Spin models for chord diagrams

by Roland BACHER

In this paper we study chord diagrams using spin models. Such a spin model yields a linear application Z (called partition function) from the vector space $\mathbf{D}$ spanned by all chord diagrams into some vector space V.

The partition function of a diagram D is the sum of "local" contributions associated to subsets of k chords D (such subsets are called sites). In the case of models with more than one spin, one has also to sum over all states (applications of the set of chords in D into a finite set Sp called the set of spins).

We search now for a subspace $R \subset V$ of "relations" which is easy to describe and which contains the vector space $\tilde{R} \subset V$ spanned by the partition functions of all 4T relations in chord diagrams. The main feature of all our examples is the fact that sites which do not contain the two special chords involved in a 4T relation contribute nothing to the partition function. This fact allows a concrete description of a space $R \subset V$ which is generated by the contribution of a site containing the two chords involved in some 4T relation r or by the contributions of two sites each of which contains exactly one of the two chords involved in r and which are identical otherwise.

Choosing a linear function $\mu : V \longrightarrow \mathbf{C}$ containing R in its kernel (we call such a function a weight) we obtain a weight system Z_μ on chord diagrams by setting $Z_\mu(D) = \mu(Z(D))$.

Of course, since the partition function Z is not necessarily surjective, not every non-zero weight on V yields a non-zero weight system on chord diagrams.

We may also miss some weight systems since $\tilde{R}$ may be strictly smaller than the intersection of R with the image $Z(\mathbf{D})$.

This paper consists of an introductory section containing standard definitions and generalities on chord diagrams which is followed by four parts.

In part I we define flower diagrams and consider spin models with values in the vector space generated by all $k-$flowers. Orientable flowers lead to a spin model with one spin whereas non-orientable flowers lead to a model on 2 spins.

Math. classification: 57M25.
Keywords: chord diagram, Vassiliev invariant, weight system

Part I is in some sense a straightforward generaization of the graphical calculus for the Lie groups $gl(n)$ (orientable flowers) and $so(n)$ (non-orientable flowers).

In part II we define spin models with values in vector spaces defined by intersection properties of chords.

In part III we define spin models with values in the algebra $\mathcal{D} = \oplus \mathbf{D}_n / <$ 4T relations $>$ of chord diagrams modulo 4T relations. We call partition functions of such models a local operators.

In part IV we consider spin models whose partition functions generalize "cabling operations".

A further example which fits well into the mainstream of this paper is given by the comultiplication $\mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D}$ of the Hopf algebra $\mathcal{D}$ on chord diagrams. This example will not be mentioned further since spin models don't suggest anything new about it.

1. Basic definitions

In the sequel we work over the field of complex numbers. Since only linear equations with integral coefficients are involved, most results hold for arbitrary (commutative) fields or even commutative rings with unit.

Notice that for knots there are some problems when working with arbitrary fields or rings: The Kontsevitch integral is then no longer defined.

Definition 1.1. An $n-$*chord diagram* (or a *diagram over n chords*) is an oriented circle (inducing a cyclic order on its points) together with a distinguished set of n unordered pairs of points on it, up to diffeomorphisms preserving the orientation of the circle. An unordered pair of points in a chord diagram is a *chord*.

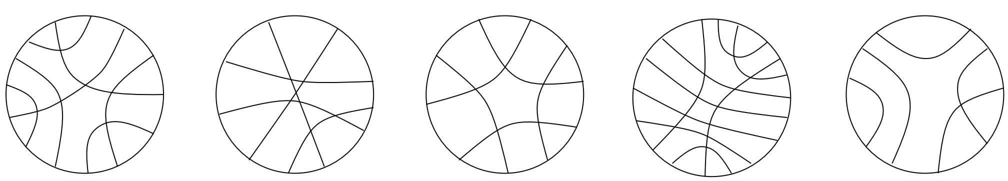

Figure 1.1. Examples of chord diagrams

We denote by $C(D)$ (or by C if the underlying chord diagram D is obvious) the set of chords in a chord diagram D.

Equivalently, an $n-$chord diagram can be defined as an involution $\overline{\alpha}$ without fixed points on a cyclically ordered set $\{b_1, \dots, b_{2n}\}$ (ie. two such involutions $\overline{\alpha}$ and $\overline{\alpha}'$ are equivalent if they are conjugated by a power of the cyclic permutation

$$b_1 \longmapsto b_2 \longmapsto \dots \longmapsto b_{2n} \longmapsto b_1 \quad) .$$

Given an $n-$chord diagram D represented by an involution $\overline{\alpha} \in \mathrm{Sym}(\{b_1, \dots, b_{2n}\})$, a chord of D is a pair of points of the the form $\{b_i, \overline{\alpha}(b_i)\}$.

Let $\mathbf{D}$ denote the graded vector space generated by all chord diagrams with grading given by the number of chords in a diagram.

Given a graded complex vector space V we denote by V_n the subspace of all elements with grading n. We denote moreover by $V[t]$ (respectively $V_n[t]$) the $\mathbf{C}[t]-$module $V \otimes_{\mathbf{C}} \mathbf{C}[t]$ (respectively $V_n \otimes_{\mathbf{C}} \mathbf{C}[t]$).

We denote hence by $\mathbf{D}_n$ the finite dimensional subspace of $\mathbf{D}$ generated by all $n-$chord diagrams and by $\mathbf{D}[t]$ (respectively $\mathbf{D}_n[t]$) the tensor product of $\mathbf{D}$ (respectively $\mathbf{D}_n[t]$) with $\mathbf{C}[t]$.

An element $r \in \mathbf{D}_n$ is a *4T relation* if r is represented by $D_1 - D_2 + D_3 - D_4$ where $D_1, \dots, D_4$ are four $n-$chord diagrams as in figure 1.2 which is drawn using the convention that there can be more chords not entering the regions enclosed by the dots but these "invisible" chords are "identical" in the four diagrams.

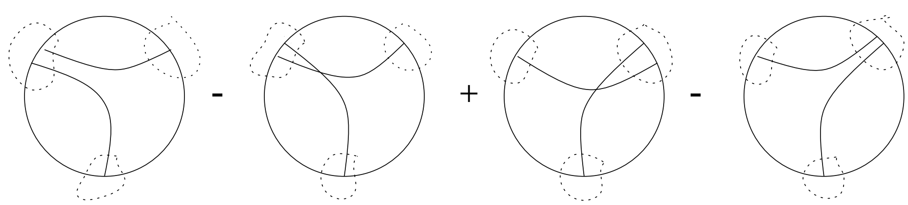

Figure 1.2. A 4T relation $r = D_1 - D_2 + D_3 - D_4$

Inspection of a 4T relation shows that such a relation is determined by a special chord c' (which is the horizontal chord in our figure) in an $(n-1)-$chord diagram $\tilde{D}$ together with a special point (the lower endpoint of the "moving" chord in our figure) which is not endpoint of a chord in $\tilde{D}$. One takes then the alternating sum of the four chord diagrams obtained by adjoining a second special chord c'' starting at the special point and ending immediately before or after the endpoints of the special chord c'.

This labelling of the two special chords in a 4T relation and identification of corresponding chords in all four diagrams of r allows us to identify the set of chords $C(D_1), \dots, C(D_4)$ with an abstract set $C(r)$ of n elements called the *chords of the relation* r.

Replacing the special chord c' by a whole bunch of chords entirely contained in a very small neighbourhood of it we get *generalized 4T relations* which we represent graphically as in figure 1.3

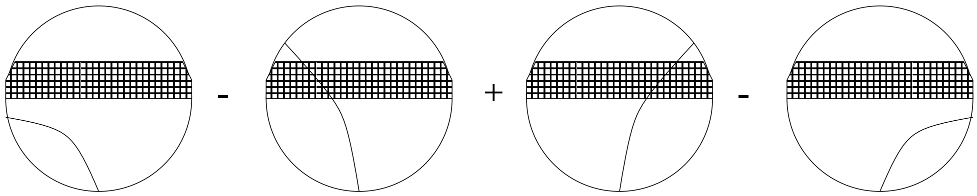

Figure 1.3. Generalized 4T relations

Any chord in such a relation has either both or none of its endpoints in the dark region of the picture. Dark regions are the same in all four diagrams and there may be many more "invisible" chords which are allowed to cross the dark region and which are also identical in all four diagrams.

Lemma 1.2. *A generalized 4T relation can be expressed as a finite sum of (ordinary) 4T relations.*

Proof. Sum up all 4T relations gotten by choosing the special chord c' among the chords in the dark region and by choosing the lower endpoint of the moving chord in figure 1.3 as the special point. For the correct sign choice a simplification occurs and yields the result. QED

A special instance of a generalized 4T relation is the case where the dark region has only "one end". A resulting generalized 4T relation simplifies then to yield the relation given by figure 1.4.

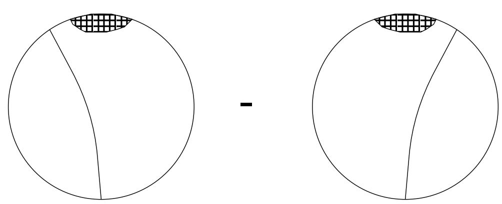

Figure 1.4. A special instance of generalized 4T relations

Relations of this last kind allow the definition of a product on the quotient space $\mathcal{D} = \mathbf{D}/ < \text{4T relations} >$ of $\mathbf{D}$ by the subspace spanned by all 4T relations turning $\mathcal{D}$ into a commutative (and associative) graded algebra called the *chord diagram algebra*. Indeed, define the product of two elements represented by chord diagrams D and D' as the "connected sum" of D and D' as illustrated by figure 1.5. More precisely, open the oriented circles S and S' supporting D and D' at arbitrarily chosen points of S, S' which are not endpoints of a chord and glue the resulting four ends two by two together in the unique way which preserves orientations and yields a chord diagram. The relation represented in figure 1.4 shows that the result is well defined modulo 4T relations. It is then easy to check that this defines a commutative and associative product on $\mathcal{D}$ with a grading induced by the grading of $\mathbf{D}$ (4T relations are homogenous). The element represented by the chord diagram without chords is the unit of this algebra.

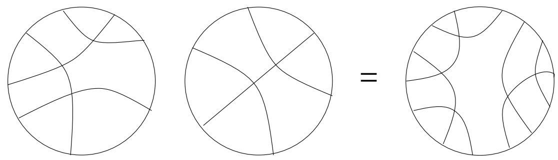

Figure 1.5. A product in $\mathcal{D}$

The algebra $\mathcal{D}$ can also be endowed with a cocommutative and coassociative coproduct. Indeed, given a subset $I \subset C(D)$ of chords in a chord diagram D we define $D(I)$ as the chord diagram obtained by deleting all chords of $C(D) \setminus I$ in D. In the sequel, given any subset $I \subset C(D)$ of chords in D, we denote by $\overline{I} = C(D) \setminus I$ its complement in $C(D)$. The coproduct $\Delta(D)$ of a chord diagram representing an element in $\mathcal{D}$ is then given by

$$\Delta(D) = \sum_{I \subset C(D)} D(I) \otimes D(\overline{I})$$

(one has to check that Δ is well defined, ie $\Delta(r) = 0$ in $\mathcal{D} \otimes \mathcal{D}$ for $r \in \mathbf{D}$ a 4T relation) and the counit is given by $\epsilon(D) = 1$ if D has no chords and by $\epsilon(D) = 0$ otherwise.

The multiplication and comultiplication introduced above turn $\mathcal{D}$ into a Hopf algebra (the compatibility between the algebra and coalgebra structures is easy to verify): see also [**B**].

Definition 1.3. An element of $\mathcal{D}_n^* = \mathrm{Hom}_{\mathbf{C}}(calD_n, \mathbf{C}) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{D}_n / < \text{4T relations} >, \mathbf{C})$ is a *weight* of degree n and an element of $\mathcal{D}^* = \prod \mathcal{D}_n^* = \mathrm{Hom}_{\mathbf{C}}(\mathcal{D}, \mathbf{C})$ is a *weight system*.

We use this terminology also in the case of the $\mathbf{C}[t]-$modules $\mathcal{D}_n[t]$ and $\mathcal{D}[t]$: a weight denotes in this case an element of $(\mathcal{D}_n[t])^*$ ie a $\mathbf{C}[t]-$linear application from $\mathcal{D}_n[t]$ into $\mathbf{C}[t]$.

In the sequel we call even a linear application from $\mathcal{D}_n$ (respectively $\mathcal{D}$) into a vector space V a *weight with values in* V. The same holds for $\mathbf{C}[t]-$linear applications from $\mathcal{D}_n[t]$ (or $\mathcal{D}[t]$) into some $\mathbf{C}[t]-$module $V[t]$.

The vector space $\oplus \mathcal{D}_n^*$ generated by all weights is also a Hopf algebra: comultiplication in $\mathcal{D}$ yields a multiplication in $\oplus \mathcal{D}_n^*$ and multiplication in $\mathcal{D}$ yields a comultiplication in $\oplus \mathcal{D}_n^*$.

Considering a completion of the tensor product turns even the space $\mathcal{D}^*$ of all weight systems into a Hopf algebra.

However, the natural object linked to knots in 3-space is not the chord diagram algebra $\mathcal{D}$ but a quotient $\tilde{\mathcal{D}}$ of it which is isomorphic to a sub-Hopf algebra in $\mathcal{D}$. The Konsevitch integral yields then a bijection of the vector space $\tilde{\mathcal{D}}_n^*$ generated by all genuine weights with the vector space of all Vassiliev invariants of type n on knots (see for instance [**B**] or [**V**] for more details on the links between the topology of knots, the Kontsevitch integral, Vassiliev invariants and weight systems).

An $n-$chord diagram D is a 1T relation if D contains an isolated chord (ie a chord not crossed by any other chord of D, see figure 1.6 for an example).

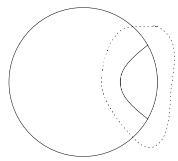

Figure 1.6. 1T relation

A 1T relation represents a product in $\mathcal{D}$ which contains (the element represented by) the chord diagram T having a unique chord as a factor. The span of 1T relations in $\mathcal{D}$ is hence the principal ideal of $\mathcal{D}$ generated by (the element represented by) T. This ideal is also a co-ideal in $\mathcal{D}$ and we get a Hopf algebra structure on the quotient

$$\tilde{\mathcal{D}} = \mathbf{D}/ < \text{1T and 4T relations} >$$

which is hence a quotient Hopf-algebra of the Hopf algebra $\mathcal{D}$. We call the subspace $\tilde{\mathcal{D}}_n^* \subset \mathcal{D}_n^*$ the space of *genuine weights* and elements of $\prod \tilde{\mathcal{D}}_n^*$ *genuine weight systems.*

There exists a natural projection

$$Rn : \mathbf{D} \longrightarrow \mathcal{D}$$

called *renormalization* (see [**CDLI**]), which contains all 4T and 1T relations in its kernel and which factors hence through $\tilde{\mathcal{D}}$. Renormalization yields an isomorphism of Hopf algebras between $\tilde{\mathcal{D}}$ and the sub-Hopf algebra $Rn(\mathbf{D})$ of $\mathcal{D}$. Identifying $\tilde{\mathcal{D}}$ with $Rn(\mathbf{D})$ we can consider the space of genuine weight systems as a quotient of the space of all weight systems. More precisely, given a weight system w, the linear form $Rn^*(w) = w \circ Rn : \mathbf{D} \longrightarrow \mathbf{C}$ defines a genuine weight system.

The renormalization operator $Rn : \mathbf{D} \longrightarrow \mathcal{D}$ is explicitely given by

$$Rn(D) = \sum_{I \subset C(D)} (-T)^{\sharp(\overline{I})} D(I)$$

with $\overline{I} = C(D) \setminus I$ (where T is the chord diagram with one chord and where all operations take place in the Hopf algebra $\mathcal{D}$, see [**CDLI**]). Equivalently, the operator Rn is given by $Rn = \mu \circ (\mathrm{id} \otimes \tau) \circ \Delta$ where $\Delta : \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D}$ is the coproduct, $\tau : \mathcal{D} \longrightarrow \mathcal{D}$ is defined by $D \longrightarrow (-T)^{\sharp(C(D))}$ for any chord diagram D and $\mu : \mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D}$ is the product.

Renormalization reduces the study of $\tilde{\mathcal{D}}$ (and of its dual, the genuine weight systems) to the study of $\mathcal{D}$ (and of its dual, the weight systems). In particular, its adjoint projects ordinary weight systems onto genuine ones.

A *framed weight system* is a $\mathbf{C}[t]-$linear application $f\ :\ \mathbf{D}[t]\ \longrightarrow\ \mathbf{C}[t]$ factoring through $\mathcal{D}[t]$ such that $f(TD)\ =\ tf(D)$ for any chord diagram D (where T stands again for the unique chord diagram with one chord). Framed weight systems deal with knots endowed with a framing, ie. with a nowhere zero section of their normal bundle.

The specialization $t=0$ turns a framed weight system into a genuine one.

Another method to get a genuine weight system out of a framed weight system is to apply the renormalization operator: Given a framed weight system f it is easy to check directly that $f\circ Rn$ given by

$$(f\circ Rn)\ (D)=\sum_{I\subset C(D)}(-t)^{\sharp(\bar{I})}f(D(I))$$

defines a genuine weight system with values in $\mathbf{C}[t]$.

A framed weight system f is *primitive* if we have $(f\circ Rn)(D)\ =\ f(D)|_{t=0}$. This is equivalent to

$$f(D)=\sum_{I\subset C(D)}t^{\sharp(\bar{I})}\left(f(D(I))|_{t=0}\right)$$

since the equality

$$f(D)=\sum_{I\subset C(D)}t^{\sharp(\bar{I})}\left(f\circ Rn(D(I))\right)$$

holds if f is an arbitrary framed weight system.

Any framed weight system f can be written as

$$f=\sum_{k=0}^{\infty}f_k t^k$$

where the functions f_k are primitive framed weight systems such that there exists a function $r\ :\ \mathbf{N}\ \longrightarrow\ \mathbf{N}$ with $f_i(D)\ =\ 0$ for any $n-$chord diagram D and any integer $i\ \geq\ r(n)$ (this ensures that $f(D)$ is a polynomial and not a formal power series). The specialization $t=0$ picks out f_0 whereas applying the adjoint Rn^* of the renormalization operator yields

$$f=\sum_{k=0}^{\infty}f_k t^k\longmapsto Rn^*(f)=\sum_{k=0}^{\infty}\left(f_k|_{t=0}\right)t^k$$

and induces a bijection between framed weight systems and genuine weight sytems with values in $\mathbf{C}[t]$.

A weight system w is *multiplicative* if we have $w(D\,D') = w(D)w(D')$ for any pair $D,\ D'$ of chord diagrams. Multiplicative weight systems are the same as characters of the algebra $\mathcal{D}$ (ie homomorphisms from $\mathcal{D}$ into the algebra $\mathbf{C}$ of complex numbers).

Remark 1.4. There exists an analogue of chord diagrams for links. This analogue has however no longer a structure of a Hopf algera but is only a graded vector space. It is fairly easy to modify the constructions of this paper in order to deal with these chord daigrams for links. We leave the details to the reader.

Part I: Flower diagrams

I.1 Orientable flowers

Let $b_1, \ldots, b_{2k}$ be a basis of a $(2k)-$dimensional vector space. Set

$$E = \{-b_1, b_1, -b_2, b_2, \ldots, -b_{2k}, b_{2k}\} \quad .$$

Definition I.1.1. An *orientable $k-$flower* is an equivalence class of two permutations $\alpha, \beta \in \mathrm{Sym}(E)$ of the set E such that

(i) α and β are involutions of E without fixed points,

(ii) α extends to a linear application of the vector space generated by E,

(iii) $\alpha(b_i) \neq -b_i$ for $i = 1, \ldots, 2k$.

(iv) $\alpha(b_i),\ \beta(b_i) \in \{-b_1, -b_2, \ldots, -b_{2k}\}$ for all i (orientability condition).

Two such pairs (α, β) and (α', β') are equivalent if they are conjugated by a power of the permutation obtained by restricting the linear application defined by

$$b_1 \longmapsto b_2 \longmapsto \ldots b_{2k-1} \longmapsto b_{2k} \longmapsto b_1$$

to the set $E = \{-b_1, b_1, -b_2, b_2, \ldots, -b_{2k}, b_{2k}\}$.

A *simple $k-$flower* is a flower having representatives $\alpha, \beta \in \mathrm{Sym}(E)$ such that $\beta(b_i) = -b_{i+1}$ for $i = 1, \ldots, 2k-1$ and $\beta(b_{2k}) = -b_1$.

We represent a $k-$flower graphically by choosing $4k$ consecutive points on a circle S which we identify with the set $E = \{-b_1, b_1, -b_2, b_2, \ldots, -b_{2k}, b_{2k}\}$. The points $-b_i,\ b_i$ should be close to each other. We join then pairs of points $\{-b_i, -\alpha(b_i)\ \}$ and $\{b_i, \alpha(b_i)\}$ by two close parallel arcs inside S and pairs $\{-b_i, \beta(-b_i)\}$ respectively $\{b_i, \beta(b_i)\}$ by arcs outside S (only the endpoints of such arcs are relevant).

Sets of the form $\{\pm b_i, \pm\alpha(b_i)\}$ are *chords*. We denote them by $C(F)$. Chords correspond graphically to pairs of close parallel arcs inside the circle S.

The set of chords in a flower F defines a $k-$chord diagram $D(F)$, called the *heart* of the flower.

The arcs of a flower past together to form closed paths. They correspond to the orbits in $\{-b_1, b_1, \ldots, -b_{2k}, b_{2k}\}$ of the group generated by α and β (as an abstract group this is always a finite dihedral group). These paths have natural orientations by orienting arcs $\{-b_i, b_j\} = \{-b_i, \alpha(-b_i)\}$ inside the heart of a flower F from $-b_i$ to b_j. This induces coherent orientations on all paths.

Given an $n-$chord diagram D with $n > 0$ drawn inside a circle S, we associate to it a simple $n-$flower as followings: Replace each chord by two close parallel arcs. Join then the $4n$ points where these arcs meet S by arcs outside S in the unique way which yields a simple $n-$flower (we call such outer arcs *trivial outer arcs*). We denote this flower by $F(D)$ (see figure I.1.1).

If D is the $0-$chord diagram we set $F(D) = tF$ where F is the $0-$chord flower and t is a variable.

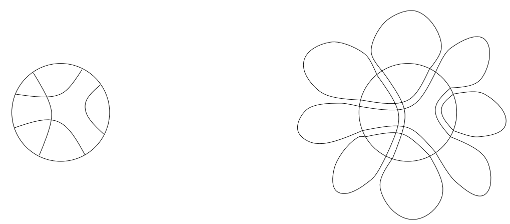

Figure I.1.1 A chord diagram D and its simple flower $F(D)$

We denote by $\mathbf{F}^+[t]$ the free $\mathbf{C}[t]-$module generated by all orientable flowers. This module is of course graded by the number of chords of a flower. We denote hence by $\mathbf{F}^+_k[t]$ the finitely generated submodule generated by all orientable $k-$flowers.

To an orientable flower F and a subset $I \subset C(F)$ of k chords in F, we associate an element $F(I) \in \mathbf{F}^+_k[t]$ as follows: Push outside the heart of F all pairs of arcs forming chords $c \notin I$. The result is a $k-$flower $\tilde{F}(I) \in \mathbf{F}^+_k$ together with ν isolated closed arcs not intersecting the heart of $\tilde{F}(I)$. Set $F(I) = t^\nu \tilde{F}(I)$. Figure I.1.2 shows an example with a set I consisting of two chords. The resulting integer ν equals 2.

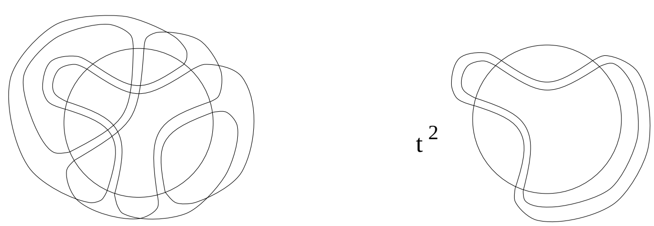

Figure I.1.2.

I.2. Weight systems for flowers

In this section we define and construct “weight systems” for orientable flowers and use these to construct weight systems for chord diagrams. This is achieved by defining relations $R(\mathbf{F}^+[t])$ in the space of orientable flowers which are preserved under a suitable projection operator $\frac{1}{(n-k)!}(\pi^+)^{n-k} : \mathbf{F}^+_n \longrightarrow \mathbf{F}^+_k$. One shows then that the linear application $D \longmapsto F(D)$ which associates to a chord diagram D the simple flower $F(D)$ having D as its heart induces a linear application $\mathcal{D} \longrightarrow \mathbf{F}^+[t]/R(\mathbf{F}^+[t])$.

We define a $\mathbf{C}[t]-$linear application $\pi^+ : \mathbf{F}^+_{k+1}[t] \longrightarrow \mathbf{F}^+_k[t]$ by

$$\pi^+(F) = \sum_{c\in C(F)} t^{\nu(c)}\tilde{F}(C \setminus \{c\}) = \sum_{c\in C(F)} F(C \setminus \{c\}) \in \mathbf{F}^+_k[t]$$

for an oriented $(k+1)-$flower F (see figure I.2.1 for an example).

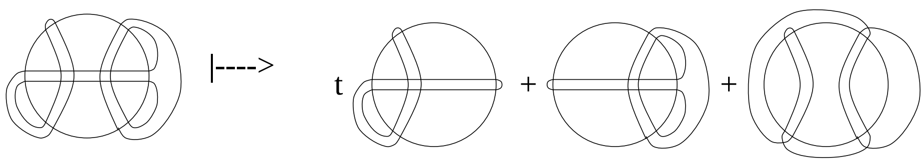

Figure I.2.1 The operator π^+

We define now relations in $\mathbf{F}^+_k[t]$ which are analogs of 4T relations for chord diagrams.

An element $r \in \mathbf{F}^+_k[t]$ is a *4T2 relation* if it is of the form $r = F_1 - F_2 + F_3 - F_4$ for four orientable $k-$flowers as in figure I.2.2. In other terms, the four $k-$chord diagrams in the hearts of the flowers form a 4T relation, the two inner arcs which are not parallel but obviously adjacent, are joined by a trivial outer arc and the four flowers agree everywhere else.

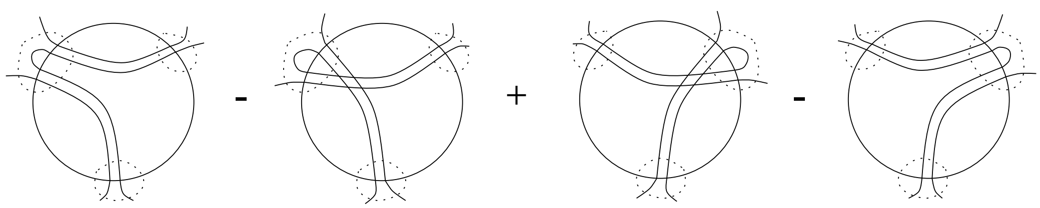

Figure I.2.2. A 4T2 relation $F_1 - F_2 + F_3 - F_4$

A *4T1 relation* is the element $r \in \mathbf{F}^+_k[t]$ obtained by applying π^+ to a 4T2 relation $\tilde{r} \in \mathbf{F}^+_{k+1}$ and by removing the $k-1$ obvious $4T2$ relations in $\mathbf{F}^+_k$ from $\pi^+(\tilde{r})$. A 4T1 relation is a $\mathbf{C}[t]-$linear combination of eight $k-$flowers as in figure I.2.4 (with the convention that isolated closed paths outside the heart correspond to a factor t).

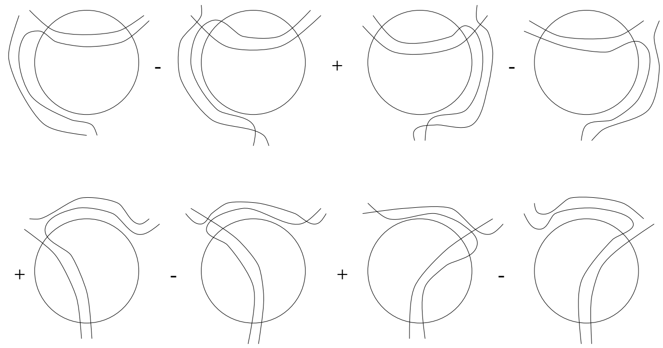

Figure I.2.4 A 4T1 relation

We denote by $R(\mathbf{F}_k^+[t])$ the $\mathbf{C}[t]-$module generated by all 4T2 and 4T1 relations in $\mathbf{F}_k^+[t]$. The *orientable flower space* $\mathcal{F}^+[t]$ is the quotient space $\oplus\mathcal{F}_k^+[t] = \oplus\mathbf{F}_k^+[t]/R(\mathbf{F}_k^+[t])$. We call the space

$$(\mathcal{F}_k^+[t])^* = \mathrm{Hom}_{\mathbf{C}[t]}(\mathcal{F}_k^+[t], \mathbf{C}[t])$$

the *weight space of orientable k-flowers.* Its elements are *weights of degree k*. Elements of $\prod(\mathcal{F}_k^+[t])^*$ are *weight systems* for orientable flowers.

Recall that we can associate to each $k-$chord diagram D an orientable $k-$flower $F(D)$ (replacing all chords of D by pairs of parallel arcs and drawing trivial outer arcs around the heart).

Proposition I.2.1. *The* $\mathbf{C}[t]-$*linear extension of the application*

$$D \longmapsto F(D)$$

induces a homomorphism $i_k^+ : \mathcal{D}_k[t] \longrightarrow \mathcal{F}_k^+[t]$.

Proof. This is obvious from the definition of $F(D)$ and from the definition of relations in $\mathbf{F}_k^+[t]$. Indeed, four diagrams forming a 4T relation in $\mathbf{D}_k$ yield flowers forming a 4T2 relation in $\mathbf{F}_k^+[t]$. QED

In section I.3 we will construct a $\mathbf{C}[t]-$linear application $\rho_k^+ : \mathcal{F}_k^+[t] \longrightarrow \mathcal{D}_k[t]$ which is a retraction of i_k^+, ie the composition $\rho_k^+ \circ i_k^+$ is the identity on $\mathcal{D}_k^+[t]$.

Proposition I.2.2. *We have* $\pi^+(R(\mathbf{F}_{k+1}^+[t])) \subset R(\mathbf{F}_k^+[t])$.

Proof: Given a 4T2 relation $r \in R(\mathbf{F}_{k+1}^+[t])$, the element $\pi^+(r)$ is (by definition of 4T1 relations) a linear combination of 4T1 and 4T2 relations.

For a 4T1 relation $r \in R(\mathbf{F}_{k+1}^+[t])$, an inspection shows that $\pi^+(r)$ is the sum of k 4T1 relations in $R(\mathbf{F}_k^+[t])$ (this follows also from the fact that pushing out the heart of the flowers both chords implied in a 4T2 relation of $\mathbf{F}^+_{k+2}[t]$ yields an alternate sum of four elements in $\mathbf{F}^+_{k+1}[t]$ which annihilate two by two). QED

This last proposition allows us to define a projection $\mathcal{F}^+_n[t] \longrightarrow \mathcal{F}^+_k[t]$ by considering

$$F \longmapsto \frac{}{(n-k)!}(\pi^+)n - k(F)$$

for any orientalbe $n-$flower F. We have hence the following result:

Corollary I.2.3. *A weight $\mu \in (\mathcal{F}^+_k[t])^*$ of orientable $k-$flowers defines a weight system Z_μ with values in $\mathbf{C}[t]$ on orientable flowers and on chord diagrams.*

Proof. Given a weight $\mu \in (\mathcal{F}^+_k[t])^*$ and $F \in \mathbf{F}^+_n$ we set $Z_\mu(F) = 0$ if $n < k$ and

$$Z_\mu(F) = \mu\big(\frac{1}{(n-k)!}(\pi^+)^{n-k}(F)\big) = \sum_{I \subset C(F),\ \sharp(I)=k} \mu(F(I))$$

otherwise. For an $n-$chord diagram $D \in \mathbf{D}_n$ we set $Z_\mu(D) = Z_\mu(F(D))$ where $F(D)$ is the simple $n-$flower associated to D.

Corollary I.2.3 is now implied by propositions I.2.1 and I.2.2. QED

We call the weight system Z_μ constructed in the proof of Corollary I.2.3 the *partition function* associated to μ.

Another immediate consequence of proposition I.2.3 and corollary I.2.3 is the following result.

Corollary I.2.4. *(i) The adjoint $(\pi^+)^*$ of the linear operator π^+ induces a homomorphism (still written) $(\pi^+)^* : (\mathcal{F}_k[t])^* \longrightarrow (\mathcal{F}_{k+1}[t])^*$ from the space of weights of orientable $k-$flowers into the space of weights of orientable $(k+1)-$flowers.*

(ii) For $F \in \mathcal{R}^+_n[t]$ with $n \geq k+1$ and $\mu \in (\mathcal{F}^+_k[t])^*$ we have

$$Z_{(\pi^+)^*(\mu)}(F) = (n-k)Z_\mu(F) \quad .$$

Remark I.2.5. One could introduce the analog of 1T relations of chord diagrams for orientable flowers. Since this is messy, useless (because of the renormalization operator) and yields much less weight systems we omit it.

I.3. Standard weights

In this section we construct a retraction $\rho^+_k : \mathcal{F}^+_k[t] \longrightarrow \mathcal{D}_k[t]$ of the homomorphism i^+_k considered in proposition I.2.1. This implies that each weight $\tilde{\mu}$ can be extended to a weight system $Z_\mu : \mathcal{D} \longrightarrow \mathbf{D}[t]$ by setting $\mu = (\rho_k^+)^*(\tilde{\mu})$ and applying corollary I.2.3. We call the weight $\mu = (\rho_k^+)^*(\tilde{\mu})$ the *standard weight* associated to $\tilde{\mu}$.

Proposition I.3.4 characterizes them and shows the relationship between them and invariants associated to $gl(n)$.

Let F be a $k-$flower represented by $\alpha, \beta \in \mathrm{Sym}(\pm b_1, \dots, \pm b_{2k})$. Let $P_1 \cup \dots \cup P_l = \{\pm b_1, \dots, \pm b_{2k}\}$ be the finest partition of the set $\{\pm b_1, \dots, \pm b_{2k}\}$ which satisfies the following conditions:

(i) $b_i \in P_j \Longleftrightarrow -b_i \in P_j$,

(ii) $\pm b_i \in P_j \Longleftrightarrow \beta(b_i), \beta(-b_i) \in P_j$,

(iii) $P_j \neq \emptyset$ for $1 \leq j \leq l$.

Set $\lambda(F) = l - 1$.

Let us recall that given an orientable $k-$flower F we denote by $D(F)$ the associated $k-$chord diagram formed by the heart of the flower. We extend the application $R \longmapsto D(F)$ which associates to an orientable flower its heart by $\mathbf{C}[t]-$ linearity to an application $D : \mathbf{F}^+[t] \longrightarrow \mathbf{D}[t]$.

Theorem I.3.1. *The* $\mathbf{C}[t]-$*linear application* $\rho_k^+ : \mathbf{F}_k^+[t] \longrightarrow \mathbf{D}_k^+[t]$ *defined by*

$$\rho_k^+(F) = t^{\lambda(F)} D(F)$$

induces a $\mathbf{C}[t]-$*linear application (still called)* $\rho_k^+ : \mathcal{F}_k^+[t] \longrightarrow \mathcal{D}_k^+[t]$.

Proof. We have to show that $\rho_k^+(r) \in R(\mathbf{D}_k[t])$ for $r \in R(\mathbf{F}_k^+[t])$ a 4T2 or 4T1 relation.

An inspection of figure I.2.2 shows that $\lambda(F_1) = \lambda(F_2) = \lambda(F_3) = \lambda(F_4)$ if $r = F_1 - F_2 + F_3 - F_4$ is a 4T2 relation. We denote this common value by $\lambda(r)$. We have hence $\rho_k^+(r) = (D(F_1) - D(F_2) + D(F_3) - D(F_4))\, t^{\lambda(r)}$. Since $D(F_1) - D(F_2) + D(F_3) - D(F_4)$ is a 4T relation of $k-$chord diagrams in $\mathbf{D}_k[t]$ we have $\rho_k^+(r) \in R(\mathbf{D}_k[t])$.

Given a 4T2 relation $\tilde{r} = F_1 - F_2 + F_3 - F_4$ of orientable $(k+1)-$flowers we consider the associated 4T1 relation of orientable $k-$flowers. Let us write $F_i' = F_i(C(F_i) \setminus \{c'\}) \in \mathbf{F}_k^+[t]$ and $F_i'' = F_i(C(F_i) \setminus \{c''\}) \in \mathbf{F}_k^+[t]$ where c', c'' are the two special chords involved in $\tilde{r}$.

The element

$$r = F_1' + F_1'' - F_2' - F_2'' + F_3' + F_3'' - F_4' - F_4'' \in \mathbf{F}_k^+[t]$$

is then a 4T1 relation in $\mathbf{F}_k[t]$ and every 4T1 relation is of this kind. One checks that $\rho_k^+(F_1') = \rho_k^+(F_2')$, $\rho_k^+(F_1'') = \rho_k^+(F_2'')$ and $\rho_k^+(F_3') = \rho_k^+(F_4')$, $\rho_k^+(F_3'') = \rho_k^+(F_4'')$ which shows that $\rho_k^+(r) = 0$ for any 4T1 relation $r \in \mathbf{F}_k^+[t]$. QED

One has $\rho_k^+(F(D)) = D(F(D))t^{\lambda(F(D))} = D$ for any chord diagram D (recall that $F(D))$ denotes the unique orientable simple flower with heart the chord diagram D). This shows that $\rho_k^+ \circ i_k^+ : \mathbf{D}_k[t] \longrightarrow \mathbf{D}_k[t]$ is the identity on $\mathbf{D}_k[t]$ and ρ_k^+ is hence a retraction of the $\mathbf{C}[t]-$linear operator $i_k^+ : \mathbf{D}_k[t] \longrightarrow \mathbf{F}_k^+[t]$ introduced in proposition I.2.1.

The adjoint operator $(\rho_k^+)^*$ induces a $\mathbf{C}[t]-$linear operator from the weights $(\mathcal{D}_k[t])^*$ on $k-$chord diagrams into the weights $(\mathcal{F}_k^+[t])^*$ on orientable $k-$flowers by setting $\mu(F) = \tilde{\mu}(\rho_k^+(F))$ for any weight $\tilde{\mu} \in (\mathcal{D}_k[t])^*$. We call the weight $\mu = (\rho_k^+)^*(\tilde{\mu})$ the *standard weight* associated to $\tilde{\mu}$.

We have hence the following result.

Proposition I.3.2. *Given a weight $\tilde{\mu} \in (\mathcal{D}_k)^*$ on $k-$chord diagrams, the partition function Z_μ defined in corollary I.2.3 with $\mu = (\rho_k^+)^*(\tilde{\mu})$ extends $\tilde{\mu}$ to a weight system with values in $\mathbf{C}[t]$.*

Example I.3.3. Let $\tilde{\mu}_0 : \mathbf{D}_0 \longrightarrow \mathbf{C}$ be the weight which sends the unique chord diagrams without chords onto 1. The weight system Z_{μ_0} constructed by the previous corollary is given by the graphical calculus associated to the Lie algebras $gl(n)$ (see [**B**] for the details).

The following proposition describes weight systems on $\mathcal{D}$ which come from standard weights.

Proposition I.3.4. *Let $\tilde{\mu} \in (\mathcal{D}_k)^*$ be a weight and let $\mu \in (\mathcal{F}_k^+[t])^*$ be the associated standard weight. The weight system $Z_\mu \in \mathcal{D}^*$ is then the product of $\tilde{\mu}$ (extended to a weight system by setting $\tilde{\mu}(D) = 0$ if $D \notin \mathbf{D}_k$) with the weight system Z_{μ_0} described in example I.3.3.*

This last result shows that standard weights yield no interesting (new) weight systems.

I.4. Spin models

In this section we introduce spin models and rephrase corollary I.2.3 in terms of spin models.

Definition I.4.1. A *spin model* with values in a vector space V is defined by the following data:

(1) A finite set Sp of *spins*.

(2) A finite set C.

(3) A finite set S of *sites*.

(4) An *energy function* $e : S \times Sp^C \longrightarrow V$.

(5) A function $f : Sp^C \longrightarrow \mathbf{C}$.

Given a spin model $SM(Sp, C, S, e, f)$, an element $\sigma \in Sp^C$ is a *state* of the model. A state defines a *spin function* $\sigma : C \longrightarrow Sp$ on the set C which assigns a spin $\sigma(c) \in Sp$ to every element $c \in C$. We identify states with their spin functions.

A state σ yields a function $I \longmapsto e(I, \sigma) \in V$ from the set S of sites into the vector space V. The element $e(I, \sigma) \in V$ is the *energy* of the site I in the state σ.

The element $f(\sigma) \sum_{I \in S} e(I, \sigma)$ is the *energy of the state* σ.

The element

$$Z_{SM} = \sum_{\sigma \in Sp^C} f(\sigma) \sum_{I \in S} e(I, \sigma) \in V$$

obtained by summing up the energies of all states is the *partition function* of the model.

Remarks I.4.2. (i) This definition of spin models is of course not well suited for statistical physics. There are however analogies with models coming from statistical physics which justify hopefully the usurpation of terminology.

(ii) We would like to have more precise and more restricted definitions. Indeed, the energy function should be “local” in some sense and the function f should depend on the spin functions in a “global” and simple way. This justifies the separation of the function f from the energy function e.

(iii) In most examples the set C will be the set of chords in a chord diagram (or in a flower) and the set I of sites will consist in all subsets $I \subset C$ containing exactly k elements (for some fixed integer k).

Given a positive integer k, we define for each orientable $n-$flower F a spin model with values in the vector space $\mathbf{F}_k^+[t]$ spanned by orientable $k-$flowers with polynomial coefficients. This model is given by the data:

(1) The set of spins is the set $\{+1\}$.

(2) The set C is the set $C(F)$ of chords in F.

(3) The set S of sites is the set $\{I \subset C(F) \mid \sharp(I) = k\}$ of all subsets with k chords in $C(F)$.

(4) A subset $I \subset C(F)$ yields an element $F(I) \in \mathbf{F}_k^+[t]$ (which is an orientable $k-$flower multiplied by a power of t). The energy $e(I)$ of the site I (in the unique state of the model) is the element $F(I) \in \mathbf{F}_k^+[t]$.

(5) Since our model has only one state the function f is the constant 1.

Since there is only one state in our model, the partition function $Z_k = Z_{SM}$ boils down to

$$Z_k(F) = \sum_{I\subset C,\ \sharp(I)=k} F(I) \in \mathbf{F}_k^+[t] \quad .$$

Extending Z_k by $\mathbf{C}[t]-$linearity, we get a linear application $Z_k : \mathbf{F}^+[t] \longrightarrow \mathbf{F}_k^+[t]$ which we continue to call a partition function.

Choosing a $\mathbf{C}[t]-$linear form $\mu \in (\mathbf{F}_k^+[t])^*$ we get a spin model with values in $\mathbf{C}[t]$ by replacing the energy function in the above model with $e(I) = \mu(F(I))$. This model has partition function $Z_\mu(F) = \mu(Z_k(F))$ which we extend $\mathbf{C}[t]-$linearly to $\mathbf{F}^+[t]$.

Theorem I.4.3. *(i) Let $r \in R(\mathbf{F}_n^+[t])$ be a 4T2 or a 4T1 relation. The partition function $Z_k(r) \in \mathbf{F}_k^+[t]$ is then an element of the subspace $R(\mathbf{F}_k^+[t])$ spanned by all 4T2 and 4T1 relations in $\mathbf{F}_k^+[t]$.*

(ii) Let $\mu \in (\mathcal{F}_k^+[t])^*$ be a weight of orientable $k-$flowers. The associated partition function

$$F \longmapsto Z_\mu(F) = \sum_{I\subset C,\ \sharp(I)=k} \mu\big(F(I)\big) \in \mathbf{C}[t]$$

defines then a weight system of orientable flowers.

Assertions (i) and (ii) of this theorem are obviously equivalent and assertion (ii) is a translation of Corollary I.2.3 into the formalism of spin models.

Proof of Theorem I.4.3. We prove only (i). Let us first consider a 4T2 relation $r = F_1 - F_2 + F_3 - F_4 \in \mathbf{F}_n^+[t]$ and compute the contribution of all sites to the partition function $Z_k(r)$.

We call a site $S \subset C(r)$ of type i if it contains exactly i chords among the special chords c', c'' involved in r. We denote by S_i the set of sites of type i. We have hence a partition $S = S_0 \cup S_1 \cup S_2$ of all sites in $C(r)$ into three subsets according to their type.

We compute the contributions of sites of type $2,\ 1$ or 0 to the partition function.

First case: Consider a site I of type 2. This implies that the element $r_I = F_1(I) - F_2(I) + F_3(I) - F_4(I)$ is a 4T2 relation (perhaps multiplied by a power of t). The contribution of a site $I \in S_2$ to Z_k is hence in $R(\mathbf{F}_k^+[t])$.

Second case: Sites of type 1. These sites exist in pairs I', I'' such that $c' \in I'$ and $c'' \in I''$ and $(I' \setminus \{c'\}) = (I'' \setminus \{c''\})$. The element $F(I') + F(I'') \in \mathbf{F}_k^+[t]$ is then a 4T1 relation (perhaps multiplied by a power of t) and the sum of the energies of both sites I' and I'' is in $R(\mathbf{F}_k^+[t])$.

Third case: Sites of type 0. An inspection of figure I.2.2 shows that pushing outside the flowers F_i all arcs associated to the two chords involved in a relation of type 4T2 yields 2 pairs $\{F_1, F_4\}$ and $\{F_2, F_3\}$ of isomorphic flowers (perhaps multplied by common powers of t). We have hence $F_1(I) = F_4(I)$ and $F_2(I) = F_3(I)$ and their contributions to the partition function cancel mutually. The contribution of a site of type 0 to the partition function is hence zero.

4T1 relations can be treated in the same way. The details are easy and left to the reader. One can also use the fact that for $F \in \mathbf{F}_n^+[t]$ with $n > k$

$$(n-k)Z_k(F) = Z_k(\pi^+(F))$$

where $\pi^+ : \mathbf{F}_n^+[t] \longrightarrow \mathbf{F}_{n-1}^+[t]$ is the linear operator introduced in section I.2. QED

I.5 Homogeneous relations

In this section we describe for a fixed positive integer n a subspace $R(\mathbf{F}_k^+[t])_n$ of the space $R(\mathbf{F}_k^+[t])$ of relations in orientable $k-$flowers such that $Z_k(r) \in R(\mathbf{F}_k^+[t])_n$ for any 4T2 relation of orientable $n-$flowers.

The results of this section may no longer hold if one works over a field or a ring which does not contain the rationals (in fact, if $(n-k)$ has no multiplicative inverse).

Let $r = F_1 - F_2 + F_3 - F_4$ be a 4T2 relation of $(k+1)-$flowers. Let c', c'' be the two special chords involved in the relation r. For any chord $c \in C(r)$ we set

$$r_c = F_1(C(F_1) \setminus \{c\}) - F_2(C(F_1) \setminus \{c\}) + F_3(C(F_1) \setminus \{c\}) - F_4(C(F_1) \setminus \{c\}) \quad .$$

We call the element

$$(n-k)(r_{c'} + r_{c''}) + \sum_{c \in C(r),\ c \neq c', c''} r_c \in \mathbf{F}_k^+[t]$$

a *homogenous 4T relation* of degree n.

A homogeneous relation of degree n is hence a sum of the 4T1 relation $(r_{c'} + r_{c''})$ (taken $(n-k)$ times) and of the $(k-1)$ 4T2 relations $r_c,\ \ c \neq c', c''$. The $\mathbf{C}[t]-$linear span $R(\mathbf{F}_k^+[t])_n$ of all homogeneous relations of degree n is hence a subspace of the space $R(\mathbf{F}_k^+[t])$ spanned by all 4T2 and 4T1 relations.

Theorem I.5.1. *For any 4T2 relation $r \in \mathbf{F}_n^+[t]$ the partition function $Z_k(r)$ is an element in the space $R(\mathbf{F}_k^+[t])_n$ of homogeneous 4T relations of degree n.*

Proof. Consider a subset $J \subset C(r)$ of $k+1$ chords containing the two special chords c', c''. In other words, J is a type 2 subset of $(k+1)$ chords in $C(r)$. Each such subset J contains two appariated sites of type 1 and $k-1$ sites of type 2. Appariated sites of type 1 are in bijection with such subsets whereas each site of type 2 belongs to $(n-k)$ different sets J as above. A simple counting argument and the fact that type 0 sites contribute nothing to $Z_k(r)$ yield then the result. QED

Corollary I.5.2. *Let $\mu : \mathbf{F}_k^+[t] \longrightarrow \mathbf{C}[t]$ be a $\mathbf{C}[t]-$linear application containing $R(\mathbf{F}_k^+[t])_n$ in its kernel.*

The restriction of the partition function Z_μ to $n-$chord diagrams yields then a weight of degree n.

I.6. Non-orientable flowers

Definition I.6.1. A *$k-$flower* (or a *non-orientable $k-$flower*) is an equivalence class of two permutations α, β on the set $E = \{-b_1, b_1, \ldots, -b_{2k}, b_{2k}\}$ such that

(i) α and β are involutions of E without fixed points,

(ii) α extends to a linear application of the vector space generated by E,

(iii) $\alpha(b_i) \neq -b_i$ for $i = 1, \ldots, 2k$.

Equivalence classes are defined as in the case of orientable flowers. The difference between flowers and orientable flowers is requirement (iv) (the orientability condition) in definition I.2.1 for orientable flowers.

Flowers have of course graphical representations.

The heart of a flower F is a chord diagram $D(F)$ together with a function $\sigma : C(D(F)) \longrightarrow \{\pm 1\}$ defined by $\sigma(c) = 1$ if a chord $c = \{\pm b_i, \pm b_j\}$ is given by $\alpha(b_i) = -b_j$ and $\sigma(c) = -1$ otherwise (ie. $\alpha(b_i) = b_j$ and $\alpha(b_j) = b_i$). We call such a function a *spin function*. A flower F with $\sigma(c) = 1$ for every chord $c \in C(F)$ is a *positive* flower. An arbitrary flower can also be considered as a positive flower together with a spin function.

Given a chord diagram D with k chords and a spin function $\sigma : C(D) \longrightarrow \{\pm 1\}$, we define the *simple flower* $F(D,\sigma)$ associated to the chord diagram D with spin function σ to be the unique flower with heart D, spin function σ and which has only trivial outer arcs (ie. if $\alpha,\ \beta \in \mathrm{Sym}(\{-b_1, b_1, \ldots, -b_{2k}, b_{2k}\})$ represent $F(D,\sigma)$ then $\beta(b_i) = -b_{i+1}$ for $i = 1, \ldots, 2k-1$ and $\beta(b_{2k}) = -b_1$).

We denote by $\mathbf{F}[t]$ the graded $\mathbf{C}[t]-$module generated by all flowers with grading given by the number of chords in a flower. We denote moreover by $\mathbf{F}^p[t]$ the graded $\mathbf{C}[t]-$module generated by all positive flowers.

Given any subset $I \subset C(F)$ of k chords in an $n-$flower F, we define the element $F(I) \in \mathbf{F}_k[t]$ by

$$F(I) = t^\nu \tilde{F}(I)$$

where $\tilde{F}(I)$ is the flower obtained by pushing out the heart of F all arcs in chords $\notin I$ and by deleting the ν isolated closed arcs created during this process outside the heart.

Given a positive flower $F \in \mathbf{F}^p_{n+1}[t]$ and a chord $c \in C(F)$ we denote by $F(C(F) \setminus \{c\}) \in \mathbf{F}^p_n[t]$ the obvious element obtained by pushing the chord c out the heart of F. We denote moreover by $F(C(F) \setminus \{\overline{c}\}) \in \mathbf{F}^p_n[t]$ the element obtained by changing first the spin of the chord c and then pushing this out of the flower.

We define a $\mathbf{C}[t]-$linear operator $\pi^p : \mathbf{F}^p_{n+1}[t] \longrightarrow \mathbf{F}^p_n[t]$ by

$$\pi(F) = \sum_{c \in C(F)} F(C(F) \setminus \{c\}) - F(C(F) \setminus \{\overline{c}\}) \quad .$$

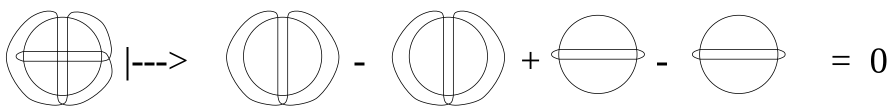

Figure I.6.1 An example for the operator π^p.

As for orientable flowers there exist relations for positive flowers.

The definition of 4T2 relations for positive flowers is exactly the same as for orientable flowers (except that all flowers involved are only positive), see figure I.2.2.

A *4T1 relation* is the element $r \in \mathbf{F}^p_k[t]$ obtained by applying π^p to a 4T2 relation $\tilde{r} \in \mathbf{F}^p_{k+1}$ and by removing the $2(k-1)$ obvious $4T2$ relations in $\mathbf{F}^p_k$ from $\pi^p(\tilde{r})$. A 4T1 relation is hence a $\mathbf{C}[t]-$linear combination of sixteen positive $k-$flowers.

We denote by $R(\mathbf{F}^p_k[t])$ the $\mathbf{C}[t]-$module generated by all 4T2 and 4T1 relations in $\mathbf{F}^p_k[t]$. The *positive flower space* $\mathcal{F}^p_k[t]$ is the quotient space $\mathcal{F}^p_k[t] = \mathbf{F}^p_k[t]/R(\mathbf{F}^p_k[t])$. The space

$$(\mathcal{F}^p_k[t])^* = \mathrm{Hom}_{\mathbf{C}[t]}(\mathcal{F}^p_k[t], \mathbf{C}[t])$$

is the *weight space of positive k-flowers*. Its elements are *weights of degree k*. Elements of $\prod(\mathcal{F}^+_k[t])^*$ are *weight systems* for positive flowers.

Given a $k-$chord diagram D, its associated simple orientable $k-$flower $F(D)$ is of course a positive flower. We have now analogs of all statements given for orientable flowers before. We list the analogous results without proofs.

Proposition I.6.2. *The $\mathbf{C}[t]-$linear extension of the application*

$$D \longmapsto F(D)$$

induces a homomorphism $i^p_k : \mathcal{D}_k[t] \longrightarrow \mathcal{F}^p_k[t]$.

Proposition I.6.3. *We have $\pi^p(R(\mathbf{F}^p_{k+1}[t])) \subset R(\mathbf{F}^p_k[t])$.*

Corollary I.6.4. *A weight $\mu \in (\mathcal{F}_k^p[t])^*$ of positive $k-$flowers defines a weight system Z_μ (given by*

$$Z_\mu(F) = \mu\Big(\frac{1}{(n-k)!}(\pi^p)^{n-k}(F)\Big)$$

for a positive $n-$chord flower F with $n \geq k$) with values in $\mathbf{C}[t]$ on positive flowers and on chord diagrams.

Corollary I.6.5. *(i) The adjoint $(\pi^p)^*$ of the linear operator π^p induces a homomorphism (still written) $(\pi^p)^* : (\mathcal{F}_k^p[t])^* \longrightarrow (\mathcal{F}_{k+1}^p[t])^*$ from the space of weights of positive $k-$flowers into the space of weights of positive $(k+1)-$flowers.*

(ii) For $F \in \mathcal{R}_n^+[t]$ with $n \geq k+1$ and $\mu \in (\mathcal{F}_k^+[t])^*$ we have

$$Z_{(\pi^+)^*(\mu)}(F) = (n-k)Z_\mu(F) \quad .$$

The same definition as in I.3 works also for positive flowers and defines a retraction $\rho_k^p : \mathcal{F}_k^p[t] \longrightarrow \mathcal{D}_k[t]$ of the homomorphism i_k^p considered in proposition I.6.2 above. We have hence also standard weight systems for positive flowers. The standard weight system associated to positive $0-$flowers is given by the graphical calculus for the Lie groups $so(n)$ and we have of course the analogue of proposition I.3.4 in this setting.

Given a positive integer k we use non-orientable positive flowers to define a Spin model on an $n-$chord diagram D with values in $\mathbf{F}_k^p[t]$ in the sense of Definition I.3.1. The spin model is then defined by:

(1) The set of spins is the set $\{\pm 1\}$.

(2) The set C is the set $C(D)$ of chords in D.

(3) The set S of sites is the set $\{I \subset C(D) \mid \sharp(I) = k\}$ of all subsets with k chords in $C(D)$.

(4) A site $s \in S$ is a subset $I \subset C(D)$ of k chords in D. A spin function $\sigma : C(D) \longrightarrow \{\pm 1\}$ defines a simple flower $F(D, \sigma)$. We define then the energie function $e : S \times \{\pm 1\}^{C(D)} \longrightarrow \mathbf{F}_k[t]$ by

$$e(I, \sigma) = F(D, \sigma)(I)$$

if $F(D, \sigma)(I)$ is a positive flower multiplied by a power of t and by $e(I, \sigma) = 0$ otherwise (we identify the subset I of chords in D with the obvious corresponding subset of chords in the simple flower $F(D, \sigma)$).

(5) The function $f : \sigma \times \mathbf{C}[t] \longrightarrow \mathbf{C}[t]$ is given by

$$f(\sigma) = \prod_{c \in C(D)} \sigma(c) \quad .$$

We have hence $f(\sigma) = -1$ if the state σ has an odd number of chords with spin -1 and $f(\sigma) = 1$ otherwise.

The partition function of this spin model is given by

$$Z_k(D) = \sum_{\sigma\in\{\pm1\}^{C(D)}} \big(\prod_{c\in C(D)} \sigma(c)\big) \sum_{I\subset C(D),\, \sharp(I)=k,\, \sigma(c)=1\ \forall c\in I} F(D,\sigma)(I) \quad .$$

It is easy to check that we have also

$$Z_k(D) = \frac{1}{(n-k)!} \sum_{\sigma\in\{\pm1\}^{C(D)}} \big(\prod_{c\in C(D)} \sigma(c)\big)(\pi^p)^{n-k}(F(D)) \quad .$$

We extend Z_k in order to get a $\mathbf{C}[t]-$linear application $\mathbf{D}[t] \longrightarrow \mathbf{F}^p_k[t]$.

The important feature of the partition function Z_k is given by the following Lemma:

I.6.6. *Let $r \in \mathbf{D}_n$ be a 4T relation. Let $I \subset C(r)$ be a site of type 0 (containing neither of the special chords c', c'' involved in r).*

Then the site I contributes nothing to $Z_k(r)$ (ie

$$\sum_{\sigma\in Sp^{C(r\setminus I)}} f(\sigma)F(D,\sigma)(I) = 0 \quad).$$

The proof is an inspection of figure I.6.2 below.

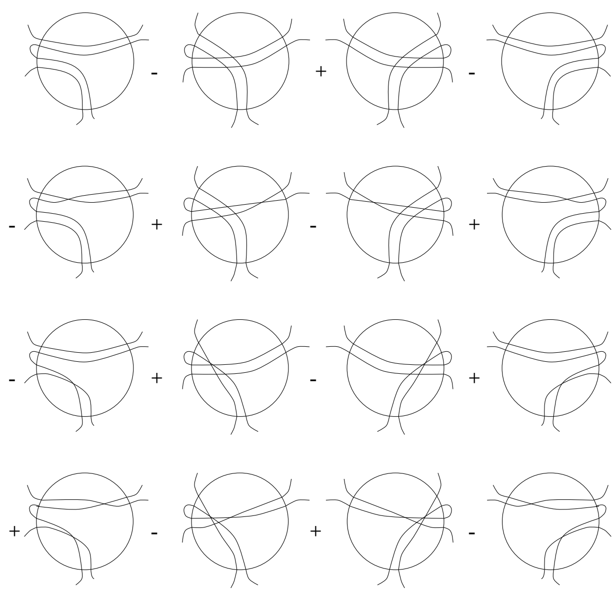

Figure I.6.2. Proof of lemma I.6.6.

As in the case of orientable flowers, a 4T2 relation is an elementary contribution of a site of type 2 to the partition function and a 4T1 relation corresponds to the contribution of two appariated type 1 sites.

In analogy with section I.5 we can describe a subspace $R(\mathbf{F}_k^p[t])_n \subset R(\mathbf{F}_k^p[t])$ such that $Z_\mu(r) \in R(\mathbf{F}_k^p[t])_n$ for any 4T relation $r \in \mathbf{D}_n$ and for any element $\mu \in (\mathbf{F}_k^p[t])^*$ containing $R(\mathbf{F}_k^p[t])_n$ in its kernel.

More precisely, given a 4T2 relation $r = F_1 - F_2 + F_3 - F_4 \in \mathbf{F}_{k+1}^p[t]$ of positive $(k+1)$ flowers, we set for any $c \in C(r)$

$$r_c = \sum_{i=1}^{4} (-1)^i \big(F_i(C(r) \setminus \{c\}) - F_i(C(r) \setminus \{\overline{c}\}) \big)$$

with the same notation as in the beginning of the section.

We call then the element

$$(n-k)(r_{c'} + r_{c''}) + \sum_{c \in C(r),\, c \neq c', c''} r_c \in \mathbf{F}_k^p[t]$$

a *homogeneous 4T relation of degree n* in $\mathbf{F}_k^p[t]$.

The space $R(\mathbf{F}_k^p[t])_n$ of all $\mathbf{C}[t]-$linear combinations of such relations has then the desired properties.

I.7. Flower diagrams for braid groups

This section gives an outline of how flowers can be used for braid groups. Only the pure braid group is considered but it is possible to deal with the ordinary braid group after some modifications.

Definition 7.1. A $k-$*chord diagram for the pure braid group* on $n-$strands is a sequence

$$\{c_1, c_1'\}, \{c_2, c_2'\}, \dots, \{c_k, c_k'\}$$

of k subsets with two elements in $\{1, 2, \dots, N\}$.

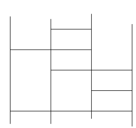

Figure I.7.1. A $5-$*chord diagram on the pure braid group with 4 strands*

Such diagrams have graphical representations as illustrated in figure I.7.1. They can be composed in the same way as braids.

The subsets $\{c_i, c_i'\}$ are the *chords* of the diagram.

4T relations of such diagrams are given by linear combinations of four diagrams as in figure I.7.2. We denote by $\mathbf{B}_k[t]$ the free $\mathbf{C}[t]-$module on all such $k-$chord diagrams and by $R(\mathbf{B}_k)$ the subspace spanned by all 4T relations.

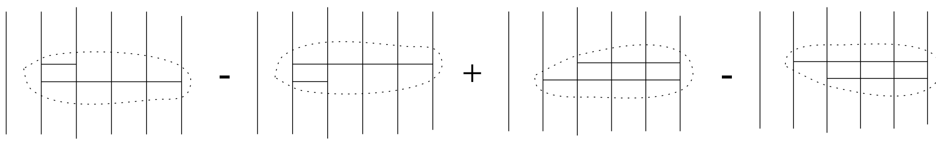

Figure I.7.2. 4T relations

We define again the *space of weights* $\mathcal{B}_k[t])^*$ of degree k where $\mathcal{B}_k[t] = \mathbf{B}_k[t]/R(\mathbf{B}_k[t])$.

Orientable pure braid flowers are defined as suggested by the examples of Figure I.7.3.

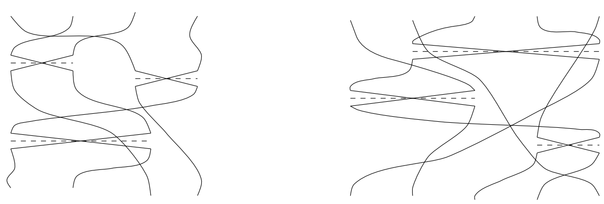

Figure I.7.3. Examples of orientable pure braid flowers

One defines also 4T2 and 4T1 relations according to figures I.7.4 and I.7.5.

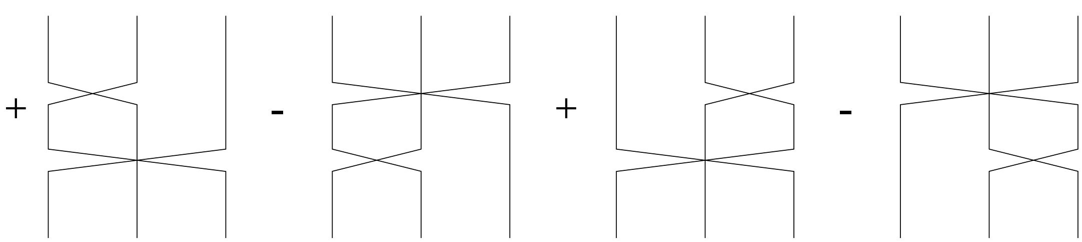

Figure I.7.4. 4T2 relations for orientable pure braid flowers

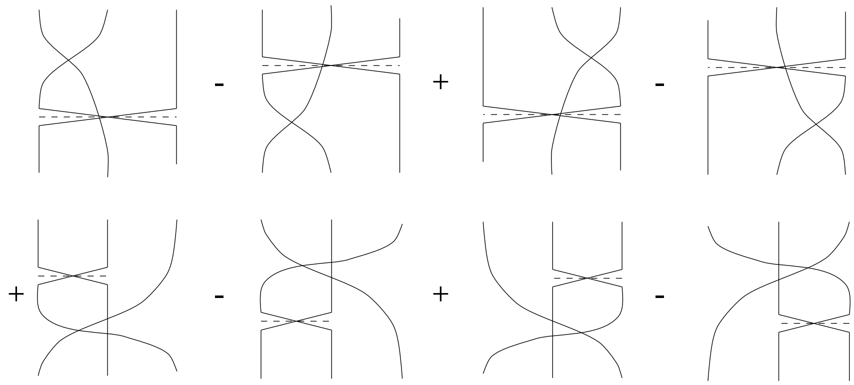

Figure I.7.5. 4T1 relations for orientable pure braid flowers

One can then define spin models, partition functions etc as discussed for chord diagrams.

Given a spin function $\sigma : C(B) \longrightarrow \{\pm 1\}$ (where $C(B)$ denotes the set of chords of such a diagram B) we associate to it a *simple (pure braid-)flower* with spin function σ by replacing all chords with two parallel or crossing arcs (according to the value of the spin function).

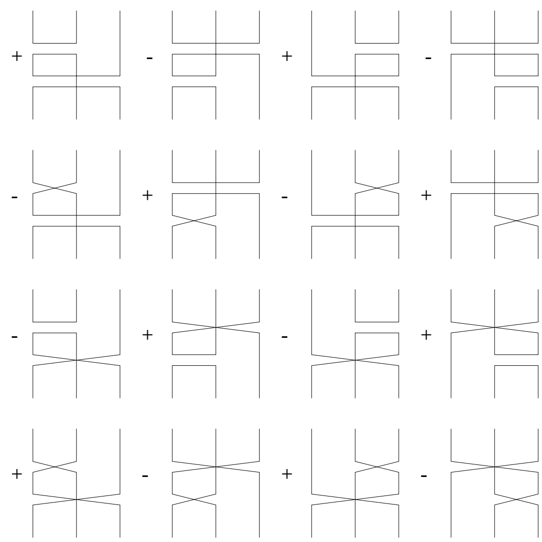

Figure I.7.6. 4T2 relations for non-orientable pure braid flowers

The obvious results for non-orientable flowers on chord diagrams adapt then and yield analogous results on pure braid flowers.

I.8. Some weight systems for small flowers

In this section we describe the weight spaces for small flowers.

Example I.8.1: Orientable $0-$**flowers.** They yield a unique weight system which is standard and corresponds to the graphical calculus associated to the Lie groups $gl(n)$.

Example I.8.2: Orientable $1-$**flowers.** There are only two orientable 1-flowers (the flowers A and B of figure I.8.4 below).

There are no 4T2 relations since they always involve at least two chords. A short computation (using symetries there is in fact only one case to check) yields that there are also no 4T1 relations. Orientable $1-$flowers yield hence two $\mathbf{C}[t]-$linearly independent weight systems.

The first one is the standard solution (given by $\mu(A) = t$ and $\mu(B) = 1$) and the second one is $(\pi^+)^*(\mu)$ where μ is the standard weight on $0-$flowers (and we have hence $(\pi^+)^*(\mu)(A) = 1$ and $(\pi^+)^*(\mu)(B) = t$).

Example I.8.3: Orientable $2-$**flowers.** (i) Let $F_1, \dots, F_{10}$ be the flowers represented by $\alpha(\pm b_1) = \mp b_3,\ \alpha(\pm b_2) = \mp b_4$ and β given by

$F_1:\quad \beta(\pm b_i) = \mp b_i \ (i = 1, 2, 3, 4),$

$F_2:\quad \beta(\pm b_1) = \mp b_1,\ \beta(\pm b_2) = \mp b_2$ and $\beta(\pm b_3) = \mp b_4,$

$F_3:\quad \beta(\pm b_1) = \mp b_1,\ \beta(\pm b_3) = \mp b_3$ and $\beta(\pm b_2) = \mp b_4,$

$F_4:\quad \beta(\pm b_1) = \mp b_1,\ \beta(-b_2) = b_3,\ \beta(b_2) = -b_4$ and $\beta(-b_3) = b_4,$

$F_5:\quad \beta(\pm b_1) = \mp b_1,\ \beta(-b_2) = b_4,\ \beta(b_2) = -b_3$ and $\beta(b_3) = -b_4,$

$F_6:\quad \beta(\pm b_1) = \mp b_2$ and $\beta(\pm b_3) = \mp b_4,$

$F_7:\quad \beta(\pm b_1) = \mp b_3$ and $\beta(\pm b_2) = \mp b_4,$

$F_8:\quad \beta(-b_1) = b_2,\ \beta(b_1) = -b_4,\ \beta(-b_2) = b_3$ and $\beta(-b_3) = b_4,$

$F_9:\quad \beta(-b_1) = b_2,\ \beta(b_1) = -b_3,\ \beta(-b_2) = b_4$ and $\beta(b_3) = -b_4,$

$F_{10}:\quad \beta(-b_1) = b_4,\ \beta(b_1) = -b_2,\ \beta(b_2) = -b_3$ and $\beta(b_3) = -b_4.$

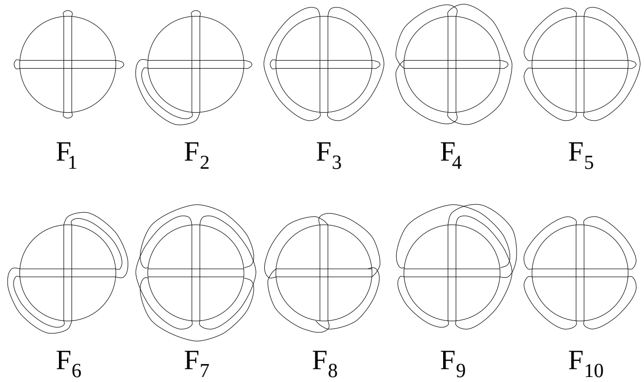

Figure I.8.1. The 10 orientable 2 flowers with crossing chords

We get a standard weight by setting $\mu(F_1) = t^3$, $\mu(F_2) = \mu(F_3) = t^2$, $\mu(F_4) = \mu(F_5) = \mu(F_6) = \mu(F_7) = t$ and $\mu(F_8) = \mu(F_9) = \mu(F_{10}) = 1$ and $\mu(F) = 0$ if F is an orientable $2-$flower with crossing chords.

(ii) Let $F_{11}, \ldots, F_{26}$ be the flowers represented by $\alpha(\pm b_1) = \mp b_4$, $\alpha(\pm b_2) = \mp b_3$ and β given by

$F_{11}:$ $\beta(\pm b_i) = \mp b_i$ $(i = 1, 2, 3, 4)$,

$F_{12}:$ $\beta(\pm b_1) = \mp b_1$, $\beta(\pm b_2) = \mp b_3$ and $\beta(\pm b_4) = \mp b_4$,

$F_{13}:$ $\beta(\pm b_1) = \mp b_1$, $\beta(\pm b_2) = \mp b_2$ and $\beta(\pm b_3) = \mp b_4$,

$F_{14}:$ $\beta(\pm b_1) = \mp b_3$, $\beta(\pm b_2) = \mp b_2$, $\beta(\pm b_4) = \mp b_4$,

$F_{15}:$ $\beta(\pm b_1) = \mp b_1$, $\beta(\pm b_2) = \mp b_4$, $\beta(\pm b_3) = \mp b_3$,

$F_{16}:$ $\beta(-b_1) = b_2$, $\beta(b_1) = -b_3$, $\beta(-b_2) = b_3$, $\beta(\pm b_4) = \mp b_4$,

$F_{17}:$ $\beta(-b_1) = b_2$, $\beta(b_1) = -b_4$, $\beta(-b_2) = b_4$, $\beta(\pm b_3) = \mp b_3$,

$F_{18}:$ $\beta(\pm b_1) = \mp b_2$, $\beta(\pm b_3) = \mp b_4$,

$F_{19}:$ $\beta(\pm b_1) = \mp b_1$, $\beta(-b_2) = b_4$, $\beta(b_2) = -b_3$, $\beta(b_3) = -b_4$,

$F_{20}:$ $\beta(-b_1) = b_3$, $\beta(b_1) = -b_2$, $\beta(b_2) = -b_3$, $\beta(\pm b_4) = \mp b_4$,

$F_{21}:$ $\beta(\pm b_1) = \mp b_3$, $\beta(\pm b_2) = \mp b_4$,

$F_{22}:$ $\beta(\pm b_1) = \mp b_4$, $\beta(\pm b_2) = \mp b_3$,

$F_{23}:$ $\beta(-b_1) = b_2$, $\beta(b_1) = -b_3$, $\beta(-b_2) = b_4$, $\beta(b_3) = -b_4$,

$F_{24}:$ $\beta(-b_1) = b_3$, $\beta(b_1) = -b_4$, $\beta(-b_2) = b_4$, $\beta(b_2) = -b_3$,

$F_{25}:$ $\beta(-b_1) = b_2$, $\beta(b_1) = -b_4$, $\beta(-b_2) = b_3$, $\beta(-b_3) = b_4$,

$F_{26}:$ $\beta(-b_1) = b_4$, $\beta(b_1) = -b_2$, $\beta(b_2) = -b_3$, $\beta(b_3) = -b_4$.

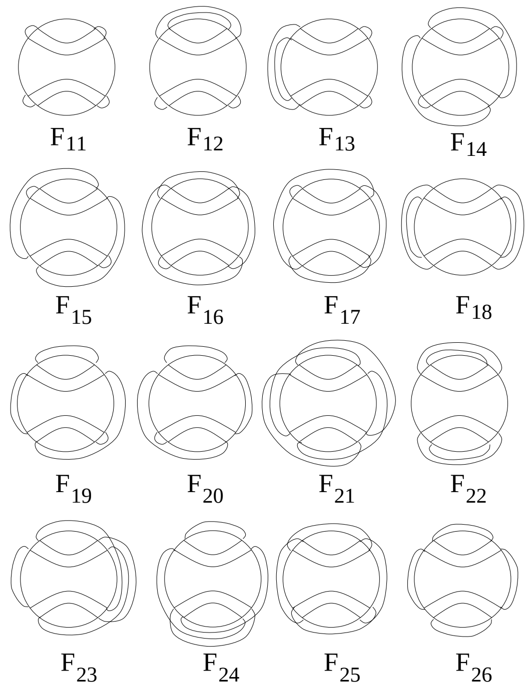

Figure I.8.2. The 16 orientable 2 flowers without crossing chords

Setting $\mu(F_{11}) = t^3$, $\mu(F_{12}) = \mu(F_{13}) = \mu(F_{14}) = \mu(F_{15}) = t^2$, $\mu(F_{16}) = \mu(F_{17}) = \mu(F_{18}) = \mu(F_{19}) = \mu(F_{20}) = \mu(F_{21}) = \mu(F_{22}) = t$ and $\mu(F_{23}) = \mu(F_{24}) = \mu(F_{25}) = \mu(F_{26}) = 1$ and $\mu(F_i) = 0$ for $i = 1 \dots 10$ we get another standard weight.

Two more $\mathbf{C}[t]-$linearly weight system on orientable $2-$flowers are of the form $(\pi^+)^*(\mu)$ where μ is a weight on orientable $1-$flowers.

There are two solutions which are only weights $\pmod t$ (ie only the constant term of $Z_\mu(r)$ is zero for a 4T relation $r \in \mathbf{D}$) and which are not standard. They are given as follows.

(iii) Setting $\mu(F_8) = 2$, $\mu(F_9) = 1$ and $\mu(F) = 0$ in all other cases and taking the result $\pmod t$ (ie considering only the constant term) one gets a weight system.

(iv) A second such solution is given by $\mu(F_2) = \mu(F_{13}) = \mu(F_{14}) = \mu(F_{15}) = \mu(F_{24}) = 1$, $\mu(F_8) = 2$ and $\mu(F) = 0$ in all other cases.

Let us compute the value of the weight systems (i)-(iv) on all $3-$chord diagrams. The associated orientable simple flowers are given by Figure I.8.3.

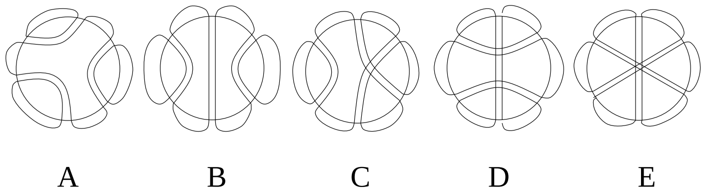

Figure I.8.3. The five orientable simple flowers associated to all 3−*chord diagrams*

Applying the operator π^+ to these five flowers and evaluating the weight systems μ_* described by (*) we get:

flower	π^+(flower)	μ_i	μ_{ii}	μ_{iii}	μ_{iv}
A	$3t\,F_{26}$	0	$3t$	0	0
B	$2t\,F_{26}+F_{22}$	0	$3t$	0	0
C	$t\,F_{10}+F_{19}+F_{20}$	t	$2t$	0	0
D	$2\,F_5+F_{18}$	$2t$	t	0	0
E	$3\,F_6$	$3t$	0	0	0

(generators of 4T-relations for 3−chord diagrams are given by $A-B$ and $C+E-2D$).

Example I.8.4: Non-orientable 0−**flowers.** They yield one solution which is standard.

Example I.8.5: Non-orientable positive 1−**flowers.**

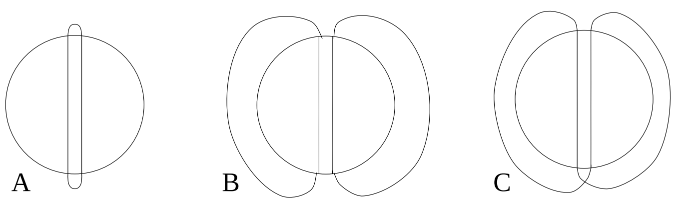

Figure I.8.4. The three non-orientable positive −flowers

There are three non-orientable positive 1-flowers. Since 4T2 relations always involve at least 2 chords there are no such relations. The computation of the subspace generated by 4T1 relations can be done by hand (using symmetries one has to consider only four different cases) and one gets the unique relation

$$2A-t(B+C)$$

with A, B, C the three positive 1-flowers of figure I.8.4.

We have two $\mathbf{C}[t]$−linearly independent weight systems. All solutions are $\mathbf{C}[t]$−linear combinations of a standard solution (given by $\mu(A)=t$ and $\mu(B)=\mu(C)=1$) and of $\pi^*(\mu)$ with μ standard (ie given by $\pi^*(\mu)(A)=0, \pi^*(\mu)(B)=t-1, \pi^*(\mu)(C)=1-t$).

II. Intersection properties

II.1. Preintersection matrices

Let $\mathbf{SD}_k$ denote the finite set of $k-$chord diagrams.

Given a $k-$chord diagram $D \in \mathbf{SD}_k$, its *automorphism group* $\mathrm{Aut}(D)$ is the group of all "rotations" which leave D invariant. More formally, let D be defined by a fixed-point free involution $\alpha \in \mathrm{Sym}(\{b_1, \dots, b_{2k}\})$ (where $\{b_1, \dots, b_{2k}\} \in S$ are the endpoints of chords in D, cyclically ordered by the orientation of S and where the orbits of α correspond to chords in D), its automorphism group $\mathrm{Aut}(D)$ is the cyclic group $\{\gamma \mid \gamma^j \alpha \gamma^{-j} = \alpha\} \subset \mathrm{Sym}(\{b_1, \dots, b_{2k}\})$ of all permutations γ which preserve the cyclic order of $\{b_1, \dots, b_{2k}\}$ and which commute with α.

Given a $k-$chord diagram $D \in \mathbf{SD}_k$, the k chords of it define (for $k \geq 1$) $2k$ open disjoint intervals on the oriented circle S supporting D (remove all endpoints of chords from the circle S). Denote these intervals by $J_1, \dots, J_{2k}$.

Recall that a subset $I \subset C(D)$ of k chords in an $n-$chord diagram D defines a $k-$chord diagram $D(I)$ obtained by erasing all chords of $C(D) \setminus I$. The endpoints of the k chords in $D(I)$ cut the circle S containing D also into $2k$ open intervals. Choosing an isomorphism ψ between $D(I)$ and the unique $k-$chord diagram $A \in \mathbf{SD}_k$ isomorphic to $D(I)$ we get a labelling $J'_1, \dots, J'_{2k}$ of these intervals by setting $J'_i = \psi(J_i)$. Since there are $|\mathrm{Aut}(A)|$ isomorphisms between $D(I)$ and A this labelling is not uniquely determined by the fixed labelling of the corresponding intervals in the diagram A.

We define *preintersection numbers* by

$$i_{st} = \sharp\{c \in C(D) \setminus I \text{ with extremities in } J'_s \text{ and } J'_t\} \quad .$$

We call the symmetric matrix $\tilde{M}_{D(I)}$ with entries i_{st} the *preintersection matrix* of $D(I)$. The chord diagram $D(I)$ is the *index* of the matrix $\tilde{M}_{D(I)}$.

The group $\mathrm{Aut}(A)$ of automorphisms of the diagram A acts on preintersection numbers by permuting their indices. It acts hence also on preintersection matrices (by $\tilde{M} \longmapsto P^t\ \tilde{M}\ P$ where P is the appropriate permutation matrix).

Example II.1.1. The following figure shows three chord diagrams D_1, D_2 and D_3 with subsets $I \subset C(D_i)$ indicated by chords which have been drawn as straight fat segments.

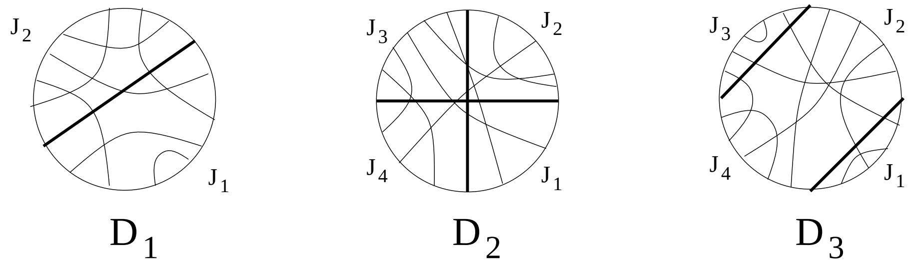

Figure II.1.1

For $I \subset D_1$ the preintersection numbers i_{st} (which are of course symmetric in s, t) are given by $i_{11} = i_{22} = 2$ and $i_{12} = 3$. The preintersection matrix is

$$\tilde{M}_{D_1(I)} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \quad .$$

The automorphism group of the unique $1-$chord diagram is cyclic of order two and acts on J_1, J_2 by transposing them. In the case of the chord diagram D_1, the automorphism group of $D_1(I)$ leaves the preintersection matrix invariant.

For $I \subset D_2$ the preintersection numbers are $i_{11} = i_{33} = i_{44} = i_{12} = i_{14} = 0,\ i_{22} = i_{23} = i_{24} = 1$ and $i_{34} = i_{13} = 2$ and we get the preintersection matrix

$$\tilde{M}_{D_2(I)} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

The automorphism group of the $2-$chord diagram $D_2(I)$ is the cyclic group of order four acting on the intervals $J_1, \ldots, J_4$ by permuting them cyclically. The orbit of preintersection matrices under $\mathrm{Aut}(D_2(I))$ is hence given by

$$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix} \quad .$$

For $I \subset D_3$ the preintersection numbers are $i_{22} = i_{14} = 0,\ i_{11} = i_{33} = i_{44} = i_{12} = i_{13} = i_{23} = i_{34} = 1$ and $i_{24} = 2$ and we get the preintersection matrix

$$\tilde{M}_{D_3(I)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

The automorphism group of the $2-$chord diagram $D_3(I)$ is the cyclic group of order two which acts on the intervals $J_1, \ldots, J_4$ by interchanging J_1, J_3 respectively J_2, J_4. The orbit of preintersection matrices under $\mathrm{Aut}(D_3(I))$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \quad .$$

II.2. A spin model

Choose a natural integer $p \geq 2$ which will be fixed in the sequel ($p = 1$ works but is uninteresting). Considering all $k-$preintersection matrices $\pmod p$ we get a finite set of symmetric $(2k) \times (2k)$ matrices with coefficients in the finite ring $\mathbf{Z}/p\mathbf{Z}$ and indices in the set of $k-$chord diagrams.

For a given $k-$chord diagram A the group $\mathrm{Aut}(A)$ of its automorphisms acts by permutation on all preintersection matrices with index A. Call two preintersection matrices *equivalent* if they are indexed by the same chord diagram A and if they are in the same orbit under $\mathrm{Aut}(A)$.

This action and the associated equivalence relation are of course compatible with the reduction $\pmod p$ of preintersection matrices.

We denote by

$$\mathbf{E}_k = \mathbf{E}_k(p)$$

the finite-dimensional vector space with basis all equivalence classes of $k-$preintersection matrices taken $\pmod p$.

Given a preintersection matrix $\tilde{M}_{D(I)}$ of index a $k-$chord diagram $D(I)$, we denote by $E_{D(I)} = E_{D(I)}(p)$ the corresponding element in $\mathbf{E}_k$.

We define a spin model values in $\mathbf{E}_k$ by considering the same settings as in section I.4 for orientable flowers except that the energy function of a site I is given by $e(I) = E_{D(I)} \subset \mathbf{E}_k$ and we extend the partition function

$$D \longmapsto Z_k(D) = \sum_{I \subset C(D),\ \sharp(I)=k} E_{D(I)} \in \mathbf{E}_k$$

$\mathbf{C}-$linearly to a partition function $Z_k : \mathbf{D} \longrightarrow \mathbf{E}_k$

We denote by $\tilde{R}(\mathbf{E}_k)$ the subspace $\tilde{R}(\mathbf{E}_k) \subset \mathbf{E}_k$ spanned by the set

$$\{\pi_k(r) \mid r \text{ a 4T relation of chord diagrams}\} \subset \mathbf{E}_k >$$

A linear form $\mu \in \mathbf{E}_k^*$ containing $\tilde{R}(\mathbf{E}_k)$ in its kernel defines a partition function

$$Z_\mu(D) = l(\pi_k(D)) = \sum_{I \subset C(D),\ \sharp(I)=k} \mu(E_{D(I)})$$

which extends linearly to a weight system on $\mathbf{D}$.

The main difficulty is of course the description of the subspace $\tilde{R}(\mathbf{E}_k)$. In the next section we exhibit a finite set of elements whose span contains $\tilde{R}(\mathbf{E}_k)$.

II.3. Relations

Let $r = D_1 - D_2 + D_3 - D_4 \in \mathbf{D}_n$ be a 4T relation of $n-$chord diagrams. Let $c', c'' \in C(r)$ be the two chords involved in r.

Let $I \subset C(D_i)$ be a $k-$chord subset of $C(r)$ which is of type 2, ie which contains c' and c''. We call the element

$$E_{D_1(I)} - E_{D_2(I)} + E_{D_3(I)} - E_{D_4(I)} \in \mathbf{E}_k$$

a *4T2 relation.*

Let I' and I'' be two appariated type 1 subsets of k chords in $C(r)$ (ie. $I' = (I' \cup I'') \setminus \{c'\}$ and $I'' = (I' \cup I'') \setminus \{c''\}$). We call the element

$$E_{D_1(I')} + E_{D_1(I'')} - E_{D_2(I')} - E_{D_2(I'')} + E_{D_3(I')} + E_{D_3(I'')} - E_{D_4(I')} - E_{D_4(I'')}$$

of $\mathbf{E}_k$ a *4T1 relation.*

Theorem II.3.1. *Let $R(\mathbf{E}_k) \subset \mathbf{E}_k$ be the vector space spanned by all 4T2 relations in $n-$chord diagrams with $n \leq k(2k+1)(p-1)+k$ and by all 4T1 relations given by $n-$chord diagrams with $n \leq (k+1)(2k+3)(p-1)+k+1$. Then the space $\tilde{R}(\mathbf{E}_k)$ is contained in $R(\mathbf{E}_k)$.*

Proof. Consider the contribution $E_{D_1(I)} - E_{D_2(I)} + E_{D_3(I)} - E_{D_4(I)} \in \mathbf{E}_k$ of a given site $I \subset C(r)$ to the partition function

$$Z(r) = \sum_{I \subset C(D_i),\ \sharp(I)=k} E_{D_1(I)} - E_{D_2(I)} + E_{D_3(I)} - E_{D_4(I)} \in \mathbf{E}_k$$

where $r = D_1 - D_2 + D_3 - D_4$ is a 4T relation of chord diagrams.

If I is a site of type 0 we have $E_{D_1(I)} = E_{D_2(I)}$ and $E_{D_3(I)} = E_{D_4(I)}$. Type zero sites contribute hence nothing to partition functions.

Consider now a site of type 2. The four preintersection matrices $\tilde{M}_{D_1(I)}, \dots, \tilde{M}_{D_4(I)}$ associated to these sites have the “same” entries (of course placed in a different manner). If such an entry is $\geq p$, we can remove the same set of p chords from the four diagrams $D_1, \dots, D_4$ without changing the matrices $\tilde{M}_{D_1(I)} \pmod p, \dots, \tilde{M}_{D_4(I)} \pmod p$ and we get finally a 4T relation $\tilde{r} = \tilde{D}_1 - \tilde{D}_2 + \tilde{D}_3 - \tilde{D}_4$ such that $E_{D_i(I)} = E_{\tilde{D}_i(\tilde{I})}$ and the partition function

$$Z(\tilde{r}) = E_{\tilde{D}_1(\tilde{I})} - E_{\tilde{D}_2(\tilde{I})} + E_{\tilde{D}_3(\tilde{I})} - E_{\tilde{D}_4(\tilde{I})}$$

of $\tilde{r}$ is a 4T2 relation. All $(2k)(2k+1)/2 = k(2k+1)$ different preintersection numbers i_{st} of this relation are $\leq p-1$ and

$$\sum_{s\leq t} i_{st} + k \leq \frac{2k(2k+1)}{2}(p-1) + k$$

equals the number of chords in the diagrams of the relation $\tilde{r}$. This implies $Z_k(\tilde{r}) = Z_k(r) \in R(\mathbf{E}_k)$.

Sites of type 1 are always associated in pairs. An argument similar to that used for sites of type 2 can be applied to the sum of contributions of two appariated type 1 sites and yields the result. QED

Remark II.2.2. (i) The estimations of the number n of chords given in Theorem II.2.1 are not optimal. Indeed they can easily be lowered by remarking that at least $2k$ of the preintersection numbers given by a site I of type 2 in a relation are zero. An analogous remark holds for sites of type I.

(ii) The set of generators described in theorem II.2.1 is highly redundant. It is indeed enough to consider 4T2 relations with respect to all subset I such that all preintersection numbers are $< p$. 4T1 relations are given by subsets J of type 2 with $k+1$ chords in a relation r and one can again assume that all preintersection numbers given by J are $< p$. Moreover, the exact position of chords in $C(D) \setminus I$ (or $C(D) \setminus J$) is irrelevant, only their relative positions with respect to chords in I (or J) matters.

(iii) The space $\mathbf{E}_k$ has a grading given by the number of chords $\pmod{p}$ in D for an element $M_{D(I)} \in \mathbf{E}_k$. Since all relations are also graded, one can solve all equations in the p different homogeneous parts.

Unfortunately, even using all simplifications implied by the above remarks awfully many variables and even more relations still remain.

II.4. Homogeneous relations

As in sections I.5 and I.8 it is possible to describe a subspace $R(\mathbf{E}_k)_n \subset R(\mathbf{E}_k)$ of homogeneous relations such that any linear function $\mu \in \mathbf{E}_k^*$ containing $R(\mathbf{E}_k)_n$ defines a partition function Z_μ which yields a weight of degree n on chord diagrams.

Let us take a 4T relation $r = D_1 - D_2 + D_3 - D_4$ with special chords c', c'' involved in the relation. Let us also choose a subset $I \subset C(r)$ of $k+1$ chords which is of type 2 (ie. $c', c'' \in I$). We define $E_c \in \mathbf{E}_k$ by

$$E_c = E_{D_1(I\setminus\{c\})} - E_{D_2(I\setminus\{c\})} + E_{D_3(I\setminus\{c\})} - E_{D_4(I\setminus\{c\})}$$

for any chord $c \in I$. We call the element

$$(n-k)(E_{c'} + E_{c''}) + \sum_{c\in I\setminus\{c',c''\}} E_c$$

a homogenous 4T relation of degre n in $\mathbf{E}_k$.

Such a relation is hence a sum of an unique 4T1 relation (taken $(n-k)$ times) and of $(k-1)$ 4T2 relations.

Theorem II.4.1. *Let $R(\mathrm{E}_k)_n$ be the subspace spanned by all homogenous 4T relations coming from $m-$chord diagrams with $m \equiv n \pmod p$ and $m \leq \min\{(k+1)(2k+3)(p-1)+k+1, n\}$.*

Then $Z_k(r) \in R(\mathbf{E_k})_n$ for any 4T relation of $n-$chord diagrams.

Proof. Given a 4T relation $r = D_1 - D_2 + D_3 - D_4$ of $n-$chord diagrams one checks that

$$(n-k)Z(r) = \sum_{J \subset C(D_i),\ \sharp(J)=k+1,\ c',c'' \in J} R_{J \subset C(r)}$$

where $R_{J \subset C(r)}$ denotes the homogeneous 4T relation of degree n associated to the subset J of $(k+1)$ chords in $C(r)$. As in the proof of theorem II.3.1 one can replace the relation r by a relation $\tilde{r}$ having less chords if n is too huge. QED

Remark II.4.2. For a fixed n, preintersection matrices coming from $n-$chord diagrams are in finite number and the positive integer p used for reducing preintersection matrices $\pmod p$ is useless if $p > n-k$.

II.5. Colours

It is possible to "colour" chords in $C(D)$ by spin functions $\sigma : C(D) \longrightarrow Sp = \{s_1, \ldots, s_r\}$ with values in a finite set Sp of spins or colours.

All definitions and results of sections II.1-4 generalize easily and lead to spin models which have states given by colorations.

In fact, different versions of such models can be set up.

The simplest spin model is defined by considering all states given by all possible colorations in $Sp^{C(D)}$

One can also restrict oneself to consider only states which have a prescribed number of chords in each colour. There are of course more relations to consider than in the first case. One works however with vector spaces having smaller dimensions.

II.6. A trivial spin model

The recipe for the construction of spin models is to consider some intrinsic "local" properties of chord diagrams which can be encoded by elements in a finite dimensional vector space and to study a spin model with values in this vector space. One has then to caracterize the subspace generated by relations (or at least a space containing it) thus getting weight systems on chord diagrams.

The easiest way to get intrinsic local properties out of preintersection matrices is to consider the sum of preintersection matrices under $\mathrm{Aut}(D(I))$ and to turn these matrices into a vector space. We define hence the *intersection matrix* $M_{D(I)}$ associated to a preintersection matrix $\tilde{M}_{D(I)}$ by

$$M_{D(I)} = \sum_{\varphi \in \mathrm{Aut}(D(I))} \varphi(\tilde{M}_{D(I)})$$

and we denote by $\mathbf{IM}_k$ the vector space spanned by all intersection matrices (the "coordinates" of $\mathbf{IM}_k$ are hence symmetric $(2k) \times (2k)$ matrices indexed by $k-$chord diagrams).

We define a partition function $Z_k : \mathbf{D}_n \longrightarrow \mathbf{IM}_k$ by

$$D \longmapsto Z_k(D) = \sum_{I \subset C(I),\ \sharp(I)=k} M_{D(I)}$$

and call the subspace $R(\mathbf{IM}_k)$ generated by all partition functions of 4T relations in $\oplus \mathbf{D}_n$ the space of relations.

Given a linear function $\mu : \mathbf{IM}_k \longrightarrow \mathbf{C}$ containing all elements of $R(\mathbf{IM}_k)$ in its kernel, we get a weight system on $\mathcal{D}$ by considering the linear application defined by the partition function

$$D \longmapsto Z_\mu(D) = \mu(Z_k(D)) = \sum_{I \subset C(I),\ \sharp(I)=k} \mu(M_{D(I)})$$

The following result shows that this construction is uninteresting. The main ingredient of its proof is the fact that for a given subset $I \subset C(D)$ the application $D \longmapsto M_{D(I)}$ is in some sense "linear" in the chords $c \in C(D) \setminus I$. We leave the details to the reader.

Theorem II.6.1. *(i) The space $R(\mathbf{E}_k)$ of relations in $\mathbf{IM}_k$ is spanned by the image of all 4T relations in $(k+1)-$chord diagrams.*

(ii) Every weight system Z_μ as above is of the form

$$D \longmapsto \sum_{I \subset C(D),\ \sharp(I)=k+1} f(D(I))$$

where $f : \mathbf{D}_{k+1} \longrightarrow \mathbf{C}$ is a suitable weight of degre $(k+1)$ on chord diagrams.

III. Local operators

III.1. Definitions

In this part we construct spin models on $\mathcal{D}$ with values in $\mathcal{D}$. Such spin models are hence simply endomorphisms of the (infinite dimensional) vector space $\mathcal{D}$.

Let $D \in \mathbf{D}_n$ be an $n-$chord diagram drawn inside an oriented circle S. Given a subset I of $k-$chords, a *neighbourhood* of I is an open subset of S having $2k$ (for $k \geq 1$) connected components which contains all endpoints of chords in I and which contains no endpoints of chords not in I. A neighbourhood of the empty chord set is given by any open connected subset in S which contains no endpoints of chords.

Definition III.1.1. A *local* $(k, l)-$*operator* is an application L which associates to a chord diagram D and a subset $I \subset C(D)$ of k chords in D an element $L(D, I) \in \mathcal{D}_{n-k+l}$ (with n the number of chords in D) such that the following conditions hold:

(i) $L(D, I)$ is given by a linear combination of $(n - k + l)-$chord diagrams which are obtained from D by erasing the chords $c \in I$ in D and by gluing l new chords with all their endpoints in a neighbourhood of the erased chords.

The coefficients and the new chord diagrams depend only on the subdiagram $D(I)$ formed by the erased set of chords.

(ii) If $r = D_1 - D_2 + D_3 - D_4$ is a 4T relation and $I \subset C(r)$ is a type 2 subset of k chords, then

$$L(D_1, I) - L(D_2, I) + L(D_3, I) - L(D_4, I) = 0$$

in $\mathcal{D}$.

(iii) If $r = D_1 - D_2 + D_3 - D_4$ is a 4T relation and $I', I'' \subset C(r)$ are two appariated type 1 subsets of k chords, then

$$L(D_1, I') + L(D_1, I'') - L(D_2, I') - L(D_2, I'') + L(D_3, I') + L(D_3, I'') - L(D_4, I') - L(D_4, I'') = 0 \blacksquare$$

in $\mathcal{D}$.

Given a local $(k, l)-$operator L as in the above definition and an $n-$chord diagram D we define a spin model on D with partition function

$$Z_L(D) = \sum_{I \subset C(D),\ \sharp(I) = k} L(D, I) \in \mathcal{D}$$

which we extend to a partition function $Z_L : \mathbf{D} \longrightarrow \mathcal{D}$.

Theorem III.1.2. *(i) The partition function Z_L factorizes through 4T relations and yields hence a linear application (still called partition function) $Z_L : \mathcal{D} \longrightarrow \mathcal{D}$.*

(ii) The vector space spanned by all partition functions Z_L with L a local operator is an algebra for the composition of such partition functions.

Proof. (i) Given a 4T relation $r = D_1 - D_2 + D_3 - D_4$ in $\mathbf{D}_n$ the element $L(D_1, I) - L(D_2, I) + L(D_3, I) - L(D_4, I)$ of $\mathcal{D}_{n-k+l}$ is obviously zero if I is a type 0 subset.

For a type 2 subset the above element is zero by requirement (ii) in definition III.1.1.

Requirement (iii) shows that

$$L(D_1, I') + L(D_1, I'') - L(D_2, I') - L(D_2, I'') + L(D_3, I') + L(D_3, I'') - L(D_4, I') - L(D_4, I'')$$

is zero if I', I'' are two associated type 1 sets. This proves assertion (i).

(ii) Given a local $(k, l)-$operator L and a local $(k', l')-$operator L', it is easy to check that $Z_L \circ Z_{L'}$ is a sum of partition functions of local $(s, t)-$operators with $s \leq k + k'$ and $t \leq l + l'$. QED

Examples III.1.3. (i) Multiplication in $\mathcal{D}$ by a fixed $l-$chord diagram is given by a partition function Z_L with L a local $(0, l)-$operator.

(ii) Given a weight $f \in (\mathcal{D}_k)^*$, the application $(\mathrm{id} \otimes f) \circ \Delta$ (with $\Delta : \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D}$ the comultiplication) is given by Z_L with L a local $(k, 0)-$operator.

The above examples exhaust the set of local $(0, l)$ and local $(k, 0)-$operators. They generate a subalgebra which is the algebra of differential operators on $\mathcal{D}$ with polynomial coefficients.

The partition function Z_L associated to a local $(k, l)-$operator has degree $l - k$ and one has of course the obvious inclusion $\mathcal{ZL}_{k,l} \subset \mathcal{ZL}_{k+1,l+1}$ if $\mathcal{ZL}_{k,l} \subset \mathrm{End}(\mathcal{D})$ denotes the vector space of all partition functions coming from local $(k, l)-$operators. It would of course be interesting to know the dimensions of the spaces $\mathcal{ZL}_{k,l}$ (they are obviously finite). In particular, we have always $\dim(\mathcal{ZL}_{0,k}) = \dim(\mathcal{ZL}_{k,0})$ since the first number is the dimension of $\mathcal{D}_k$ and the second the dimension of $(\mathcal{D}_k)^*$.

III.2. Local $(1, *)-$operators

We state here without proof two easy properties of local $(1, *)-$operators.

Proposition III.2.1. *(i) Given k local $(1, l_i)-$operators $L_1, \ldots, L_k$, we get a local $(k, \sum l_i)-$ operator by applying the operators $L_1, \ldots, L_k$ in the $k!$ possible ways to the k chords of a $k-$chord set I (applying a different operator on each chord).*

(ii) Denote the operator constructed above by $L\{L_1, \ldots, L_k\}$. The local operator associated to the partition function

$$Z_{L_1} \circ Z_{L_2} - Z_{L\{L_1, L_2\}}$$

is a local $(1, l_1 + l_2 - 1)-$operator.

This proposition allows the construction of many $(1,*)-$operators by iteration.

The following result (whose proof is easy) shows that local $(1,*)-$operators are also interesting from an algebraic point of view.

Proposition III.2.2. *(i) The partition function Z_L associated to a local $(1,*)-$operator defines a derivation on the algebra $\mathcal{D}$ (ie we have $Z_L(D\cdot D') = (L_Z(D))\cdot D' + D\cdot(L_Z(D'))$).*

(ii) Given a local $(1,l)-$operator L, let C_L denote the application defined by applying the local operator L to any chord of a chord diagram. The operator C_L defines then an endomorphism of $\mathcal{D}$ which is moreover a homomorphism for the algebra structure on $\mathcal{D}$ (ie $C_L(D\cdot D') = (C_L(D))\cdot(C_L(D'))$).

Examples III.2.3. (i) Let L_1 be defined by figure III.2.1 (only the chord I and a neighbourhood of it have been drawn).

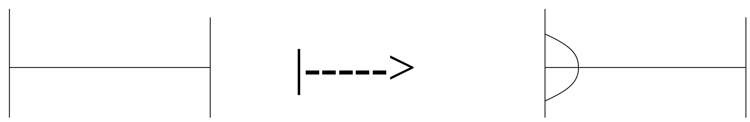

Figure III.2.1. The local $(1,2)-$operator L_1

We claim that this defines a local $(1,2)-$operator. The proof is by contemplation of the sum of two 4T relation given by the following figure (and by remembering that 4T relations imply generalized 4T relations).

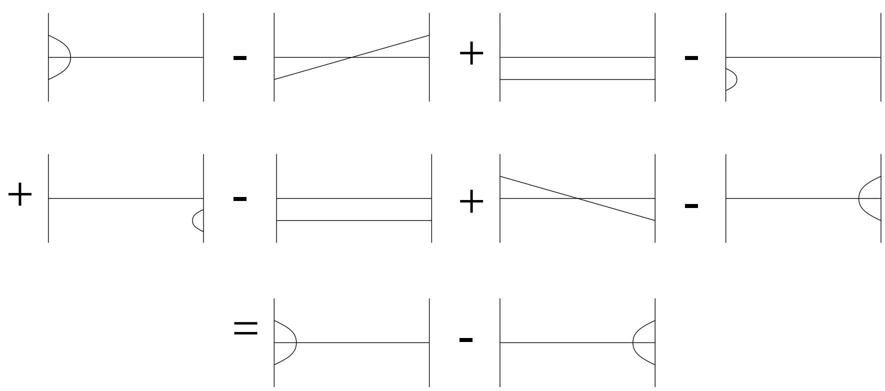

Figure III.2.2. Proof

(ii) The local $(1,3)-$operator associated to $Z_{L_1}\circ Z_{L_1} - Z_{L\{L_1,L_1\}}$ with L_1 as above is then given by

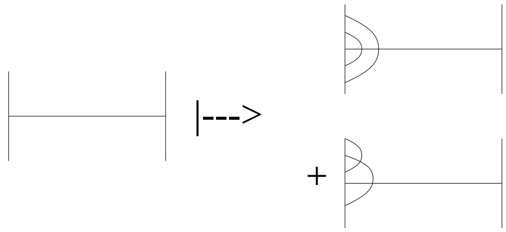

Figure III.2.3. A local $(1,3)-$*operator*

Example III.2.4. The operator of example III.2.2 (i) can be modified to yield local $(1,l)-$operators by replacing the single additional chord by a bunch of $(l-1)$ parallel chords as shown in figure III.2.4 with $l=4$.

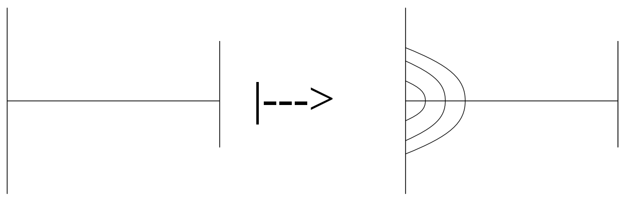

Figure III.2.4. A local $(1,4)-$*operator*

Proposition III.2.5. *Let L be a local* $(1,*)-$*operator from example III.2.4 and let* C_L *be the associated algebra homomorphism defined in proposition III.2.2 (ii).*

Considering C_L *as a homomorphism from* $\mathcal{D}$ *to* $\tilde{\mathcal{D}}$ *the element* C_L *is even a Hopf algebra homomorphism (ie we have also* $(C_L \otimes C_L) \circ \Delta = \Delta \circ C_L : \mathcal{D} \longrightarrow \tilde{\mathcal{D}}$*).*

IV. Liftings and $q-$podal chord diagrams

IV.1. Definitions

In this section we define spin models on $\mathbf{D}_n$ with values in $\mathbf{D}$ which yield linear applications $\mathcal{D}_n \longrightarrow \mathcal{D}$. These models generalize a well-known construction on chord diagrams which corresponds to cablings of knots.

Given an integer $q \geq 1$ and a chord diagram $D \in \mathbf{D}_n$ one defines $\psi^q(D) \in \mathbf{D}_n$ as the sum of all liftings of D to the $q-$cover of the oriented circle supporting D (see definition 3.11 in [**B**]). The element $\psi^q(D)$ is hence a sum of q^{2n} diagrams.

We give the principal properties of the linear operator ψ^q in the following (well-known) proposition:

Proposition IV.1.1. *(i) The operator* ψ^q *induces an operator (still called)* $\psi^q : \mathcal{D} \longrightarrow \mathcal{D}$ *(ie* $\psi^q(r) \in <$ 4T relations $>$ *for any 4T relation r).*

(ii) $\psi^q \circ \psi^p = \psi^{qp}$.

(iii) $\Delta \circ \psi^q = (\psi^q \otimes \psi^q) \circ \Delta$ *(ie* ψ^q *defines a morphism of coalgebras).*

The operator ψ^q can be generalized to an operator $\psi^{q_1,\dots,q_l} : \mathcal{D}_n \longrightarrow \mathcal{D}_n^l$ (where $\mathbf{D}_n^l$ denotes the set of chord diagrams associated to singular links having n singularities and whose desingularisations have l components) in the following way: Choose l (non-zero) integers $q_1, \dots, q_l$ and consider a covering of an oriented circle which has l connected components and the degree of the $i-$th component is $|q_i|$. Orient the $i-$th component such that the projection operator preserves the orientation if $q_i > 0$ and reverses the orientation if $q_i < 0$. As before take the sum of all liftings of chords in D to this covering but multiply an element by -1 if the number of endpoints of chords on negatively oriented circles (circles corresponding to a negative integer q_i) is odd.

In the sequel we will mainly consider the simpler operator ψ^q but all constructions can also be done with the operators $\psi^{q_1,\dots,q_l}$. The necessary modifications are only roughly outlined.

Definition IV.1.2. (i) A *$q-$podal chord diagram* is a chord diagram D inside a circle S together with a set of q disjoint open intervals on S which contain all endpoints of chords in D.

(ii) A *$q-$podal subdiagram* is a subset of chords $I \subset C(D)$ of a chord diagram D such that there exists q disjoint open intervals in S containing all endpoints of chords in I and no endpoints of chords in $C(D) \setminus I$.

A *$q-$podal relation* is an alternating sum of $2q$ chord diagrams as suggested in figure IV.1.1.

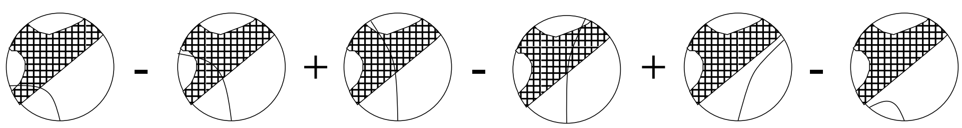

Figure IV.1.1. A $3-$*podal relation*

It is easy to check that 4T relations imply all $q-$podal relations (the proof is the same as for Lemma 1.2).

IV.2. A spin model

Let $q \geq 2$ be a fixed integer and let G be a fixed $q-$podal chord diagram.

Given three positive integers n, k_1, k_2 with $n \geq k_1 + k_2$, we construct for each $n-$chord diagram $D \in \mathbf{D}_n$ a spin model with values in $\mathbf{D}$ as follows:

(1) Given a chord c the finite set of spins which it can take are the q^2 possible liftings of c to the connected covering $p_q : S \longrightarrow S$ of degree q of the circle underlying D.

(2) The set C is the set of chords in D.

(3) A site $I = (I_1^o \cup I_2)$ is given by the choice of k_1 chords I_1 in $C(D)$ together with one of their endpoints and by the choice of k_2 more chords I_2 (there are hence $2^{k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}$ sites). The choosen endpoint of the chords in I_1 endows them with an orientation. We denote hence by I_1^o the set of these oriented chords.

(4) The energy $e(I,\sigma) \in \mathbf{D}$ of the site $I = (I_1^o, I_2)$ in the state σ is the sum of the $q^{k_1+2k_2}$ chord diagrams defined as follows:

Lift first the $n - k_1 - k_2$ chords in $C(D) \setminus \{I_1 \cup I_2\}$ according to the state σ to the connected $q-$th covering of S.

Given an endpoint $\alpha \in I_1^o$ glue the fixed $q-$podal diagram G onto an $\epsilon-$neighbourhood of $p_q^{-1}(\alpha)$ (there are q such gluings corresponding to the action of the cyclic group $\mathbf{Z}/q\mathbf{Z}$ on $p_q^{-1}(\alpha)$).

For chords $c \in I_2$ do the same with both endpoints of c.

Finally define $e(I,\sigma)$ as the sum of all $q^{k_1+2k_2}$ possible chord diagrams obtained in this way.

(5) The function f is independent of the state σ (take for instance the function $f(\sigma) = 1$ for all σ).

The partition function $Z(D)$ is then a sum of

$$2^{k_1}\binom{n}{k_1}\binom{n-k_1}{k_2} q^{2n+k_1+2k_2-2k_1-2k_2} = q^{2n}\left(\frac{2}{q}\right)^{k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}$$

diagrams.

Example IV.2.1. (i) The choice $k_1 = k_2 = 0$ (no $q-$pode is then involve) leads to $Z(D) = \psi^q(D)$ with $\psi^q : \mathbf{D}_n \longrightarrow \mathbf{D}_n$ as in section IV.1.

(ii) The choice $q = 2$, G the unique $2-$pode with a unique chord joining the two intervals of G, $k_1 = 0$, $k_2 = n$ leads to $Z(D) = 2^{2n}A$ where $A \in \mathbf{D}_{2n}$ is the $2n-$chord diagram having $2n$ chords joining $2n$ pairs of diagonally opposite points.

Extending the partition function Z to a linear application $Z : \mathbf{D}_n \longrightarrow \mathbf{D}$ we get the following result.

Theorem IV.2.2. *Let $r \in \mathbf{D}_n$ be a 4T relation. Then $Z(r) \in R(\mathbf{D})$ (ie $Z(r)$ is a sum of 4T relations in $\mathbf{D}$).*

This theorem shows that the partition function Z defines a linear application (still denoted) $Z : \mathcal{D}_n \longrightarrow \mathcal{D}$.

Proof. Sites not containing any of the special chords involved in the relation r contribute sums of 4T relations. Sites containing exactly one of the special chords contribute either zero or sums of $q-$podal relations and sites containing both special chords contribute always zero. QED

Remarks IV.2.3. (i) If the $q-$podal diagram G used in the above construction has automorphisms (as a $q-$podal diagram; the definition is the obvious one), the energy $e(I, \sigma)$ of a site and the partition function are easier to compute since all involved chord diagrams have huge multiplicities.

(ii) The construction of this section can easily be generalized as follows: instead of gluing the same fixed $q-$podal diagram G it is possible to consider a set of $k_1 + 2k_2$ fixed $q-$podal diagrams $G_1, \dots, G_{k_1+2k_2}$ and to glue them in all $(k_1 + 2k_2)!q^{k_1+2k_2}$ possible ways in order to define the energy $e(I, \sigma)$ of a site I.

(iii) The construction can of course be generalized to arbitrary finite coverings of a circle S. There is an obvious definition of $(q_1, \dots, q_l)-$podal chord diagrams (for links). The main difference is the fact that the function f involved in the definition of the partition function is no longer constant but is a sign function depending on the number of endpoints of lifted chords ending in negatively oriented circles.

Let us finish this section with a possible application of the construction using $q-$podal chord diagrams.

One of the main problems concerning chord diagrams is the question if a chord diagram is always equivalent modulo 4T relations to its image in a mirror (obtained by reflection along a line). This would indeed imply that Vassiliev invariants are not complete.

So far, no weight system distinguishing some chord diagram from its mirror has been found.

The $q-$podal construction outlined in this section sends a chord diagram and its mirror to (seemingly) very different sums of diagrams (at least if the $q-$podal diagram G is not equal or equivalent modulo 4T relation to its mirror $q-$podal diagram).

If a weight system distinguishing some chord diagram from its mirror exists, it is perhaps possible to construct such a weight system by using the spin model outlined in this section (with suitable parameters) together with some known weight system.

I thank Luis Funar very much for many discussions, explications and references.

Bibliography

[**B**] Bar-Natan, *On the Vassiliev knot invariants*, Topology, **34** (1995), 514-568.

[**CD**] S.V.Chmutov, S.V. Duzhin, *An upper bound for the number of Vassiliev knot invariants*, J. Knot Theory and Ramifications, Vol. 3 No. 2 (1994), 141-151.

[**CDLI**] S.V.Chmutov, S.V. Duzhin, S.K.Lando, *Vassiliev knot invariants I*, Soviet Mathematics, Volume 21 (Singularities and Bifurcations, ed. V.I.Arnold), 1994 (AMS), 117-126.

[**F**] L.Funar, *Vassiliev invariants I: Braid groups and rational homotopy theory*, Rev. Roumaine Math. Pures Appl., 42, 3-4 (1997), 245-272.

[**V**] Vogel, Invariants de Vassiliev des noeuds, Séminaire Bourbaki, exposé 769 (1993).

Roland BACHER

Université de Grenoble I

Institut Fourier

UMR 5582 CNRS-UJF

38402 St. Martin d'Hères Cedex (France)

e-mail: Roland.Bacher@ujf-grenoble.fr