GENERALIZED MEYER SETS AND THUE-MORSE QUASICRYSTALS WITH TORIC INTERNAL SPACES

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ABSTRACT. We show that one-dimensional aperiodic sets of points having the Delaunay property can be associated with generalized Meyer sets, for which 1) the internal space is toric, $\mathbb{R}/\lambda\mathbb{Z}$, with a selection rule based on a congruence mode with respect to the frequencies λ producing punctuated windows, 2) a scaling exponent function, having values in $[\,0\,;1\,]$, can be uniquely defined on the window, is related to the scaling properties of the intensity function and the point density measure on canonical 1-dimensional sublattices of period λ, where a scaling exponent of 1 corresponds to Bragg-peaks, 3) the projection mappings are adapted to the global average lattice and are not orthogonal. The case of the Thue-Morse quasicrystal is explicitely developed. We prove also that it is a Meyer set and it is harmonious.

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1. Introduction

The notion of model set in the context of the cut-and-project scheme developed by Y. Meyer [1],[2] in the 1970s has been successfully and extensively applied to the study of the structure of quasicrystals for more than ten years. Its simplicity, which is expressed by the existence of a suitable lattice in a periodization space and suitable windows in the internal space, and its power when refinements of structure models [3] are in cause make model sets exciting new tools for studying aperiodic sets of points in the quasiperiodic context and quasicrystals.

As far as the authors know it, the constructions of Y. Meyer were developed in the context of locally compact Abelian groups such as finitely-dimensional Euclidean spaces $\mathbb{R}^m$, compact subgroups of the infinite torus $\mathbb{T}^\infty$, discrete or finite groups, and all the mixed possibilities (see Rudin, [12], chap. 2), whereas the applications to quasicrystals were using only $\mathbb{R}^m$ spaces, for a certain integer $m \geq 1$. A natural and fundamental question to be asked is therefore to understand when toric internal or physical spaces appear naturally in the study of point sets and for which purposes they have to be used. The present contribution is reporting a new vision of the (CPS) cut-and-project scheme for which a toric internal space is bearing information about the way the intensity is scaled with the number of diffracting sites in the point sets at each element of the spectrum and its geometrical representation with the corresponding scaling exponents at each element of the windows.

Indeed, when aperiodic point sets are not any more quasiperiodic, the search for the sets supporting the components of the spectrum, Bragg, singular continuous and absolutely continuous becomes more complicated. Model sets in the quasiperiodic context, the context of the (CPS) *normal* cut-and-project scheme with Euclidean spaces as physical space and internal space, give only pure Bragg spectra. Many other possibilities could of course occur, and the full generality of the theory calls new concepts to deal with the scaling exponents associated with the singular continuous component of the spectrum.

On the other hand, Hof [4] has developed a mathematical modelization of diffraction theory adapted to any Delaunay set in $\mathbb{R}^n, n \geq 1$. He shows that the knowledge of the autocorrelation measure of this set is basic since it gives rise, by its Fourier transform, to the diffraction patterns of the set, known to physicists. This autocorrelation can for instance be computed directly by hand in the case of interpenetrating G-clusters, where G is a finite non-crystallographic symmetry point group and when G-clusters are arranged quasiperiodically or not [5]. When arranged quasiperiodically in a suitable way, the system of interpenetrating G-clusters is composing a quasicrystal [6], and this approach is another view to the diffraction of the quasicrystal and its decomposition through the existence of the local clusters of atoms it contains.

Of importance is the search of the localization of the Bragg-peaks, the peaks belonging to the singular continuous component of the spectrum. This can be carried out by a suitable Fourier-Bohr analysis [7, 8, 9], or, in some extent, with the average unit cell approach [10],[11] typical of finite systems.

In the following, in order to justify our claim that a toric internal space is important for describing the behaviour of the intensity function, the autocorrelation measure or any function linked, by duality, to the point set considered, we will treat the problem in dimension 1.

We are then interested in the following by (non-periodic) Delaunay sequence of points on the real line given for instance by the image $\{ f(n) \ / \ n \in \mathbb{N} \}$ of a strictly increasing function $f : \mathbb{N} \to \mathbb{R}$, $f(0) = 0$, that we will consider modulo a lattice $\lambda\mathbb{Z}$, where $\lambda > 0$ is a real number, and such that the set of values $\Omega_\lambda = \{ f(n) \,\text{modulo}\, \lambda\mathbb{Z} \ / \ n \in \mathbb{N} \}$ is *finite*. This means that we assume that, by some arithmetical spectral computation - through the definition of f -, we know which frequencies are such that Ω_λ is finite. Under this hypothesis, we show that we can associate scaling exponents to each element of Ω_λ in a natural way if some regularity assumptions are satisfied by the sequence : at first, we express Bochner's theorem [12] on the interval $]-\lambda/2\,;\,+\lambda/2]$ by splitting up the extremities $\{\pm\lambda/2\}$ of the torus, second, we make a suitable Lebesgue decomposition of the structure factor. The exponents which appear by this method are shown to be correlated to fractal rates of occupancy at infinity of some 1-dimensional lattices associated with f. The case of the Thue-Morse quasicrystal is presented in this context. We show in particular that a Thue-Morse quasicrystal has an average lattice, is a Delone set, satisfies the Meyer property, but, since it does possess a non empty singular continuous component of the spectrum, is not a 'model set'. We show that the Thue-Morse quasicrystal is an harmonious set.

In the last paragraph, we present a generalized cut-and-project scheme built in order to extend the normal CPS to sequences which are not quasiperiodic, where Ω_λ behaves as a *window* in the one-dimensional torus. It allows more general spectra than only pure Bragg spectra and enables to understand the origin of the scaling exponents for the intensity. We examplify some results about the Thue-Morse in this context.

All the considerations below for one-sided sequences can be transposed easily, without gaining in generality, to two-sided sequences, extended for instance by inversion with respect to the origin, and we will leave this aside. Another generalization, far from trivial, will consist in developing the present approach to non punctuated windows in one-dimensional tori, and in m-dimensional tori, m greater than one. This will be the subject of another contribution. In the following, we will denote by $\mathbb{Z}y$

or $y\mathbb{Z} \subset \mathbb{R}$ the lattice $\{ y\, m \mid m \in \mathbb{Z} \}$.

2. Definitions

In the following, we will consider sequences $\{f\}$ of points satisfying the following assumptions :

(H) : f is a strictly increasing function defined on $\mathbb{N}$ taking values in $\mathbb{R}$. $\{f(n) \,/\, n \in \mathbb{N}\}$ represents an infinite sequence of points on the real line, satisfying the crystallographic hypothesis (Delaunay) :
(D1) (uniformly discrete) $\exists r > 0$ such that, for all $n \in \mathbb{N}$, each interval $]-r+f(n); +r+f(n)[$ contains only the point $f(n)$ of the sequence,
(D2) (relatively dense) $\exists R > 0$ such that $\forall x \geq f(0)$, $\exists n \in \mathbb{N}$ such that $|x - f(n)| < R$.

Each time we will consider such a sequence f, we will (arbitrarily) set the origin at $f(0)$, so that $f(0) = 0$. Let us consider a sequence f satisfying the hypothesis (H) above, $\lambda > 0$ a real number, and

$$\Omega_\lambda = \{u \in]-\lambda/2; +\lambda/2] \ /\exists n \in \mathbb{N}, f(n) = u + p\lambda \text{ for a certain } p = p(n)\}$$

the set of values which are reached by the sequence $\{f(n)\}$ modulo $\mathbb{Z}\lambda$. Denote

$$\Omega_\lambda^+ = \Omega_\lambda \cap \mathbb{R}^{+*} \qquad\qquad \Omega_\lambda^- = \Omega_\lambda \cap \mathbb{R}^-$$

For each $u \in]-\lambda/2; +\lambda/2]$, we call

$$\mathbb{N}_\lambda(u) = \{n \in \mathbb{N} \,/\, \exists p \in \mathbb{N}, -p\lambda + f(n) = u\}$$

By definition, all the sets $\mathbb{N}_\lambda(u)$ are empty when $u \in]-\lambda/2; +\lambda/2] \setminus \Omega_\lambda$, and non empty when $u \in \Omega_\lambda$.

Lemma 1.- *The following union is disjoint :*

$$\mathbb{N} = \bigsqcup_{u \in \Omega_\lambda} \mathbb{N}_\lambda(u) \tag{1}$$

This lemma represents just a partitioning of the set $\mathbb{N}$ with respect to the 'lattices' $u + \mathbb{Z}\lambda$.

Some of these lattices are strongly occupied and others very few. In order to understand their roles in the diffraction process of the sequence f, we introduce asymptotic rates of occupancy to each of them and study fractional exponents associated with them.

For any interval $[a;b]$ of the real line, we denote by $\chi_{[a;b]}$ its characteristic function. Let denote by μ the measure on $\mathbb{N}$ which is counting points : i.e. for any $A \subset \mathbb{N}$, $\mu(A) = \text{Card}(A)$. For any $\lambda > 0$, $q \in \mathbb{N}$, N an integer ≥ 1 and $v \in]-\lambda/2; +\lambda/2]$, we call

$$\mathbb{P}_\lambda(q) = \{n \in \mathbb{N} / f(n) \in [0; q\lambda]\} \tag{2}$$

and

$$\delta_{\lambda,N,q}(v) = \int_{\mathbb{N}_\lambda(v) \cap \mathbb{P}_\lambda(q)} \chi_{[0;N-1]}(n) \, d\mu(n) \tag{3}$$

When λ, q, v are fixed, the sequence $N \to \delta_{\lambda,N,q}(v)$ is stationnary : it can be calculated with q and N such that

$$f(N-1) \leq q\lambda < f(N) \tag{4}$$

and $q = \left[\dfrac{f(N)}{\lambda}\right]$ the greatest integer less than $\dfrac{f(N)}{\lambda}$. We denote by

$$\delta_{\lambda,q}(v) = \lim_{N \to \infty} \delta_{\lambda,N,q}(v) = \#(\mathbb{N}_\lambda(v) \cap \mathbb{P}_\lambda(q)) \tag{5}$$

For couples q, N such that Eq. [4] is valid, we have $\delta_{\lambda,N,q}(v) = \delta_{\lambda,q}(v)$. For each $v \in \Omega_\lambda$, we now consider the sets of exponents

$$\{\beta \in [0;1] / \liminf_{q \to \infty} \frac{\delta_{\lambda,q}(v)}{q^\beta} > 0\} \tag{6}$$

and

$$\{\gamma \in [0;1] / \limsup_{q \to \infty} \frac{\delta_{\lambda,q}(v)}{q^\gamma} < +\infty\} \tag{7}$$

Since $\mathbb{N}_\lambda(v) \neq \emptyset$, the first set of exponents is not empty : it contains at least $\beta = 0$. We call

$$\underline{\alpha}_\lambda(v) = \sup\{\beta \in [0;1] / \liminf_{q \to \infty} \frac{\delta_{\lambda,q}(v)}{q^\beta} > 0\} \tag{8}$$

Similarly, the second set of exponents above is not empty since it contains $\beta = 1$: this comes from the fact that $\#(\mathbb{N}_\lambda(v) \cap \mathbb{P}_\lambda(q)) \leq q$. Let us denote

$$\overline{\alpha}_\lambda(v) = \inf\{\gamma \in [0;1] / \limsup_{q \to \infty} \frac{\delta_{\lambda,q}(v)}{q^\gamma} < +\infty\} \tag{9}$$

Proposition 1.- *For all $v \in \Omega_\lambda$, we have*

$$0 \leq \underline{\alpha}_\lambda(v) \leq \overline{\alpha}_\lambda(v) \leq 1 \tag{10}$$

Proof : If $\underline{\alpha}_\lambda(v) = 0$ or if $\overline{\alpha}_\lambda(v) = 1$, the proposition is proved. Assume $\underline{\alpha}_\lambda(v) > 0$ and $0 \leq \overline{\alpha}_\lambda(v) < \underline{\alpha}_\lambda(v) \leq 1$. Then there exists two real numbers β, β' such that

$$\overline{\alpha}_\lambda(v) < \beta' < \beta < \underline{\alpha}_\lambda(v)$$

We have $\underline{\alpha}_\lambda(v) > \beta - \beta' > 0$, hence $\liminf_{q\to+\infty} \frac{\delta_{\lambda,q}(v)}{q^{\beta-\beta'}} > 0$. Let us denote by L this $\liminf$. It is strictly positive and there exists a subsequence $\{q_i / i \in \mathbb{N}\}$ such that $\lim_{i\to+\infty} \frac{\delta_{\lambda,q_i}(v)}{q_i^{\beta-\beta'}} = L$. Therefore, $\lim_{i\to+\infty} \frac{\delta_{\lambda,q_i}(v)}{q_i^{\beta}} = \lim_{i\to+\infty} L q_i^{\beta'}$ and tends to infinity. But $\overline{\alpha}_\lambda(v) < \beta$ and hence $\lim_{i\to+\infty} \frac{\delta_{\lambda,q_i}(v)}{q_i^{\beta}} \leq \limsup_{q\to+\infty} \frac{\delta_{\lambda,q}(v)}{q^{\beta}} < +\infty$. Contradiction $\square$.

Definition 1.- *Given $\lambda > 0$; for each $v \in \Omega_\lambda$, an average sublattice in v of period λ, $v + \mathbb{Z}\lambda$, of the sequence $\{f(n)\}$ is a lattice in $\mathbb{R}$ which contains v and such that its period λ satisfies*

$$\underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v) = 1$$

Definition 2.- *Given $v \in \Omega_\lambda$. When $\underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v)$ and $\liminf_{q\to\infty} \frac{\delta_{\lambda,q}(v)}{q^{\underline{\alpha}_\lambda(v)}}$ and $\limsup_{q\to\infty} \frac{\delta_{\lambda,q}(v)}{q^{\overline{\alpha}_\lambda(v)}}$ are strictly positive, exist and are equal, we denote*

$$\alpha_\lambda(v) = \underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v) \tag{11}$$

and

$$\delta_\lambda^{(\alpha_\lambda(v))}(v) = \liminf_{q\to\infty} \frac{\delta_{\lambda,q}(v)}{q^{\alpha_\lambda(v)}} = \limsup_{q\to\infty} \frac{\delta_{\lambda,q}(v)}{q^{\alpha_\lambda(v)}} \tag{12}$$

In particular, when

$$\alpha_\lambda(v) = 1$$

we call this common limit $\delta_\lambda(v)$. It is by definition the average number of points of the sequence $\{f(n)\}$ per period of the lattice $v + \mathbb{Z}\lambda$, with $f(n) \equiv v \pmod{\mathbb{Z}\lambda}$.

The notation of Eq. [12] is such that the quantity $(\alpha_\lambda(v))$ is a superscript and not an exponent. When $\alpha_\lambda(v) = 1$, the occupancy of the lattice $v + \mathbb{Z}\lambda$ is fairly regular. In this case, we always have $\delta_\lambda(v) \in [0; 1]$. The occupancy is fractional.

When $\underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v) = 1$ and $\delta_\lambda(v) = 1$, it means that we have exactly one point of the sequence, in average, congruent to v, per cell of the lattice $v + \mathbb{Z}\lambda$. This does not mean that we have a full occupancy of the lattice $v + \mathbb{Z}\lambda$. Owing to the assumption

(D2), the number of successive cells of the lattice $v + \mathbb{Z}\lambda$ receiving no points of the sequence at all cannot be arbitrarily large.

Definition 3.- *Given $\lambda > 0$; a sequence $\{f(n)\}$ which is such that for each $v \in \Omega_\lambda$,*

$$\overline{\alpha}_\lambda(v) \;<\; 1$$

is called singular in λ.

Conversely, for sequences of points that are subsets of lattices, we have the following result.

Proposition 2.- *If $\{f(n)\}$ is a subset of the lattice $\mathbb{Z}2r$, $r > 0$, consisting of the points of the lattice indexed by the positive integers, except possibly a finite number of them, then :*

i) if $\lambda = 2r$, then $\Omega_\lambda = \{0\}$. The lattice $\mathbb{Z}2r$ is the average sublattice in $v = 0$ of period $2r$ of the sequence $\{f(n)\}$, with $\delta_\lambda(0) = 1$,
ii) if λ is such that $0 \neq \lambda/2r = t/w \in \mathbb{Q}$, $gcd(t,w) = 1$, then Ω_λ is the set of residues of $0, 1 \times 2r, 2 \times 2r, \ldots, (t-1) \times 2r$ in $]-\lambda/2; +\lambda/2]$ modulo $\mathbb{Z}\lambda$, and, for all $v \in \Omega_\lambda$, $\underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v) = 1$, with $\delta_\lambda(v) = 1/w$,
iii) if λ is such that $\lambda/2r \notin \mathbb{Q}$, then Ω_λ is the uniformly dense set of residues of $\mathbb{Z}2r$ in $]-\lambda/2; +\lambda/2]$ modulo $\mathbb{Z}\lambda$ and $\underline{\alpha}_\lambda(v) = \overline{\alpha}_\lambda(v) = 0$.

From case ii), we see that each time λ is an integral multiple of $2r$, then $w = 1$, ie each lattice $v + \mathbb{Z}\lambda$ is an average sublattice in v, for all $v \in \Omega_\lambda$ (see Definition 1.-). Such collections of points are singular for any λ which are incommensurate with the period $2r$, from iii).

Corollary .- *For any sequence $\{f(n)\}$, satisfying the hypothesis (H), if r_b denotes the maximal bound of r such that the intervals : $]-r + f(n); +r + f(n)[, n \in \mathbb{N}$, are mutually disjoint, then the sequence $\{f(n)\}$ is such that for any $\lambda < 2r_b$ and any $v \in]-\lambda/2; +\lambda/2]$, we have $\delta_\lambda(v) \in [0; 1[$ if it exists.*

The value 1 cannot be reached in this case, sites being too dispersed on each sublattice $v + \mathbb{Z}\lambda, v \in \Omega_\lambda$.

3. Fourier Transform Decomposition by Sublattices

Let $\lambda > 0$ and assume that Ω_λ is finite. Let $k = 2\pi/\lambda$ the corresponding wave vector and $N \geq 1$ an integer. We will make a Lebesgue-type decomposition of the structure factor of the sequence f below, gathering diffracting sites by sublattices setting 1 to each site as individual scattering factor. The structure factor considered for N diffracting sites is equal to:

$$\sum_{n=0}^{N-1} 1\, e^{ikf(n)} = \int_{I\!N} \chi_{[0;N-1]}(n) e^{ikf(n)}\, d\mu(n) \tag{13}$$

Lemma 1 implies

$$= \int_{\bigsqcup_{u\in\Omega_\lambda} I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n) e^{ikf(n)}\, d\mu(n)$$

$$= \sum_{u\in\Omega_\lambda} \int_{I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n) e^{ikf(n)}\, d\mu(n)$$

$$= \int_{\Omega_\lambda} \left(\int_{I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n) e^{ikf(n)}\, d\mu(n) \right) d\mu(u) \tag{14}$$

where $\mu = \sum_{u\in\Omega_\lambda} \delta_u$ is the measure on Ω_λ, which counts points : i.e., $\mu(A) = \mathrm{Card}(A)$ for any $A \subset \Omega_\lambda$. Since $k\lambda = 2\pi$, and that $f(n) - p(n)\lambda = u$ for a certain $u \in \Omega_\lambda$ and a certain (unique) integer $p(n)$ associated with n, we have

$$= \int_{\Omega_\lambda} \left(\int_{I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n) e^{ik[f(n)-p(n)\lambda]}\, d\mu(n) \right) d\mu(u)$$

$$= \int_{\Omega_\lambda} \left(\int_{I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n) e^{iku}\, d\mu(n) \right) d\mu(u)$$

$$= \int_{\Omega_\lambda} e^{iku} \left(\int_{I\!N_{\lambda}(u)} \chi_{[0;N-1]}(n)\, d\mu(n) \right) d\mu(u) \tag{15}$$

Let us assume now that the value λ is such that the sequence $\{f(n)/n \geq 1\}$ satisfies the equality conditions (Eq. [11,12]) of Definition 2 for all $v \in \Omega_\lambda$. Then, the expression of the structure factor can be written, $N >> 1$:

$$\approx \sum_{v\in\Omega_\lambda} e^{ikv} \left(\left[\frac{f(N)}{\lambda} \right] \right)^{\alpha_\lambda(v)} \delta_\lambda^{(\alpha_\lambda(v))}(v) \tag{16}$$

owing to Eq. [3,4,12]. We now classify the elements $v \in \Omega_\lambda$ by lexicographic order in the following way : if v and w are any two elements of Ω_λ, we say that $v \succ w$ if $\alpha_\lambda(v) > \alpha_\lambda(w)$, or, when $\alpha_\lambda(v) = \alpha_\lambda(w)$, $v \leq w$. Therefore, there exists a stationary sequence of integers $n_0 = 1, n_1, \ldots$ such that

$$v_{n_0=1} \succ v_2 \succ \ldots \succ v_{n_1-1} \succ v_{n_1} \succ v_{n_1+1} \succ \ldots \succ v_{n_2-1} \succ v_{n_2} \succ v_{n_2+1} \succ \ldots$$

with

$$1 \geq \alpha_\lambda(v_{n_0=1}) = \alpha_\lambda(v_2) = \ldots = \alpha_\lambda(v_{n_1-1}) > \alpha_\lambda(v_{n_1}) = \alpha_\lambda(v_{n_1+1}) = \cdots$$

$$\cdots = \alpha_\lambda(v_{n_2-1}) > \alpha_\lambda(v_{n_2}) = \alpha_\lambda(v_{n_2+1}) = \cdots$$

and

$$v_1 < v_2 < \ldots < v_{n_1-1}$$

$$v_{n_1} < v_{n_1+1} < \ldots < v_{n_2-1}$$

$$\vdots$$

corresponding to the jumps of the scaling exponent function α_λ on Ω_λ. This sequence is finite since Ω_λ is assumed finite. With the convention, for any integer $i \geq 0$, that $\sum_{n_i}^{n_i-1} = 0$ the structure factor is equal to (the first summation is finite)

$$\sum_{l=0}^{+\infty} \left[\sum_{j=n_l}^{n_{l+1}-1} e^{2i\pi v_j/\lambda} \, \delta_\lambda^{(\alpha_\lambda(v_{n_l}))}(v_j) \right] \left(\left[\frac{f(N)}{\lambda} \right] \right)^{\alpha_\lambda(v_{n_l})} \tag{17}$$

Let us denote by

$$c_\lambda(l) = \sum_{j=n_l}^{n_{l+1}-1} e^{2i\pi v_j/\lambda} \, \delta_\lambda^{(\alpha_\lambda(v_{n_l}))}(v_j)$$

the l-th coefficient. The index l is called the *level index* of the scaling exponent. The *level* l is constituted by $n_{l+1} - n_l$ elements in the window Ω_λ. The intensity $I(2\pi/\lambda)$ produced by an infinite number of diffracting sites is given by the limit (thermodynamic limit) when $N \to +\infty$

$$I(2\pi/\lambda) = \left| \sum_{j=1}^{n_1-1} e^{2i\pi v_j/\lambda} \, \delta_\lambda^{(\alpha_\lambda(v_1))}(v_j) \, \lambda^{-\alpha_\lambda(v_1)} \right|^2 \left(\lim_{N\to+\infty} \frac{f(N)^{2\alpha_\lambda(v_1)}}{N} \right) \tag{18}$$

the other terms being negligible if the first coefficient $c_{\lambda(0)}$ is not equal to zero and if the limit exists in the second term of the product ; here we have made the approximation, for $N >> 1$, and all $i \geq 0$

$$\frac{1}{\sqrt{N}} \left(\left[\frac{f(N)}{\lambda} \right] \right)^{\alpha_\lambda(v_{n_i})} \simeq \frac{1}{\sqrt{N}} \left(\frac{f(N)}{\lambda} \right)^{\alpha_\lambda(v_{n_i})} \tag{19}$$

which is justified by the fact that $f(N) - f(N-1) \leq 2R$, uniformly for all N, the Delaunay constant of assumption (D2). More generally

Proposition 3.- *If the first $h+1$ terms ($h \geq 1$) $c_\lambda(l)$ are such that, for any $l = 0, 1, \ldots, h-1$*

$$\limsup_{N\to+\infty} \frac{c_\lambda(l)}{\sqrt{N}} \left(\frac{f(N)}{\lambda} \right)^{\alpha_\lambda(v_{n_l})} = 0 \tag{20}$$

and

$$\lim_{N\to+\infty} \frac{c_\lambda(h)}{\sqrt{N}} \left(\frac{f(N)}{\lambda}\right)^{\alpha_\lambda(v_{n_h})} \textit{ exists and is} \neq 0 \tag{21}$$

then, we have

$$I(2\pi/\lambda) = \left| \sum_{j=n_h}^{n_{h+1}-1} e^{2i\pi v_j/\lambda}\, \delta_\lambda^{(\alpha_\lambda(v_{n_h}))}(v_j)\, \lambda^{-\alpha_\lambda(v_{n_h})} \right|^2 \left(\lim_{N\to+\infty} \frac{f(N)^{2\alpha_\lambda(v_{n_h})}}{N} \right) \tag{22}$$

Thus, we have an explicit expression of the scaling factor for intensities with N in the general case, given by the second term in the product.

Corollary.- *Under the above assumptions, the scaling exponent of the intensity with the size N of diffracting sites is given by*

$$\left(\frac{f(N)^{2\alpha_\lambda(v_{n_h})}}{N} \right) \tag{23}$$

In particular, if the sequence $\{f(n)/n \geq 1\}$ has an average lattice, i.e. when the limit $\lim_{N\to+\infty} f(N)/N = 1$, *$\delta$-peaks can be obtained if and only if the scaling exponent is 1, that is if and only if*

$$\alpha_\lambda(v_{n_h}) = 1, \quad \textit{that is for} \quad h = 0, \ n_0 = 1 \tag{24}$$

and

$$\lim_{N\to+\infty} \frac{c_\lambda(0)}{\sqrt{N}} \left(\frac{f(N)}{\lambda}\right)^{\alpha_\lambda(v_1)} \textit{ exists and is} \neq 0 \tag{25}$$

The intensity of the δ-peak at $k = 2\pi/\lambda$, per diffracting site, is then

$$\lim_{N\to+\infty} \frac{I(2\pi/\lambda)}{N} = \left| \sum_{j=1}^{n_1-1} e^{2i\pi v_j/\lambda}\, \delta_\lambda(v_j)\, \lambda^{-1} \right|^2 \tag{26}$$

The lexicographical classification of the elements v of Ω_λ by the values of the scaling exponent $\alpha_\lambda(v)$ can be reported naturally on the torus : indeed, the present situation means that, if we identify Ω_λ with the torus $\mathbb{R}/\lambda\mathbb{Z}$, and denote $\overline{v}$ the corresponding elements in the torus, we obtain *levels* which are the different values of the scaling exponent function α_λ. This defines naturally a scaling exponent function $\overline{\alpha_\lambda}$ on the torus by, for any $v \in \Omega_\lambda$:

$$\overline{\alpha_\lambda}(\overline{v}) = \alpha_\lambda(v)$$

The levels are attached to the 1-dimensional lattices $v + \lambda\mathbb{Z} \equiv \overline{v}$.

Assume $\lambda_1 > 0, \Omega_{\lambda_1}$ finite and $\lambda_2 = m\lambda_1$, with $m \geq 1$ an integer.

Proposition 4.- *a) If m is odd, there exists an integer $M = M(\lambda_1, m) \leq m-1$ such that :*

$$\Omega_{\lambda_2} \subset \bigcup_{j=0}^{m-1} \left[\left(\frac{2j-m+1}{2} \right) \lambda_1 + \Omega_{\lambda_1} \right] \subset \bigcup_{k=-M}^{M} (\Omega_{\lambda_2} + k\lambda_1)$$

b) If m is even, there exists an integer $M' = M'(\lambda_1, m) \leq m-1$ such that :

$$\Omega_{\lambda_2} \subset \left[\left(-\frac{m}{2} + \Omega^+_{\lambda_1} \right) \cup \bigcup_{j=1}^{m-1} \left[\left(\frac{2j-m}{2} \right) \lambda_1 + \Omega_{\lambda_1} \right] \cup \left(\frac{m}{2}\lambda_1 + \Omega^-_{\lambda_1} \right) \right]$$

$$\subset \bigcup_{k=-M'}^{M'} (\Omega_{\lambda_2} + k\lambda_1)$$

Proof: a) For any $u \in \Omega_{\lambda_2}$, there exists $v \in]-\lambda_1/2; +\lambda_1/2]$ such that $u \equiv v$ modulo $\mathbb{Z}\lambda_1$. Therefore, there exists $j \in \{0, 1, \ldots, m-1\}$ such that $u = -\lambda_2/2 + j\lambda_1 + \lambda_1/2 + v = \left(\frac{2j-m+1}{2}\right)\lambda_1 + v$. Now, since $u \in \Omega_{\lambda_2}$, there exists $q, p(q)$ integers such that $u = f(q) - p(q)\lambda_2$. The fact that m is odd implies that $\frac{2j-m+1}{2}$ is an integer and that

$$v = u - \left(\frac{2j-m+1}{2} \right) \lambda_1 = f(q) - p(q)m\lambda_1 - \left(\frac{2j-m+1}{2} \right) \lambda_1$$

can be written $f(q) - p'(q)\lambda_1$ for certain integers $q, p'(q)$; therefore $v \in \Omega_{\lambda_1}$ and

$$\Omega_{\lambda_2} \subset \bigcup_{j=0}^{m-1} \left[\left(\frac{2j-m+1}{2} \right) \lambda_1 + \Omega_{\lambda_1} \right]$$

Conversely, assume $u \in]-\lambda_2/2; +\lambda_2/2]$ and that there exists $j \in \{0, 1, \ldots, m-1\}$ such that $u \in \left(\frac{2j-m+1}{2}\right)\lambda_1 + \Omega_{\lambda_1}$. Then there exists $s, q(s)$ integers such that

$$u = \left(\frac{2j-m+1}{2} \right) \lambda_1 + f(s) - q(s)\lambda_1$$
$$= f(s) - (\lambda_2/2 - j\lambda_1 + q(s)\lambda_1 - \lambda_1/2)$$

But $q(s)\lambda_1$ lies within an interval $]r\lambda_2 - \lambda_2/2; r\lambda_2 + \lambda_2/2]$; hence, for a certain integer $r = r(s)$ and $j' \in \{0, 1, \ldots, m-1\}$, we have

$$q(s)\lambda_1 = r\lambda_2 - \frac{\lambda_2}{2} + j'\lambda_1 + \frac{\lambda_1}{2}$$

and

$$u = f(s) - r\lambda_2 + (j-j')\lambda_1$$

with $j-j' \in \{-m+1, -m+2, \ldots, m-2, m-1\}$. M denotes the greatest (reached necessarily) value of $|j-j'|$, over all the elements $u \in \Omega_{\lambda_1}$.

b) Similarly when m is even$\square$.

Corollary.- *If $\lambda_1 > 0$ is such that Ω_{λ_1} is finite, then, for any integer $m \geq 1$, Ω_{λ_2} is finite.*

In the following, we will be interested in sequences $f\{n\}$ and values of λ_1 for which $M(\lambda_1, m) = 0$, that is for which the set Ω_{λ_2} is completely decomposed, ie : if m is odd,

$$\Omega_{\lambda_2} = \bigcup_{j=0}^{m-1} \left[\left(\frac{2j-m+1}{2}\right)\lambda_1 + \Omega_{\lambda_1}\right]$$

if m is even,

$$\Omega_{\lambda_2} = \left(-\frac{m}{2} + \Omega_{\lambda_1}^{+}\right) \cup \bigcup_{j=1}^{m-1} \left[\left(\frac{2j-m}{2}\right)\lambda_1 + \Omega_{\lambda_1}\right] \cup \left(\frac{m}{2}\lambda_1 + \Omega_{\lambda_1}^{-}\right)$$

We call this case of complete decomposition *the assumption (SD).*

Proposition 5.- *Under the assumption (SD), we have :*

a) m odd : for any $v \in \Omega_{\lambda_2}$, $\exists j \in \{0, 1, \ldots, m-1\}$ with $v - \frac{2j-m+1}{2}\lambda_1 \in]-\frac{\lambda_1}{2}; \frac{\lambda_1}{2}]$ and

$$I\!N_{\lambda_2}(v) \subset I\!N_{\lambda_1}\left(v - \frac{2j-m+1}{2}\lambda_1\right) \tag{27}$$

For any $w \in \Omega_{\lambda_1}$, $\exists j' \in \{0, 1, \ldots, m-1\}$ with $w + \frac{2j-m+1}{2}\lambda_1 \in \Omega_{\lambda_2}$ and

$$I\!N_{\lambda_1}(w) \subset I\!N_{\lambda_2}(w + \frac{2j'-m+1}{2}\lambda_1) \tag{28}$$

b) m even : for any $v \in \Omega_{\lambda_2}$, $\exists j' \in \{0, 1, \ldots, m\}$ with $v - \frac{2j-m}{2}\lambda_1 \in]-\frac{\lambda_1}{2}; \frac{\lambda_1}{2}]$ and

$$I\!N_{\lambda_2}(v) = I\!N_{\lambda_1}\left(v - \frac{2j-m}{2}\lambda_1\right) \tag{29}$$

For any $w \in \Omega_{\lambda_1}$, $\exists j' \in \{0, 1, \ldots, m\}$ with $w + \frac{2j'-m}{2}\lambda_1 \in \Omega_{\lambda_2}$ and

$$I\!N_{\lambda_1}(w) \subset I\!N_{\lambda_2}(w + \frac{2j'-m}{2}\lambda_1) \tag{30}$$

Proof.- a) Assume $n \in \mathbb{N}_{\lambda_2}(v)$. Then, for a certain integer $p = p(n)$, we have

$$f(n) = v + p\lambda_2$$

But $\exists j \in \{0, 1, \ldots, m-1\}$ and $w \in]\lambda_1/2; +\lambda_1/2]$ such that

$$v = w + \frac{-\lambda_2}{2} + \frac{\lambda_1}{2} + j\lambda_1$$

Since $\dfrac{2j - m + 1}{2}$ is an integer, we have

$$\begin{aligned} f(n) &= v - \frac{2j-m+1}{2}\lambda_1 + \frac{2j-m+1}{2}\lambda_1 + p\lambda_2 \\ &= w + \frac{2j-m+1}{2}\lambda_1 + pm\lambda_1 \end{aligned}$$

hence

$$n \in \mathbb{N}_{\lambda_1}\left(v - \frac{2j-m+1}{2}\lambda_1\right)$$

The second inclusion is obvious using assumption (SD). The two other assertions follow in the way.

4. The Thue-Morse Case

Let $a, b > 0$ two real numbers with $a \neq b$ and $n \in \mathbb{N}, n \geq 1$. The nth-tile t_n of the Thue-Morse sequence (for instance [7]) is given by

$$t_n = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)(-1)^{S_2(n)}$$

where $S_2(n)$ is the sum of the 2-digits in the binary expansion of n. In other terms, if

$$n = a_0 + a_1 2^1 + a_2 2^2 + a_3 2^3 + \ldots$$

then

$$S_2(n) = a_0 + a_1 + a_2 + a_3 + \ldots$$

each sum being obviously finite.

Another sequence produced by the Thue-Morse automaton is the following, but it is obvious that its spectrum is much simpler since it is periodical: its nth-tile t'_n is given by

$$t'_n = \frac{1}{2}(a+b) + \frac{1}{2}(a-b)\pi_n$$

where $\pi_n = \epsilon_0\epsilon_1 \ldots \epsilon_n$, and $\epsilon_n = (-1)^n, n \in I\!N$. The aperiodic sequence $f(n)$, resp. $f'(n)$, with $n \in I\!N$, is given by

$$f(0) = 0 = f'(0)$$

$$f(n) = \sum_{0 \le m \le n-1} t_m$$

$$f'(n) = \sum_{0 \le m \le n-1} t'_m$$

for $n \geq 1$. We will analyze the rates of occupancy of sublattices at infinity of each sequence as in Proposition 3 and the stability under multiplication by integers as in Proposition 4. We will show that both sequences will behave in a very similar way, at least for a collection of Bragg peaks, for which the intensities are exactly the same.

Lemma 2.- *For all $n \geq 1$, we have*

$$f'(n) = \frac{n}{2}(a+b) + \frac{1}{2}(a-b) \sum_{m=0}^{n-1} (-1)^{m(m+1)/2}$$

Consequently, when $n = 2p$, p integer, $p \geq 1$, then

$$f'(2p) = p(a+b)$$

$$f'(2p-1) = f'(2p) - (a+b)/2 - \frac{1}{2}(a-b)(-1)^{(2p-1)p}$$

It can be easily checked that $(-1)^{(2p-1)p}$ is equal to $+1$ when p is even, and to -1 when p is odd. Therefore, $f'(2p) - f'(2p-1) = b$ or a with equal probability, when p lies in $I\!N$. With the above notations, leaving the superscript ' for the quantities associated with the sequence f', we deduce

Proposition 6.- *With $\lambda = a + b$, we have, relatively to the sequence f' :*

$$\Omega'_\lambda = \{ v'_1 = -b, v'_2 = 0, v'_3 = +b \}$$
$$\alpha'_\lambda(-b) = \alpha'_\lambda(0) = \alpha'_\lambda(+b) = 1$$
$$\delta'_\lambda(-b) = \delta'_\lambda(+b) = \frac{1}{4}$$
$$\delta'_\lambda(0) = \frac{1}{2}$$
$$n'_0 = 1, \; n'_1 = 4 = n'_2 = n'_3 = \ldots = Card(\Omega'_\lambda) + 1$$

Lemma 3.- *For all $n \geq 1$, we have*

$$f(n) \;=\; \frac{n}{2}(a+b) + \frac{1}{2}(a-b)\sum_{m=0}^{n-1}(-1)^{S_2(m)}$$

with

$$\sum_{m=0}^{n-1}(-1)^{S_2(m)} \;\in\; \{-1, 0, +1\}$$

Proof : If n is even, the first coefficient a_0 in its binary expansion is equal to 0. Therefore, going from n to $n+1$ leads to just adding 1 to $S_2(n)$ to find $S_2(n+1)$. We have :

- if $S_2(n)$ is even, then $S_2(n+1)$ is odd,
- if $S_2(n)$ is odd, then $S_2(n+1)$ is even.

In other terms, if n is even :

$$(-1)^{S_2(n)}(-1)^{S_2(n+1)} \;=\; -1$$

Now, we prove inductively that, for any $n \in 2\mathbb{N}$,

$$\sum_{m=0}^{n-1}(-1)^{S_2(m)} \;=\; 0$$

If $n = 2$, the result is true. Assume the result for $n \geq 2$ even. We have :

$$\sum_{m=0}^{n-1}(-1)^{S_2(m)} \;=\; 0$$

and

$$\sum_{m=0}^{n+1}(-1)^{S_2(m)} \;=\; \sum_{m=0}^{n-1}(-1)^{S_2(m)} + (-1)^{S_2(n)} + (-1)^{S_2(n+1)} \;=\; (-1)^{S_2(n)} + (-1)^{S_2(n+1)}$$

but the sum of these two quantities is zero. Hence, the result. □

Proposition 7.- *With $\lambda = a + b$, we have, for the Thue-Morse sequence :*

$$\Omega_\lambda \;=\; \{v_1 = -b, v_2 = 0, v_3 = +b\}$$

$$\alpha_\lambda(-b) \;=\; \alpha_\lambda(0) \;=\; \alpha_\lambda(+b) \;=\; 1$$

$$\delta_\lambda(-b) \;=\; \delta_\lambda(+b) \;=\; \frac{1}{4}$$

$$\delta_\lambda(0) \; = \; \frac{1}{2}$$

$$n_0 \; = \; 1, \;\; n_1 \; = \; 4 \; = \; n_2 \; = \; n_3 \; = \; \ldots \; = \; Card(\Omega_\lambda) + 1$$

Proof : Since, for each $n \in 2I\!N$, we have

$$\sum_{m=0}^{n-1} (-1)^{S_2(m)} \; = \; 0$$

then $\alpha_\lambda(0) \; = \; 1, \; \delta_\lambda(0) \; = \; \frac{1}{2}$. Denote now

$$\mathcal{A}^- \; = \; \{ \, n \in 2I\!N \, | \, (-1)^{S_2(n)} \; = \; -1 \, \}$$

$$\mathcal{A}^+ \; = \; \{ \, n \in 2I\!N \, | \, (-1)^{S_2(n)} \; = \; +1 \, \}$$

We have $2I\!N \; = \; \mathcal{A}^- \cup \mathcal{A}^+$ as a disjoint union. The injective application $\phi : x \to 2x+2$ defined on $2I\!N$ sends $\mathcal{A}^-$ to $\mathcal{A}^+$ and $\mathcal{A}^+$ to $\mathcal{A}^-$. Therefore, for the distribution of points $f(n)$, with $n \in 1+2I\!N$, on the sublattices $\pm b + \lambda Z\!\!Z$, we have : $\alpha_\lambda(-b) \; = \; \alpha_\lambda(+b) \; = \; 1$, and $\delta_\lambda(-b) \; = \; \delta_\lambda(+b) \; = \; \frac{1}{4}$. Hence, $n_0 \; = \; 1$ with the other values n_j equal to 4, for $j \geq 1$.□

Corollary 1.- *The intensity per diffracting site of the sequence f' or the Thue-Morse sequence f, at the wave vector $k \; = \; 2\pi/(a+b)$, is given by*

$$\frac{1}{4} \left| 1 + \cos\left(\frac{2\pi b}{a+b}\right) \right|^2$$

Proof : Counting the intensity per diffracting site leads to divide by N before taking the limit in Proposition 3. This gives immediately (abbreviating the notation in a non-correct but understandable form)

$$\lim_{N \to +\infty} \frac{I(2\pi/(a+b))}{N} \; = \; \left| e^{2i\pi(-b)/(a+b)} \frac{1}{4} \; + \; \frac{1}{2} \; + \; e^{2i\pi(+b)/(a+b)} \frac{1}{4} \right|^2$$

hence the result □.

It can be easily checked that, for the Thue-Morse sequence f and for the sequence f', the assumption (SD) [Corollary of Proposition 4] is satisfied for $\lambda \; = \; a+b$ with any integer $m \geq 2$ and that we have equalities in the equations Eq. [27, 28] and Eq. [29, 30]. Hence, with the notations of Proposition 5, we deduce, for m odd

$$\lim_{q \to +\infty} \frac{\delta_{\lambda_1,q}(v - \frac{2j-m+1}{2})}{q} \; = \; m \lim_{q \to +\infty} \frac{\delta_{\lambda_2,q}(v)}{q}$$

for m even

$$\lim_{q\to+\infty}\frac{\delta_{\lambda_1,q}(v-\frac{2j'-m}{2})}{q} = m\lim_{q\to+\infty}\frac{\delta_{\lambda_2,q}(v)}{q}$$

Similarly, for the $'$ quantities associated with f'. Therefore, going from Ω_{λ_1} to Ω_{λ_2}, resp. Ω'_{λ_1} to Ω'_{λ_2}, leads to divide all the coefficients $c_{\lambda_1}(l)$, resp. $c'_{\lambda_1}(l)$, by m. For instance, with $\lambda_1 = a+b, m=3$,

$$\Omega_{3(a+b)} = \Omega'_{3(a+b)} = \{-a-2b,-a-b,-a,-b,0,+b,+a,+a+b,+a+2b\}$$

and the intensity per diffracting site is, for both sequences :

$$\lim_{N\to+\infty}\frac{I(2\pi/3(a+b))}{N} = \Big|\frac{1}{12}e^{2i\pi(-a-2b)/(3(a+b))}+\frac{1}{6}e^{2i\pi(-a-b)/(3(a+b))}+$$
$$+\frac{1}{12}e^{2i\pi(-a)/(3(a+b))}+\frac{1}{12}e^{2i\pi(-b)/(3(a+b))}+\frac{1}{6}+\frac{1}{12}e^{2i\pi(+b)/(3(a+b))}+$$
$$+\frac{1}{12}e^{2i\pi(+a)/(3(a+b))}+\frac{1}{6}e^{2i\pi(+a+b)/(3(a+b))}+\frac{1}{12}e^{2i\pi(+a+2b)/(3(a+b))}\Big|^2$$

These expressions allow to consider the case when $N = +\infty$.

Corollary 2.- *The Thue-Morse sequence gives rise to a lattice of δ-peaks at the integral multiples of $\lambda_1 = a+b$.*

Proof : It is a consequence of the previous expression of the intensity .□

This result was also obtained according to another approach, by Kolar et al [17] for substitutional systems of length 2 formed with two tiles. The authors report there other Bragg-peaks.

5. Generalized Meyer Sets

Before showing that the Thue-Morse sequence can be formulated within the context of Meyer sets, as generalized Meyer sets under generalized cut-and-project schemes (generalized CPS), what will be explained below, we recall at first the basic facts of Meyer's constructions with locally compact abelian groups, following Meyer[1], Moody[2] and Baake and Moody[13].

Definition 4.- [classical CPS : cut-and-project scheme] *Let G and H be two locally compact abelian groups, and $\pi_1 : G\times H \to G, \pi_2 : G\times H \to H$ the canonical*

projections. We say that G produces H if there exists
(**A.s**) *a* <u>*closed subgroup*</u> *L of $G \times H$ satisfying :*

- (**A.a**) *L is discrete in $G \times H$*
- (**A.b**) *L is relatively dense in $G \times H$* [property **H**-(D2)]
- (**A.c**) *$L \cap \{0\} \times H = \{0, 0\}$ where 0 denotes the neutral element of G, resp. H*
- (**A.d**) *$\pi_2(L)$ is dense in H*

The structure of locally compact groups is well known, see for instance Rudin [12], chap. 2. A l.c.a group G contains an open (also closed) subset G_1 of the type $\mathbb{R}^m \times K$ such that K is a compact in the infinite torus $\mathbb{T}^\infty$ and that the quotient G/G_1 is a discrete group. In particular, K may be a space like a finitely-dimensional torus $(\mathbb{R}/\mathbb{Z})^l$, for $l \geq 1$ any integer. Up to the knowledge of one of the authors, for all the applications concerning the crystallography and structure models of quasicrystals, particularly icosahedral quasicrystals and decagonal quasicrystals, only the euclidean part was used up till now in cut-and-project schemes. Very recent results communicated to the authors also make use of p-adic internal spaces [14] in the spirit of the previous works of Meyer [1] and Schreiber [15], but, they seem not being used as such by experimentalists up till now. The normal CPS is then a collection of mappings and euclidean spaces :

$$\begin{array}{ccccc} \mathbb{R}^m & \stackrel{\pi_1}{\longleftarrow} & \mathbb{R}^m \times \mathbb{R}^n & \stackrel{\pi_2}{\longrightarrow} & \mathbb{R}^n \\ & & \cup & & \\ & & L & & \end{array}$$

where $L \subset \mathbb{R}^m \times \mathbb{R}^n$ is a lattice, π_1 and π_2 the orthogonal projection mappings onto $\mathrm{Im}(\pi_1) :=$ the physical space $= \mathbb{R}^m$, and $\mathrm{Im}(\pi_2) :=$ the internal space $= \mathbb{R}^n$. L is assumed such that with L the physical space *produces* the internal space, $\pi_2(L)$ is dense in $\mathbb{R}^n$ and $\pi_1|_L$ is injective. Let $\Upsilon := \pi_1(L)$. The application

$$(.)^* \ := \ \pi_2 \circ (\pi_1|_L)^{-1}$$

is well defined on Υ and has values in the internal space. It is extended on the $\mathbb{Q}$-span $\mathbb{Q}\Upsilon$ of Υ. In the context of structure models of quasicrystals, we normally choose lattices L which are invariant under a finite symmetry group (the icosahedral group, cyclic groups, ...) and one or several windows [2][13] in the internal space $\mathbb{R}^n$ to select points of L. If $W \subset \mathbb{R}^n$ is a window, it satisfies the following assumptions :

W1 The window $W \subset \mathbb{R}^n$ is compact,

W2 $W \ = \ \overline{\mathrm{int}(W)} \ \neq \ \emptyset$,

W3 The boundary ∂W of W has Lebesgue measure 0.

and a model set is given by

$$\Lambda = \{ x \in \Upsilon \mid x^* \in W \} \subset \mathbb{R}^m$$

Some properties of model sets are the following :

M1 Λ is a Delone set [property **H**] : it is relatively dense and uniformly discrete.

M2 Λ is a *Meyer* set : Λ is discrete and relatively dense and there exists a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$

M3 Λ has a well-defined point density d [see Rogers [16] for definitions for instance], i.e.

$$d = \lim_{R \to +\infty} \frac{\sharp\, (\Lambda \cap B(0, R))}{\mathrm{Vol}(B(0, R))}$$

where $B(0, R)$ is the ball centred at the origin of radius $R > 0$ in $\mathbb{R}^m$. Its volume is $\pi^{m/2} R^m / \Gamma\left(\frac{m+2}{2}\right)$.

M4 Λ has a well-defined spectrum composed of Bragg-peaks.

We now show that the toric part $(\mathbb{R}/\mathbb{Z})^l$, with $l = 1$, in the internal space, plays naturally a role in the representation of the Thue-Morse sequence for frequencies λ such that Ω_λ is finite.

Lemma 4.- *The Thue-Morse sequence and the sequence f' satisfy the properties* **M1, M2** *and* **M3**.

Proof : The physical space has dimension $m = 1$ here. If $\Lambda := \{ f(n) \mid n \in \mathbb{N} \}$, resp. $\Lambda' := \{ f'(n) \mid n \in \mathbb{N} \}$, denotes the subset of $\mathbb{R}$ composed of the points of the Thue-Morse sequence, resp. of the sequence f', we clearly see that **M1** is satisfied for Λ, resp Λ'. The fact that **M2** is satisfied for f follows from Lemma 3 with $F := \{ 0, \pm(a-b)/2, \pm(a-b), \pm 3(a-b)/2 \}$ since, for any $m > n \geq 0$, we have $f(m) - f(n) - f(m-n) \in F$. Similarly, the algebraic representation of $\{f'(n)\}$, given by Lemma 2, yields, for m, n integers, $m > n$,

$$f'(m) - f'(n) = \frac{m-n}{2}(a+b) + \frac{1}{2}(a-b) \left[\sum_{p=0}^{m-1} (-1)^{p(p+1)/2} - \sum_{p=0}^{n-1} (-1)^{p(p+1)/2} \right]$$

which can be written

$$= f'(m-n) + \frac{1}{2}(a-b)\left[\sum_{p=0}^{m-1}(-1)^{p(p+1)/2} - \sum_{p=0}^{n-1}(-1)^{p(p+1)/2} - \sum_{p=0}^{m-n-1}(-1)^{p(p+1)/2}\right]$$

But, since each sum in the last equation is equal to ± 1 or 0, we have

$$f'(m) - f'(n) - f'(m-n) \subset \frac{1}{2}(a-b)\{\pm 3, \pm 2, \pm 1, 0\}$$

The set $F' := \{0, \pm(a-b)/2, \pm(a-b), \pm 3(a-b)/2\}$ is finite and we have

$$\Lambda' - \Lambda' \subset \Lambda' + F'$$

proving **M2**. The property **M3** is clearly satisfied for f and f' since

$$\Lambda_{av} = \Lambda'_{av} := \frac{1}{2}(a+b)\,\mathbb{Z}$$

is the global average lattice for them. There is one point of Λ, resp. Λ', per node of Λ_{av} and the point density d, resp. d', of the Thue-Morse sequence is equal to 1. □

Proposition 8.- *The Thue-Morse sequence is harmonious.*

Proof : This is a consequence of Lemma 3 and Theorem X in chap. II in Meyer [1].□

We will analyze somewhere else characters on the Thue-Morse sequence, with the notions of duality following this proposition.

Now, since it is well-known that the Thue-Morse sequence has a spectrum which is not only composed of Bragg-peaks [for instance Kolar et al [17], Queffelec [8]], we should remove some assumptions from the normal cut-and-project scheme in order to get more general spectra than Bragg-spectra, as given by **M4**. We will do this only in a minimal way, sticking to the formalism of the previous paragraphs and we will have to join to this geometrical approach and framework the need to define simultaneously scaling exponents for the intensity function, for the singular continuous component of the spectrum.

We suggest the following scheme :

S1 Take $H := \mathbb{R}/\lambda\mathbb{Z}$ the one-dimensional torus, as internal space and $G := \mathbb{R}$ the physical space.

S2 Let denote by

$$\pi_\Lambda : \frac{1}{2}(a+b)\mathbb{N} \longrightarrow \Lambda$$

the bijective mapping from the average lattice $\Lambda_{av} \cap I\!R^+$ to Λ such that, for any $n \in I\!N$,

$$f(n) \;=\; \pi_\Lambda(\frac{a+b}{2}\,n)$$

We have, for any integer n :

$$|\,\pi_\Lambda(\frac{a+b}{2}\,n) \;-\; \frac{n}{2}(a+b)\,| \;\leq\; \frac{a-b}{2}$$

and we denote by π_Λ^{-1} its inverse mapping defined on the set of the elements $\{f(n)\,/\,n \in I\!N\}$, is valued in the average lattice Λ_{av}. Call u an element in $]-\lambda/2\,;\,+\lambda/2\,[$ and $\overline{u}$ its canonical image in H.
Take

$$L := \{\,(x,\overline{u})\;\in G \times H \mid x \in \Lambda_{av},\; \overline{u}\;\text{ such that } u \in \Omega_\lambda,\; u \equiv f(\frac{2x}{a+b})\;(\text{modulo}\lambda Z\!\!\!Z)\,\}$$

Lemma 5.- *L is discrete in $G \times H$.*

S3 Let denote by $\tilde{\pi_\Lambda} : I\!R \to I\!R$ any strictly increasing function satisfying

$$\tilde{\pi_\Lambda}|_{\Lambda_{av}} \;=\; \pi_\Lambda$$

We can take it continuous but there is no reason a priori to do so. Then we have a new CPS consisting of a collection of spaces and mappings :

$$\begin{array}{ccccc} I\!R & \stackrel{\tilde{\pi_\Lambda}\circ\pi_1}{\longleftarrow} & I\!R \times I\!R/\lambda Z\!\!\!Z & \stackrel{\pi_2}{\longrightarrow} & I\!R/\lambda Z\!\!\!Z \\ & & \cup & & \\ & & L & & \end{array}$$

Lemma 6.- *$\tilde{\pi_\Lambda} \circ \pi_1$ is uniformly bounded with respect to π_1 in the sense that its restriction to L satisfies :*

$$\|\,\tilde{\pi_\Lambda} \circ \pi_{1|_L}\,\| \;:=\; \sup_{z \in L}\,|\,\tilde{\pi_\Lambda} \circ \pi_1(z) \;-\; \pi_1(z)\,| \;\leq\; \frac{a-b}{2}$$

We see that $(\tilde{\pi_\Lambda} \circ \pi_1)|_L \;=\; (\pi_\Lambda \circ \pi_1)|_L$ is injective, and that the selection mode on the closed subset L is not based on a projection mode but on a *congruent mode through f and the frequency* λ which is such that Ω_λ is finite. Clearly, L is closed in $G \times H$, and, since we have assumed that $f(0) = 0$, the properties (**A**.**a**), (**A**.**b**), (**A**.**c**) in Definition 4 are satisfied. L is a priori not a subgroup in $G \times H$ and assumption **(A.s)** has no reason to be satisfied. $\pi_2(L)$ is discrete in H by construction. We have a $(.)^*$ operation as in a the normal CPS :

$$(.)^* \;:=\; \pi_2 \circ ((\pi_\Lambda \circ \pi_1)|_L)^{-1} \;:\; \Lambda \;\to H = I\!R/Z\!\!\!Z$$

S4 We can now choose windows as in the normal CPS : if W is a window, W is a subset of $\{\overline{u} \in \mathbb{R}/\mathbb{Z} \mid u \in \Omega_\lambda\}$. It is a compact set for which the boundary has Lebesgue measure 0 (properties **W1** and**W3** are satisfied). It is not the adherence of its interior, and property **W2** is not satisfied.

The generalized Meyer sets we can form from the Thue-Morse sequence f with respect to the frequency λ such that Ω_λ is finite are given, similarly to the normal CPS, by

$$\Lambda_W := \{x \in \Lambda \mid x^* \in W\}$$

Of course, the property **M4** is not any more valid and the spectrum displays more peaks than only Bragg-peaks. If the window is maximal, we obtain the full Thue-Morse sequence as defined algebraically by f. If the window is smaller and contains only some points inside the torus $\mathbb{R}/\lambda\mathbb{Z}$, we obtain a subset of the Thue-Morse sequence and we have only to consider, for the scaling exponent of the diffracting intensity of the reduced system of points to consider the values of the *levels* for the elements which are selected by the window. we have seen that the scaling exponents and the rates of occupancy for the intensity function are attached to the lattices $v + \lambda\mathbb{Z}$, that is to the elements $\overline{v} \in H = \mathbb{R}/\lambda\mathbb{Z}$ and can be classified according to a lexicographical order and that the dominant scaling exponent is given by Eq. [23].

S5 The question whether there exists a substitute to **(A.s)**, that is, an algebraic structure on L can be partially overcome by recent results obtained by Gazeau and Miekisz [18] who have proved that there exists a canonical symmetry group on the Thue-Morse quasicrystal. By the $(.)^*$-operation, this can be reported to the elements of the window, and globally on L. However, the operations of this group have no reason to be stable by classes inside the toric internal space. So, this operation is not well-defined and cannot be used in this case.

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