Best constants for a Real Schwarz Lemma

Andrea Sambusetti
Institut Fourier de Mathematiques BP 74
38402 Saint Martin d'Heres
e-mail: asambuse@mozart.ujf-grenoble.fr

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Abstract

A sharp version of the Real Schwarz Lemma of G. Besson, G. Courtois and S. Gallot is proved. Namely we show that every homotopy class of maps $[f] : (Y, g) \to (X, g_0)$ between riemannian manifolds of dimension $n \geq 3$ contains a family of mappings which reduce volumes of a factor tending to $\mathrm{Ent}_H(g)^n/(n-1)^n$, provided that X has sectional curvature $k(g_0) \leq -1$. The constant $\mathrm{Ent}_H(g)$ is the exponential growth rate of volumes of balls in the covering of Y determined by the class $[f]$. Moreover, in case volumes are both globally and infinitesimally preserved (i.e. if $\mathrm{Vol}(Y, g) = \deg(f) \cdot \mathrm{Vol}(X, g_0)$ and the reduction factor is precisely 1) then $[f]$ contains a riemannian covering. This result is also shown to hold for non-orientable manifolds. Then, we consider the case of a connected sum and we discuss optimality of our reduction factor.

1 Statement of the results

By the term "Schwarz lemma" one commonly means, in differential geometry, a result of the kind: given manifolds Y, X endowed with some metric structures g, g_0 (for instance, hermitian metrics), normalized in order that suitable hypotheses on their curvature are satisfied, point out a class of maps $f : Y \to X$ with the property of contracting volumes [1]; moreover, possibly characterize the case where volumes are preserved.

The protoype of these results is the classical Schwarz lemma: the derivative of any holomorphic map f of the unitary disk $D \subset \mathbf{C}$ into itself has norm smaller than 1 (equivalently, it contracts the natural hyperbolic distance of the disk) and, if $| f' |= 1$ somewhere, then f is a conformal automorphism of D (i.e. a hyperbolic isometry).

Several generalizations (due to Yau, Ahlfors, Pick et al.) are presented in [14]. In [4] G. Besson, G. Courtois and S. Gallot proved a similar result (a "Real Schwarz Lemma") for riemannian manifolds $(Y, g), (X, g_0)$: under the assumption that the sectional curvature $k(g_0) \leq 1$, every homotopy class of maps

[1] this means that $| Jac_y f | \leq 1$ for every y, or, equivalently, that $\mathrm{Vol}(U, g) \geq \mathrm{Vol}(f(U), g_0)$ for any sufficiently small neighbourhood U of a regular point of f; as a consequence, $\mathrm{Vol}(Y, g) \geq | \deg(f) | \mathrm{Vol}(X, g_0)$

$[f] : Y \to X$ contains an explicit family of mappings which reduce volumes of a factor tending to $\mathrm{Ent}(g)^n/(n-1)^n$, where n is the dimension of Y and X, and $\mathrm{Ent}(g)$ is the volume entropy of g (an asymptotic riemannian invariant depending on the curvature of g, which will be discussed hereafter). Moreover, provided that f has nonvanishing degree and that the metric g is normalized so that $\mathrm{Ent}(g) = (n-1)$, they completely described the case where volume is preserved: i.e. $\mathrm{Vol}(Y, g) = \deg(f)\mathrm{Vol}(X, g_0)$ if and only if f is homotopic to a riemannian covering. An even better result holds when g_0 is moreover assumed to be locally symmetric, in which case it has been shown (see [3]) that the family of mappings under consideration reduces volumes of a factor $\mathrm{Ent}(g)^n/\mathrm{Ent}(g_0)^n$.

Hereby we are mainly concerned about the problem of sharpening (when possible) the reduction factor. The purpose of the paper is to prove a sharper version of the "Real Schwarz Lemma", by replacing the entropy $\mathrm{Ent}(g)$ with another invariant of g whose value is generally smaller: the exponential growth rate of volumes of balls in the riemannian covering $\overline{Y}$ associated to the subgroup $H = \ker(f_*) \lhd \pi_1(Y)$ (where $f_* : \pi_1(Y) \to \pi_1(X)$ denotes the homomorphism induced by f). Moreover, we show that this result also holds for non-orientable manifolds (the extension being almost straightforward), provided that we consider the notion of absolute degree $\mathrm{Adeg}(f)$ of a map f, instead of the usual degree.

Definition 1.1 *Entropy relative to a covering.*
Given a compact riemannian manifold (Y, g) and a regular covering $\overline{Y} \to Y$, associated to a normal subgroup H of $\pi_1(Y)$, the *(volume) entropy of g relative to $\overline{Y}$* (or *relative to H*) is defined as

$$\mathrm{Ent}_H(g) \doteq \lim_{R\to+\infty} \tfrac{1}{R}\log \mathrm{Vol}(\overline{B}(\overline{y}, R)),$$

where $\mathrm{Vol}(\overline{B}(\overline{y}, R))$ denotes the volume of the ball of radius R centered at $\overline{y}$ in the covering $\overline{Y}$ (with respect to the riemannian distance d induced by the lift of g to $\overline{Y}$). It is classical that this limit always exists and it does not depend on the choice of $\overline{y} \in Y$.
If $H = (1)$ (i.e. $\overline{Y} = \tilde{Y}$, the universal covering of Y), the above formula defines the usual (volume) entropy $\mathrm{Ent}(g)$ of g. Clearly, one always has $\mathrm{Ent}_H(g) \leq \mathrm{Ent}(g)$.
Another fundamental characterization of $\mathrm{Ent}_H(g)$ is given by the formula:

$$\mathrm{Ent}_H(g) = \inf\{c > 0 \setminus \int_{\overline{Y}} e^{-cd(\overline{y},\overline{y}')} dv_g(\overline{y}') < \infty\} \tag{1}$$

(see for instance [15], where the entropy is discussed in the general setting of a discrete cocompact group Γ acting on a metric space endowed with a Γ-invariant measure).

Namely, we will prove

Theorem 1.2 (Strong Real Schwarz Lemma)
Let (Y, g) and (X, g_0) be compact riemannian manifolds of dimension $n \geq 3$, and let $f : Y \to X$ be a continuous map. Let H denote the kernel $ker(f_*)$ of the homomorphism induced by f between the fundamental groups.
Assume that the sectional curvature of g_0 satisfies $k(g_0) \leq -1$. Then, in the homotopy class of f there exists a family of C^1 mappings f_ϵ which verify:

$$(2) \qquad | Jacf_\epsilon | \leq \left(\frac{Ent_H(g) + \epsilon}{n-1} \right)^n .$$

In particular,

$$(3) \qquad Vol(Y, g) \geq \left(\frac{n-1}{Ent_H(g)} \right)^n \cdot Adeg(f) \cdot Vol(X, g_0).$$

Moreover, in case $Adeg(f) \neq 0$, equality holds in the above formula (3) if and only if f is homotopic to a riemannian covering, and both (Y, g) and (X, g_0) have constant curvature -1 (up to rescaling g).

Theorem 1.3 (Strong Real Schwarz Lemma, locally symmetric case)
Let (Y, g) and (X, g_0) be compact riemannian manifolds of dimension $n \geq 3$, and let $f : Y \to X$ be a continuous map. Let H denote the kernel $ker(f_*)$ of the homomorphism induced by f between the fundamental groups.
Assume that g_0 is a negatively curved locally symmetric metric. Then, in the homotopy class of f there exists a family of C^1 mappings f_ϵ which verify:

$$(4) \qquad | Jacf_\epsilon | \leq \left(\frac{Ent_H(g) + \epsilon}{Ent(g_0)} \right)^n .$$

In particular,

$$(5) \qquad Vol(Y, g) \geq \left(\frac{Ent(g_0)}{Ent_H(g)} \right)^n \cdot Adeg(f) \cdot Vol(X, g_0).$$

Moreover, in case $Adeg(f) \neq 0$, equality holds in the above formula (5) if and only if f is homotopic to a riemannian covering (up to rescaling g).

Recall that the *absolute degree* [2] of a continuous map $f : Y \to X$ between compact n-dimensional manifolds is the non-negative integer

$$\text{Adeg}(f) = \inf\{G(f') \setminus f' : Y \to X \text{ homotopic to } f\},$$

where $G(f')$ denotes the *geometric degree* of f' (that is the smallest number of connected components of $f'^{-1}(D)$, when D varies among the $n-$cells of

[2] By definition, the absolute degree is a homotopy invariant and it coincides with the absolute value $| \deg(f) |$ of the usual degree for maps between oriented manifolds. In any case, it is congruent mod 2 to the modulo 2 degree of f (denoted $\deg_2(f)$). If f is a covering, $\text{Adeg}(f)$ is equal to the number of sheets of the covering.

X such that $f'^{-1}(D) \to D$ is a covering). By Sard's theorem, for any sufficiently regular map $f : Y \to X$ (e.g. a C^1 map) one has that $G(f) = \inf\{\#f^{-1}(x) \setminus x$ regular value of $f\}$, hence $\#f^{-1}(x) \geq \mathrm{Adeg}(f)$ for almost every $x \in X$.
In [7], D.B.A. Epstein gave an explicit criterion (mainly due to H. Hopf) to compute $\mathrm{Adeg}(f)$, disregarding the whole homotopy class of f: it essentially consists in computing the usual degree of some lift of f to suitable oriented coverings of Y and X.

In section 2 we will prove inequalities (2) and (4) simultaneously, while in section 3 we will prove the rigidity statements in the equality cases. The method is largely founded on the works [4] and [3] of G. Besson, G. Courtois and S. Gallot, with the following main differences:

a) the universal covering $\tilde{Y}$ of Y is replaced by the intermediate covering $\overline{Y} \to Y$ associated to H (which is the smallest covering of Y such that f can be lifted to a map $\overline{f} : \overline{Y} \to \tilde{X}$, whose arrival space is the universal covering of X), as well as the lift $\tilde{f} : \tilde{Y} \to \tilde{X}$ is replaced by $\overline{f}$, and every geometric object defined in [4] on the space $\tilde{Y}$ can correctly be built on $\overline{Y}$;

b) in [4] the authors essentially develop the proof of the Real Schwarz Lemma when Y is homotopically equivalent to X (and $k(g) < 0$ also), in which case an explicit mapping $f_0 : Y \to X$ is defined with the property that $|\mathrm{Jac} f_0| \leq \mathrm{Ent}(g)^n/(n-1)^n$ exactly, and f_0 is an isometry if $\mathrm{Ent}(g)^n \cdot \mathrm{Vol}(Y, g) = \mathrm{Adeg}(f) \cdot \mathrm{Ent}(g_0)^n \cdot \mathrm{Vol}(X, g_0)$. The general case is only sketched using a family of probability measures, supported by the *geometric boundary* $\partial\tilde{X}$ of $\tilde{X}$, defined by the convolution (and renormalization) of the measures $\tilde{f}_*[e^{-cd(y,y')}dv_g(y')]$ with the measure $e^{-h_0 B_\theta(x)}d\theta$ of $\partial\tilde{X}$.
In contrast, we simply use the family of measures, supported by $\tilde{X}$, given by the direct image via $\overline{f}$ of measures $e^{-cd(y,y')}dv_g(y')$ (which are defined on $\overline{Y}$). This makes the exposition more elementary, and it leads us to consider a notion of *barycenter* of a measure defined on a negatively curved simply connected space (compare with [5]). On the other hand, we pay this simplification by some difficulties which arise because of the necessary renormalizations of the quadratic forms which express the estimates on $|\mathrm{Jac} f_\epsilon|$ (as these forms are integrals on $\overline{Y}$, which is non-compact, differently from $\partial\tilde{X}$).

c) as to the equality case, the proof is more delicate than in [4], since we build the isometry $f_0 : Y \to X$ as a limit of the f_ϵ's. We explicitely followed the lines of [3] (in particular for lemmas 3.5 and 3.6).

In section 4, in order to show that the new estimates of theorems 1.2 & 1.3 are generally strictly sharper than those of [4], we will analyze explicitely the case of a connected sum $Y = X\sharp M$. Finally, we discuss optimality of the reduction factor $\mathrm{Ent}(g)^n/\mathrm{Ent}(g_0)^n$.

For some other applications of our strong version of the Real Schwarz Lemma see [16].

2 Proof of the inequalities

Let $\overline{Y} \to Y$ be the covering associated to H. We lift g and g_0 to metrics on $\overline{Y}$ and $\tilde{X}$ (which will be still denoted by g, g_0, in order to avoid cumbersome notations), and let d, d_0 be the induced riemannian distances. As $f_*(H) = (1)$, the map $f : Y \to X$ can be lifted to a map $\overline{f} : \overline{Y} \to \tilde{X}$, by the theory of covering spaces. The groups of deck transformations $\mathrm{Aut}(\overline{Y})$ and $\mathrm{Aut}(\tilde{X})$ can be identified to $\Gamma \doteq \pi_1(Y)/H$ and $\pi_1(X)$ respectively, and $\overline{f}$ is equivariant with respect to the homomorphism $\phi : \Gamma \hookrightarrow \pi_1(X)$ induced by f_*, i.e. $\overline{f}(\gamma \cdot y) = \phi(\gamma) \cdot \overline{f}(y)$ [3] for every $y \in \overline{Y}$ and $\gamma \in \Gamma$.
Let us consider the spaces $\mathcal{M}(\overline{Y})$ and $\mathcal{M}(\tilde{X})$ of positive and finite Borel measures on $\overline{Y}$ and $\tilde{X}$. The groups $\mathrm{Aut}(\overline{Y})$ and $\mathrm{Aut}(\tilde{X})$ naturally act on $\mathcal{M}(\overline{Y})$ and $\mathcal{M}(\tilde{X})$ by pushing forward measures (i.e., $\gamma \cdot \mu \doteq \gamma_* \mu$, if $\gamma \in \mathrm{Aut}(Z)$ and $\mu \in \mathcal{M}(Z)$).
The idea is to embed $\overline{Y}$ "as much isometrically as possible" in $\mathcal{M}(\overline{Y})$ by means of maps $y \mapsto \mu_{\epsilon,y}$ (for $\epsilon \to 0$), then to take the push-forward $\overline{f}_* \mu_{\epsilon,y}$ of the measures $\mu_{\epsilon,y}$ via $\overline{f}$, and to come back on $\tilde{X}$ by the barycenter map (described hereafter). One therefore obtains maps $\overline{f}_\epsilon : \overline{Y} \to \tilde{X}$, defined by the composition $\overline{f}_\epsilon = \mathrm{bar}(\overline{f}_* \mu_{\epsilon,y})$, which are "more isometric" than the initial map $\overline{f}$:

$$\begin{array}{ccc} \mathcal{M}(\overline{Y}) & \xrightarrow{\overline{f}_*} & \mathcal{M}(\tilde{X}) \\ \mu_{\epsilon,y} \uparrow & & \downarrow \mathrm{bar} \\ \overline{Y} & \underset{\overline{f}}{\overset{\overline{f}_\epsilon}{\rightrightarrows}} & \tilde{X} \end{array}$$

The measures $\mu_{\epsilon,y}$ are defined on $\overline{Y}$ by $\mu_{\epsilon,y} \doteq e^{-(Ent_H(g)+\epsilon)d(y,y')} dv_g(y')$, and they are *finite* by the characterization (1) of the invariant $\mathrm{Ent}_H(g)$.
The *barycenter* of a measure μ on a simply connected riemannian manifold of negative curvature is defined by the following

Proposition-Definition 2.1 Barycenter.
Let $(\tilde{X}, g_0)$ be a simply connected riemannian manifold of negative curvature, and let $\mu \in \mathcal{M}(\tilde{X})$. If the integral $\int_{\tilde{X}} d(x_0, x')^2 \mu(x')$ converges for some $x_0 \in \tilde{X}$, then the function $\mathcal{D}_\mu(x) = \int_{\tilde{X}} d_0(x, x')^2 \mu(x')$ is well defined and C^∞ for every $x \in \tilde{X}$. It admits a unique critical point, which is an absolute minimum and which is called the barycenter of μ *(and denoted $bar[\mu]$).*
The barycenter of μ is univoquely characterized by the implicit equation

$$(6) \quad (d\mathcal{D}_\mu)_{bar[\mu]}(v) = 2 \int_{\tilde{X}} \rho_{x'}(bar[\mu]) \cdot (d\rho_{x'})_{bar[\mu]}(v) \; \mu(x') \; = 0, \; \forall v \in T_{bar[\mu]}\tilde{X}$$

where $\rho_{x'}$ denotes the function of x defined by $\rho_{x'}(x) = d_0(x, x')$. [4]

[3] given a covering $\overline{Z} \to Z$, we will write $\gamma \cdot z$ instead of $\gamma(z)$, if $z \in \overline{Z}$ and $\gamma \in \mathrm{Aut}(\overline{Z})$ or $\gamma \in \mathrm{Isom}(\overline{Z})$ (in case $\overline{Z}$ is riemannian).

[4] the expression $(d\rho_{x'})_z$ will always denote, in the following, the differential of $d_0(x, x')$ with respect to the first variable, at $x = z$.

The barycenter satisfies the following properties:

(a) for every $\sigma \in Isom(\tilde{X}, g_0)$ one has $bar[\sigma_* \mu] = \sigma \cdot bar[\mu]$

(b) let l_x be the visual measure of $(\tilde{X}, g_0)$ from x, i.e. $l_x = (exp_x)_*(L_x)$ (where $L_x = dx_1...dx_n$ denotes the usual Lebesgue measure on $T_x\tilde{X}$), and let $\lambda_x = F(d_0(x,\cdot))l_x$, for some function F such that $\int_{\tilde{X}} d_0(x,x')^2 \lambda_x(x') < \infty$. Then, one has $bar[\lambda_x] = x$.

Remark 2.2 Remark that the measures $\overline{f}_* \mu_{\epsilon,y}$ satisfy the assumptions of the above proposition, as there exist positive constants A, B such that $d_0(\overline{f}(y_1), \overline{f}(y_2)) \leq A \cdot d(y_1, y_2) + B$ for every $y_1, y_2 \in \overline{Y}$ (since the quotient spaces $Y = \overline{Y}/\Gamma$ and $X = \tilde{X}/\pi_1(X)$ are compact).

Proof of proposition 2.1.

If $\mathcal{D}_\mu(x_0) < \infty$, then clearly also $\mathcal{D}_\mu(x) < \infty$ for any $x \in \tilde{X}$. As $\tilde{X}$ is simply connected and negatively curved, the function $d(x,x')^2$ is C^∞, and the function $t \mapsto \rho^2_{x'}(c(t)) = d(c(t),x')^2$ is strictly convex, for any choice of a point $x' \in \tilde{X}$ and of a geodesic c. By Lebesgue's dominated convergence theorem, also $\mathcal{D}_\mu$ is C^∞ and strictly convex, hence it admits exactly one critical point which is an absolute minimum (as obviously $\mathcal{D}_\mu(x) \to \infty$ if $x \to \infty$).

Still by Lebesgue's theorem, one can differentiate under the integral and deduce formula (6), as $(d\rho^2_{x'})_x(v) = 2\rho_{x'}(x)(d\rho_{x'})_x(v)$ a.e. x'.

Property (a) immediately follows from the fact that $\mathcal{D}_{\sigma_*\mu}(\sigma \cdot x) = \mathcal{D}_\mu(x)$ for all $\sigma \in \text{Isom}(\tilde{X}, g_0)$. To verify (b), notice that, as $(\tilde{X}, g_0)$ is negatively curved and simply connected, if $x' = \exp_x(u)$ one has

$$d_0(x, \exp_x(u)) = \| u \| \quad \text{and} \quad \text{grad}\rho_{x'}(x) = -u/\| u \|$$

so that $(d\mathcal{D}_{\lambda_x})_x(v) = -\int_{T_x\tilde{X}} \| u \| \, g_0(u/\| u \|, v) F(\| u \|) L_x(u) = 0$ for every $v \in T_x\tilde{X}$ (as it is the integral of an odd function of u). Hence $x = \text{bar}[\lambda_x]$. □

Lemma 2.3 Equivariance of the maps $\overline{f}_\epsilon$.

1) The maps $\overline{f}_\epsilon$ induce maps between the quotient spaces $f_\epsilon : Y \to X$;

2) the maps f_ϵ are homotopic to the initial map f.

Proof of lemma 2.3.

1. It is enough to show that the maps $\overline{f}_\epsilon$ are $\phi-$equivariant, i.e. that $\text{bar}[\overline{f}_* \mu_{\epsilon,\gamma\cdot y}] = \phi(\gamma) \cdot \text{bar}[\overline{f}_* \mu_{\epsilon,y}]$ for all $\gamma \in \Gamma \subset \text{Isom}(\overline{Y}, g)$

First, we notice that $\mu_{\epsilon,\gamma\cdot y} = \gamma_* \mu_{\epsilon,y}$ for every $\gamma \in \text{Isom}(\overline{Y}, g)$ because

$\int_{\overline{Y}} h(y') e^{-(Ent_H(g)+\epsilon) d(\gamma\cdot y, y')} dv_g(y') = \int_{\overline{Y}} h(\gamma \cdot y') e^{-(Ent_H(g)+\epsilon) d(\gamma\cdot y, \gamma\cdot y')} dv_g(y')$.

This implies that $\overline{f}_* \mu_{\epsilon,\gamma\cdot y} = \phi(\gamma)_* \overline{f}_* \mu_{\epsilon,y}$ so, by property *(a)* of proposition 2.1, one gets the $\phi-$equivariance of the $\overline{f}_\epsilon$'s.

2. Let L_x, l_x and λ_x be as in proposition 2.1*(b)*. One easily checks that $\lambda_{\sigma\cdot x} = \sigma_* \lambda_x$ for all $\sigma \in \text{Isom}(\tilde{X})$. Then, the map $\overline{\Theta} : \overline{Y} \times I \to \tilde{X}$ defined by $\overline{\Theta}(y,t) = \text{bar}[(1-t)\lambda_{\overline{f}(y)} + t\overline{f}_* \mu_{\epsilon,y}]$ satisfies $\overline{\Theta}(\gamma \cdot y, t) = \phi(\gamma) \cdot \overline{\Theta}(y,t)$, by the equivariance properties of the measures $\mu_{\epsilon,y}, \lambda_x$ and by proposition 2.1.*(a)*.

Therefore, $\overline{\Theta}$ induces on quotient spaces a map $\Theta : Y \times I \to X$, which is a homotopy between f_ϵ and f, since $\overline{\Theta}(y,1) = \overline{f}_\epsilon(y)$ and $\overline{\Theta}(y,0) = \mathrm{bar}\lambda_{\overline{f}(y)} = \overline{f}(y)$ (by proposition 2.1.*(b)*). □

Lemma 2.4 Implicit equations for $\overline{f}_\epsilon$.
The maps $\overline{f}_\epsilon$ are univoquely defined by the equation $G_\epsilon(\overline{f}_\epsilon(y), y) = 0$, if $G_\epsilon = (G^i_\epsilon) : \tilde{X} \times \overline{Y} \to \mathbf{R}^n$ is defined by

$$G^i_\epsilon(x,y) = \frac{1}{2}\int_{\overline{Y}} (d\rho^2_{\overline{f}(y')})_x(E_i)\, e^{-(Ent_H(g)+\epsilon)d(y,y')} dv_g(y')$$

where $\{E_i\}$ is a global g_0-orthonormal frame of the (trivial) tangent bundle $T\tilde{X} \simeq \tilde{X} \times \mathbf{R}^n$, and where $\rho_{\overline{f}(y')}$ denotes the function (of x) $d_0(\overline{f}(y'), x)$.

Proof of lemma 2.4.
Remark that $T\tilde{X}$ is trivial since $\tilde{X}$ is negatively curved, hence diffeomorphic to $\mathbf{R}^n$. By proposition 2.1, $\overline{f}_\epsilon(y)$ is the unique point of $\tilde{X}$ which satisfies, for all i,

$$0 = (d\mathcal{D}_{\mu_{\epsilon,y}})_{\overline{f}_\epsilon(y)}(E_i) = \left(d \textstyle\int_{\tilde{X}} \rho_{x'}(x)^2 \overline{f}_* \mu_{\epsilon,y}(x')\right)_{\overline{f}_\epsilon(y)}(E_i) = 2G^i_\epsilon(\overline{f}_\epsilon(y), y)$$

where differentiation under the integral is allowed by Lebesgue's dominated convergence theorem since, if $x(t) \to x$, one has

$$\left|\frac{d_0(x(t),\overline{f}(y'))^2 - d_0(x,\overline{f}(y'))^2}{t}\right| \cdot e^{-cd(y,y')} \le (2+\delta)\cdot d_0(x,\overline{f}(y'))\cdot e^{-cd(y,y')}$$

which belongs to $L^1(\overline{Y}, dv_g)$, if $c > \mathrm{Ent}_H(g)$. □

Lemma 2.5 Formulas for $d\overline{f}_\epsilon$.
The map $G_\epsilon : \tilde{X} \times \overline{Y} \to \mathbf{R}^n$ is C^1 and its differential $(dG_\epsilon)_{(x,y)} \mid_{T_x\tilde{X}}$ is non-singular, for every $y \in \overline{Y}$ and $x = \overline{f}_\epsilon(y) \in \tilde{X}$.
As a consequence, $\overline{f}_\epsilon : \overline{Y} \to \tilde{X}$ is C^1 and satisfies

$$(7) \qquad (d\overline{f}_\epsilon)_y = -(dG_\epsilon)_{(x,y)} \mid^{-1}_{T_x\tilde{X}} \circ (dG_\epsilon)_{(x,y)} \mid_{T_y\overline{Y}} .$$

Namely, if $u \in T_y\overline{Y}$ and $v \in T_x\tilde{X}$ one has [5]*:*

$$(8)\, (dG^i_\epsilon)_{(x,y)}(u) = -(Ent_H(g)+\epsilon)\int_{\overline{Y}} (d\rho_{y'})_y(u)(d\rho_{\overline{f}(y')})_x(E_i)\, d_0(\overline{f}(y'),x)\, \mu_{\epsilon,y}(y')$$

$$(9) \qquad (dG^i_\epsilon)_{(x,y)}(v) = \frac{1}{2}\int_{\overline{Y}} \left(Dd\rho^2_{\overline{f}(y')}\right)_x (v, E_i)\, \mu_{\epsilon,y}(y')$$

Proof of lemma 2.5.
Let y and $x = \overline{f}_\epsilon(y)$ be fixed and, for any unitary $v \in T_x\tilde{X}$, let $x(t)$ be the geodesic such that $x(0) = x, x'(0) = v$. Then,

$$(dG^i_\epsilon)_{(x,y)}(v) = \lim_{t\to 0}\int_{\overline{Y}} \left[\frac{(d\rho^2_{\overline{f}(y')})_{x(t)}(E_i) - (d\rho^2_{\overline{f}(y')})_x(E_i)}{2t}\right] e^{-cd(y,y')} dv_g(y')$$

[5] notice that, for fixed $y \in \overline{Y}$ and $u \in T_y\overline{Y}$, $(d\rho_{y'})_y(u)$ esists if $y \notin \mathrm{Cut}(y')$ (that is, if $y' \notin \mathrm{Cut}(y)$). As $\mathrm{Cut}(y)$ has zero Lebesgue measure, formula (8) makes sense.

if $c = \mathrm{Ent}_H(g) + \epsilon$. As the curvature of $(\tilde{X}, g_0)$ is bounded from below by a constant $-k^2$ (X being compact), by Rauch's theorem (cf. [6]) it follows that the Hessian $\| Dd\rho_z \| \leq k/\mathrm{tgh}(k\rho) \leq k$. Since $\frac{1}{2}Dd\rho^2 = \rho Dd\rho + d\rho \otimes d\rho$, one gets $\frac{1}{2} \| (Dd\rho^2_{\overline{f}(y')})_x \| \leq kd_0(\overline{f}(y'), x) + 1$. Thus, by Lagrange's theorem, the term $\frac{1}{2t}\left|(d\rho^2_{\overline{f}(y')})_{x(t)}(E_i) - (d\rho^2_{\overline{f}(y')})_x(E_i)\right|$ is dominated, for small t, by $1 + (k + \| D_v E_i \|) d_0(\overline{f}(y'), x) + \delta$.

Then, by Lebesgue's dominated convergence theorem, and by using the fact that x is the barycenter of $\overline{f}_* \mu_{\epsilon,y}$, i.e.

$\int_{\overline{Y}} (d\rho^2_{\overline{f}(y')})_x (D_v E_i)\, \mu_{\epsilon,y}(y') = \int_{\tilde{X}} (d\rho^2_{x'})_x (D_v E_i) \overline{f}_* \mu_{\epsilon,y}(x') = 0 \quad \forall i = 1, ..., n$

one deduces formula (9) and that $\partial_v G_\epsilon$ is continuous.

Analogously, if u is a unitary tangent vector at y and $y(t)$ is the geodesic such that $y(0) = y$, $y'(0) = u$, one has, for small t and $c > \mathrm{Ent}_H(g)$

$\left|\left(\frac{e^{-cd(y(t),y')} - e^{-cd(y,y')}}{t}\right)(d\rho^2_{\overline{f}(y')})_x(E_i)\right| \leq 4(c+\delta) e^{-cd(y,y')} d_0(\overline{f}(y'), x) \in L^1(\overline{Y}, dv_g)$

which allows to differentiate the G^i_ϵ's with respect to u under the sign of the integral, and to obtain (8).

Finally, we already remarked that the function $\rho_{\overline{f}(y')}(x)^2$ is strictly convex (since $(\tilde{X}, g_0)$ has negative curvature), hence $Dd\rho^2_{\overline{f}(y')}(v,v) > 0$ for every $v \in T_x\tilde{X}$. Therefore $(dG_\epsilon)_{(x,y)} |_{T_x\tilde{X}}$ is non-singular. By the implicit functions theorem it then follows that $\overline{f}_\epsilon$ is C^1, as well as formula (7). □

Lemma 2.6 Estimate of $\mathrm{Jac}\overline{f}_\epsilon$.
For every fixed $y \in \overline{Y}$, let $\nu_{\epsilon,y}$ be the measure on $\overline{Y}$ given by

$$\nu_{\epsilon,y}(y') \doteq d_0(\overline{f}_\epsilon(y), \overline{f}(y'))\mu_{\epsilon,y}(y').$$

Put $x = \overline{f}_\epsilon(y) \in \tilde{X}$, and let $k^X_{\epsilon,y}, h^X_{\epsilon,y}$ and $h^Y_{\epsilon,y}$ denote respectively the positive definite quadratic forms [6] *defined on $T_x\tilde{X}$, $T_x\tilde{X}$ and $T_y\overline{Y}$ by*

$$k^X_{\epsilon,y}(v,v) \doteq \frac{1}{\nu_{\epsilon,y}(\overline{Y})} \int_{\overline{Y}} \frac{1}{2}\left(Dd\rho^2_{\overline{f}(y')}\right)_x (v,v)\, \mu_{\epsilon,y}(y')\ , \quad \forall v \in T_x\tilde{X}$$

$$h^X_{\epsilon,y}(v,v) \doteq \frac{1}{\nu_{\epsilon,y}(\overline{Y})} \int_{\overline{Y}} (d\rho_{\overline{f}(y')})_x(v)^2\, \nu_{\epsilon,y}(y')\ , \quad \forall v \in T_x\tilde{X}$$

$$h^Y_{\epsilon,y}(u,u) \doteq \frac{1}{\nu_{\epsilon,y}(\overline{Y})} \int_{\overline{Y}} (d\rho_{y'})_y(u)^2\, \nu_{\epsilon,y}(y')\ , \quad \forall u \in T_y\overline{Y}$$

We will moreover write $K^X_{\epsilon,y}, H^X_{\epsilon,y}$ and $H^Y_{\epsilon,y}$ respectively for the endomorphisms of $T_x\tilde{X}$, $T_x\tilde{X}$ and $T_y\overline{Y}$ associated to $k^X_{\epsilon,y}, h^X_{\epsilon,y}$ and $h^Y_{\epsilon,y}$ (with respect to the metrics g_0, g). One then has:

[6] the notations $(\cdot)^X$, $(\cdot)^Y$ indicate on which spaces the forms are defined (respectively on $T\tilde{X}$ and $T\overline{Y}$), whereas the subscript $(\cdot)_{\epsilon,y}$ stresses the dependance of the forms on y, via $\overline{f}_\epsilon$.

(10) $$Tr_g H^Y_{\epsilon,y} = 1\ , \quad Tr_{g_0} H^X_{\epsilon,y} = 1$$

and the eigenvalues of $H^X_{\epsilon,y}, H^Y_{\epsilon,y}$ *are strictly included between* 0 *and* 1. *Moreover:*

(11) $$k^X_{\epsilon,y}((d\overline{f}_\epsilon)_y(u), v) \le (Ent_H(g)+\epsilon) h^Y_{\epsilon,y}(u,u)^{1/2} h^X_{\epsilon,y}(v,v)^{1/2}$$

for all $u \in T_y\overline{Y}$, $v \in T_x\tilde{X}$, *and*

(12) $$\left|Jac_y \overline{f}_\epsilon\right| \le \frac{(Ent_H(g)+\epsilon)^n}{n^{n/2}} \cdot \frac{\left(det H^X_{\epsilon,y}\right)^{1/2}}{det K^X_{\epsilon,y}} \qquad \forall y \in \overline{Y}.$$

Proof of lemma 2.6.
Since $\| \mathrm{grad}\, \rho_{x'} \| = 1$, one has $\sum_{i=1}^n d\rho_{\overline{f}(y')}(E_i)^2 = 1$, and so $\mathrm{Tr}_{g_0} H^X_{\epsilon,y} = 1$ (and analogously $\mathrm{Tr}_g H^Y_{\epsilon,y} = 1$). As $h^X_{\epsilon,y}$ and $h^Y_{\epsilon,y}$ are positive definite, the eigenvalues are positive and strictly smaller than 1.
From formulas (7), (8) and (9) of lemma 2.5 it straightforwardly follows that
$$k^X_{\epsilon,y}\left((d\overline{f}_\epsilon)_y(u), v\right) = \frac{Ent_H(g)+\epsilon}{\nu_{\epsilon,y}(\overline{Y})} \int_{\overline{Y}} (d\rho_{y'})_y(u)(d\rho_{\overline{f}(y')})_x(v)\nu_{\epsilon,y}(y')$$
for all $u \in T_y\overline{Y}$, $v \in T_x\tilde{X}$. The Cauchy-Schwarz inequality then yields (11).
To show that (12) holds, let us first notice that $K^X_{\epsilon,y}$ is non-singular (as $(Dd\rho^2_{x'})_x$ is positive definite), and that (12) is trivially satisfied when $\mathrm{Jac}_y\overline{f}_\epsilon = 0$. On the other hand, at a point y where $(d\overline{f}_\epsilon)_y$ is invertible, inequality (12) directly follows from (11) algebraically. In fact, let $F : (U,g) \to (V,g_0)$ be a linear isomorphism $F : (U,g) \to (V,g_0)$ between euclidean spaces of dimension n, and let h^U and h^V, k^V be positive definite bilinear forms respectively on U and V (represented by endomorphisms H^U, K^V, H^V) which verify
$$\mid k^V(F(u),v) \mid \le C \cdot h^U(u,u)^{1/2} \cdot h^V(v,v)^{1/2}.$$
Since K^V is an isomorphism, if one chooses a g_0-orthonormal basis $\{v_i\}$ of V which diagonalizes h^V, one can consider the basis $\{u_i\}$ of U which is obtained by orthonormalization of the basis $\{(K^V \circ F)^{-1}(v_i)\}$. Then $K^V \circ F(u_i)$ is a linear combination $\sum_{j\le i} a_{ij}v_j$, hence the matrix of $K^V \circ F$ with respect to the bases $\{u_i\}$ and $\{v_i\}$ is superior triangular. It follows that
$$\mid \mathrm{det} K^V \mid \cdot \mid \mathrm{det} F \mid = \prod_{i=1}^n \left| g_0\left(K^V \circ F(u_i), v_i\right)\right| = \prod_{i=1}^n \left| k^V\left(F(u_i), v_i\right)\right| \le$$
$$\le C^n \left(\prod_{i=1}^n h^U(u_i,u_i)^{1/2}\right)\cdot\left(\prod_{i=1}^n h^V(v_i,v_i)^{1/2}\right) \le C^n \left(\tfrac{1}{n}\mathrm{Tr} H^U\right)^{n/2} \cdot \left(\mathrm{det} H^V\right)^{1/2} \ \square$$

Now the proofs of inequalities (2) and (4) differ about the computation of the bilinear forms $k^X_{\epsilon,y}$.

Lemma 2.7 Estimate of $K^X_{\epsilon,y}$.
Let $x_0 \in (\tilde{X}, g_0)$ *and put* $\rho(x) = d(x_0, x)$.
1) Assume that $k(g_0) \le -1$. *Then,*

(13) $$k^X_{\epsilon,y} \ge g_0 - h^X_{\epsilon,y} \quad and \quad det K^X_{\epsilon,y} \ge det(I - H^X_{\epsilon,y})\ .$$

2) Assume that g_0 is locally symmetric of negative curvature [7]. *If $d = dim_{\mathbf{R}}\mathbf{K}$, let $J_1, ..., J_{d-1}$ denote the orthogonal endomorphisms of $T\tilde{X}$ such that $J_k^2 = -id$, induced by multiplication by the imaginary units of the algebra $\mathbf{K}$. Let moreover be $J_d = id$. Then,*

$$k^X_{\epsilon,y}(v,v) \geq g_0(v,v) - h^X_{\epsilon,y}(v,v) + \sum_{k=1}^{d-1} h^X_{\epsilon,y}(J_k v, J_k v) \tag{14}$$

and

$$det K^X_{\epsilon,y} \geq det(I - \sum_{k=1}^{d} J_k H^X_{\epsilon,y} J_k) \ . \tag{15}$$

Proof of lemma 2.7.
Let $c_u(\rho)$ be the unitary geodesic issuing from x_0 with $c'_u(0) = u$, and put $\frac{\partial}{\partial\rho} = c'_u(\rho)$. Let $S(\rho) = \{x \in \tilde{X} \ \backslash d(x_0, x) = \rho\}$ and let $II_{S(\rho)}(\cdot,\cdot)$ be the second fundamental form of $S(\rho)$. One has $II_{S(\rho)}(v,v) = (Dd\rho)(v,v)$ for every $v \in T_x S(\rho)$, while $Dd\rho(\cdot, \frac{\partial}{\partial\rho}) = 0$.

1. For any Jacobi field $Y(\rho) = (d\exp_{x_0})_{tu}(tv)$ along c_u, orthogonal to c_u, one has $II_{S(\rho)}(Y,Y) = g_0(D_Y \frac{\partial}{\partial\rho}, Y) = g_0(Y', Y)$. As $k(g_0) \leq -1$, by Rauch's comparison theorem (cf. [6]) it follows that

$$\frac{Dd\rho(Y,Y)}{g_0(Y,Y)} = \frac{g_0(Y',Y)}{g_0(Y,Y)} \geq \frac{\overline{g}_0(\overline{Y}', \overline{Y})}{\overline{g}_0(\overline{Y}, \overline{Y})} = \frac{1}{\mathrm{tgh}\rho} \tag{16}$$

if $\overline{Y}$ denotes a comparison Jacobi field in the hyperbolic space $(H^n(\mathbf{R}), \overline{g}_0)$ along a geodesic $c_{\overline{u}}$, orthogonal to $c_{\overline{u}}$, with $\overline{Y}(0) = 0$, $\| \overline{Y}'(0) \| = \| Y'(0) \|$, and $\overline{g}_0(\overline{Y}'(0), \overline{u}) = g_0(Y'(0), u)$. .
This shows that $Dd\rho \geq \frac{1}{tgh\rho}(g_0 - d\rho \otimes d\rho)$ (since $d\exp_{x_0}$ is everywhere non-singular), and therefore

$$\tfrac{1}{2} Dd\rho^2 = \rho Dd\rho + d\rho \otimes d\rho \geq \rho(g_0 - d\rho \otimes d\rho)$$

By the very definition of $k^X_{\epsilon,y}$ and $h^X_{\epsilon,y}$, it then follows that $k^X_{\epsilon,y} \geq g_0 - h^X_{\epsilon,y}$, and hence that $\det K^X_{\epsilon,y} \geq \det(I - H^X_{\epsilon,y})$.

2. One knows that the complex tangent planes $< v, J_k v >$ of the spaces $H^m(\mathbf{K})$ have sectional curvature equal to -4 (and $R(v, J_k v)v // J_k(v)$), whereas totally real planes (i.e. planes $< v_1, v_2 >$ with $v_2 \perp < J_k v_1 >_{k=1,\ldots,d}$, have sectional curvature equal to -1 (and $R(v_1, v_2)v_1 // v_2$).
So let $\{v_i\}$ an orthonormal basis at x_0 such that $v_i = J_i u$ for $1 \leq i \leq d$, and such that $v_i \perp < J_k u >_{k=1,\ldots,d}$ if $i > d$. Let $V_i(\rho)$ denote the parallel displacement of the v_i's along $c_u(\rho)$.

[7] this amounts to say that $(\tilde{X}, g_0)$ is the hyperbolic space $H^m(\mathbf{K})$ over the algebra $\mathbf{K}$ (where $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) or the Cayley hyperbolic plane $H^2(\mathbf{Ca})$. We will assume g_0 normalized so that $k(g_0) = -1$ if $\mathbf{K} = \mathbf{R}$, and $-4 \leq k(g_0) \leq -1$ otherwise.

Since $DR = 0$, one finds that the Jacobi fields Y_i along c_u, whose initial data are $Y_i(0) = 0$, $Y_i'(0) = v_i$, satisfy:

$v_i \perp < J_k u >_{k=1,...,d} \;\Rightarrow\; k(V_i, \frac{\partial}{\partial\rho}) = -1$ along $c_u \;\Rightarrow\; Y_i(\rho) = \text{senh}\rho \cdot V_i(\rho)$

and $v_i = J_i u$ (for $i \neq d$) $\;\Rightarrow\; k(V_i, \frac{\partial}{\partial\rho}) = -4$ along $c_u \;\Rightarrow\; Y_i(\rho) = \frac{1}{2}\text{senh}2\rho \cdot V_i(\rho)$.

Hence the $Y_i(\rho)$'s are eigenvectors of the second fundamental form of $S(\rho)$ for every ρ (because $D_{Y_i}\frac{\partial}{\partial\rho} = D_{\frac{\partial}{\partial\rho}} Y_i = Y_i' // Y_i$), and the principal curvatures are

$$\eta_i = \frac{g_0(Y_i'(R), Y_i(R))}{g_0(Y_i(R), Y_i(R))} = \begin{cases} \frac{1}{tgh\rho} & \text{if } v_i \perp < J_k u >_{k=1,...,d} \\ \frac{2}{tgh2\rho} & \text{if } v_i = J_i u \quad (\text{for } i < d). \end{cases}$$

One then deduces that, for any $v \in T_x\tilde{X}$,

$$Dd\rho(v,v) = \frac{1}{tgh\rho}\left\{g_0(v,v) - d\rho(v)^2\right\} + (tgh\rho)\sum_{k=1}^{d-1} d\rho(J_k v)^2$$

and also the estimate $Dd\rho(v,v) \geq g_0(v,v) - d\rho(v)^2 + \sum_{k=1}^{d-1} d\rho(J_k v)^2$.

As $\frac{1}{2}Dd\rho^2 \geq \rho Dd\rho$, one therefore obtains formula (14) for $k_{\epsilon,y}^X$. As the form $\sum_{k=1}^{d-1} h_{\epsilon,y}^X(J_k\cdot, J_k\cdot)$ is represented by the operator $-\sum_{k=1}^{d-1} J_k H_{\epsilon,y}^X J_k$ with respect to g_0, formula (15) also immediately follows. □

Lemma 2.8 (Algebraic lemma)
Let $V = \mathbf{K}^m$ (where $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{Ca}$) and $d = dim_{\mathbf{R}}\mathbf{K}$. Let $J_1, ..., J_{d-1}$ denote the endomorphisms of V which are the multiplication by the imaginary units of $\mathbf{K}$, and let $J_d = id_V$. Assume moreover that $dim_{\mathbf{R}}V = n \geq 3$.
Let $\mathcal{K}$ be the compact set made up of all symmetric endomorphisms H of V (with respect to the standard metric of V) whose eigenvalues h_i satisfy $0 \leq h_i \leq 1$ and $Tr(H) = \sum h_i = 1$. Then, one has

$$\Phi(H) \doteq \frac{(det H)^{1/2}}{det(I - \sum_{k=1}^{d} J_k H J_k)} \leq \frac{n^{n/2}}{(n+d-2)^n} \qquad \textit{for every } H \in \mathcal{K}^o$$

and the value $\frac{n^{n/2}}{(n+d-2)^n}$ is attained at the unique point of absolute maximum $H = I/n$. Moreover, one has $\limsup_{H\to\partial\mathcal{K}} \Phi(H) < \Phi(I/n)$.

Proof: see *Appendice B* of [3].

Conclusion: by introducing the estimates (13) and (15) of lemma 2.7 for $\det K_{\epsilon,y}^X$ in formula (12), which expresses the jacobian of f_ϵ, one obtains:

$$|\,\text{Jac}_y f_\epsilon\,| \leq \frac{(\text{Ent}_H(g)+\epsilon)^n}{n^{n/2}} \cdot \frac{\left(\det H_{\epsilon,y}^X\right)^{1/2}}{\det\left(I - \sum_{k=1}^{d} J_k H_{\epsilon,y}^X J_k\right)} \tag{17}$$

(where clearly one means that $d = 1$ and $\sum_{k=1}^{d} J_k H_{\epsilon,y}^X J_k = H_{\epsilon,y}^X$ when $k(g_0) \leq -1$ but g_0 is not assumed to be locally symmetric).

Since $\text{Ent}(g_0) = n + d - 2$ if g_0 is locally symmetric [8] (with curvature equal to -1 if g_0 is real hyperbolic, and normalized between -4 and -1 otherwise), the above formula and the previous algebraic lemma yield:

$$|\text{Jac}_y f_\epsilon| \leq \begin{cases} \left(\frac{Ent_H(g)+\epsilon}{n-1}\right)^n & \text{if } k(g_0) \leq -1 \\ \left(\frac{Ent_H(g)+\epsilon}{Ent(g_0)}\right)^n & \text{if } g_0 \text{ is locally symmetric.} \end{cases}$$

By using the coarea formula $\int_Y \mid Jac_y f_\epsilon \mid dv_g \geq \text{Adeg}(f_\epsilon) \cdot \text{Vol}(X, g_0)$, and by taking the limit as $\epsilon \to 0$, one also deduces inequalities (3) and (5) (as $\text{Adeg}(f_\epsilon) = \text{Adeg}(f)$, since the f_ϵ's are homotopic to f).

3 The equality case

In order to treat both cases simultaneously, we let $d = 1$ and $h_0 = n - 1$ in the general case (theorem 1.2), while h_0 will denote the constant $\text{Ent}(g_0) = n+d-2$ and $d = \dim_{\mathbf{R}}\mathbf{K}$ when g_0 is assumed to be locally symmetric (with sectional curvature equal to -1 if g_0 is real hyperbolic, and normalized between -4 and -1 otherwise). By rescaling the metric g on Y, one may suppose moreover that also $\text{Ent}_H(g) = h_0$.

So, we have to show that *if Adeg*$(f) \neq 0$ *and Vol*$(Y, g) =$ *Adeg*$(f) \cdot$ *Vol*(X, g_0), *the homotopy class of f contains a riemannian covering* $f_0 : (Y, g) \to (X, g_0)$.

In particular, in case h_0 is assumed to be equal to $n-1$, this will show that also g_0 is a metric with $\text{Ent}(g_0) = n-1$ (and sectional curvature bounded from above by -1): therefore g_0 necessarily has constant sectional curvature $k(g_0) = -1$, by a well-known result due to U. Hamenstädt (see [10]).

Now, by the coarea formula

$$\text{Adeg}(f_\epsilon)\cdot\text{Vol}(X, g_0) \leq \int_Y |\text{Jac}_y f_\epsilon|\, dv_g \leq \left(\tfrac{h_0+\epsilon}{h_0}\right)^n \cdot\text{Vol}(Y, g) \xrightarrow{\epsilon\to 0} \text{Vol}(Y, g)$$

and by the hypothesis $\text{Vol}(Y, g) = \text{Adeg}(f)\text{Vol}(X, g_0)$, one deduces that if $\epsilon \to 0$ then $|\text{Jac}_y f_\epsilon| \xrightarrow{L^1} 1$. We will extract, from the f_ϵ's, a sequence f_{ϵ_i} which tends to a riemannian covering f_0 as $i \to \infty$.

As $|\text{Jac}_y f_\epsilon| \xrightarrow{L^1} 1$, we can consider a decreasing sequence $\epsilon_i \searrow 0$, such that the maps f_{ϵ_i} satisfy $|\text{Jac}_y f_{\epsilon_i}| \to 1$ a.e. y, if $i \to \infty$. For simplicity, we will replace, from now on, every index ϵ_i with i.

Remark 3.1 Let $\overline{\pi} : \overline{Y} \to Y$ e $\tilde{\pi} : \tilde{X} \to X$ be the coverings of Y and X under consideration. If $\overline{y} \in \overline{Y}$ and $y = \overline{\pi}(\overline{y})$, by the equivariance properties of the measures $\mu_{i,\overline{y}} \in \mathcal{M}(\overline{Y})$ proved in lemma 2.3, it follows that the bilinear form

[8] although classical, this result directly follows from the computation of the second fundamental form of the geodesic sphere $S(\rho) \subset H^m(\mathbf{K})$ developed in lemma 2.7.2, since, by L'Hôpital's rule, the entropy $\text{Ent}(g_0)$ is

$\lim_{\rho\to+\infty} \text{Vol}S(\rho)^{-1} \cdot \frac{d}{d\rho}\text{Vol}S(\rho) = \lim_{\rho\to+\infty} TrII_{S(\rho)} = \lim_{\rho\to+\infty} \left[\frac{n-d}{tgh\,\rho} + \frac{2(d-1)}{tgh\,2\rho}\right]$.

$h^Y_{i,\overline{y}}$ of $T_{\overline{y}}\overline{Y}$ induces on T_yY a bilinear form (still denoted by $h^Y_{i,y}$), defined as $h^Y_{i,y}(u,u) = h^Y_{i,\overline{y}}(\overline{u},\overline{u})$, for every $\overline{u} \in T_{\overline{y}}\overline{Y}$ such that $(d\overline{\pi})_{\overline{y}}(\overline{u}) = u$.
Analogously, because of the equivariance properties of the maps $\overline{f}_i : \overline{Y} \to \tilde{X}$ with respect to the homomorphism $\phi : \Gamma \to \pi_1(X)$, also the quadratic forms $h^X_{i,\overline{y}}$ and $k^X_{i,\overline{y}}$, defined on the spaces $T_{\overline{f}_i(\overline{y})}\tilde{X}$ for every $\overline{y} \in \overline{Y}$, induce quadratic forms on $T_{f_i(y)}X$, which again will be denoted by $h^X_{i,y}, k^X_{i,y}$ (if $y = \overline{\pi}(\overline{y})$).
We will equally use $H^X_{i,y}$ and $K^X_{i,y}$ to indicate the symmetric endomorphisms of $T_{f_i(y)}X$ induced by the endomorphisms $H^X_{i,\overline{y}}$, $K^X_{i,\overline{y}}$.

Lemma 3.2 Estimates for $H^X_{i,y}, K^X_{i,y}$ when $i \to \infty$.
1) $H^X_{i,y} \to I/n$ *a.e.* y*, as* $i \to \infty$ *;*
2) $K^X_{i,y} \to h_0 I/n$ *a.e.* y*, as* $i \to \infty$ *.*

Proof of lemma 3.2.
Inequalites (12), (13) & (15) and lemma 2.8 yield

$$\frac{n^{n/2}}{(h_0+\epsilon_i)^n} \cdot |\mathrm{Jac}_y f_i| \leq \frac{\left(\det H^X_{i,y}\right)^{1/2}}{\det K^X_{i,y}} \leq \frac{\left(\det H^X_{i,y}\right)^{1/2}}{\det\left(I - \sum_{k=1}^d J_k H^X_{i,y} J_k\right)} \leq \frac{n^{n/2}}{h_0^n}$$

therefore $(\det H^X_{i,y})^{1/2}/\det(I - \sum_{k=1}^d J_k H^X_{i,y} J_k) \to n^{n/2}/h_0^n$ a.e. y necessarily, as $i \to \infty$, since $|\,\mathrm{Jac}_y f_i\,| \to 1$ a.e. y by assumption. Still by lemma 2.8 it follows that $H^X_{i,y} \to I/n$ a.e. y.
On the other hand, the previuos chain of inequalities shows that, if $i \to \infty$,

$$|\det K^X_{i,y} - \det(I - \textstyle\sum_{k=1}^d J_k H^X_{i,y} J_k)\,| \to 0 \quad \text{a.e. } y.$$

Moreover, as we have, by formulas (13) & (14) of lemma 2.7,

$$g_0(K^X_{i,y}v, v) \geq g_0((I - \textstyle\sum_{i=1}^d J_k H^X_{i,y} J_k)v, v)$$

we deduce that, if $i \to \infty$,

$$\|\, K^X_{i,y} - (I - \textstyle\sum_{i=1}^d J_k H^X_{i,y} J_k)\,\| \to 0 \text{ a.e. } y$$

which proves the second assertion, since $I - \sum_{i=1}^d J_k H^X_{i,y} J_k \to h_0 I/n$. □

Lemma 3.3 Estimate of $\|\, df_i \,\|$ when $i \to \infty$.
1) There exists $\epsilon_0 > 0$ *such that if* $\|\, H^X_{i,y} - I/n \,\| \leq \epsilon_0$ *then* $\|\,(df_i)_y\,\| \leq 4n \;\; \forall i$*;*
2) $\|\,(df_i)_y\,\| \to 1$ *a.e.* y*, as* $i \to \infty$ *(not necessarily uniformly in* y*).*

Proof of lemma 3.3.
1. Taking the infimum in inequality (11) of lemma 2.6 over $v \in T_{f_i(y)}X$, $\|\, v\,\| \leq 1$, one obtains

$$(18)\; \|\, k^X_{i,y} \circ (df_i)_y(u)\,\| \leq (h_0+\epsilon_i)\; \|\, H^X_{i,y}\,\|^{1/2} \left(\frac{1}{\nu_{i,\overline{y}}(\overline{Y})} \int_{\overline{Y}} (d\rho_{y'})_{\overline{y}}(\overline{u})^2 \nu_{i,\overline{y}}(y') \right)^{1/2}$$

for all $u \in T_yY$, and for all $\overline{y} \in \overline{Y}, \overline{u} \in T_{\overline{y}}\overline{Y}$ such that $\overline{\pi}(\overline{y}) = y, d\overline{\pi}(\overline{u}) = u$.

Assume $\| H^X_{i,y} - I/n \| \leq \epsilon$. As $g_0(K^X_{i,y}v, v) \geq g_0((I - \sum_{i=1}^d J_k H^X_{i,y} J_k)v, v)$, we deduce that the smallest eigenvalue of $K^X_{i,y}$ is greater than $1 - d(\frac{1}{n} + \epsilon)$.
By (18) it then follows that, for all $u \in T_y Y$, $\| u \| \leq 1$,

$$\| df_i(u) \| \leq \| K^X_{i,y} \circ (df_i)_y(u) \| / (1 - d(\tfrac{1}{n} + \epsilon)) \leq \tfrac{n+d-2+\epsilon_1}{1-d(1/n+\epsilon)} \sqrt{1/n + \epsilon} \leq 4n$$

if ϵ is smaller than some ϵ_0 (assuming that ϵ_1 was chosen smaller than 1).

2. It will suffice to show that the pullback $\left(K^X_{i,y} \circ (df_i)_y\right)^* g_0$ tends to $(h_0/n)^2 g$ a.e. y, since we already showed (lemma 3.2.2) that $K^X_{i,y} \to h_0 I/n$ a.e. y.
Now, on the one hand

$$\lim_{i \to \infty} \det \left(K^X_{i,y} \circ (df_i)_y\right)^* g_0 = \left(\frac{h_0}{n}\right)^{2n} \quad \text{a.e. } y \tag{19}$$

since, by assumption and by lemma 3.2.2,

$$\det(K^X_{i,y} \circ (df_i)_y)^* g_0 = | \operatorname{Jac}_y f_i | \cdot \det(K^X_{i,y})^* g_0 \to 1 \cdot \det(h_0 I/n)^2 \text{ a.e. } y.$$

On the other hand, if $\overline{\pi}(\overline{y}) = y$ and $\{\overline{u}_i\}$ is a $g-$orthonormal basis of $T_{\overline{y}}\overline{Y}$, formula (18) yields

$$\frac{1}{(h_0+\epsilon_i)^2} \operatorname{Tr}_g (K^X_{i,y} \circ (df_i)_y)^* g_0 \leq \frac{\| H^X_{i,y} \|}{\nu_{i,\overline{y}}(\overline{Y})} \int_{\overline{Y}} \sum_{i=1}^n (d\rho_{y'})_{\overline{y}}(\overline{u}_i)^2 \nu_{i,\overline{y}}(y') = \| H^X_{i,y} \|$$

so

$$\lim_{i \to \infty} \operatorname{Tr}_g \left(K^X_{i,y} \circ (df_i)_y\right)^* g_0 \leq \frac{h_0^2}{n} \quad \text{a.e. } y. \tag{20}$$

From formulas (19) and (20) it then follows, algebraically, that $\left(K^X_{i,y} \circ (df_i)_y\right)^* g_0$ tends to $(h_0/n)^2 g$ for almost every y. □

Remarks 3.4
1) As $\operatorname{Adeg}(f) \neq 0$, *the map* $\overline{f} : (\overline{Y}, g) \to (\tilde{X}, g_0)$ *is a quasi-isometry*, i.e. there exist positive costants a, b, A, B such that

$$a \cdot d(y_1, y_2) - b \leq d_0(\overline{f}(y_1), \overline{f}(y_2)) \leq A \cdot d(y_1, y_2) + B \quad \forall y_1, y_2 \in \overline{Y}. \tag{21}$$

In fact, the spaces $(\overline{Y}, g)$ and $(\tilde{X}, g_0)$ are quasi-isometric respectively to the groups Γ and $\pi_1(X)$ (endowed by the word metrics induced by some finite sets of generators, cf. [9]) and, in turn, Γ is quasi-isometric to $\pi_1(X)$, since it may be identified to a subgroup of finite index of $\pi_1(X)$ via ϕ.
2) *There exists* $\nu_0 > 0$ *such that* $\| \nu_{i,y} \| \geq \nu_0$ *for all* i *and for all* $y \in \overline{Y}$ (where $\| \nu_{i,y} \| \doteq \int_{\overline{Y}} \nu_{i,y}$ is the norm of $\nu_{i,y}$ as a finite Borel measure on $\overline{Y}$).
Actually, there exists $\nu_0 > 0$ such that the measures

$$\nu_{\epsilon,x,y} \doteq \int_{\overline{Y}} d_0(x, \overline{f}(y')) e^{-(Ent_H(g)+\epsilon) d(y,y')} dv_g(y') \geq \nu_0$$

for all $x \in \tilde{X}$, $y \in \overline{Y}$ and for all $\epsilon > 0$.
In fact, let $y_0 \in \overline{Y}$ be given: as $Y = \overline{Y}/\Gamma$ is compact, and by (21), we can find some $R_0 > 0$ such that for every $x \in \tilde{X}$ one has $\overline{f}^{-1}(B_{g_0}(x, 1)) \subset B_g(\gamma \cdot y_0, R_0)$, for some $\gamma \in \Gamma$. Let moreover $\gamma' \in \Gamma$ such that $d(\gamma' \cdot y_0, y) \leq \operatorname{diam}(Y, g)$. Then,

$$\| \nu_{\epsilon,y} \| \geq \int_{\overline{Y} \setminus B_g(\gamma y_0, R_0)} e^{-(Ent_H(g)+\epsilon)d(y,y')} dv_g(y') \geq$$
$$\geq e^{-(Ent_H(g)+\epsilon)diam(Y,g)} \int_{\overline{Y} \setminus B_g(\gamma y_0, R_0)} e^{-(Ent_H(g)+\epsilon)d(\gamma' y_0, y')} dv_g(y') \geq$$
$$\geq e^{-(Ent_H(g)+1)diam(Y,g)} \int_{\overline{Y} \setminus B_g(y_0, R_0)} e^{-(Ent_H(g)+1)d(y_0,y')} dv_g(y')$$

because of the $\Gamma-$invariance of dv_g and of the metric g on $\overline{Y}$.

Lemma 3.5 Variation of $H^X_{i,y}$ with respect to y, when i is fixed.
Let $y_1, y_2 \in Y$, let $x_1 = f_i(y_1), x_2 = f_i(y_2)$ and let β be a minimizing g_0-geodesic from x_1 to x_2. If P_{x_2} denotes the parallel displacement from x_1 to x_2 along β, one has:

$$\| H^X_{i,y_1} - P^{-1}_{x_2} \circ H^X_{i,y_2} \circ P_{x_2} \| \leq M \{ d(y_1,y_2) + d_0(x_1,x_2) \}$$

for some constant M which does not depend on i, y_1, y_2.

Proof of lemma 3.5.
Let $\overline{y}_1, \overline{y}_2 \in \overline{Y}$ e $\tilde{x}_1, \tilde{x}_2 \in \tilde{X}$ be points which are projected respectively onto y_1, y_2, x_1, x_2, and such that $d(y_1,y_2) = d(\overline{y}_1, \overline{y}_2)$ and $d_0(x_1,x_2) = d_0(\tilde{x}_1, \tilde{x}_2)$. Let $\tilde{\beta}$ be the lift of β from $\tilde{x}_1$ to $\tilde{x}_2$, and $P_{\tilde{\beta}(t)}$ the parallel displacement from $\tilde{x}_1$ to $\tilde{\beta}(t)$ along $\tilde{\beta}$. Let us compute $\left| h^X_{i,\overline{y}_1} - h^X_{i,\overline{y}_2} \circ P_{\tilde{x}_2}(v_1,v_1) \right|$ for $v_1 \in T_{\tilde{x}_1}\tilde{X}$, $\| v_1 \| \leq 1$. Adding the terms $\pm \int_{\overline{Y}} (d\rho_{\overline{f}(y')})_{\tilde{x}_1}(v_1)^2 \cdot \frac{d_0(\tilde{x}_2, \overline{f}(y')) e^{-(h_0+\epsilon_i)d(\overline{y}_2,y')}}{\| \nu_{i,\overline{y}_2} \|} dv_g(y')$ one gets:

$$\left| h^X_{i,\overline{y}_1} - h^X_{i,\overline{y}_2} \circ P_{\tilde{x}_2}(v_1,v_1) \right| \leq$$
$$\leq \frac{1}{\| \nu_{i,\overline{y}_2} \|} \int_{\overline{Y}} \left| (d\rho_{\overline{f}(y')})_{\tilde{x}_1}(v_1)^2 - (d\rho_{\overline{f}(y')})_{\tilde{x}_2}(v_2)^2 \right| \nu_{i,\overline{y}_2}(y') + \tag{22}$$
$$+ \left\| \frac{\nu_{i,\overline{y}_1}}{\| \nu_{i,\overline{y}_1} \|} - \frac{\nu_{i,\overline{y}_2}}{\| \nu_{i,\overline{y}_2} \|} \right\| .$$

As the curvature of $(\tilde{X}, g_0)$ is bounded from below by a constant $-k^2$, Rauch's comparison theorem yields (as in lemma 2.5) $\| Dd\rho_z \| \leq M_1$ for all $z \in \tilde{X}$.
Then, by Lagrange's theorem, the value of $\left| (d\rho_{\overline{f}(y')})_{\tilde{x}_1}(v_1)^2 - (d\rho_{\overline{f}(y')})_{\tilde{x}_2}(v_2)^2 \right|$ (and, in turn, of the integral in (22)) is smaller than $2M_1 d_0(\tilde{x}_1, \tilde{x}_2)$.
Let us now consider the last term of (22). One has

$$\left\| \frac{\nu_{i,\overline{y}_1}}{\| \nu_{i,\overline{y}_1} \|} - \frac{\nu_{i,\overline{y}_2}}{\| \nu_{i,\overline{y}_2} \|} \right\| \leq \frac{2}{\| \nu_{i,\overline{y}_2} \|} \| \nu_{i,\overline{y}_1} - \nu_{i,\overline{y}_2} \| \leq$$
$$\leq \frac{2}{\| \nu_{i,\overline{y}_2} \|} \int_{\overline{Y}} \left| d_0(\tilde{x}_1, \overline{f}(y')) - d_0(\tilde{x}_2, \overline{f}(y')) \right| e^{-(h_0+\epsilon_i)d(\overline{y}_1,y')} dv_g(y') + \tag{23}$$
$$+ \frac{2}{\| \nu_{i,\overline{y}_2} \|} \int_{\overline{Y}} d_0(\tilde{x}_2, \overline{f}(y')) \left| e^{-(h_0+\epsilon_i)d(\overline{y}_1,y')} - e^{-(h_0+\epsilon_i)d(\overline{y}_2,y')} \right| dv_g(y') .$$

By the triangular inequality, and since $\| \nu_{i,y} \| \geq \nu_0$ for every i, y (cf. remark 3.4.2), the first integral is not greater than

$$2e^{-(h_0+\epsilon_i)d(\overline{y}_1,\overline{y}_2)} \cdot d_0(\tilde{x}_1, \tilde{x}_2) \cdot \frac{1}{\|\nu_{i,\overline{y}_2}\|} \int_{\overline{Y}} e^{-(h_0+\epsilon_i)d(\overline{y}_2,y')} dv_g(y') \leq$$

$$\leq 2e^{-(h_0+\epsilon_i)d(\overline{y}_1,\overline{y}_2)} \cdot d_0(\tilde{x}_1, \tilde{x}_2) \cdot \left(1+\frac{1}{\nu_0} \int_{\overline{f}^{-1} B_{g_0}(\tilde{x}_2,1)} e^{-(h_0+\epsilon_i)d(\overline{y}_2,y')} dv_g(y')\right)$$

which is smaller than $d_0(\tilde{x}_1, \tilde{x}_2) \cdot M_2$ (because of the fact that $\overline{f}^{-1}(B_{g_0}(\tilde{x}_2, 1))$ is included in some ball of $\overline{Y}$ of radius R_0, whose value does not depend on i and $\tilde{x}_2$, by (21), and since volumes of balls of fixed radius in $(\overline{Y}, g)$ are uniformly bounded from above).

Lastly, the second integral in (23) is easily seen to be not greater than

$$2(h_0 + \epsilon_i)e^{(h_0+\epsilon_i)d(\overline{y}_1,\overline{y}_2)} d(\overline{y}_1, \overline{y}_2) \leq M_3 \cdot d(\overline{y}_1, \overline{y}_2).$$

Concluding,

$$\left| h^X_{i,\overline{y}_1} - h^X_{i,\overline{y}_2} \circ P_{\tilde{x}_2}(v_1, v_1) \right| \leq (2M_1 + M_2) d_0(\tilde{x}_1, \tilde{x}_2) + M_3 d(\overline{y}_1, \overline{y}_2). \ \square$$

Lemma 3.6 Uniform convergence of the $H^X_{i,y}$'s.
The endomorphisms $H^X_{i,y}$ converge uniformly to I/n on Y, as $i \to \infty$.

Proof of lemma 3.6.
Let us prove the lemma by contradiction, by supposing that $H^X_{i,y}$ do not converge to I/n uniformly. Then, there exists $\epsilon > 0$, smaller than the ϵ_0 of lemma 3.3, and points $\{y_i\}$ so that, for arbitrarily large i,

$$\| H^X_{i,y_i} - I/n \| > \epsilon . \tag{24}$$

As $H^X_{i,y}$ converges to I/n a.e. y, by Egoroff's theorem $H^X_{i,y}$ converges uniformly to I/n on subsets E of measure arbitrarily close to $\mathrm{Vol}(Y, g)$. Since volumes of balls with fixed radius in (Y, g) are uniformly bounded from below (because Y is compact), we can choose a $\delta = \delta(\epsilon) > 0$ and a subset $E_\delta \subset Y$ which verify

$$\sup\{\delta, \delta + M(1 + 4n)\delta\} < \epsilon < \epsilon_0 \tag{25}$$

$$d(y, E_\delta) \leq \delta \quad \text{for all } y \in Y \tag{26}$$

and such that $H^X_{i,y}$ converges uniformly to I/n on E_δ, as $i \to \infty$.
Let then $N = N(\delta)$ be such that

$$\| H^X_{i,y'} - I/n \| \leq \delta \quad \text{for all } i > N \quad \text{and for all } y' \in E_\delta . \tag{27}$$

Consider a point $y = y_i$, for some fixed $i > N$, and let $y' \in E_\delta$ be so that $d(y, y') \leq \delta$. As $\| H^X_i(y') - I/n \| \leq \delta < \epsilon$, by (27), and since $\| H^X_{i,y} - I/n \| > \epsilon$, by (24), there exists by continuity a first point y'', along a minimizing geodesic α of (Y, g) joining y' to y, where

$$\| H^X_{i,y''} - I/n \| = \epsilon. \tag{28}$$

Since, then, at every point of α between y' and y'' one has $\| H^X_{i,y''} - I/n \| \leq \epsilon < \epsilon_0$, by lemma 3.3 it follows that $\| (df_i) \| \leq 4n$ along α, between y' and y''.
Hence, $d_0(f_i(y'), f_i(y'')) \leq 4n \cdot d(y', y'') \leq 4n\delta$ and, by lemma 3.5, if $P_{f_i(y'')}$ is the parallel displacement from $f_i(y')$ to $f_i(y'')$ along a minimizing g_0-geodesic, one obtains $\| H^X_{i,y'} - P^{-1}_{f_i(y'')} \circ H^X_{i,y''} \circ P_{f_i(y'')} \| \leq M(1+4n)\delta$. This yields $\| H^X_{i,y''} - I/n \| \leq \| H^X_{i,y'} - I/n \| \leq \delta + M(1+4n)\delta < \epsilon$, by the choice (25) of δ, which contradicts (28). □

Lemma 3.7 Limit f_0 of the f_i's.
A subsequence of the $\{f_i\}$*'s converges uniformly to a map* $f_0 : Y \to X$ *which contracts distances and is homotopic to the initial map* f.

Proof of lemma 3.7.
As $H^X_{i,y} \to I/n$ uniformly on Y, lemma 3.3.1 ensures that $\| df_i \|$ is uniformly bounded on Y. By Ascoli-Arzelà's theorem there exists a subsequence of the $\{f_i\}$'s which converges uniformly to a continuous map f_0. This map is homotopic to f, being a uniform limit of maps homotopic to f.
As $\| (df_i)_y \| \to 1$ a.e. y (by lemma 3.3.2), there exists an open set $E \subset Y$ of full measure where every $\| (df_i)_y \|$ is defined and where $\| (df_i)_y \| \to 1$.
For every pair of points $y_1, y_2 \in Y$ one can choose, by Fubini's theorem, a point y_2' arbitrarily close to y_2 and a minimizing geodesic γ from y_1 to y_2' which only meets $Y \setminus E$ on some set of zero measure. Thus,

$$d_0(f_i(y_1), f_i(y_2')) \leq \int_\gamma \| df_i \| \, ds \overset{i\to\infty}{\longrightarrow} \ell(\gamma) = d(y_1, y_2')$$

by the dominated convergence theorem. Taking the limit for $i \to \infty$ and, then, for $y_2' \to y_2$, we deduce that f_∞ contracts distances. □

Conclusion: we have a map $f_0 : (Y, g) \to (X, g_0)$ which contracts distances and is homotopic to f, and, by assumption, $\mathrm{Vol}(X, g) = \mathrm{Adeg}(f)\mathrm{Vol}(X, g_0)$. Then f_0 is a riemannian covering because of the following

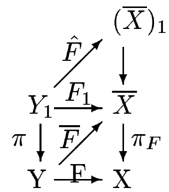
Proposition 3.8 *Let* $F : (Y, g) \to (X, g_0)$ *be a map between compact riemannian manifolds. Assume that* $Adeg(F) = d \neq 0$ *and that the metrics are normalized so that* $Vol(Y, g) = d \cdot Vol(X, g_0)$. *If* F *contracts distances, then* F *is a riemannian covering (of degree d).*

This result was proved in [3] (*Proposition C.1, Appendice C*) for *oriented* manifolds Y and X. We report here a brief proof for non-orientable manifolds, asssuming that it holds in the oriented case.

Proof of Proposition 3.8.
Let us first notice that F is a lipschitz map, thus differentiable almost everywhere, and that the condition of contracting distances is equivalent to $\| (dF)_y \| \leq 1$ a.e. y . This condition, being local, is satisfied by any (possible) lift $\hat{F} : \hat{Y} \to \hat{X}$ of F to riemannian coverings $\hat{Y}, \hat{X}$ of Y and X.

Let $\pi_F : \overline{X} \to X$ be the riemannian covering associated to the subgroup $F_*(\pi_1(Y)) \triangleleft \pi_1(X)$, let i be the number of sheets of this covering and let $\overline{F} : Y \to \overline{X}$ be a lift of F. We will use the notation $\pi : Z_1 \to Z$ for the canonical double oriented covering of a non-orientable manifold Z, and $\mathcal{O}_Z$ for the subgroup of $\pi_1(Z)$ made up of all elements which preserve orientation (i.e. the subgroup associated to the covering π).
Now, three cases have to be checked.
(i) If Y and $\overline{X}$ are orientable, then $\text{Adeg}(F) = i \cdot \mid \deg(\overline{F}) \mid$ and by applying *Proposition C.1* of [3] to the map $\overline{F}$ one deduces that $\overline{F}$ and, as a consequence, F are riemannian coverings.

(ii) If Y is non-orientable, if $F_*(\mathcal{O}_Y) \subset \mathcal{O}_X$ but $F_*(\pi_1(Y)) \not\subset \mathcal{O}_X$, then $\overline{X}$ is non-orientable and the composition map $F_1 \doteq \overline{F} \circ \pi$ can be lifted to a map $\hat{F} : Y_1 \to (\overline{X})_1$:



in this case, Epstein's criterion (cf. [7]) yields $\text{Adeg}(F) = i \cdot \mid \deg(\hat{F}) \mid$. By applying *Proposition C.1* of [3] to $\hat{F} : Y_1 \to (\overline{X})_1$, one still deduces that F is a riemannian covering.

(iii) If Y is non-orientable, if $F_*(\mathcal{O}_Y) = F_*(\pi_1(Y))$ and if $\deg_2(\overline{F}) \neq 0$, then Epstein's criterion tells us that $\text{Adeg}(F) = i$. Any other case is excluded by the condition $\text{Adeg}(F) \neq 0$ (see [7]). In this case, our asssumptions imply that $\text{Vol}(Y, g) = \text{Vol}(\overline{X}, g_0)$.
It will therefore suffice to show that if $\overline{F}$ is a map which contracts distances and preserve volumes, with non-vanishing modulo 2 degree, and such that $\overline{F}_*$ is surjective, then $\overline{F}$ is an isometry.
First of all, from the condition $|\,\text{Jac}_y(\overline{F})\,| \leq \| (d\overline{F})_y \|^n \leq 1$ a.e. y, one obtains

$$\text{Vol}(Y, g) \geq \int_Y |\,\text{Jac}_y(\overline{F})\,|\, dv_g(y) = \int_X \#\overline{F}^{-1}(x) dv_{g_0}(x) \geq \text{Vol}(X, g_0),$$

by the coarea formula (since $\overline{F}$ is surjective, because $\deg_2(\overline{F}) \neq 0$). Thus, by the condition of preserving volumes, one necessarily has $|\,\text{Jac}_y(\overline{F})\,| = \| (d\overline{F})_y \| = 1$ a.e. y, and $\#\overline{F}^{-1}(x) = 1$ a.e. x. Therefore, $\text{Vol}(\overline{F}^{-1}(A), g) = \text{Vol}(A, g_0)$ for every measurable subset $A \subset X$.
Starting from this inequality, let us show that the preimage of one point is exactly one point. Indeed, if $\overline{F}(y_1) = \overline{F}(y_2) = x$ for some $y_1 \neq y_2$, then one should have

$$\overline{F}^{-1}(B_{g_0}(x, r)) \supset B_g(y_1, r) \bigcup B_g(y_2, r)$$

since $\overline{F}$ contracts distances. Thus, for balls of radius r small enough:

$$\frac{\text{Vol}\, B_{g_0}(x,r)}{\text{Vol}\, B_g(y_1,r) + \text{Vol}\, B_g(y_1,r)} \geq 1$$

which gives a contradiction as $r \to 0$, since the volume of balls tends to the euclidean volume when $r \to 0$. Hence, $\overline{F}^{-1}$ is well-defined.
Let us now show that $\overline{F}^{-1}$ is a lipschitz map of lipschitz constant smaller than 2. In fact, otherwise, for any sufficiently small r there should exist points x, x' in X such that $d_0(x,x') = r$ and $d(\overline{F}^{-1}(x), \overline{F}^{-1}(x')) > 2r$. Then, the intersection of the balls $B_{g_0}(x,r)$ e $B_{g_0}(x',r)$ would contain the ball of radius $r/2$ centered at the middle point $\overline{x}$ of the geodesic joining x to x', while the balls of radius r centered at $\overline{F}^{-1}(x)$ and $\overline{F}^{-1}(x')$ would not intersect. So, from the contracting condition we would deduce that

$$\mathrm{Vol}B_g(\overline{F}^{-1}(x),r) + \mathrm{Vol}B_g(\overline{F}^{-1}(x'),r) \leq \mathrm{Vol}\,(B_{g_0}(x,r) \bigcup B_{g_0}(x',r)) \leq$$
$$\leq \mathrm{Vol}B_{g_0}(x,r) + \mathrm{Vol}B_{g_0}(x',r) - \mathrm{Vol}B_{g_0}(\overline{x}, r/2)\ ;$$

again, since the volume of riemannian balls of dimension n goes to zero as $\mathrm{Vol}(B^n, \mathrm{eucl}) r^n$ when $r \to 0$ (uniformly with respect to the centers, the sectional curvature of X and Y being bounded), dividing by r^n and taking the limit for $r \to 0$, we would obtain the contradiction $2 \leq 2 - (1/2)^n$. So $\overline{F}^{-1}$ is lipschitz.
The map $\overline{F}^{-1}$ is therefore almost everywhere differentiable and $\| (d\overline{F}^{-1})_x \| = \| (d\overline{F})_{\overline{F}^{-1}(x)} \| = 1$ a.e. x (since the image, via $\overline{F}$, of the zero measure set where $\overline{F}$ is not differentiable still has zero measure), which proves that also $\overline{F}^{-1}$ contracts distances. This implies that $\overline{F}$ is an isometry. □

4 Best constants on connected sums

A typical example where the reduction factor $\mathrm{Ent}(g)/\mathrm{Ent}(g_0)$ provided by the version of the Real Schwarz Lemma of [4] is inadequate is the case of a connected sum $Y = X \sharp M$ with a negatively curved manifold X. In contrast, we will show that the factor $\mathrm{Ent}_H(g)/\mathrm{Ent}(g_0)$ is "generally" optimal, in a sense that will be specified later on.

If $Y = X \sharp M$ is obtained by removing two open $n-$cells B_X and B_M respectively from X and M and then by pasting together (canonically) the resulting boundaries [9], one has a natural map $P_X : Y \to X$ defined by sending $X \setminus \overline{B_X}$ diffeomorphically onto $X \setminus \{x_0\}$, and $M \setminus B_M$ onto x_0, for some $x_0 \in B_X$. As $\mathrm{Adeg}(P_X) \leq G(P_X)$ and $\mathrm{Adeg}(P_X) \equiv \deg_2(P_X) \bmod 2$, we deduce that $\mathrm{Adeg}(P_X) = 1$. Let H be the kernel of the homomorphism induced by P_X between the fundamental groups.

In order to make our assertions more precise, we need estimates on the minimal value of the invariants $\mathrm{Ent}(g) \cdot \mathrm{Vol}(Y,g)^{1/n}$ and $\mathrm{Ent}_H(g) \cdot \mathrm{Vol}(Y,g)^{1/n}$ when g varies among all differentiable metrics on Y. We will call these lower

[9] the connected sum Y of two $n-$dimensional manifolds X, M is univoquely defined if at least one of them is non-orientable, otherwise Y is defined with respect to the choice of fixed orientations of X and M. The resulting manifold Y is orientable iff both X and M are orientable, in which case Y can be endowed of a natural orientation which coincides with those of X and M respectively on $X \setminus B_X$ and on $M \setminus B_M$ (see for instance [12] for details).

bounds – as the minimal values may not be attained by any metric– the *minimal entropy of* Y and the *minimal entropy of* Y *relative to* H (respectively denoted by $\mathrm{MinEnt}(Y)$ and $\mathrm{MinEnt}_H(Y)$).

So, let X, M be compact manifolds, and let g_0, g be metrics respectively on X and M. Consider points $x_0 \in X$, $m_0 \in M$ and a real number r_0 smaller than the injectivity radii of g_0 and g at x_0, m_0. Fix an isometry j between the tangent spaces $T_{x_0}X \to T_{m_0}M$ and let $J : T_{x_0}X \setminus \{0\} \to T_{m_0}M$ be defined by $J(v) = (r_0 - \|j(v)\|) \cdot j(v)/\|j(v)\|$.
Let us see the connected sum $Y = X \sharp M$ as the gluing of $X \setminus \{x_0\}$ to $M \setminus \{m_0\}$ by means of the map $\exp^g_{m_0} \circ J \circ (\exp^{g_0}_{x_0})^{-1} : B^*_{g_0}(x_0, r_0) \to B^*_g(m_0, r_0)$, where the * means that the center has been excised from the open geodesic balls $B_{g_0}(x_0, r_0), B_g(m_0, r_0)$. We will see $X \setminus \{x_0\}, M \setminus \{m_0\}$ as open subsets of Y (endowed with the respective metrics g_0, g), whose intersection is $B^*_{g_0}(x_0, r_0) \cong B^*_g(m_0, r_0)$. Let d_0 be the distance induced by g_0 on X (hence, on $X \setminus \{x_0\}$).
Consider real differentiable functions $h_k : [0, r_0] \to [0, 1]$ which are null for $t \leq \frac{1}{k}$ and equal to 1 for $t \geq \frac{2}{k}$. Then $h_k \circ \rho$, where $\rho \doteq d_{g_0}(x_0, \cdot)$, is a function on $X \setminus \{x_0\}$ which can be extended to Y as 1 on $X \setminus B^*_{g_0}(x_0, r_0)$ and as 0 on $M \setminus B^*_g(m_0, r_0)$. Let us define on Y the metrics

$$g_k = (h_k \circ \rho)^2 \cdot g_0 + (1 - h_k \circ \rho)^2 \cdot \frac{g}{k^2}, \tag{29}$$

so that $g_k = g_0$ on $X \setminus B^*_{g_0}(x_0, \frac{2}{k})$, and $g_k = \frac{1}{k^2} g$ on $M \setminus B^*_g(m_0, r_0 - \frac{1}{k})$. Let d_k be the distance functions induced by the metrics g_k on Y.
The main properties of the above metrics g_k are summarized by the following

Proposition 4.1 *Let X, M be $n-$dimensional compact manifolds, and let H denote the kernel of the homomorphism induced by $P_X : Y = X \sharp M \to X$ between their fundamental groups. For every metric g_0 on X, the metrics g_k defined by (29) satisfy*
1) $Ent_H(g_k)^n \cdot Vol(g_k) \stackrel{k\to\infty}{\longrightarrow} Ent(g_0)^n \cdot Vol(g_0)$;
2) $diam(g_k) \stackrel{k\to\infty}{\longrightarrow} diam(g_0)$;
3) $(Y, d_k) \stackrel{k\to\infty}{\longrightarrow} (X, d_0)$ *with respect to the Gromov-Hausdorff distance.* [10]

Corollary 4.2 *Let (X, g_0) be a compact locally symmetric manifold of negative curvature, of dimension $n \geq 3$, and let $Y = X \sharp M$ be the connected sum of X with any manifold M of same dimension. Then, one has $MinEnt_H(Y) = MinEnt(X)$ and, if Y is not diffeomorphic to X, the value $MinEnt_H(Y)$ is never attained by any metric on Y.*

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[10] Recall that (cf. [9]) the *Gromov-Hausdorff distance* $d_H(X_1, X_2)$ between two metric spaces (X_1, d_1), (X_2, d_2) is defined as

$$\inf \left\{ d^Z_H(X_1, X_2) \setminus X_1 \subset Z, X_2 \subset Z \text{ isometric immersions in some metric space } Z \right\}$$

where $d^Z_H(X_1, X_2)$ denotes the usual Hausdorff distance between subsets of a metric space (Z, d), that is $d^Z_H(X_1, X_2) = \inf\{r \setminus d(X_1, x_2) \leq r,\ d(x_1, X_2) \leq r \text{ for all } x_1 \in X_1, x_2 \in X_2\}$.

Remark 4.3 Corollary 4.2 incidentally shows that *the minimal entropy problem is completely solved for $Y = X\sharp M$ when M is simply connected*: one has $\mathrm{MinEnt}(Y) = \mathrm{MinEnt}(X)$ and this value is not attained by any metric on Y iff Y is not diffeomorphic to X (e.g. if M is not a homotopy sphere).
Notice that the number $\mathrm{MinEnt}(X)$ is precisely equal to $\mathrm{Ent}(g_0)\cdot\mathrm{Vol}(X, g_0)^{1/n}$, as a corollary of the Real Schwarz Lemma proved by G. Besson, G. Courtois and S. Gallot in the locally symmetric case (cf. [3]), and it is explicitely computable from the Euler characteristic of X in even dimension, by the Hirzebruch-Gauss-Bonnet proportionality formula)

To prove proposition 4.1, we need a preliminary lemma.
Endow the universal covering $\tilde{X}$ of X with the lifted metric $\tilde{g}_0$ (and the induced distance $\tilde{d}_0$) and, as previously done for Y, perform, around all the preimages $\{\gamma\cdot\tilde{x}_0\}_{\gamma\in\pi_1(X)}$ of x_0 in $\tilde{X}$, the gluing of (infinite) copies of $M\setminus\{m_0\}$ to $\tilde{X}\setminus\{\gamma\cdot\tilde{x}_0\}_{\gamma\in\pi_1(X)}$. Let $\overline{Y}$ denote the resulting manifold: $\overline{Y}$ contains $\tilde{X}\setminus\{\gamma\cdot\tilde{x}_0\}_{\gamma\in\pi_1(X)}$ as an open set, and it naturally defines a covering $\overline{\pi}:\overline{Y}\to Y$. It is easily verified that this is the covering of Y associated to the kernel H of the homomorpism induced by $P_X : Y \to X$ (since the lift of any element $\gamma\in\pi_1(Y)$ is a closed path if and only if $\gamma\in H$).
Finally, let $\overline{g}_k$ be the lift of the g_k's to $\overline{Y}$, and let $\overline{d}_k$ denote the distance induced by $\overline{g}_k$ on $\overline{Y}$.

Lemma 4.4 *For every positive ϵ small enough, there exists $K = K(\epsilon)$ such that for all $k \geq K$ one has*
$$(1-\epsilon)\tilde{d}_0(x,y) \leq \overline{d}_k(x,y) \leq (1+\epsilon)\tilde{d}_0(x,y)$$
if $x, y \in \tilde{X}\setminus\{\gamma\tilde{x}_0\}_{\gamma\in\pi_1(X)}$ and if $y \notin B_{\tilde{g}_0}(x,\epsilon)$.

Proof of the lemma 4.4.
Let $B_i(\frac{2}{k})$ denote the closed $\tilde{g}_0$-balls of radius $\frac{2}{k}$ centered at the preimages x_i of x_0 in $\tilde{X}$, and $B_i^*(\frac{2}{k}) = B_i(\frac{2}{k})\setminus\{x_i\}$. As before, we may consider that $\tilde{X}\setminus\bigcup_i B_i^*(\frac{2}{k})$ is a subset of $\overline{Y}$ on which the metrics $\overline{g}_k$ and $\tilde{g}_0$ coincide.

We start proving the first inequality: let γ be a $\overline{d}_k$-minimizing curve joining x to y in $\widetilde{X\sharp M}$. If γ entirely lies in $\tilde{X}\setminus\bigcup_i B_i^*(\frac{2}{k})$, the inequality $\overline{d}_k(x,y)\geq\tilde{d}_0(x,y)$ is trivially satisfied. Otherwise, if γ meets $B_1^*(\frac{2}{k}),\dots,B_N^*(\frac{2}{k})$ in this order, let $\tilde{\gamma}$ denote the part of γ which lies in $\tilde{X}\setminus\bigcup_i B_i^*(\frac{2}{k})$, and let $[u_i, v_{i+1}]$ be the $\overline{d}_k$-geodesic segment of $\tilde{\gamma}$ which joins $B_i^*(\frac{2}{k})$ to $B_{i+1}^*(\frac{2}{k})$. As $\tilde{d}_0(x_i, x_{i+1})$ is greater than twice the injectivity radius of (X, g_0), we deduce that

$$\frac{\tilde{d}_0(v_{i+1},u_{i+1})}{\tilde{d}_0(u_i,u_{i+1})} \leq \frac{4/k}{2r_0 - 4/k}\,, \tag{30}$$

which implies, by the triangle inequality, that
$$\overline{d}_k(x,y) \geq \ell_{\overline{g}_k}(\tilde{\gamma}) = \ell_{\tilde{g}_0}(\tilde{\gamma}) \geq \tilde{d}_0(x,u_1) - 4/k +$$

$$+\sum_{i=1}^{N-2}\Big(\tilde{d}_0(u_i,u_{i+1})-\tilde{d}_0(v_{i+1},u_{i+1})\Big)+\tilde{d}_0(u_{N-1},v_N)+\tilde{d}_0(v_N,y)-4/k\geq$$
$$\geq \tilde{d}_0(x,y)(1-\tfrac{4/k}{2r_0-4/k})-8/k\geq \tilde{d}_0(x,y)(1-\tfrac{4}{2r_0k-4}-\tfrac{8}{\epsilon k}).$$

This proves the first inequality.

The proof of the fact that $\overline{d}_k(x,y)\leq(1+\epsilon)\tilde{d}_0(x,y)$ is obtained by mimicking the above proof, exchanging the roles of $\tilde{d}_0$ and $\overline{d}_k$ and considering a $\tilde{d}_0$-minimizing geodesic γ from x to y in $\tilde{X}$. In this case, by the definition (29) of g_k, the distance $\overline{d}_k(u,v)$ in $\overline{Y}$ between any two points u,v of the same ball $B_i^*(\frac{2}{k})$ is bounded by $(\frac{4}{k}+\frac{D}{k})$, where D is an upper bound of the g-diameter of $M\setminus B_g(m_0,r_0)$. By the triangle inequality, we thus deduce that $\overline{d}_k(x,u_1)-(\frac{4+D}{k})$ and $\overline{d}_k(v_N,y)-(\frac{4+D}{k})$ bound from below $\overline{d}_k(x,B_1^*(\frac{2}{k}))$ and $\overline{d}_k(B_N^*(\frac{2}{k}),y)$ respectively. Moreover, as the first inequality of lemma 4.4 is valid, we also deduce that

$$\frac{\overline{d}_k(v_{i+1},u_{i+1})}{\overline{d}_k(u_i,u_{i+1})}\leq\frac{(4+D)/k}{(1-\epsilon)\tilde{d}_0(u_i,u_{i+1})}\leq\frac{(4+D)/k}{(1-\epsilon)(2r_0-4/k)}.$$

Replacing inequality (30) by this estimate in the end of the proof, we get in the same way

$$\tilde{d}_0(x,y)\geq\ell_{\overline{g}_k}(\tilde{\gamma})\geq\overline{d}_k(x,y)(1-\tfrac{(4+D)/k}{(1-\epsilon)(2r_0-4/k)})-2(4+D)/k,$$

therefore

$$\overline{d}_k(x,y)\leq\tilde{d}_0(x,y)\left(1+\tfrac{2(4+D)}{\epsilon\,k}\right)\left(1-\tfrac{4+D}{(1-\epsilon)(2r_0k-4)}\right)^{-1}.\ \square$$

Proof of proposition 4.1.

1) Consider $x\in\tilde{X}\setminus\{\gamma\cdot\tilde{x}_0\}_{i\in\pi_1(X)}\subset\overline{Y}$. For every $\gamma\in\mathrm{Aut}(\overline{Y})\cong\pi_1(Y)/H\simeq\pi_1(X)$, one has $\tilde{d}_0(x,\gamma x)>2r_0$ and therefore, by lemma 4.4:

$$(1-\epsilon)\tilde{d}_0(x,\gamma x)\leq\overline{d}_k(x,\gamma x)\leq(1+\epsilon)\tilde{d}_0(x,\gamma x)$$

if $\epsilon\leq 2r_0$ e $k>K(\epsilon)$. This implies

$$\mathrm{Ent}(g_0)/(1+\epsilon)\leq\mathrm{Ent}_H(g_k)\leq\mathrm{Ent}(g_0)/(1-\epsilon)$$

that is $\mathrm{Ent}_H(g_k)\overset{k\to\infty}{\longrightarrow}\mathrm{Ent}(g_0)$.

On the other hand, the concavity of the function $\log(\det A)$ (on positive definite matrices) and the fact that $a^{1-t}b^t\leq a+b$ for $a,b\in\mathbf{R}^+$ imply that $dv_{g_k}\leq dv_{g_0}+dv_{g/k^2}$ in $B_{g_0}^*(x_0,\frac{2}{k})$. By integration, this yields

$$\mathrm{Vol}\left(X\setminus B_{g_0}^*(x_0,\tfrac{2}{k})\,,\,g_0\right)\leq\mathrm{Vol}(Y,g_k)\leq\mathrm{Vol}(X,g_0)+\mathrm{Vol}(M,g/k^2).$$

This shows that also $\mathrm{Vol}(Y,g_k)\overset{k\to\infty}{\longrightarrow}\mathrm{Vol}(X,g_0)$, thus proving 1).

2) First notice that, if $x,y\in X\setminus\{x_0\}\subset Y$,

$$d_0(x,y)=\inf\{\tilde{d}_0(\overline{x},\overline{y})\ \setminus\ \overline{x},\overline{y}\in\overline{Y},\ \overline{\pi}(\overline{x})=x,\ \overline{\pi}(\overline{y})=y\},$$
$$d_k(x,y)=\inf\{\overline{d}_k(\overline{x},\overline{y})\ \setminus\ \overline{x},\overline{y}\in\overline{Y},\ \overline{\pi}(\overline{x})=x,\ \overline{\pi}(\overline{y})=y\};$$

hence lemma 4.4 holds for $x,y\in\tilde{X}\setminus\{\gamma\cdot\tilde{x}_0\}_{\gamma\in\pi_1(X)}$ as well as for $x,y\in X\setminus\{x_0\}$. So, $\mathrm{diam}(X\setminus B_{g_0}(x_0,\frac{2}{k}),d_k)\overset{k\to\infty}{\longrightarrow}\mathrm{diam}(X,d_0)$, while it is easy to see that $\mathrm{diam}(M\setminus B_g(m_0,r_0-\frac{2}{k}),d_k)\overset{k\to\infty}{\longrightarrow}0$.

By the decomposition $Y=X\setminus B_{g_0}(x_0,\frac{2}{k})\bigcup M\setminus B_g(m_0,r_0-\frac{2}{k})$ it follows that $\mathrm{diam}(Y,d_k)\overset{k\to\infty}{\longrightarrow}\mathrm{diam}(X,d_0)$.

3) For any fixed ϵ_0, let $N = \{n_i\}_{i\in I}$ be a maximal $(\epsilon_0/2)$-separated set of $(X\setminus\{x_0\}, d_0)$, i.e.

$$\bigcup_i B_{g_0}(n_i, \tfrac{\epsilon_0}{2}) = X \setminus \{x_0\} \quad \text{and} \quad d(n_i, n_j) \geq \epsilon_0/2.$$

Thus, N is ϵ_0-net of X. Since $\mathrm{diam}(M \setminus B_g(m_0, r_0 - \frac{2}{k}), d_k) \stackrel{k\to\infty}{\longrightarrow} 0$, by lemma 4.4 (applied to $X \setminus \{x_0\}$) we deduce that $N \subset X \setminus \{x_0\}$ also is a ϵ_0-net of (Y, d_k), if $k > K(\epsilon_0)$, and that $\mid d_k(n_i, n_j)/d_0(n_i, n_j) - 1 \mid < \epsilon$ if $k > K(\epsilon)$. Since $\sup_k \mathrm{diam}(Y, d_k) < \infty$ and ϵ_0 is arbitrarily small, this is equivalent to the Gromov-Hausdorff convergence $(Y, d_k) \stackrel{k\to\infty}{\longrightarrow} (X, d_0)$ (cf. [9]). □

Proof of corollary 4.2.

The inequality $\mathrm{MinEnt}_H(Y) \leq \mathrm{MinEnt}(X)$ is given by proposition 4.1. On the other hand, by using inequality (4) of theorem 1.3 in the case $f = P_X : Y \to X$, $\mathrm{Adeg}(f) = 1$, we get

$$\left(\frac{Ent_H(g)+\epsilon}{Ent(g_0)}\right)^n \mathrm{Vol}(g) \geq \int_Y |\mathrm{Jac}(f_\epsilon)|\, dv_g \geq \mathrm{Vol}(g_0)$$

by the coarea formula, hence the opposite inequality $\mathrm{MinEnt}_H(Y) \geq \mathrm{MinEnt}(X)$.

Finally, if the value $\mathrm{MinEnt}_H(Y)$ was attained by some metric g on Y, we should have $\mathrm{Ent}_H(g)^n \mathrm{Vol}(Y, g) = \mathrm{Ent}(g_0)^n \mathrm{Vol}(X, g_0)$, so Y would be diffeomorphic to X by the rigidity statement of the strong version of the Real Schwarz Lemma 1.3. □

Remarks 4.5

1) One generally has $\mathrm{Ent}_H(g) < \mathrm{Ent}(g)$, hence theorems 1.2 and 1.3 strictly sharpen the result of G. Besson, G. Courtois and S. Gallot of [4]. Actually, from Proposition 4.1 one can deduce in some cases an uniform lower bound on $\mathrm{Ent}(g) - \mathrm{Ent}_H(g)$ (provided some normalization condition on the metrics g is assumed, for instance $\mathrm{Vol}(Y, g) = 1$).

In fact, a general inequality is known for the minimal entropy of a compact $n-$dimensional manifold Y, namely

$$\mathrm{MinEnt}(Y)^n \geq \frac{n^{n/2}}{n!} \,\| Y \|$$

where $\| Y \|$ is a homotopy invariant (introduced by M. Gromov in order to study the minimal volume problem for riemannian manifolds, cf. [8]) which is called the ***simplicial volume*** of Y.

In the case of a connected sum $Y = X \sharp M$ with a negatively curved manifold X, the additivity of the simplicial volume on connected sums and Proposition 4.1 then yield, if $\| M \| \gg 0$,

$$\mathrm{MinEnt}(Y)^n \geq \frac{n^{n/2}}{n!} (\| X \| + \| M \|) > \mathrm{MinEnt}(X)^n \geq \mathrm{MinEnt}_H(Y)^n \,.$$

An interesting question is whether $\mathrm{MinEnt}(X \sharp M)$ always is greater than $\mathrm{MinEnt}(X)$, for any non-simply connected M, or if $\| M \| \neq 0$.

2) Also the case of dimension 2 is enlightening [11] to get convinced that generally $\mathrm{Ent}_H(g) < \mathrm{Ent}(g)$. As a simple example, let us consider two compact surfaces $Y = \Sigma_{k'}$ and $X = \Sigma_k$ of genera $k' > k \geq 2$ respectively, and let $f : Y \to X$ be the map which shrinks some handles of Y to a point of X. As the minimal entropy of a surface is precisely attained by hyperbolic metrics (cf. [11], [3]), given hyperbolic metrics g' and g on Y and X one has:
$\mathrm{Ent}(g') = \mathrm{Ent}(g) = 1, \ \mathrm{Vol}(Y, g') = 2\pi(2k' - 2), \ \mathrm{Vol}(X, g) = 2\pi(2k - 2)$
hence $\mathrm{MinEnt}(Y)^2 = 2\pi(2k' - 2) > 2\pi(2k - 2) = \mathrm{MinEnt}(X)^2 \geq \mathrm{MinEnt}_H(Y)^2$.

3) Remark also that not even the estimate $\mathrm{MinEnt}(Y) \geq \mathrm{Adeg}(f)\cdot\mathrm{MinEnt}(X)$ can be generally obtained from the version of the Real Schwarz Lemma proved in [4], in case $f : Y \to X$ is a map between *non-oriented* manifolds.
For instance, when $Y = X\sharp M$, where X is orientable and M not, the lift $\overline{P}_X : \overline{Y} \to X$ of P_X to any oriented covering $\overline{Y}$ of Y has zero degree (since it factors through the canonical double oriented covering of Y). This shows that the above inequality cannot be generally obtained by simply applying the result of G. Besson, G. Courtois and S. Gallot to the lift of f to some suitable oriented covering of Y.

Remark 4.6 *About the optimality of the factor $Ent_H(g)^n/Ent(g_0)^n$.*
Given the geometric data of a C^1 map $f : (Y, g) \to (X, g_0)$ between riemannian manifolds (normalized so that f globally preserves volumes, i.e. $\mathrm{Vol}(Y, g) = \mathrm{Adeg}(f)\cdot\mathrm{Vol}(X, g_0)$), one may wonder to what extent f can be deformed, within its homotopy class, in order that volumes are infinitesimally preserved as much as possible. That is, what is the value of the best reduction factor

$$\mathcal{R}([f], g, g_0) = \inf\{\sup_{y\in Y} |Jac_y f'| \ \ \backslash \ f' \text{ homotopic to } f\} = ?$$

Of course the answer depends on the particular geometric problem under consideration, but the coarea formula yields the general lower bound $\mathcal{R}([f], g, g_0) \geq 1$. On the other hand, when g_0 is locally symmetric and negatively curved, by theorem 1.3 one has $1 \leq \mathcal{R}([f], g, g_0) \leq \mathrm{Ent}_H(g)^n/\mathrm{Ent}(g_0)^n$.
When moreover $f = P_X : X\sharp M \to X$ and g is $\epsilon-$close to the minimal value of the functional $\mathrm{Ent}_H(\cdot)^n\mathrm{Vol}(X\sharp M, \cdot)$, Proposition 4.1 shows that the factor $\mathrm{Ent}_H(g)^n/\mathrm{Ent}(g_0)^n$ is $o(\epsilon)-$close to the solution $\mathcal{R}([f], g, g_0)$, since $\left|\frac{Ent_H(g)^n}{Ent(g_0)^n} - 1\right| \leq o(\epsilon)$.
In other words, the factor $\mathrm{Ent}_H(g)^n/\mathrm{Ent}(g_0)^n$ tends to be optimal as long as g tends to be minimal for the volume-entropy functional $\mathrm{Ent}_H(\cdot)^n\mathrm{Vol}(X\sharp M, \cdot)$.
Let us finally remark that the case that we have considered (where $Y = X\sharp M$ and $f = P_X : Y \to X$) is not so a particular case, as combining our strong version of the Real Schwarz Lemma with some results of I. K. Babenko ([1])

[11] even if it is not pertinent to show that theorems 1.2 and 1.3 are strictly sharper than the version of the Real Schwarz Lemma proved in [4], since they hold in dimension greater than 2. However, it should be pointed out at least that also the inequality $\mathrm{MinEnt}(Y) \geq \mathrm{Adeg}(f)\cdot\mathrm{MinEnt}(X)$, found by G. Besson, G. Courtois and S. Gallot in [3] for a map $f : Y \to X$ of non-vanishing degree between compact surfaces, can be replaced by the sharp equality $\mathrm{MinEnt}_H(Y) = \mathrm{Adeg}(f) \cdot \mathrm{MinEnt}(X)$ (see [16]).

one can prove that the equality $\mathrm{MinEnt}_H(Y) = \mathrm{Adeg}(f)\cdot\mathrm{MinEnt}(X)$ holds in general for maps $f : Y \to X$ onto a locally symmetric manifold of negative curvature, (see [16]).

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