AN ANDREOTTI-VESENTINI SEPARATION THEOREM ON REAL HYPERSURFACES

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Abstract

Using Serre duality in CR manifolds and integral operators for the solution of the tangential Cauchy-Riemann equation with compact support, we prove a separation theorem of Andreotti-Vesentini type for the $\overline{\partial}_b$-cohomology in q-concave real hypersurfaces.

Let X be an n-dimensional complex manifold and E a holomorphic vector bundle over X. If X is q-concave in the sense of Andreotti-Grauert (*i.e.* $(n-q)$-concave in the sense of [6]), the Andreotti-Grauert finiteness theorem [1] says that

$$\dim H^{0,r}(X,E) < \infty \qquad \text{if} \quad \mathrm{r} \leqslant \mathrm{n-q-1}.$$

Moreover Andreotti and Vesentini [2] proved that the cohomology group $H^{0,n-q}(X,E)$ is separated. Other proofs of this separability are given in [6],[9] and [12].

Let us consider now the case of CR manifolds. Let M be a CR generic manifold embedded in an n-dimensional complex manifold X and E a holomorphic vector bundle over X. Assume that the Levi form of M restricted to the complex tangent space has at least q negative eigenvalues in all directions at each point of M, this condition ensures the local solvability of the $\overline{\partial}_b$-equation up to bidegree $(0,q)$ (cf. [4]).

If M is compact, Henkin [4] obtained the finitness of the $\overline{\partial}_b$-cohomology groups of bidegree $(0,r)$ for $r \leqslant q-1$ and the separability of the group $H^{0,q}(X,E)$. This result has been generalized by Hill and Nacinovich [7] to the case of compact abstract CR manifolds.

If M is no more compact but admits an exhausting function, whose Levi form restricted to the complex tangent space to M has at least q negative eigenvalues at each point outside a compact subset of M then (cf. [7], and [8] for the hypersurface case)

$$\dim H^{0,r}(M,E) < \infty \qquad \text{if} \quad \mathrm{r} \leqslant \mathrm{q-2}.$$

The purpose of the present paper is to prove that the $\overline{\partial}_b$-cohomology group $H^{0,q-1}(M,E)$ is separated in this situation, when M is of real codimension 1. The proof follows the ideas of [9], where we replace the integral operators of the Grauert-Henkin-Lieb type by the operators constructed

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in [8] to get the finitness of some $\overline{\partial}_b$-cohomology group with compact support. Then the separation theorem follows from a careful study of Serre duality in the CR setting, which involves some propagation property of the exactness in bidegree $(0, q-1)$.

To get the same result in higher codimension, we would need some analogue of the operators of [8] and some theorems on the propagation of the exactness in bidegree $(0, q-1)$, which do not exist at this moment.

Note that the possibility to prove a separation theorem of this type by means of integral formulas was pointed out for the first time by Henkin in his survey article [5].

1. Notations and definitions

Let X be an n-dimensional complex manifold. For $\xi \in X$, we denote by $T^{1,0}_\xi(X)$ the holomorphic tangent space of X at ξ, *i.e.* if $z_1, \dots, z_n$ are holomorphic coordinates in a neighborhood of ξ, then $T^{1,0}_\xi(X)$ consists of all tangent vectors t of the form

$$t = \sum_{j=1}^{n} t_j \frac{\partial}{\partial z_j}(\xi) \tag{1.1}$$

where $t_1, \dots, t_n$ are complex numbers.

If M is a real $\mathscr{C}^2$-submanifold of X and $\xi \in M$, then we denote by $T^{1,0}_\xi(M)$ the subspace of all vectors in $T^{1,0}_\xi(X)$ which are tangential to M.

Let $\xi \in X$ and ρ a real $\mathscr{C}^2$-function defined in a neighborhood of ξ. Then we denote by $L^X_\xi(\rho)$ the Levi form of ρ at ξ, *i.e.* the hermitian form on $T^{1,0}_\xi(X)$ defined by

$$L^X_\xi(\rho)t = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\xi)}{\partial z_j \partial \overline{z}_k} t_j \overline{t}_k$$

if $t \in T^{1,0}_\xi(X)$ is written in the form (1.1). If M is a real $\mathscr{C}^2$-submanifold of X and $\xi \in M$, then we denote by $L^M_\xi(\rho)$ the restriction of $L^X_\xi(\rho)$ to $T^{1,0}_\xi(M)$.

DEFINITION 1.1. — *Let M be a real $\mathscr{C}^2$-hypersurface in some n-dimensional complex manifold X, and let q be an integer with $0 \leqslant q \leqslant (n-1)/2$. M will be called q-*convex-concave *at a point $\xi \in M$ if, for each real $\mathscr{C}^2$-function ρ in some X-neighborhood U of ξ with $M \cap U = \{\rho = 0\}$ and $d\rho(\xi) \neq 0$, $L^M_\xi(\rho)$ has at least q positive and q negative eigenvalues. M will be called q-*convex-concave *if it is q-convex-concave at each point in M.*

DEFINITION 1.2. — *Let M be a real $\mathscr{C}^2$-hypersurface in an n-dimensional complex manifold X, and let q be an integer with $0 \leqslant q \leqslant (n-1)/2$. A real $\mathscr{C}^2$-function φ on M will be called* $(q+1)$-convex *at a point $\xi \in M$ if there exist an X-neighborhood U of ξ and real $\mathscr{C}^2$-functions $\tilde{\varphi}, \rho_+, \rho_-$ on U with the following properties:*

- $\tilde{\varphi} = \varphi$ *on* $U \cap M$
- $M \cap U = \{\rho_+ = 0\} = \{\rho_- = 0\}$

- $d\rho_\pm(\zeta) \neq 0$ *for all* $\zeta \in U$ *and* $\rho_+\rho_- < 0$ *on* $U \setminus M$

- *for each* $\lambda \in [0,1]$ *and for each* $\zeta \in U$, *the forms* $L^X_\zeta(\lambda\tilde{\varphi} + (1-\lambda)\rho_\pm)$ *have at least* $(q+1)$ *positive eigenvalues.*

Proposition 1.3. — (see Proposition I.2.3 in[8]). *Let* M *be a real* $\mathscr{C}^2$*-hypersurface in some* n*-dimensional complex manifold* X*, and let* q *be an integer with* $0 \leqslant q \leqslant (n-1)/2$*. A real* $\mathscr{C}^2$*-function* φ *on* M *is* $(q+1)$*-convex at a point* $\xi \in M$ *if and only if* M *is* q*-convex-concave at* ξ *and, for any real* $\mathscr{C}^2$*-extension* ψ *of* φ *to an* X*-neighborhood of* ξ*,* $L^M_\xi(\psi)$ *has at least* q*-positive eigenvalues.*

Definition 1.4. — *Let* M *be a real* $\mathscr{C}^2$*-hypersurface in an* n*-dimensional complex manifold* X*, and let* q *be an integer with* $0 \leqslant q \leqslant (n-1)/2$*. A real* $\mathscr{C}^2$*-function* φ *on* M *will be called* $(q+1)$-concave at a point $\xi \in M$ *if* $-\varphi$ *is* $(q+1)$*-convex at* ξ*. A* $(q+1)$-concave function *on* M *is, by definition, a real* $\mathscr{C}^2$*-function on* M *which is* $(q+1)$*-concave at all points in* M*.*

Remark : If M is a q-convex-concave real $\mathscr{C}^2$-hypersurface in an n-dimensional complex manifold X, $0 \leqslant q \leqslant (n-1)/2$, and ψ is a strictly plurisubharmonic function on X, then the restriction of $-\psi$ to M is $(q+1)$-concave on M in the sense of the previous definition.

Definition 1.5. — *Let* M *be a* q*-convex-concave real* $\mathscr{C}^2$*-hypersurface in an* n*-dimensional complex manifold* X*,* $1 \leqslant q \leqslant \frac{(n-1)}{2}$*. In the following definitions, for subsets* W *of* M*, we denote by* $\overline{W}$ *the closure of* W *in* M*, and by* ∂W *the boundary of* W *in* M*.*

(i) A q-concave extension in M *is an ordered couple* $[D,\Omega]$ *of open subsets* $D \subset \Omega$ *of* M *satisfying the following condition:* ∂D *is compact, every connected component of* Ω *has a non-empty intersection with* D*, and there exist an open* M*-neighborhood* $U_{\partial D}$ *of* ∂D *and a* $(q+1)$*-concave* $\mathscr{C}^2$ *function* φ *defined on* $U_\varphi := U_{\partial D} \cup (\Omega \setminus D)$ *such that for some* $c_0, c_\infty \in \mathbb{R} \cup \{\infty\}$*, with* $c_0 < c_\infty$*,*

(a) $D \cap U_\varphi = \{\varphi < c_0\}$ *and* $d\varphi(z) \neq 0$ *for* $z \in \partial D$

(b) the sets $(\Omega \setminus D) \cap \{\varphi \leqslant c\}$*,* $c_0 \leqslant c < c_\infty$*, are compact.*

(ii) *M will be called* q-concave *if there exists a relatively compact open subset* D *in* M *such that* $[D, M]$ *is a* q*-concave extension in* M*.*

Let E be a holomorphic vector bundle over X, M a real $\mathscr{C}^2$-hypersurface in X (not necessarily closed) and D an open subset with $\mathscr{C}^2$-boundary in M, and let $\overline{D}$ be the closure of D in M.

We denote by $\mathscr{C}^\alpha_{n,r}(\overline{D},E)(0 \leqslant r \leqslant n, 0 \leqslant \alpha < 1)$ the space of continuous (if $\alpha = 0$), *resp.* Hölder continuous with exponent α (if $\alpha > 0$), E-valued differential forms of bidegree (n,r) on $\overline{D}$.

If $\overline{D}$ is compact, then $\mathscr{C}^\alpha_{n,r}(\overline{D},E)$ will be considered as *Banach space* endowed with the max-norm (if $\alpha = 0$), *resp.* the Hölder norm with exponent α (if $\alpha > 0$).

If $\overline{D}$ is not compact, then $\mathscr{C}^\alpha_{n,r}(\overline{D},E)$ will be considered as *Fréchet space* endowed with the topology defined by the Banach spaces $\mathscr{C}^\alpha_{n,r}(\overline{W},E)$ where W runs over all open sets $W \subseteq D$ with $\mathscr{C}^2$-boundary such that the closure $\overline{W}$ of W in $\overline{D}$ is compact.

The forms which are Hölder continuous with exponent $1/2-\varepsilon$ for all $\varepsilon > 0$ are of particular interest in this paper. Therefore we introduce also the spaces

$$\mathscr{C}^{<1/2}_{n,r}(\overline{D},E) := \bigcap_{\varepsilon>0} \mathscr{C}^{1/2-\varepsilon}_{n,r}(\overline{D},E).$$

These spaces will be considered as Fréchet spaces endowed with the topology defined by the topologies of the spaces $\mathscr{C}^{1/2-\varepsilon}_{n,r}(\overline{D},E)$, $\varepsilon > 0$, *i.e.* a map with values in $\mathscr{C}^{<1/2}_{n,r}(\overline{D},E)$ is continuous if and only if it is continuous as a map with values in each $\mathscr{C}^{1/2-\varepsilon}_{n,r}(\overline{D},E)$, $\varepsilon > 0$.

We denote by $Z^{\alpha}_{n,r}(\overline{D},E)$ $(0 \leqslant r \leqslant n, 0 \leqslant \alpha < 1)$ and $Z^{<1/2}_{n,r}(\overline{D},E)$ the subspaces of closed forms in $\mathscr{C}^{\alpha}_{n,r}(\overline{D},E)$, *resp.*$\mathscr{C}^{<1/2}_{n,r}(\overline{D},E)$. For $1 \leqslant r \leqslant n$, we set

$$E^{<1/2}_{n,r}(\overline{D},E) = Z^{0}_{n,r}(\overline{D},E) \cap d\mathscr{C}^{<1/2}_{n,r-1}(\overline{D},E) \quad \text{and} \quad H^{n,r}_{<1/2}(\overline{D},E) = Z^{0}_{n,r}(\overline{D},E)/E^{<1/2}_{n,r}(\overline{D},E).$$

2. Finiteness of some $\overline{\partial}_b$-cohomology group with compact support in q-concave real hypersurfaces

This section is devoted to the proof of the following result :

THEOREM 2.1. — *Let X be an n-dimensional complex manifold, $n \geqslant 5$, E a holomorphic vector bundle over X and M a real $\mathscr{C}^2$-hypersurface in X (not necessarily closed). Assume M is q-concave,* $2 \leqslant q \leqslant (n-1)/2$, *then*

$$dim\, H^{n,n-q+1}_{c,0}(M,E) < \infty \tag{2.1}$$

where $H^{n,n-q+1}_{c,0}(M,E)$ denotes the $\overline{\partial}_b$-cohomology group of bidegree $(n, n-q+1)$ for continuous E-valued forms with compact support in M (see (2.2)*).*

First we introduce some notations and prove some lemmas.

Let X be a complex manifold, E a holomorphic vector bundle over X and M a real $\mathscr{C}^2$-hypersurface in X. If $D \subset\subset M$ is open, then we denote by $\mathscr{C}^{\alpha}_{n,r}(\overline{D};M,E)$, $0 \leqslant \alpha < 1$, the Banach space of forms $f \in \mathscr{C}^{0}_{n,r}(M,E)$ with

$$\text{supp}\, f \subseteq \overline{D} \qquad \text{and} \qquad f|_{\overline{D}} \in \mathscr{C}^{\alpha}_{n,r}(\overline{D},E).$$

$\mathscr{C}^{\alpha}_{n,r}(M,E)$ denotes the Fréchet space of forms $f \in \mathscr{C}^{0}_{n,r}(M,E)$ such that $f|_{\overline{D}} \in \mathscr{C}^{\alpha}_{n,r}(\overline{D},E)$ for each open $D \subset\subset M$, endowed with the topology of convergence in each $\mathscr{C}^{\alpha}_{n,r}(\overline{D},E)$.

If Y is an arbitrary subset of M, then we denote by $\mathscr{C}^{\alpha}_{n,r}(Y;M,E)$ the subspace of all $f \in \mathscr{C}^{\alpha}_{n,r}(M,E)$ with supp $f \subseteq Y$ endowed with the Fréchet topology of $\mathscr{C}^{\alpha}_{n,r}(M,E)$. We set

$$Z^{\alpha}_{n,r}(Y;M,E) = Z^{\alpha}_{n,r}(M,E) \cap \mathscr{C}^{\alpha}_{n,r}(Y;M,E).$$

$Z^{\alpha}_{n,r}(Y;M,E)$ will be considered also as Fréchet space endowed with the topology of $\mathscr{C}^{\alpha}_{n,r}(M,E)$. Note that if Y is compact, then $\mathscr{C}^{\alpha}_{n,r}(Y;M,E)$ and $Z^{\alpha}_{n,r}(Y;M,E)$ are Banach spaces.

$\mathscr{C}^\alpha_{n,r}(c; M, E)$ denotes the linear subspace of $\mathscr{C}^\alpha_{n,r}(M, E)$ which consists of the forms with compact support. Set

$$Z^\alpha_{n,r}(c; M, E) = Z^\alpha_{n,r}(M, E) \cap \mathscr{C}^\alpha_{n,r}(c; M, E).$$

These two spaces will be considered as topological vector spaces endowed with the inductive limit topology of the spaces $\mathscr{C}^\alpha_{n,r}(K; M, E)$, $K \subset\subset M$ compact.

Further, we denote by $E^0_{n,r}(c; M, E)$ the space of all $\varphi \in \mathscr{C}^0_{n,r}(c; M, E)$ of the form $\varphi = \overline{\partial}\psi$ with $\psi \in \mathscr{C}^0_{n,r-1}(c; M, E)$ if $r > 0$, and we set $E^0_{n,r}(c; M, E) = \{0\}$ if $r = 0$. With these notations,

$$H^{n,r}_{c,0}(M, E) = Z^0_{n,r}(c; M, E)/E^0_{n,r}(c; M, E). \tag{2.2}$$

Lemma 2.2. — *Let X be an n-dimensional complex manifold, $n \geqslant 5$, E a holomorphic vector bundle over X, M a q-convex-concave real $\mathscr{C}^2$-hypersurface in X and ψ a real $\mathscr{C}^2$ function on M whose Levi form $L^M_\xi(\psi)$ has at least $q+1$ positive eigenvalues at every point ξ in M, $2 \leqslant q \leqslant (n-1)/2$, such that if*

$$a: = \inf_{\zeta \in M} \psi(\zeta) \quad \text{and} \quad b: = \sup_{\zeta \in M} \psi(\zeta),$$

then, for all $\alpha, \beta \in]a, b[$, the set $\{\alpha \leqslant \psi \leqslant \beta\}$ is compact. Then, for all $\alpha, \beta \in]a, b[$ with $\alpha < \beta$ and for any $\delta > 0$, the following assertion holds :

There exists a continuous linear operator

$$T^\alpha_{n-q+1} : Z^0_{n,n-q+1}(\{\alpha \leqslant \psi\}; M, E) \longrightarrow \mathscr{C}^{<1/2}_{n,n-q}(\{\alpha - \delta \leqslant \psi \leqslant \beta + \delta\}; M, E)$$

such that

$$\overline{\partial}_b T^\alpha_{n-q+1} f = f \qquad \text{on} \quad \{\psi < \beta\}$$

for all $f \in Z^0_{n,n-q+1}(\{\alpha \leqslant \psi\}; M, E)$.

Proof. — Lemmas I.6.6 (ii) and I.7.6 in [8] immediately imply the following statement: If $f \in Z^0_{n,n-q+1}(\{\alpha \leqslant \psi\}; M, E)$, then there exists $u \in \mathscr{C}^0_{n,n-q}(M, E)$ with $\overline{\partial}_b u = f$ on $\{\psi < \beta\}$. Moreover, the proof of Lemma I.6.6 (ii) in [8] shows that this solution has support in $\{\alpha - \delta \leqslant \psi\}$ and can be given by an operator T^α_{n-q+1} as required. □

Theorem 2.3. — *Let M be a q-concave real $\mathscr{C}^2$-hypersurface in an n-dimensional complex manifold X, $2 \leqslant q \leqslant (n-1)/2$, E a holomorphic vector bundle over X, and let φ, c_0, c_∞ be as in Definition 1.5. Further, let $\alpha_0, \alpha \in \mathbb{R}$ be given such that: $\alpha_0 < \alpha < c_0$ and α_0 such that if $\xi \in M$ with $\varphi(\xi) \geqslant \alpha_0$ then the Levi form $L^M_\xi(\varphi)$ has at least $q+1$ positive eigenvalues.*

Then, for all $\delta > 0$, there exist continuous linear operators

$$T_{n-q+1} : Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E) \longrightarrow \mathscr{C}^{<1/2}_{n,n-q}(\{\varphi \leqslant \alpha + \delta\}; M, E)$$

and

$$K_{n-q+1} : Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E) \longrightarrow Z^{<1/2}_{n,n-q+1}(\{\varphi \leqslant \alpha_0\}; M, E)$$

such that

$$\overline{\partial} T_{n-q+1} f = f + K_{n-q+1} f$$

for all $f \in Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)$.

Proof. — By Lemma 2.2, if δ is sufficiently small, there exist continuous linear operators

$$T^{\alpha}_{n-q+1} : Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E) \longrightarrow \mathscr{C}^{<1/2}_{n,n-q}(\{\alpha_0 - \delta < \varphi < \alpha + \delta\}; M, E)$$

such that

$$\overline{\partial}_b T^{\alpha}_{n-q+1} f = f \qquad \text{on} \quad \left\{\alpha_0 - \frac{\delta}{2} < \varphi\right\}$$

Take a $\mathscr{C}^\infty$ partition of unity $\chi_\alpha, \chi_1, \dots, \chi_N$ on M such that

(a) $\chi_\alpha \equiv 1$ in a neighborhood of $\{\alpha_0 \leqslant \varphi\}$ and $\chi_\alpha \equiv 0$ in a neighborhood of $\{\varphi \leqslant \alpha_0 - \delta/2\}$;

(b) for $1 \leqslant j \leqslant N$, the support of χ_j is contained in certain open subset $U_j \subset\subset \{\varphi < \alpha_0\}$ of M, which is sufficiently small. Then we have continuous linear operators (e.g.[4] or TheoremI.4.2 in [8]).

$$T^j_{n-q+1} : Z^0_{n,n-q+1}(M, E) \longrightarrow \mathscr{C}^{<1/2}_{n,n-q}(U_j, E), \qquad 1 \leqslant j \leqslant N,$$

such that

$$\overline{\partial}_b T^j_{n-q+1} f = f|_{U_j}$$

for all $f \in Z^0_{0,q+1}(M, E)$.

Now the operators

$$T_{n-q+1} := \chi_\alpha T^\alpha_{n-q+1} + \sum_{j=1}^{N} \chi_j T^j_{n-q+1}$$

and

$$K_{n-q+1} := \overline{\partial}\chi_\alpha \wedge T^\alpha_{n-q+1} + \sum_{j=1}^{N} \overline{\partial}\chi_j \wedge T^j_{n-q+1}$$

have the required properties. □

Proof of Theorem 2.1. — Let $\varphi, c_0, c_\infty, \alpha, \alpha_0, \delta, T_r, K_r$ be as is Theorem 2.3 where δ is so small that $inf\,\varphi < \alpha - \delta$. Then $Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)$ is a Banach space,

$$T_{n-q+1}\left(Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)\right) \subseteq \mathscr{C}^0_{n,n-q}(c; M, E) \tag{2.3}$$

and, by Ascoli's theorem, K_{n-q+1} is compact as an operator acting from $Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)$ into itself. Since $\overline{\partial}_b T_{n-q+1} = id + K_{n-q+1}$ on this space, it follows that $\overline{\partial}_b T_{n-q+1}$ is a Fredholm operator in $Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)$. Hence

$$\overline{\partial}_b T_{n-q+1}\left(Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)\right)$$

is of finite codimension in $Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)$. Since, by (2.3),

$$\overline{\partial}_b T_{n-q+1}\left(Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)\right) \subseteq E^0_{n,n-q+1}(c; M, E),$$

this implies that

$$\dim\left[Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)\Big/ \left(E^0_{n,n-q+1}(c; M, E) \cap Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\}; M, E)\right)\right] < \infty. \tag{2.4}$$

Moreover, Theorem 2.3 implies the relation

$$Z^0_{n,n-q+1}(c;M,E) = \text{linear hull of} \quad E^0_{n,n-q+1}(c;M,E) \cup Z^0_{n,n-q+1}(\{\varphi \leqslant \alpha\};M,E). \tag{2.5}$$

Now (2.1) follows from (2.4) and (2.5). □

3. Serre duality in CR manifolds

In this section we need not to restrict ourselves to the hypersurface case. For the main properties of CR manifolds and of the tangential Cauchy-Riemann complex, which are useful in this section, the reader may consult for example the book of Boggess [3]

Let X be an n-dimensional complex manifold, E a holomorphic vector bundle over X, M a $\mathscr{C}^\infty$-smooth CR generic submanifold of X of real codimension k, $p,q \in \mathbb{N}$ and $l \in \mathbb{N}\cup\infty$. We denote by $\Lambda^{p,q}_{X,E}$ the vector bundle over X of (p,q)-forms in X with values in E and by $\Lambda^{p,q}_{M,E}$ the vector bundle over M of (p,q)-forms in M with values in E. $\mathscr{C}^l_{p,q}(M,E)$ is the space of $\mathscr{C}^l$-smooth sections of $\Lambda^{p,q}_{M,E}$ over M and $\mathscr{C}^l_{p,q}(c;M,E)$ the space of compactly supported elements of $\mathscr{C}^l_{p,q}(M,E)$. Note that $\Lambda^{p,q}_{M,E} = 0$ if either $p > n$ or $q > n-k$ and consequently $\mathscr{C}^l_{p,q}(M,E) = \mathscr{C}^l_{p,q}(c;M,E) = 0$ for such p and q.

We put on $\mathscr{C}^l_{p,q}(M,E)$ the topology of uniform convergence on compact sets of the sections and all their derivatives.

Let K be a compact subset of M, let $\mathscr{C}^l_{p,q}(K;M,E)$ the closed subspace of $\mathscr{C}^l_{p,q}(M,E)$ of forms with support in K endowed with the induced topology. Choose $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of compact subsets of M. Then $\mathscr{C}^l_{p,q}(c;M,E) = \bigcup_{n\geqslant 0} \mathscr{C}^l_{p,q}(K_n;M,E)$. We put on $\mathscr{C}^l_{p,q}(c;M,E)$ the strict inductive limit topology defined by the Fréchet spaces $\mathscr{C}^l_{p,q}(K_n;M,E)$.

The space of currents on M with values in E^* of bidegree $(n-p,n-k-q)$ is the dual of the space $\mathscr{C}^\infty_{p,q}(c;M,E)$ and is denoted by $\mathscr{D}'^{n-p,n-k-q}(M,E^*)$. An element of $\mathscr{D}'^{n-p,n-k-q}(M,E^*)$ can be identified with a distribution section of $\Lambda^{n-p,n-k-q}_{M,E^*}$. The dual of $\mathscr{C}^\infty_{p,q}(M,E)$, denoted by $\mathscr{E}'^{n-p,n-k-q}(M,E^*)$, is the space of currents of bidegree $(n-p,n-k-q)$ with compact supports.

We shall use the following notations:

$$\begin{aligned}
Z^l_{p,q}(M,E) &= \mathscr{C}^l_{p,q}(M,E)\cap \operatorname{Ker}\overline{\partial}_b, & E^l_{p,q}(M,E) &= \mathscr{C}^l_{p,q}(M,E)\cap \operatorname{Im}\overline{\partial}_b\\
Z^l_{p,q}(c;M,E) &= \mathscr{C}^l_{p,q}(c;M,E)\cap \operatorname{Ker}\overline{\partial}_b, & E^l_{p,q}(c;M,E) &= \mathscr{C}^l_{p,q}(c;M,E)\cap \operatorname{Im}\overline{\partial}_b\\
Z^{\mathrm{cur}}_{p,q}(M,E^*) &= \mathscr{D}'^{p,q}(M,E^*)\cap \operatorname{Ker}\overline{\partial}_b, & E^{\mathrm{cur}}_{p,q}(M,E^*) &= \mathscr{D}'^{p,q}(M,E^*)\cap \operatorname{Im}\overline{\partial}_b\\
Z^{\mathrm{cur}}_{p,q}(c;M,E^*) &= \mathscr{E}'^{p,q}(M,E^*)\cap \operatorname{Ker}\overline{\partial}_b, & E^{\mathrm{cur}}_{p,q}(c;M,E^*) &= \mathscr{E}'^{p,q}(M,E^*)\cap \operatorname{Im}\overline{\partial}_b\\
H^{p,q}_l(M,E) &= Z^l_{p,q}(M,E)/E^l_{p,q}(M,E), & H^{p,q}_{c,l}(M,E) &= Z^l_{p,q}(c;M,E)/E^l_{p,q}(c;M,E)\\
H^{p,q}_{\mathrm{cur}}(M,E^*) &= Z^{p,q}_{\mathrm{cur}}(M,E^*)/E^{p,q}_{\mathrm{cur}}(M,E^*), & H^{p,q}_{c,\mathrm{cur}}(M,E^*) &= Z^{\mathrm{cur}}_{p,q}(c;M,E^*)/E^{\mathrm{cur}}_{p,q}(c;M,E^*).
\end{aligned}$$

In [10], using the fact that for all (p,q), $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n-k$, $\mathscr{C}^\infty_{p,q}(M,E)$ is a FS-space and its dual $\mathscr{E}'^{n-p,n-k-q}(M,E^*)$ is a DFS-space, we get the following result :

THEOREM 3.1. — *Let M be a CR generic $\mathscr{C}^\infty$-submanifold of real codimension k in an n-dimensional complex manifold X. Let $p,q \in \mathbb{N}$ with $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n-k$, then the following assertions are equivalent :*

(i) $E^{\mathrm{cur}}_{n-p,n-k-q}(c;M,E^*) = \Big\{T \in \mathscr{E}'_{n-p,n-k-q}(M,E^*) \mid \langle T,\varphi\rangle = 0,\ \forall \varphi \in Z^\infty_{p,q}(M,E)\Big\}$

(ii) $H^{n-p,n-k-q}_{c,\mathrm{cur}}(M,E^*)$ *is separated;*

(iii) $E^\infty_{p,q+1}(M,E) = \Big\{f \in \mathscr{C}^\infty_{p,q+1}(M,E) \mid \langle T,f\rangle = 0,\ \forall T \in Z^{\mathrm{cur}}_{n-p,n-k-q-1}(c;M,E^*)\Big\}$

(iv) $H^{p,q+1}_\infty(M,E)$ *is separated.*

Moreover, if these assertions hold, then

a) the natural linear map $H^{n-p,n-k-q}_{c,\mathrm{cur}}(M,E^*) \longrightarrow \big(H^{p,q}_\infty(M,E)\big)'$ *is a topological isomorphism;*

b) the natural linear map $H^{p,q+1}_\infty(M,E) \longrightarrow \big(H^{n-p,n-k-q-1}_{c,\mathrm{cur}}(M,E^*)\big)'$ *is an algebraic isomorphism.*

Now the question arises about what happens if we assume that $H^{p,q+1}_{c,\infty}(M,E)$ is separated. Would it imply that the group $H^{p,q+1}_{\mathrm{cur}}(M,E^*)$ is separated ? In this case several difficulties appears : $\mathscr{C}^\infty_{p,q}(c;M,E)$ is no more a Fréchet space but only a strict inductive limit of Fréchet spaces and a closed subspace of a strict inductive limit of Fréchet spaces is not a strict inductive limit of Fréchet spaces, which gives troubles in the application of the open mapping theorem, and also $\mathscr{D}'^{n-p,n-k-q}(M,E^*)$ is not metrizable. Nevertheless, under additional assumptions on the $\overline{\partial}_b$ operator, we can get some duality theorem.

DEFINITION 3.2. — *Let M be a CR generic $\mathscr{C}^\infty$-submanifold of real codimension k in an n-dimensional complex manifold X and $p,q \in \mathbb{N}$ with $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n-k$. The $\overline{\partial}_b$ operator* is regular in bidegree (p,q), *if for T a $(p,q-1)$ current on an open subset Ω of M, with values in a holomorphic vector bundle E over X, such that $\overline{\partial}_b T$ is defined by a $\mathscr{C}^\infty$-smooth (p,q)-form on Ω, their exists a $(p,q-1)$ form u of class $\mathscr{C}^\infty$ on Ω, with values in E, such that $\overline{\partial}_b u = \overline{\partial}_b T$.*

DEFINITION 3.3. — *Let M be a CR generic $\mathscr{C}^\infty$-submanifold of real codimension k in an n-dimensional complex manifold X and $p,q \in \mathbb{N}$ with $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n-k$. We shall say that M satisfies the (p,q)*-exactness $\mathscr{C}^l$-propagation property *if the following assertion holds :*

There exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of M such that $K_n \subset\subset \overset{\circ}{K}_{n+1}$ for all $n \in \mathbb{N}$, $M = \bigcup_{n\in\mathbb{N}} K_n$ and if φ is a (p,q)-form of class $\mathscr{C}^l$ on M with values in a holomorphic vector bundle E over X which satisfies $\overline{\partial}_M \psi_n = \varphi$ on $\overset{\circ}{K}_n$, where ψ_n is a form of class $\mathscr{C}^l$ on $\overset{\circ}{K}_n$, one can find a form ψ_{n+1} of class $\mathscr{C}^l$ on $\overset{\circ}{K}_{n+1}$ such that $\overline{\partial}_M \psi_{n+1} = \varphi$ on $\overset{\circ}{K}_{n+1}$ and $\psi_{n+1} = \psi_n$ on $\overset{\circ}{K}_{n-1}$.

THEOREM 3.4. — *Let M be a CR generic $\mathscr{C}^\infty$-submanifold of real codimension k in an n-dimensional complex manifold X, and $p,q\in\mathbb{N}$ with $0\leqslant p\leqslant n$ and $0\leqslant q\leqslant n-k$. Assume $\overline{\partial}_b$ is regular in bidegree $(n-p,n-k-q)$ and satisfies the $(n-p,n-k-q)$-exactness $\mathscr{C}^l$-propagation property for some $l\in\mathbb{N}\cup\infty$.*

Consider the following assertions

(i) $E^l_{p,q+1}(c;M,E)=\Big\{f\in\mathscr{C}^l_{p,q+1}(c;M,E)\mid\langle f,\varphi\rangle=0,\ \forall\varphi\in Z^l_{n-p,n-k-q-1}(M,E^*)\Big\}$;

(ii) $H^{p,q+1}_{c,l}(M,E)$ *is separated;*

(iii) $E^\infty_{n-p,n-k-q}(M,E^*)=\Big\{\varphi\in\mathscr{C}^\infty_{n-p,n-k-q}(M,E^*)\mid\langle\varphi,f\rangle=0,\ \forall f\in Z^l_{p,q}(c;M,E)\Big\}$;

(iv) $H^{n-p,n-k-q}_\infty(M,E^*)$ *is separated;*

then $(i)\Longrightarrow(ii)\Longrightarrow(iii)\Longrightarrow(iv)$.

Moreover if (i) holds for $l=\infty$, then the natural map

$$H^{n-p,n-k-q}_\infty(M,E^*)\longrightarrow\big(H^{p,q}_{c,\infty}(M,E)\big)'$$

is an algebraic isomorphism.

Theorem 3.4 will be deduced from several lemmas.

LEMMA 3.5. — *Let M, p, q be as in Theorem 3.4, we consider the following assertions*

(ii) $H^{p,q+1}_{c,l}(M,E)$ *is separated;*

$(ii)'$ *For each compact subset K in M, $\mathscr{C}^l_{p,q+1}(K;M,E)\cap E^l_{p,q+1}(c;M,E)$ is topologically closed in $\mathscr{C}^l_{p,q+1}(K;M,E)$.*

$(ii)''$ *For each compact subset K in M, there exists a compact subset $\widetilde{K}$ in M with*

$$\mathscr{C}^l_{p,q+1}(K;M,E)\cap E^l_{p,q+1}(c;M,E)=\mathscr{C}^l_{p,q+1}(K;M,E)\cap\overline{\partial}_b\mathscr{C}^l_{p,q}(\widetilde{K};M,E)$$

then $(ii)\Longrightarrow(ii)'\Longrightarrow(ii)''$.

Proof. — The assertion (ii) says that $E^l_{p,q+1}(c;M,E)$ is topologically closed in $\mathscr{C}^l_{p,q+1}(c;M,E)$, this implies $(ii)'$ by definition of the inductive limit topology on $\mathscr{C}^l_{p,q+1}(c;M,E)$.

Assume now that $(ii)'$ is fulfilled. We denote by $(K_n)_{n\in\mathbb{N}}$ an exhaustive sequence of compact subsets in M, then for a fixed compact subset K in M

$$\mathscr{C}^l_{p,q+1}(K;M,E)\cap E^l_{p,q+1}(c;M,E)=\cup_{n\in\mathbb{N}}\left(\mathscr{C}^l_{p,q+1}(K;M,E)\cap\overline{\partial}\mathscr{C}^l_{p,q}(K_n;M,E)\right).$$

Since, by $(ii)'$, $\mathscr{C}^l_{p,q+1}(K;M,E)\cap E^l_{p,q+1}(c;M,E)$ is a Fréchet space, for a certain n_0, the space $\mathscr{C}^l_{p,q+1}(K;M,E)\cap\overline{\partial}\mathscr{C}^l_{p,q}(K_{n_0};M,E)$ is of second Baire category. Then $\overline{\partial}_b$ is a closed operator with domain of definition $\{\varphi \in \mathscr{C}^l_{p,q}(K_{n_0}; M, E) \mid \overline{\partial}_M\varphi \in \mathscr{C}^l_{p,q}(K; M, E)\}$ between the Fréchet spaces $\mathscr{C}^l_{p,q}(K_{n_0}; M, E)$ and $\mathscr{C}^l_{p,q+1}(K; M, E) \cap E^l_{p,q+1}(c; M, E)$ whose range is of second Baire category in $\mathscr{C}^l_{p,q+1}(K; M, E) \cap E^l_{p,q+1}(c; M, E)$; it follows by the open mapping theorem that this operator is onto and open. Setting $\widetilde{K} = K_{n_0}$, $(ii)''$ is proven. □

LEMMA 3.6. — *Let M, p, q be as in Theorem 3.4 and K a fixed compact subset in M. Assume*

$(ii)'_K$ $\mathscr{C}^l_{p,q+1}(K; M, E) \cap E^l_{p,q+1}(c; M, E)$ is topologically closed in $\mathscr{C}^l_{p,q+1}(K; M, E)$ and let $\widetilde{K}$ be the compact subset associated to K by $(ii)''$ in Lemma 3.5. If φ is a $(n-p, n-k-q)$-form of class $\mathscr{C}^\infty$ on M with values in E^* which satisfies $\langle\varphi, f\rangle = 0$ for all $f \in Z^l_{p,q}(c; M, E) \cap \mathscr{C}^l_{p,q}(\widetilde{K}; M, E)$, then there exists a form ψ of class $\mathscr{C}^\infty$ on M such that $\overline{\partial}_b\psi = \varphi$ on $\mathring{K}$.

Proof. — We define a linear form L_φ on $\mathscr{C}^l_{p,q+1}(K; M, E) \cap E^l_{p,q+1}(c; M, E)$ by setting for $\varphi \in \mathscr{C}^\infty_{n-p,n-k-q}(M, E^*)$ as in the lemma

$$L_\varphi(\beta) = \langle\varphi, \alpha\rangle \text{ if } \beta = \overline{\partial}_M\alpha \text{ with } \operatorname{supp}\alpha \subset \widetilde{K}.$$

By Lemma 3.5 and the orthogonality condition on φ, L_φ is well defined, moreover by $(ii)'_K$ and the open mapping theorem, L_φ is continuous. We may apply the Hahn-Banach theorem and extends L_φ to a continuous linear form on $\mathscr{C}^l_{p,q+1}(M, E)$.

This extension defines a current S of order l on M such that

$$\langle S, \overline{\partial}_M\alpha\rangle = \langle T, \alpha\rangle \text{ for } \alpha \in \mathscr{C}^\infty_{p,q}(K; M, E)$$

which implies $\overline{\partial}_M S = \varphi$ on $\mathring{K}$. As the operator $\overline{\partial}_b$ is regular in bidegree $(n-p, n-k-q)$, there exists a a form ψ of class $\mathscr{C}^\infty$ on M such that $\overline{\partial}_M\psi = \varphi$ on $\mathring{K}$. □

Proof of Theorem 3.4. — It is clear that $(i) \Longrightarrow (ii)$ and $(iii) \Longrightarrow (iv)$. It remains to prove $(ii) \Longrightarrow (iii)$.

Assume (ii) is fulfilled and fix $\varphi \in \mathscr{C}^\infty_{n-p,n-k-q}(M, E^*)$ which satisfies $\langle\varphi, f\rangle = 0$ for all $f \in Z^l_{p,q}(c; M, E)$.Then by Lemma 3.5, $(ii)'$ and $(ii)''$ are satisfied and it follows from Lemma 3.6 that, for any compact subset K of M, there exists a form ψ of class $\mathscr{C}^\infty$ on M such that $\overline{\partial}_M\psi = \varphi$ on $\mathring{K}$.

By the $(n-p, n-k-q)$-exactness $\mathscr{C}^l$-propagation property, there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of M such that $K_n \subset\subset \mathring{K}_{n+1}$ for all $n \in \mathbb{N}$, $M = \bigcup_{n\in\mathbb{N}} K_n$ and if φ is a $(n-p, n-k-q)$-form of class $\mathscr{C}^l$ on M which satisfies $\overline{\partial}_M\psi_n = \varphi$ on $\mathring{K}_n$, where ψ_n is a form of class $\mathscr{C}^l$ on $\mathring{K}_n$, one can find a form ψ_{n+1} of class $\mathscr{C}^l$ on $\mathring{K}_{n+1}$ such that $\overline{\partial}_M\psi_{n+1} = \varphi$ on $\mathring{K}_{n+1}$ and $\psi_{n+1} = \psi_n$ on $\mathring{K}_{n-1}$. Taking $K = K_1$, we can define by induction a sequence $(\psi_n)_{n\in\mathbb{N}}$ of $(n-p, n-k-q-1)$-form of class $\mathscr{C}^l$ such that $\psi_{n+1} = \psi_n$ on $\mathring{K}_{n-1}$. Then the $\mathscr{C}^l$-smooth $(n-p, n-k-q-1)$-form $\psi = \lim_{n\to\infty}\psi_n$ satisfies $\overline{\partial}_b\psi = \varphi$ on M. Moreover by regularity of the $\overline{\partial}_b$-operator in bidegree $(n-p, n-k-q)$ and since φ is of class $\mathscr{C}^\infty$, there exists a $\mathscr{C}^\infty$-smooth $(n-p, n-k-q-1)$-form $\widetilde{\psi}$ on M such that $\overline{\partial}_b\widetilde{\psi} = \varphi$ on M. □

4. An Andreotti-Vesentini separation theorem on q-concave hypersurfaces

From the previous sections we can deduce a separation theorem of Andreotti-Vesentini type for the $\overline{\partial}_b$ cohomology in q-concave hypersurfaces.

THEOREM 4.1. — *Let X be an n-dimensional complex manifold, E a holomorphic vector bundle over X and M a real $\mathscr{C}^\infty$-hypersurface in X (not necessarily closed). Assume M is q-concave, then $H_\infty^{0,q-1}(M,E)$ is separated.*

Proof. — If $q = 1$, $H_\infty^{0,q-1}(M,E)$ is the space of CR sections of E in M, and therefore separated.

If $q \geqslant 2$, in Theorem 2.1 we have proved that under the hypotheses of Theorem 4.1

$$\dim H_{c,0}^{n,n-q+1}(M,E^*) < \infty,$$

which implies the condition (ii)' of Lemma 3.5 in bidegree $(n, n-q+1)$ for $l = 0$.

As a q-concave real hypersurface is q-convex-concave, it follows from the De Rham-Weil isomorphism and the Poincaré lemma for the $\overline{\partial}_b$ operator in bidegree $(0, q-1)$ (cf. [4], [11]) that the $\overline{\partial}_b$ operator is regular in bidegree $(0, q-1)$. Moreover Theorem 9.5 in [8] says that a q-concave real hypersurface has the $(0, q-1)$-exactness $\mathscr{C}^0$-propagation property. Consequently the hypotheses of Theorem 3.4 are fulfilled and we deduce from the proof of this theorem that the condition (ii)' of Lemma 3.5 in bidegree $(n, n-q+1)$ for $l = 0$ implies that the $\overline{\partial}_b$-cohomology group $H_\infty^{0,q-1}(M,E)$ is separated. □

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