5.5):

$$\begin{aligned} &\mathfrak{f}(s\alpha x) = \psi(s)\mathfrak{f}(x) \text{ for all } x \in G_\mathbf{A} \\ &\text{for } s \in F_\mathbf{A}^\times \text{ (the center of } G_\mathbf{A}), \text{ and } \alpha \in G_\mathbf{Q}. \end{aligned} \tag{5.3}$$

If we let $\psi_0 : (\mathcal{O}_F/\mathfrak{c})^\times \to \mathbf{C}^\times$ denote the $\mathfrak{c}$-part of the character ψ, and then extend the definition of ψ over W by the formula

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \psi_0(a_\mathfrak{c} \bmod \mathfrak{c}),$$

($a_\mathfrak{c}$ being the $\mathfrak{c}$-part of a) then for all $x \in G_\mathbf{A}$

$$\mathfrak{f}(x\, w) = \psi(w^\iota)\mathfrak{f}(x) \text{ for } w \in W_\mathfrak{c} \text{ with } w_\infty = 1. \tag{5.4}$$

If $w = w(\theta) = (w_1(\theta_1), \cdots, w_n(\theta_n))$ where

$$w_\nu(\theta_\nu) = \begin{pmatrix} \cos\theta_\nu & -\sin\theta_\nu \\ \sin\theta_\nu & \cos\theta_\nu \end{pmatrix},$$

then

$$\mathfrak{f}(x\, w(\theta)) = \mathfrak{f}(x) e^{-ik\{\theta\}} \quad (x \in G_{\mathbf{A}}). \tag{5.5}$$

An automorphic form $\mathfrak{f}$ is called a *cusp form* if

$$\int_{F_{\mathbf{A}}/F} \mathfrak{f}\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) dt = 0 \text{ for all } g \in G_{\mathbf{A}}. \tag{5.6}$$

The vector space $\mathcal{M}_k(\mathfrak{c}, \psi)$ of Hilbert automorphic forms of *holomorphic type* is defined as the set of functions satisfying (5.3) – (5.5) and the following *holomorphy condition* (5.7): for any $x \in G_{\mathbf{A}}$ with $x_\infty = 1$ there exists a holmorphic function $g_x : \mathfrak{H}^n \to \mathbf{C}$, such that for all $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$ we have

$$\mathfrak{f}(x\, y) \;=\; (g_x|_k y)(\mathbf{i}) \tag{5.7}$$

(in the case $F = \mathbf{Q}$ we must also require that the functions g_x be holomorphic at the cusps). The property (5.7) enables one to describe the automorphic forms $\mathfrak{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ more explicitly in terms of Hilbert modular forms on $\mathfrak{H}^n$. For this purpose we put $f_\lambda = g_{x_\lambda^{-\iota}}$ where $x_\lambda^{-\iota} = \begin{pmatrix} t_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$, then $f_\lambda(z) \in \mathcal{M}_k(\Gamma_\lambda, \psi_0)$ for the congruence subgroup

$$\Gamma_\lambda = \Gamma_\lambda(\mathfrak{c}) \subset G_{\mathbf{Q}}^+,$$

$$\Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap G_{\mathbf{Q}} =$$

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}}^+ | b \in \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1},\ c \in \tilde{t}_\lambda \mathfrak{d}\mathfrak{c}, a, d \in \mathcal{O}_F,\ ad - bc \in \mathcal{O}_F^\times\right\}.$$

This means that for all $\gamma \in \Gamma_\lambda(\mathfrak{c})$ the following condition (5.8) is satisfied:

$$f_\lambda|_k\gamma = \psi(\gamma) f_\lambda \quad\text{and}\quad f_\lambda(z) = \sum_\xi a_\lambda(\xi) e_F(\xi z), \tag{5.8}$$

where $0 \ll \xi \in \tilde{t}_\lambda$ or $\xi = 0$ in the sum over ξ (see [Shi78] for a more detailed discussion of Fourier expansions). The map $\mathfrak{f} \mapsto (f_1, \cdots, f_h)$ defines a vector space isomorphism

$$\mathcal{M}(\mathfrak{c}, \psi) \cong \oplus_\lambda \mathcal{M}_k(\Gamma_\lambda, \psi)$$

Put

$$C(\mathfrak{m}, \mathfrak{f}) = \begin{cases} a_\lambda(\xi)\mathcal{N}(\tilde{t}_\lambda)^{-k/2}, & \text{if the ideal } \mathfrak{m} = \xi\tilde{t}_\lambda^{-1} \text{ is integral;} \\ 0, & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases} \tag{5.9}$$

We have the following *Fourier expansion*:

$$\mathfrak{f}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{0 \ll \zeta \in F, \zeta = 0} C(\zeta\tilde{y}, \mathfrak{f})|y|^{k/2} e_F(\zeta \mathbf{i} y_\infty)\chi(\zeta x), \tag{5.10}$$

where $\chi_F : F_{\mathbf{A}}/F \to \mathbf{C}^\times$ is a fixed additive character with the condition $\chi_F(x_\infty) = e_F(x_\infty)$ (see [Shi78], p. 650).

Let $\mathcal{S}_k(\mathfrak{c}, \psi) \subset \mathcal{M}(\mathfrak{c}, \psi)$ be the subspace of cusp forms and $\mathfrak{f} \in \mathcal{S}_k(\mathfrak{c}, \psi)$ then $a_\lambda(0) = 0$ for all $\lambda = 1, \cdots, h$.

5.3. Hecke operators (see [Shi78]). — They are introduced by means of double cosets of the type WyW for y in the semigroup

$$Y_{\mathfrak{c}} = G_{\mathbf{A}} \cap (G_{\infty}^{+} \times \prod Y_{\mathfrak{c}}(\mathfrak{p})),$$

where

$$Y_{\mathfrak{c}}(\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) \mid a\mathcal{O}_{\mathfrak{p}} + \mathfrak{c}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}},\ b \in \mathfrak{d}_{\mathfrak{p}}^{-1}, c \in \mathfrak{c}_{\mathfrak{p}}\mathfrak{d}_{\mathfrak{p}}, d \in \mathcal{O}_{\mathfrak{p}} \right\}. \quad (5.11)$$

The Hecke algebra $\mathcal{H}_{\mathfrak{c}}$ consists of all formal finite sums of the type $\sum_y c_y WyW$ with $y \in Y_{\mathfrak{c}}, c_y \in \mathbf{C}$ and with the standard multiplication law defined by means of decomposition of double cosets into a disjoint union of a finite number of left cosets. By definition, $T_{\mathfrak{c}}(\mathfrak{m})$ is an element of the ring $\mathcal{H}_{\mathfrak{c}}$ obtained by taking the sum of all different WyW with $y \in Y_{\mathfrak{c}}$ such that $\det(y) = \mathfrak{m}$. Let

$$T'_{\mathfrak{c}} = \mathcal{N}(\mathfrak{m})^{(k-2)/2} T_{\mathfrak{c}}(\mathfrak{m}) \quad (5.12)$$

be the normalized Hecke operator, whose action on the Fourier coefficients of an automorphic form (of the holomorphic type) $\mathfrak{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ is given by the usual formula

$$C(\mathfrak{m}, \mathfrak{f} | T'_{\mathfrak{c}}(\mathfrak{m})) = \sum_{\mathfrak{m}+\mathfrak{n}=\mathfrak{a}} \psi^*(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{k-1} C(\mathfrak{a}^{-2}\mathfrak{m}\mathfrak{n}, \mathfrak{f}) \quad (5.13)$$

If $\mathfrak{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ is an eigenfunction of all Hecke operators $T'_{\mathfrak{c}}(\mathfrak{m})$ with $\mathfrak{f}|_k T'_{\mathfrak{c}}(\mathfrak{m}) = \lambda(\mathfrak{m})\mathfrak{f}$ then we have that $C(\mathfrak{m}, \mathfrak{f}) = \lambda(\mathfrak{m}) C(\mathcal{O}_F, \mathfrak{f})$. If we normalize the form $\mathfrak{f}$ by the condition $C(\mathcal{O}_F, \mathfrak{f}) = 1$ then the L-function has the following Euler product expansion:

$$\begin{aligned} L(s, \mathfrak{f}) &= \sum_{\mathfrak{n}} C(\mathfrak{n}, \mathfrak{f}) \mathcal{N}(\mathfrak{n})^{-s} = \sum_{\mathfrak{n}} \lambda(\mathfrak{n}) \mathcal{N}(\mathfrak{n})^{-s} = \\ &\prod_{\mathfrak{p}} [1 - C(\mathfrak{p}, \mathfrak{f}) \mathcal{N}(\mathfrak{p})^{-s} + \psi^*(\mathfrak{p}) \mathcal{N}(\mathfrak{p})^{k-1-2s}]^{-1}. \end{aligned} \quad (5.14)$$

In this case the coefficients $C(\mathfrak{n}, \mathfrak{f})$ of the form $\mathfrak{f}$ are algebraic integers.

5.4. — The Petersson inner product is defined for $\mathfrak{f} = (f_1, \cdots, f_h) \in \mathcal{S}_k(\mathfrak{c}, \psi)$ and $\mathfrak{g} = (g_1, \cdots, g_h) \in \mathcal{M}_k(\mathfrak{c}, \psi)$ by setting

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathfrak{c}} = \sum_{\lambda=1}^{h} \int_{\Gamma_\lambda(\mathfrak{c}) \backslash \mathfrak{H}^n} \overline{f_\lambda(z)} g_\lambda(z) \mathcal{N}(y)^k \, d\mu(z), \quad (5.15)$$

where

$$d\mu(z) = \prod_{\nu=1}^{n} y_\nu^{-2} \, dx_\nu \, dy_\nu$$

is a G_∞^+ -invariant measure on $\mathfrak{H}^n$.

If $\mathfrak{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$ and $\mathfrak{f}|_k T'_{\mathfrak{c}}(\mathfrak{m}) = \lambda(\mathfrak{m})\mathfrak{f}$ for all $\mathfrak{m}$ with $\mathfrak{m} + \mathfrak{c} = \mathbf{Q}_F$ then

$$\begin{aligned} &\lambda(\mathfrak{m}) = \psi^*(\mathfrak{m}) \overline{\lambda(\mathfrak{m})}, \\ &\psi^*(\mathfrak{m}) \langle \mathfrak{f}|_k T'_\lambda(\mathfrak{m}), \mathfrak{g} \rangle_{\mathfrak{c}} = \langle \mathfrak{f}, \mathfrak{g}|_k T'_\lambda(\mathfrak{m}) \rangle_{\mathfrak{c}} \end{aligned} \quad (5.16)$$

(ψ-hermitian property of the Hecke operators). Let $\mathfrak{q}$ be an integral ideal and $\mathfrak{f} \in \mathcal{M}_k(\mathfrak{c}, \psi)$. Let us define the operators $\mathfrak{f}|\mathfrak{q}$, $\mathfrak{f}|U(\mathfrak{q})$ by their action on Fourier coefficients:

$$C(\mathfrak{m}, \mathfrak{f}|\mathfrak{q}) = C(\mathfrak{q}^{-1}\mathfrak{m}, \mathfrak{f}), \;\; C(\mathfrak{m}, \mathfrak{f}|U(\mathfrak{q})) = C(\mathfrak{q}\mathfrak{m}, \mathfrak{f}). \tag{5.17}$$

Here is the explicit description of these operators: for a finite idele $q \in F_{\mathbb{A}}^{\times}$ with $\tilde{q} = q$

$$(\mathfrak{f}|\mathfrak{q})(x) \;\; = \;\; \mathcal{N}(\mathfrak{q})^{-k/2}\mathfrak{f}(x \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}), \tag{5.18}$$

$$(\mathfrak{f}|U(\mathfrak{q}))(x) = \mathcal{N}(\mathfrak{q})^{k/2-1} \sum_{v \in \mathcal{O}_F/\mathfrak{q}} \mathfrak{f}(x \begin{pmatrix} 1 & v \\ 0 & q \end{pmatrix}). \tag{5.19}$$

We recall now the definition of *Eisenstein series* in the Hilbert modular case. Let $\mathfrak{a}, \mathfrak{b}$ be arbitrary fractional ideals, m a positive integer, $q \in (q_1, \cdots, q_n) \in \mathbf{Z}^n$, $q_v \geq 0$, η a Hecke character of finite order modulo an integral ideal $\mathfrak{e} \subset \mathcal{O}_F$ such that $\eta^*((x)) = \mathrm{sign}\mathcal{N}(x)^m$ for $x \equiv 1 \bmod {}^{\times}\mathfrak{e}$, $x \in \mathcal{O}_F$. We put (for Re $(s) > 2 - m$)

$$\begin{aligned} K_m^q(z, s; \mathfrak{a}, \mathfrak{b}; \eta) &= (2\pi i)^{-\{q\}}(z - \overline{z})^{-q} \times \\ \times \sum_{c,d} \mathrm{sign}\mathcal{N}(d)^m\, \eta^*(d\mathfrak{b}^{-1}) &\left(\frac{c\overline{z} + d}{cz + d}\right)^q \mathcal{N}(cz + d)^{-m}|\mathcal{N}(cz + d)|^{-2s}, \end{aligned} \tag{5.20}$$

$$\begin{aligned} L_m^q(z, s; \mathfrak{a}, \mathfrak{b}; \eta) &= (2\pi i)^{-\{q\}}(z - \overline{z})^{-q} \times \\ \times \sum_{c,d} \mathrm{sign}\mathcal{N}(c)^m\, \eta^*(c\mathfrak{a}^{-1}) &\left(\frac{c\overline{z} + d}{cz + d}\right)^q \mathcal{N}(cz + d)^{-m}|\mathcal{N}(cz + d)|^{-2s}, \end{aligned} \tag{5.21}$$

where $z^q = \prod_v z_v^{q_v}$, $\mathcal{N}(z) = z_1 \cdots z_n$, the summation in (5.20) and (5.21) is taken over a system of representatives (c, d) of $\mathcal{O}_F^{\times}$-equivalence classes of non-zero elements in $\mathfrak{a} \times \mathfrak{b}$ ($(c, d) \sim (uc, ud)$ for $u \in \mathcal{O}_F^{\times}$).

Gauss sums and the twist operator. Let χ be a Hecke character of finite order with a conductor $\mathfrak{m}$ and $\chi(x_\infty) = \mathrm{sign}(x_\infty)^r$ for $r = (r_1, \cdots, r_n) \in (\mathbf{Z}/2\mathbf{Z})^n$ (the parity of χ). Let χ^* be the character of the group of fractional ideals prime to $\mathfrak{m}$ which is associated to χ. Let us set $\chi^*(\mathfrak{a}) = 0$ for those $\mathfrak{a}$ which are not coprime to $\mathfrak{m}$ and define the Gauss sum by

$$\tau(\chi) = \sum_{x \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}} \mathrm{sign}(x)^r \chi^*((x)\mathfrak{m}\mathfrak{d}) e_F(x)$$

Then $|\tau(\chi)|^2 = \mathcal{N}(\mathfrak{m})$. The series (5.20) and (5.21) can be extended to functions on the adelic group $G_{\mathbb{A}}$ in such a way that

$$K_m^q(s; \mathfrak{a}, \mathfrak{b}; \eta)_\lambda(z) = \mathcal{N}(\tilde{t}_\lambda)^{s+(m/2)}\mathcal{N}(y)^s K_m^q(z, s; \tilde{t}_\lambda\mathfrak{d}\mathfrak{a}, \mathfrak{b}; \eta), \tag{5.22}$$

$$L_m^q(s; \mathfrak{a}, \mathfrak{b}; \eta)_\lambda(z) = \mathcal{N}(\tilde{t}_\lambda)^{-s-(m/2)}\mathcal{N}(y)^s L_m^q(z, s; \mathfrak{a}, \mathfrak{b}\tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}; \eta). \tag{5.23}$$

The functions (5.20)–(5.23) admit analytic continuation onto the entire complex plane with respect to the parameter $s \in \mathbf{C}$, and they satisfy the following functional equation (see [Shi78], p.672, where it is given in a slightly different notation) under the assumption that η is *primitive* modulo $\mathfrak{e}$:

$$\Delta_m^q(z, 1-m-s)K_m^q(1-m-s; \mathfrak{a}, \mathfrak{b}; \eta) = \\ \tau(\eta)\mathcal{N}(\mathfrak{d}\mathfrak{a}\mathfrak{b}\mathfrak{e})^{m+2s-1}\Delta_m(s)^q L_m^q(s; \mathfrak{a}, \mathfrak{b}\mathfrak{e}; \overline{\eta}), \tag{5.24}$$

with the Γ-factor $\Delta_m^q(z,s) = \pi^{-ns}y^{(m+s)}\prod_v \Gamma(s+m+q_v)$.

We need Fourier expansions of the Eisenstein series which can be explicitly written in terms of the Whittaker function $W(y,\alpha,\beta)$. This function is defined by the integral

$$W(y,\alpha,\beta) = \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-yu}du$$

which is absolutely convergent for $\mathrm{Re}\,(\alpha+\beta) > 1$, and

$$W(y,\alpha,-r) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}y^{r-i} \text{ for } r \in \mathbf{Z}, r \geq 0$$

5.5. Proposition. *(On Fourier expansions of the Eisenstein series). Let s be an integer such that $s - q_v \leq 0$ for all v, $\eta(\mathrm{mod}\,\mathfrak{e})$ be a Hecke character of finite order as above (not necessarily primitive) such that $\mathfrak{e} \neq \mathcal{O}$. Then under the above notation there is the following Fourier expansion:*

$$\frac{D_F^{1/2}\mathcal{N}(\tilde{t}_\lambda)\prod_v \Gamma(s+m+q_\lambda)}{(-2\pi i)^{n(m+2s)}(-1)^{ns+\{q\}}}L_m^q(z, 0; \mathcal{O}_F, \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}; \eta) = \\ (4\pi y)^{-q}\sum_{0 \ll \xi \in \tilde{t}_\lambda} a_\lambda(\xi, s, y, \eta)e_F(\xi z),$$

where

$$a_\lambda(\xi, s, y, \eta) = \\ \sum_{\substack{\tilde{\xi}=\tilde{b}\tilde{c} \\ c \in \mathcal{O}_F, b \in \tilde{t}_\lambda}} \mathrm{sign}\mathcal{N}(\tilde{b})^{m-1}\mathcal{N}(\tilde{b})^{m+2s-1}\eta^*(\tilde{c})\prod_v W(4\pi\xi_v y_v, m+s+q_v, s-q_v),$$

and $W(y,\alpha,\beta)$ is the Whittaker function.

Proof. Using Fourier transform, one has

$$D_F^{1/2}\mathcal{N}(\tilde{t}_\lambda)\sum_{t \in \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}}\left(\frac{\overline{z}+t}{z+t}\right)^q \mathcal{N}(z+t)^{-m}|\mathcal{N}(z+t)|^{-2s} = \\ \sum_{b \in \tilde{t}_\lambda}\prod_v \int_{\mathbf{R}} \frac{\exp(-2\pi i b_v t_v)\, dt_v}{(z_v+t_v)^{-(m+2q_v)}(\overline{z_v}+t_v)^{-2(s-q_v)}},$$

where

$$\int_{\mathbf{R}} \frac{\exp(-2\pi i b_\nu t_\nu)\, dt_\nu}{(z_\nu + t_\nu)^{-(m+2q_\nu)}(\overline{z_\nu} + t_\nu)^{-2(s-q_\nu)}} =$$

$$(-2\pi i)^{m+2s}(-1)^{s+q_\nu} b_\nu^{m+2s-1}\Gamma(s+m+q_\nu)^{-1} W(4\pi b_\nu y_\nu, m+s+q_\nu, s-q_\nu) e_F(bz).$$

Indeed, if we consider the integral

$$f(t) = \int_{-\infty}^{\infty} z^{-\alpha}\overline{z}^{-\beta} \exp(-2\pi i t z)\, dz,$$

which is absolutely convergent for Re $(\alpha+\beta) > 1$, then application of contour integration shows that

$$f(t) = (2\pi)^{\alpha+\beta} i^{\beta-\alpha}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} e^{2\pi t y} \times$$

$$\times \begin{cases} t^{\alpha+\beta-1} e^{-4\pi t y} W(4\pi t y, \alpha, \beta) & \text{if } t > 0 \\ |t|^{\alpha+\beta-1} W(4\pi |t| y, \beta, \alpha) & \text{if } t < 0 \end{cases}$$

(see [Shi75], pp.84 –85). Therefore

$$D_F^{1/2}\mathcal{N}(\tilde{t}_\lambda) \sum_{t \in \tilde{t}_\lambda^{-1}\mathfrak{d}^{-1}} \left(\frac{\overline{z}+t}{z+t}\right)^q \mathcal{N}(z+t)^{-m} |\mathcal{N}(z+t)|^{-2s} =$$

$$(-2\pi i)^{n(m+2s)}(-1)^{ns+\{q\}} \prod_\nu \Gamma(s+m+q_\nu)^{-1} \times$$

$$\times \sum_{b \in \tilde{t}_\lambda} \mathcal{N}(b)^{m+2s-1} \prod_\nu W(4\pi b_\nu y_\nu, m+s+q_\nu, s-q_\nu) e_F(bz),$$

and the proposition is then easily deduced from the last equality (see the analogous calculation in [Ka78]).

Remark. We need only a special case of 5.2, when $q = 0$, $m = k - l$ since the weights k and l are scalars; however, the proposition is applicable for the study of Rankin convolutions of Hilbert modular forms $\mathfrak{f}$ and $\mathfrak{g}$ of arbitrary integer vector weights $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n)$ satisfying the following condition

$$k_1 \equiv k_2 \equiv \cdots \equiv k_n \bmod 2, \quad l_1 \equiv l_2 \equiv \cdots \equiv l_d \bmod 2.$$

5.6. The integral representation. We set

$$G_1 \in \mathcal{S}_k(\mathfrak{m}_1, \psi), \quad G_2 \in \mathcal{S}_l(\mathfrak{m}_2, \omega).$$

Then the following *integral representation of Rankin type* holds (see [Shi78], (4.32)):

$$\Psi(s, G_1, G_2) = D_F^{1/2}\Gamma(s+1-l)^n \pi^{-ns} \langle G_1^\rho, V(s-k+1, \psi\omega)\rangle_{\mathfrak{m}_1\mathfrak{m}_2}, \tag{5.25}$$

where

$$V(s,\psi\omega) = G_2 \cdot K^0_{k-l}(s; \mathfrak{m}_1\mathfrak{m}_2, \mathcal{O}_F; \psi\omega).$$

More precisely,

$$\Psi(s, G_1, G_2) = D_F^{1/2}\Gamma(s+1-l)^n \pi^{-ns} \sum_{\lambda=1}^{h} \mathcal{N}(\tilde{t}_\lambda)^{s+1-(k+l)/2} \times$$
$$\times \int_{\Gamma_\lambda(\mathfrak{m}_1\mathfrak{m}_2)\backslash\mathfrak{H}^n} \overline{G^\rho_{1,\lambda}(z)}\, G_{2,\lambda}(z)\, K^0_{k-l}(z, s-k+1; \tilde{t}_\lambda\mathfrak{d}, \mathcal{O}_F; \psi\omega)\mathcal{N}(y)^{s-1}\, dx\, dy. \quad (5.26)$$

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