Hyperbolicity of Generic Surfaces of High Degree in Projective 3-Space

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0. Introduction

The goal of this paper is to study the celebrated conjecture of S. Kobayashi on the hyperbolicity of generic hypersurfaces in projective space. Recall that, by a well-known criterion due to Brody [Bro78], a compact complex space X is hyperbolic in the sense of Kobayashi [Ko70] if and only if there is no noncontant holomorphic map from $\mathbb{C}$ to X. Kobayashi proposed the following conjecture: *A generic n-dimensional hypersurface of large enough degree in $\mathbb{P}^{n+1}_{\mathbb{C}}$ is hyperbolic.* This is of course obvious in the case of curves, the uniformization theorem shows that a smooth curve is hyperbolic if and only if it has genus at least 2, which is the case if the degree is at least 4.

However, the picture is not at all clear in dimension $n \geq 2$. In view of results by Zaidenberg [Zai87], the most optimistic lower bound for the degree of hyperbolic n-dimensional hypersurfaces would be $2n+1$ for $n \geq 2$. In the case of a surface X, the bound is strongly expected to be equal to 5, which is precisely the lowest possible degree for X to be of general type. In fact, Green-Griffiths [GG80] have formulated the following much stronger conjecture: *If X is a variety of general type, every entire curve $f \to \mathbb{C}$ is algebraically degenerate, and* (optimistic version of the conjecture) *there is a proper algebraic subset $Y \subset X$ containing all images of non constant entire curves.* As a (very) generic surface of degree at least 5 does not contain rational or elliptic curves by the results of H. Clemens ([Cl86], [CKM88]) and G. Xu [Xu94], it would then follow that such a surface is hyperbolic. However, almost nothing was known before for the case of transcendental curves drawn on a (very) generic surface or hypersurface. Only rather special examples of hyperbolic hypersurfaces have been constructed in higher dimensions, thanks to a couple of techniques due to Brody-Green [BG78], Nadel [Na89], Masuda-Noguchi [MN94], Demailly-El Goul [DEG97] and Siu-Yeung [SY97]. The related question of complements of curves in $\mathbb{P}^2$ has perhaps been more extensively investigated, see Zaidenberg [Zai89, 93], Dethloff-Schumacher-Wong [DSW92, 94], Siu-Yeung [SY95], Dethloff-Zaidenberg [DZ95a,b].

Here, we obtain a confirmation of Kobayashi's conjecture in dimension 2, for the case of surfaces of degree at least 42. Our analysis is based on the following more general result, which also applies to surfaces not necessarily embedded in $\mathbb{P}^3$.

Main Theorem. — *Let X be a nonsingular surface of general type satisfying the following conditions:*

(a) $\mathrm{Pic}(X) = \mathbb{Z}$;

(b) $c_1^2 - \frac{9}{10}c_2 > 0$;

(c) $H^0(X, S^m T_X^\star) = 0$ *for all* $m > 0$;

(d) $H^0(X, E_{2,3}T_X^\star \otimes \mathcal{O}(-tK_X)) = 0$ *for any rational number* $t > 3/4$ *such that* tK_X *is an integral divisor.*

Then every nonconstant holomorphic map $f : \mathbb{C} \to X$ *is a leaf of an algebraic multi-foliation on* X.

Let us explain in more detail our terminology. The notation $E_{2,3}T_X^\star$ stands for the sheaf of "invariant" jet differentials of order 2 and degree 3, namely germs of differential operators of the form

$$P(f) = a(f)\, f_1'^3 + b(f)\, f_1'^2 f_2' + c(f)\, f_1' f_2'^2 + d(f)\, f_1'^3 + e(f)\, (f_1' f_2'' - f_1'' f_2')$$

acting on germs of curves $f : (\mathbb{C}, 0) \to X$, expressed as $f = (f_1, f_2)$ in local coordinates. An *algebraic multi-foliation* on a surface X is by definition associated with a rank 1 subsheaf $\mathcal{F} \subset S^m T_X^\star$, where $\mathcal{F}$ is locally generated by a section $s \in \Gamma(U, S^m T_X^\star)$ of the form

$$s(z) = \sum_{0 \le j \le m} a_j(z_1, z_2)(dz_1)^{m-j}(dz_2)^j,$$

vanishing at only finitely many points, and such that

$$s(z) = \prod_{1 \le j \le m} (c_{1,j}(z)dz_1 + c_{2,j}(z)dz_2)$$

factorizes as a product of generically distinct linear forms. Equivalently, the foliation is defined by a collection of m-lines in $T_{X,z}$ at each generic point z, so that it is associated with a (possibly singular) surface $Y \subset P(T_X)$ which is m-sheeted over X. Of course, if $\widetilde{Y}$ is a desingularization of Y, then $\widetilde{Y}$ carries an associated (possibly singular) foliation, that is, a rank 1 subsheaf of $T_{\widetilde{Y}}^\star$. A *leaf* of the multi-foliation on X is just the projection to X of a leaf of the corresponding foliation on $\widetilde{Y}$.

It is easy to check that the above conditions (a), (b), (c), (d) are met for a very generic surface in $\mathbb{P}^3$ of sufficiently high degree. Here, the terminology "generic" (resp. "very generic") will be used to indicate that the exceptional set is possibly a finite (resp. countable) union of algebraic subsets in the moduli space of surfaces in $\mathbb{P}^3$. We check the following basic properties (see sections 3 and 6).

Proposition. — *Let X be a nonsingular surface of degree d in $\mathbb{P}^3$. Then*

(a) $\mathrm{Pic}(X) = \mathbb{Z}$ *if X is very generic (Noether-Lefschetz theorem).*

(b) $10\,c_1^2 - 9\,c_2 = d(d^2 - 44\,d + 104) > 0$ *for* $d \ge 42$.

(c) $H^0(X, S^m T_X^\star \otimes \mathcal{O}(k)) = 0$ *for all* $m > 0$ *and* $k \in \mathbb{Z}$, $k \le m$.

(d) *Assume that* $d \ge 11$ *and that* X *is generic. Then*

$$H^0(X, E_{2,3}T_X^\star \otimes \mathcal{O}(-tK_X)) = 0$$

for all rational numbers $t > 1/2$ *such that* tK_X *is integral.*

According to recent results of M. McQuillan (see section 6), this solves Kobayashi's conjecture – at least in the case of surfaces.

Corollary 1. — *A very generic surface* X *in* $\mathbb{P}^3$ *of degree* $d \ge 42$ *is Kobayashi hyperbolic, that is, there is no nonconstant holomorphic map from* $\mathbb{C}$ *to* X.

As a consequence of the proof, we also get

Corollary 2. — *The complement of a very generic curve in* $\mathbb{P}^2$ *is hyperbolic and hyperbolically imbedded for all degrees* $d \ge 42$.

Our strategy is based on a careful analysis of the geometry of Semple jet bundles, as proposed in [Dem95]. We use some natural Riemann-Roch calculations to prove the existence of suitable 2-jet differentials of sufficiently large degree, and we combine this with a non existence result for degree 3 differentials. The non existence result only holds true for generic surfaces. It is used to avoid low degree components in the 2-jet differentials base locus, which appear to be an obstacle for the application of the semi-stability inequalities. We finally derive from Riemann-Roch again the existence of an algebraic multi-foliation, as stated in the Main Theorem.

Our hope is that a suitable generalization of the present techniques to higher order jets will soon lead to a solution of the Green-Griffiths conjecture: every holomorphic map from $\mathbb{C}$ to a surface of general type is algebraically degenerate. We would like to thank Gerd Dethloff and Steven Lu for sharing generously their views on these questions, and Bernie Shiffman for interesting discussions on related subjects.

1. Semple jet bundles

Let X be a complex n-dimensional manifold. According to Green-Griffiths [GG80], we let $J_k \to X$ be the bundle of k-jets of germs of parametrized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coincide for $0 \le j \le k$, when computed in some local coordinate system of X near x. The projection map $J_k \to X$ is simply $f \mapsto f(0)$. Thanks to Taylor's formula, the fiber $J_{k,x}$ can be identified with the set of k-tuples of vectors $(f'(0), \ldots, f^{(k)}(0)) \in (\mathbb{C}^n)^k$. It follows that J_k is a holomorphic fiber bundle with typical fiber $(\mathbb{C}^n)^k$ over X (however, J_k is not a vector bundle for $k \ge 2$, because of the nonlinearity of coordinate changes). In the terminology of [Dem95], a directed manifold is a pair (X,V) where X is a complex manifold and $V\subset T_X$ a subbundle. Let (X,V) be a complex directed manifold. We define $J_kV\to X$ to be the bundle of k-jets of curves $f:(\mathbb{C},0)\to X$ which are tangent to V, i.e., such that $f'(t)\in V_{f(t)}$ for all t in a neighborhood of 0, together with the projection map $f\mapsto f(0)$ onto X. It is easy to check that J_kV is actually a subbundle of J_k. One of the essential tools used here are the projectivized jet bundles $X_k\to X$ introduced in [Dem95]. Let $\mathbb{G}_k$ be the group of germs of k-jets of biholomorphisms of $(\mathbb{C},0)$, that is, the group of germs of biholomorphic maps

$$t\mapsto\varphi(t)=a_1t+a_2t^2+\cdots+a_kt^k,\qquad a_1\in\mathbb{C}^\star,\ a_j\in\mathbb{C},\ j\ge 2,$$

in which the composition law is taken modulo terms t^j of degree $j>k$. The group $\mathbb{G}_k$ acts on the left on J_kV by reparametrization, $(\varphi,f)\mapsto f\circ\varphi$. The bundle X_k can then be seen as a natural compactification of the quotient of the open subset of regular jets $J_kV^{\rm reg}\subset J_kV$ by the action of $\mathbb{G}_k$. We recall here briefly the basic construction.

To a directed manifold (X,V), one associates inductively a sequence of directed manifolds (X_k,V_k) as follows. Starting with $(X_0,V_0)=(X,V)$, one sets inductively $X_k=P(V_{k-1})$ [$P(V)$ stands for the projectivized bundle of lines in the vector bundle V], where V_k is the subbundle of T_{X_k} defined at any point $(x,[v])\in X_k$, $v\in V_{k-1,x}$, by

$$V_{k,(x,[v])}=\Big\{\xi\in T_{X_k,(x,[v])}\ ;\ (\pi_k)_\star\xi\in\mathbb{C}\cdot v\Big\},\quad \mathbb{C}\cdot v\subset V_{k-1,x}\subset T_{X_{k-1},x}\ .$$

Here $\pi_k:X_k\to X_{k-1}$ denotes the natural projection. We denote by $\mathcal{O}_{X_k}(-1)$ the tautological line subbundle of $\pi_k^\star V_{k-1}$, such that

$$\mathcal{O}_{X_k}(-1)_{(x,[v])}=\mathbb{C}\cdot v,$$

for all $(x,[v])\in X_k=P(V_{k-1})$. By definition, the bundle V_k fits in an exact sequence

$$0\longrightarrow T_{X_k/X_{k-1}}\longrightarrow V_k\xrightarrow{\pi_{k\star}}\mathcal{O}_{X_k}(-1)\longrightarrow 0,$$

and the Euler exact sequence of $T_{X_k/X_{k-1}}$ yields

$$0\longrightarrow\mathcal{O}_{X_k}\longrightarrow\pi_k^\star V_{k-1}\otimes\mathcal{O}_{X_k}(1)\longrightarrow T_{X_k/X_{k-1}}\longrightarrow 0.$$

From these sequences, we infer

$$\operatorname{rank}V_k=\operatorname{rank}V_{k-1}=\cdots=\operatorname{rank}V=r,\qquad \dim X_k=n+k(r-1).$$

We say that (X_k,V_k) is the k-jet directed manifold associated with (X,V), and we let

$$\pi_{k,j}=\pi_{j+1}\circ\cdots\circ\pi_{k-1}\circ\pi_k:X_k\longrightarrow X_j,$$

be the natural projection.

Now, let $f:\Delta_r\to X$ be a non constant tangent trajectory to V. Then f lifts to a well defined and unique trajectory $f_{[k]}:\Delta_r\to X_k$ of X_k tangent to V_k. Moreover, the derivative $f'_{[k-1]}$ gives rise to a section

$$f'_{[k-1]}:T_{\Delta_r}\to f^\star_{[k]}\mathcal{O}_{X_k}(-1).$$

With any section σ of $\mathcal{O}_{X_k}(m)$, $m \geq 0$, on any open set $\pi_{k,0}^{-1}(U)$, $U \subset X$, we can associate a holomorphic differential operator Q of order k acting on k-jets of curves $f : (\mathbb{C}, 0) \to U$ tangent to V, by putting

$$Q(f)(t) = \sigma(f_{[k]}(t)) \cdot f'_{[k-1]}(t)^{\otimes m} \in \mathbb{C}.$$

In order to understand better this correspondence, let us use locally a coordinate chart and the associated trivialization $T_X \simeq \mathbb{C}^n$, so that the projection $\mathbb{C}^n \to \mathbb{C}^r$ onto the first r-coordinates gives rise to admissible coordinates on V. Then $f', f'', \ldots, f^{(k)}$ are in one to one correspondence with the r-tuples

$$(f'_1, \ldots, f'_r), \quad (f''_1, \ldots, f''_r), \ \ldots \quad (f^{(k)}_1, \ldots, f^{(k)}_r).$$

1.1. Proposition ([Dem95]). — *The direct image sheaf* $(\pi_{k,0})_\star \mathcal{O}_{X_k}(m)$ *on* X *coincides with the (locally free) sheaf* $E_{k,m}V^\star$ *of* k*-jet differentials of weighted degree* m*, that is, by definition, the set of germs of polynomial differential operators*

$$Q(f) = \sum_{\alpha_1 \ldots \alpha_k \in \mathbb{N}^r} a_{\alpha_1 \ldots \alpha_k}(f)\, (f')^{\alpha_1} (f'')^{\alpha_2} \cdots (f^{(k)})^{\alpha_k}$$

on J_kV *[in multi-index notation,* $(f')^{\alpha_1} = (f'_1)^{\alpha_{1,1}} (f'_2)^{\alpha_{1,2}} \ldots (f'_r)^{\alpha_{1,r}}$*], which are moreover invariant under arbitrary changes of parametrization: a germ of operator* $Q \in E_{k,m}V^\star$ *is characterized by the condition that, for every germ* $f \in J_kV$ *and every germ* $\varphi \in \mathbb{G}_k$*,*

$$Q\big(f \circ \varphi\big) = \varphi'^m\, Q(f) \circ \varphi.$$

Observe that the weighted degree m is taken with respect to weights 1 for f', 2 for f'', etc., thus counts the total numbers of "primes" in each monomial of the expansion of Q.

A basic result, relying on the Ahlfors-Schwarz lemma, is that any entire curve $f : \mathbb{C} \to X$ tangent to V must automatically satisfy all algebraic differential equations $Q(f) = 0$ arising from global jet differential operators

$$Q \in H^0(X, E_{k,m}V^\star \otimes \mathcal{O}(-A))$$

which vanish on some ample divisor A. More precisely, we have the following.

1.2. Theorem ([GG80], [Dem95], [SY97]). — *Assume that there exist integers* $k, m > 0$ *and an ample line bundle* A *on* X *such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes (\pi_{k,0})^\star A^{-1}) \simeq H^0(X, E_{k,m}V^\star \otimes A^{-1})$$

has non zero sections $\sigma_1, \ldots, \sigma_N$*. Let* $Z \subset X_k$ *be the base locus of these sections. Then every entire curve* $f : \mathbb{C} \to X$ *tangent to* V *is such that* $f_{[k]}(\mathbb{C}) \subset Z$*. In other words, for every global* $\mathbb{G}_k$*-invariant polynomial differential operator* Q *with values in* A^{-1}*, every entire curve* f *tangent to* V *must satisfy the algebraic differential equation* $Q(f) = 0$*.*

By definition, a line bundle L is big if there exists an ample divisor A on X such that $L^{\otimes m} \otimes \mathcal{O}(-A)$ admits a non trivial global section when m is large (then there are lots of sections, namely $h^0(X, L^{\otimes m} \otimes \mathcal{O}(-A)) \gg m^n$ with $n = \dim X$).

As a consequence, Theorem 1.2 can be applied when $\mathcal{O}_{X_k}(1)$ is big. In the sequel, we will be concerned only with the standard case $V = T_X$.

A conjecture by Green-Griffiths and Lang states that every entire curve drawn on a variety of general type is algebraically degenerate, i.e. contained in a proper algebraic subvariety. In view of this conjecture and of Theorem 1.2, it is especially interesting to compute the base locus of the global sections of jet differentials, sometimes referred to in the litterature as the Green-Griffiths locus of X. According to the definition of invariant k-jets given in [Dem95], we introduce instead the base locus B_k of invariant k-jets, that is, the intersection

$$B_k := \bigcap_{m>0} B_{k,m} \subset X_k$$

of the base loci $B_{k,m}$ of all line bundles $\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^\star \mathcal{O}(-A)$, where A is a given arbitrary ample divisor over X (clearly, B_k does not depend on the choice of A). Although the subvariety

$$Y := \bigcap_{k>0} \pi_{k,0}(B_k) \subset X$$

might in general project on the whole base variety X, we suspect that one should have $\dim B_k \le \dim X$ whenever X is of general type (and possibly $\dim Y < \dim X$ if in addition T_X is stable).

2. Base locus of 1-jets

From now on, we suppose that X is a nonsingular surface of general type (in particular, X must be algebraic, see [BPV84]), and let c_1 and c_2 be the Chern classes of X. We first describe some known facts about surfaces of general type with $c_1^2 > c_2$, in connection with the existence of "symmetric differentials", i.e., sections in $E_{1,m}T_X^\star = S^m T_X^\star$. Section 3 will be devoted to refinements of these results in the case of order 2 jets.

The starting point is Hirzebruch's Riemann-Roch formula [Hi66]

$$\chi(X, S^m T_X^\star) = \frac{m^3}{6}(c_1^2 - c_2) + O(m^2).$$

On the other hand, Serre duality implies

$$h^2(X, S^m T_X^\star) = h^0(X, S^m T_X \otimes K_X).$$

A vanishing theorem due to Bogomolov [Bo79] (see also e.g. [Dem95], § 14) implies that, on a surface X of general type,

$$h^0\big(X, S^p T_X \otimes K_X^{\otimes q}\big) = 0 \qquad \text{for all } p,\ q \text{ such that } p - 2q > 0.$$

In particular, $h^0(X, S^m T_X \otimes K_X) = 0$ whenever $m \ge 3$ and we get

$$h^0(P(T_X), \mathcal{O}_{P(T_X)}(m)) = h^0(X, S^m T_X^\star) \ge \chi(X, S^m T_X^\star) \ge \frac{m^3}{12}(c_1^2 - c_2) + O(m^2).$$

As a consequence, the line bundle $\mathcal{O}_{X_1}(1)$ is big when $c_1^2 > c_2$, and the base locus

$$B_1 = \bigcap_{m>0} \mathrm{Bs}\left|\mathcal{O}_{X_1}(m) \otimes \mathcal{O}(-A)\right|,$$

(which is equal in this case to the Green-Griffiths locus) is a proper algebraic subset of $X_1 = P(T_X)$.

Let Z be an irreducible component of B_1 which is a horizontal surface, i.e. such that $\pi_{1,0}(Z) = X$. Then the subbundle $V_1 \subset T_{X_1}$ defines on the desingularization $\widetilde{Z}$ of Z an algebraic foliation by curves, such that the tangent bundle to the leaves is given by $T_Z \cap V_1$ at a general point. Indeed, at any regular point $x_1 = [v] \in Z$, $v \in T_{X,x}$, at which $\pi_{1,0}$ is a local biholomorphism onto X, V_{1,x_1} consists of those vectors in T_{X_1} which project to the line $\mathbb{C}v \subset T_{X,x}$, and $T_{Z,x_1} \cap V_1$ is the lifting of that line by the isomorphism $(\pi_{1,0})_\star : T_{Z,x_1} \to T_{X,x}$.

By Theorem 1.2, for any nonconstant entire curve $f : \mathbb{C} \to X$, the curve $f_{[1]}$ must lie in some component Z if B_1. If Z is not horizontal, i.e. if $C = \pi_{1,0}(Z)$ is a curve in X, then $f(\mathbb{C}) \subset C$. Otherwise, we know by the above that Z carries a canonical algebraic foliation, and that the image of $f_{[1]}$ lies either in the singular set of Z or of the projection $\pi_{1,0} : Z \to X$ (which both consist of at most finitely many curves), or is a leaf of the foliation. Combining these observations with a theorem of A. Seidenberg [Se68] on desingularization of analytic foliations on surfaces, F. Bogomolov [Bo77] obtained the following finiteness theorem.

2.1. Theorem (Bogomolov). — *There are only finitely many rational and elliptic curves on a surface of general type with $c_1^2 > c_2$.*

Theorem 2.1 can now be seen (see [M-De78]) as a direct consequence of the following theorem of J.-P. Jouanolou [Jo78] on algebraic foliations, and of the fact that a surface of general type cannot be ruled or elliptic.

2.2. Theorem (Jouanolou). — *Let L be a subsheaf of the cotangent bundle of a projective manifold defining an analytic foliation of codimension 1. Let H be the dual distribution of hyperplanes in T_X. If there is an infinite number of hypersurfaces tangent to H, then H must be the relative tangent sheaf to a meromorphic fibration of X onto a curve.*

The above result of Bogomolov does not give information on transcendental curves. As observed by Lu and Yau [LY90], one can say more if the topological index $c_1^2 - 2c_2$ is positive, using the following result of Y. Miyaoka [Mi82] on the almost ampleness of $T_X^\star$. We recall here their proof in order to point out the analogy with results of section 3 (see [ScTa85] for the general case of semi-stable vector bundles).

Fisrt recall that a line bundle L on a projective manifold is called numerically effective (nef) if the intersection $L \cdot C$ is nonnegative for all curve C in X. A surface X of general type is called minimal if its canonical bundle K_X is nef.

2.3. Theorem (Miyaoka). — *Let X be a minimal surface of general type with*

$c_1^2 - 2c_2 > 0$, then the restriction $\mathcal{O}_{X_1}(1)_{|Z}$ is big for every horizontal irreducible 2-dimensional subvariety Z of X_1.

Proof. — The Picard group of X is given by

$$\mathrm{Pic}(X_1) = \mathrm{Pic}(X) \oplus \mathbb{Z}[u]$$

where $u := \mathcal{O}_{X_1}(1)$, and the cohomology ring $H^\bullet(X_1)$ is given by

$$H^\bullet(X_1) = H^\bullet(X)\,[u]/(u^2 + (\pi^\star c_1)u + \pi^\star c_2)$$

[u denoting rather $c_1(\mathcal{O}_{X_1}(1))$ in that case]. In particular,

$$u^3 = u \cdot \pi^\star(c_1^2 - c_2) = c_1^2 - c_2, \qquad u^2 \cdot \pi^\star K_X = u \cdot \pi^\star c_1^2 = c_1^2.$$

Let Z be an horizontal irreducible 2-dimensional subvariety. In $\mathrm{Pic}(X_1)$, we have

$$Z \sim mu - \pi^\star F$$

for some $m > 0$ and some divisor F on X. In order to study $\mathcal{O}_{X_1}(1)_{|Z}$, we compute the Hilbert polynomial of this bundle. The coefficient of the leading term is

$$(\dagger) \qquad (u_{|Z})^2 = u^2 \cdot (mu - \pi^\star F) = m(c_1^2 - c_2) + c_1 \cdot F,$$

by the above Chern class relations. The main difficulty is to control the term $c_1 \cdot F$. For this, the idea is to use a *semi-stability inequality.* The multiplication morphism by the canonical section of $\mathcal{O}(Z)$ defines a sheaf injection $\mathcal{O}(\pi^\star F) \hookrightarrow \mathcal{O}_{X_1}(m)$. By taking the direct images on X, we get

$$\mathcal{O}(F) \hookrightarrow \pi_\star \mathcal{O}_{X_1}(m) = S^m T_X^\star.$$

Using the K_X-semi-stability of $T_X^\star$ (see [Yau78] or [Bo79]), we infer

$$F \cdot (-c_1) \le \frac{c_1(S^m T_X^\star) \cdot (-c_1)}{m+1} = \frac{m}{2} c_1^2.$$

From (†), we get

$$(u_{|Z})^2 \ge \frac{m}{2}(c_1^2 - 2\,c_2) > 0,$$

and Riemann-Roch implies that either $\mathcal{O}_{X_1}(1)_{|Z}$ or $\mathcal{O}_{X_1}(-1)_{|Z}$ is big. To decide for the sign, we observe that K_X is big and nef and compute

$$(\dagger\dagger) \qquad u_{|Z} \cdot \pi^\star K_X = u \cdot (mu - \pi^\star F) \cdot (-c_1) = mc_1^2 + c_1 \cdot F\,;$$

from this we get $u_{|Z} \cdot \pi^\star K_X \ge \frac{m}{2} c_1^2 > 0$ by the semi-stability inequality. It follows that $\mathcal{O}_{X_1}(1)_{|Z}$ is big. □

By applying the above theorem of Miyaoka to the horizontal components Z of B_1, we infer as in Theorem 1.2 that every nonconstant entire curve $f : \mathbb{C} \to X$ is contained in the base locus of $\mathcal{O}_{X_1}(k) \otimes \mathcal{O}(-A)_{|Z}$ for k large, if A is a given ample divisor. Therefore f is algebraically degenerate.

2.4 Remark. — Unfortunately, the "order 1" techniques developped in this section are insufficient to deal with surfaces in $\mathbb{P}^3$, because in this case

$$c_1^2 = d(d-4)^2 < c_2 = d(d^2 - 4d + 6).$$

Lemma 3.4 below shows in fact that $H^0(X, S^m T_X^\star) = 0$ for all $m > 0$.

3. Base locus of 2-jets

The theory of directed manifolds and Semple jet bundles makes it possible to extend the techniques of section 2 to the case of higher order jets. The existence of suitable algebraic foliations is provided by the following simple observation, once sufficient information on the base locus B_k is known.

3.1. Lemma. — *Let (X_k, V_k) be the bundle of projectivized k-jets associated with a surface X and $V = T_X$. For any irreducible "horizontal hypersurface" $Z \subset X_k$* (i.e. *such that* $\pi_{k,k-1}(Z) = X_{k-1}$), *the intersection $T_Z \cap V_k$ defines a distribution of lines on a Zariski open subset of Z, thus inducing a* (*possibly singular*) *1-dimensional foliation on a desingularization of Z.*

Proof. — We have $\operatorname{rank} V_k = 2$ and an exact sequence

$$0 \longrightarrow T_{X_k/X_{k-1}} \longrightarrow V_k \longrightarrow \mathcal{O}_{X_k}(-1) \longrightarrow 0$$

which follows directly from the inductive definition of V_k. Thus the intersection $T_Z \cap V_k$ defines a distribution of lines on the Zariski open subset of Z equal to the set of regular points at which $\pi_{k,k-1} : Z \to X_{k-1}$ is étale (at such points, V_k contains the vertical direction and T_Z does not, thus V_k and T_Z are transverse). □

For general order k, it is hard to get a simple decomposition of the jet bundles $E_{k,m}T_X^\star$, and thus to calculate their Euler caracteristic. However, for $k = 2$ and $\dim X = 2$, it is observed in [Dem95] that one has the remarkably simple filtration

$$\operatorname{Gr}^\bullet E_{2,m}T_X^\star = \bigoplus_{0\le j\le m/3} S^{m-3j}T_X^\star \otimes K_X^{\otimes j}.$$

An elementary interpretation of this filtration consists in writing an invariant polynomial differential operator as

$$Q(f) = \sum_{0\le j\le m/3} \ \sum_{\alpha\in\mathbb{N}^2,\,|\alpha|=m-3j} a_{\alpha,j}(f)\,(f')^\alpha(f'\wedge f'')^j$$

where

$$f = (f_1, f_2), \qquad (f')^\alpha = (f_1')^{\alpha_1}(f_2')^{\alpha_2}, \qquad f'\wedge f'' = f_1'f_2'' - f_1''f_2'.$$

As suggested by Green-Griffiths [GG80], we now use the Riemann-Roch formula to derive the existence of global jet differentials. A Riemann-Roch calculation based on the above filtration of $E_{2,m}T_X^\star$ yields

$$\chi\big(X, E_{2,m}T_X^\star\big) = \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

On the other hand,

$$H^2(X, E_{2,m} \otimes \mathcal{O}(-A)) = H^0(X, K_X \otimes E_{2,m}T_X^\star \otimes \mathcal{O}(A))$$

by Serre duality. Since $K_X \otimes (E_{2,m}T_X^\star) \otimes \mathcal{O}(A)$ admits a filtration with graded pieces

$$S^{m-3j}T_X \otimes K_X^{\otimes 1-j} \otimes \mathcal{O}(A),$$

and $h^0\big(X, S^pT_X \otimes K_X^{\otimes q}\big) = 0$, $p - 2q > 0$, by Bogomolov's vanishing theorem on the general type surface X, we find

$$h^2\big(X, E_{2,m}T_X^\star \otimes \mathcal{O}(-A)\big) = 0$$

for m large. In the special case when X is a smooth surface of degree d in $\mathbb{P}^3_{\mathbb{C}}$, we take $A = \mathcal{O}(1)_{|X}$. Then we have $c_1 = (4-d)h$ and $c_2 = (d^2 - 4d + 6)h^2$ where $h = c_1(\mathcal{O}(1)_{|X})$, $h^2 = d$, thus

$$\chi\big(E_{2,m}T_X^\star \otimes \mathcal{O}(-A)\big) = d(4\,d^2 - 68\,d + 154)\frac{m^4}{648} + O(m^3).$$

A straightforward computation shows that the leading coefficient $4\,d^2 - 68\,d + 154$ is positive if $d \geq 15$, and a count of degrees implies that the H^2 group vanishes whenever $\big((m-3j)+2(j-1)\big)(d-4)-1 > 0$ for all $j \leq m/3$. For this, it is enough that $2(m/3-1)(d-4)-1 > 0$, which is the case for instance if $d \geq 5$ and $m \geq 5$. Consequently we get the following

3.2. Theorem ([Dem95]). — *If X is an algebraic surface of general type and A an ample line bundle over X, then*

$$h^0(X, E_{2,m}T_X^\star \otimes \mathcal{O}(-A)) \geq \frac{m^4}{648}(13\,c_1^2 - 9\,c_2) - O(m^3).$$

In particular, every smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 15$ admits non trivial sections of $E_{2,m}T_X^\star \otimes \mathcal{O}(-A)$ for m large, and every entire curve $f : \mathbb{C} \to X$ must satisfy the corresponding algebraic differential equations.

We now recall a few basic facts from [Dem95]. As $X_2 \to X_1 \to X$ is a tower of $\mathbb{P}^1$-bundles over X, the Picard group $\mathrm{Pic}(X_2) = \mathrm{Pic}(X) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ consists of all isomorphism classes of line bundles

$$\pi_{2,1}^\star \mathcal{O}_{X_1}(a_1) \otimes \mathcal{O}_{X_2}(a_2) \otimes \pi_{2,0}^\star L$$

where $L \in \mathrm{Pic}(X)$. For simplicity of notation, we set

$$u_1 = \pi_{2,1}^\star \mathcal{O}_{X_1}(1), \qquad u_2 = \mathcal{O}_{X_2}(1),$$
$$\mathcal{O}_{X_2}(a_1, a_2) := \pi_{2,1}^\star \mathcal{O}_{X_1}(a_1) \otimes \mathcal{O}_{X_2}(a_2)$$

for any pair of integers $(a_1, a_2) \in \mathbb{Z}^2$. The canonical injection $\mathcal{O}_{X_2}(-1) \hookrightarrow \pi_2^\star V_1$ and the exact sequence

$$0 \longrightarrow T_{X_1/X} \longrightarrow V_1 \xrightarrow{(\pi_1)_\star} \mathcal{O}_{X_1}(-1) \longrightarrow 0$$

yield a canonical line bundle morphism

$$\mathcal{O}_{X_2}(-1) \xhookrightarrow{(\pi_2^\star)\circ(\pi_1)_\star} \pi_2^\star\, \mathcal{O}_{X_1}(-1)$$

which admits precisely the hyperplane section $D_2 := P(T_{X_1/X}) \subset X_2 = P(V_1)$ as its zero divisor. Hence we find $\mathcal{O}_{X_2}(-1) = \pi_2^\star\, \mathcal{O}_{X_1}(-1) \otimes \mathcal{O}(-D_2)$ and

$$\mathcal{O}_{X_2}(-1, 1) \simeq \mathcal{O}(D_2)$$

is associated with an effective divisor in X_2.

3.3. Lemma. — *With respect to the projection $\pi_{2,0} : X_2 \to X$, the weighted line bundle $\mathcal{O}_{X_2}(a_1, a_2)$ is*

(a) *relatively effective if and only if $a_1 + a_2 \geq 0$ and $a_2 \geq 0$;*

(a′) *relatively big if and only if $a_1 + a_2 > 0$ and $a_2 > 0$;*

(b) *relatively nef if and only if $a_1 \geq 2a_2 \geq 0$;*

(b′) *relatively ample if and only if $a_1 > 2a_2 > 0$.*

Moreover, the following properties hold.

(c) *For $m = a_1 + a_2 \geq 0$, there is an injection*

$$\left(\pi_{2,0}\right)_\star\left(\mathcal{O}_{X_2}(a_1, a_2)\right) \hookrightarrow E_{2,m}T^\star_X,$$

and the injection is an isomorphism if $a_1 - 2a_2 \leq 0$.

(d) *Let $Z \subset X_2$ be an irreducible divisor such that $Z \neq D_2$. Then in* $\mathrm{Pic}(X_2)$ *we have*

$$Z \sim a_1u_1 + a_2u_2 + \pi^\star_{2,0}L, \qquad L \in \mathrm{Pic}(X),$$

where $a_1 \geq 2a_2 \geq 0$.

(e) *Let $F \in \mathrm{Pic}(X)$ be any divisor or line bundle. In $H^\bullet(X_2) = H^\bullet(X)[u_1, u_2]$, we have the intersection equalities*

$$u_1^4 = 0, \quad u_1^3u_2 = c_1^2 - c_2, \quad u_1^2u_2^2 = c_2, \quad u_1u_2^3 = c_1^2 - 3c_2, \quad u_2^4 = 5c_2 - c_1^2,$$
$$u_1^3 \cdot F = 0, \quad u_1^2u_2 \cdot F = -c_1 \cdot F, \quad u_1u_2^2 \cdot F = 0, \quad u_2^3 \cdot F = 0.$$

Proof. — The exact sequence defining V_1 shows that V_1 has splitting type

$$V_{1|F_1} = \mathcal{O}(2) \oplus \mathcal{O}(-1)$$

along the fibers $F_1 \simeq \mathbb{P}^1$ of $X_1 \to X$, since $T_{X_1/X|F_1} = \mathcal{O}(2)$. Hence the fibers F_2 of $X_2 \to X$ are Hirzebruch surfaces $P(\mathcal{O}(2) \oplus \mathcal{O}(-1)) \simeq P(\mathcal{O} \oplus \mathcal{O}(-3))$ and

$$\mathcal{O}_{X_2}(1)_{|F_2} = \mathcal{O}_{P(\mathcal{O}(2)\oplus\mathcal{O}(-1))}(1).$$

It is clear that the condition $a_2 \geq 0$ is necessary for $\mathcal{O}_{X_2}(a_1, a_2)_{|F_2}$ to be nef or to have non trivial sections. In that case, by taking the direct image by $\pi_{2,1} : X_2 \to X_1$, global sections of $\mathcal{O}_{X_2}(a_1, a_2)_{|F_2}$ can be viewed as global sections over $F_1 \simeq \mathbb{P}^1$ of

$$S^{a_2}\big(\mathcal{O}(-2) \oplus \mathcal{O}(1)\big) \otimes \mathcal{O}(a_1) = \bigoplus_{0 \leq j \leq a_2} \mathcal{O}(a_1 + a_2 - 3j).$$

The extreme terms of the summation are $\mathcal{O}(a_1 + a_2)$ and $\mathcal{O}(a_1 - 2a_2)$. Claims (a)–(b) follow easily from this, and (a)′, (b)′ are also clear since “being big” or “being ample” is an open condition in $\mathrm{Pic}(X_2)$.

(c) We have $\mathcal{O}_{X_2}(a_1, a_2) = \mathcal{O}_{X_2}(m) \otimes \mathcal{O}(-a_1D_2)$, thus $\mathcal{O}_{X_2}(a_1, a_2) \subset \mathcal{O}_{X_2}(m)$ if $a_1 \geq 0$ and $\mathcal{O}_{X_2}(a_1, a_2) \supset \mathcal{O}_{X_2}(m)$ if $a_1 \leq 0$. In the first case, it is immediately clear that we get an injection

$$(\pi_{2,0})_\star\mathcal{O}(a_1, a_2) \subset (\pi_{2,0})_\star\mathcal{O}_{X_2}(m) \xrightarrow{\simeq} E_{2,m}T^\star_X.$$

In the second case, we a priori have

$$(\pi_{2,0})_\star\mathcal{O}(a_1,a_2)\supset(\pi_{2,0})_\star\mathcal{O}_{X_2}(m)\ \xrightarrow{\simeq}\ E_{2,m}T^\star_X,$$

but the above splitting formula shows that $(\pi_{2,0})_\star\mathcal{O}(a_1,a_2)$ is already largest possible when $a_1-2a_2\le 0$ (which is the case e.g. if $(a_1,a_2)=(0,m)$). Hence we have an isomorphism in that case.

(d) If $a_1<2a_2$, we have an injection

$$\mathcal{O}_{X_2}(a_1+1,a_2-1)=\mathcal{O}_{X_2}(a_1,a_2)\otimes\mathcal{O}_{X_2}(-D_2)\subset\mathcal{O}_{X_2}(a_1,a_2)$$

which induces the same space of sections over each fibre F_2. This shows that every divisor Z in the linear system $|\mathcal{O}_{X_2}(a_1,a_2)\otimes\pi^\star_{2,0}L|$ contains D_2 as an irreducible component, and therefore cannot be irreducible unless $Z=D_2$.

(e) More general calculations are made in [Dem95]. Our formulas are easy consequences of the relations $u_1^2+c_1u_1+c_2=0$ and $u_2^2+c_1(V_1)u_2+c_2(V_1)=0$, where

$$c_1(V_1)=c_1+u_1,\qquad c_2(V_1)=c_2-u_1^2=2c_2+c_1u_1.\qquad\square$$

Under the condition $13c_1^2-9c_2>0$, Theorem 3.2 shows that the order 2 base locus B_2 is a proper algebraic subset of X_2. In order to improve Miyaoka's result 2.3, we are going to study the restriction of the line bundle $\mathcal{O}_{X_2}(1)$ to any 3-dimensional component of B_2 (if such components exist). We get the following

3.4. Proposition. — *Let X be a minimal surface of general type. If $c_1^2-\frac{9}{7}c_2>0$, then the restriction of $\mathcal{O}_{X_2}(1)$ to every irreducible 3-dimensional component Z of $B_2\subset X_2$ which projects onto X_1 ("horizontal component") and differs from D_2 is big.*

Proof. — Write

$$Z\sim a_1u_1+a_2u_2-\pi^\star_{2,0}F,\qquad (a_1,a_2)\in\mathbb{Z}^2,\quad a_1\ge 2a_2>0,$$

where F is some divisor in X. Our strategy is to show that $\mathcal{O}_{X_2}(2,1)_{|Z}$ is big. By Lemma 3.3 (e), we find

$$(\dagger\dagger\dagger)\qquad (2u_1+u_2)^3\cdot Z=(a_1+a_2)(13\,c_1^2-9\,c_2)+12\,c_1\cdot F.$$

Now, the multiplication morphism by the canonical section of $\mathcal{O}(Z)$ defines a sheaf injection

$$\mathcal{O}(\pi^\star_{2,0}F)\hookrightarrow\mathcal{O}_{X_2}(a_1,a_2).$$

By taking direct images onto X, $\mathcal{O}(F)$ can thus be viewed as a subsheaf of

$$(\pi_{2,0})_\star\big(\mathcal{O}_{X_2}(a_1,a_2)\big)\subset E_{2,m}T^\star_X$$

where $m=a_1+a_2$. Looking at the filtration of $E_{2,m}T^\star_X$, we infer that there is a non trivial morphism

$$\mathcal{O}(F)\longrightarrow S^{m-3j}T^\star_X\otimes K_X^{\otimes j}$$

for some $j \le \frac{m}{3}$. As in § 2, the semistability inequality implies

$$F \cdot K_X \le \Big(\frac{m-3j}{2} + j\Big) K_X^2 \le \frac{m}{2} c_1^2, \qquad \text{thus} \qquad -c_1 \cdot F \le \frac{m}{2} c_1^2.$$

Formula (†††) combined with the assumption $7\,c_1^2 - 9\,c_2 > 0$ implies

$$(2u_1 + u_2)^3 \cdot Z \ge m(7\,c_1^2 - 9\,c_2) > 0.$$

The latter inequality still holds if we replace $\mathcal{O}_{X_2}(2,1)$ by $\mathcal{O}_{X_2}(2+\varepsilon,1)$ with a fixed sufficiently small positive rational number ε. By Riemann-Roch, either

$$h^0(Z, \mathcal{O}_{X_2}((2+\varepsilon)p, p)_{|Z}) \quad \text{or} \quad h^2(Z, \mathcal{O}_{X_2}((2+\varepsilon)p, p)_{|Z})$$

grows fast as p goes to infinity. We want to exclude the second possibility. For this, we look at the exact sequence

$$0 \to \mathcal{O}(-Z) \otimes \mathcal{O}_{X_2}((2+\varepsilon)p, p) \to \mathcal{O}_{X_2}((2+\varepsilon)p, p) \to \mathcal{O}_Z \otimes \mathcal{O}_{X_2}((2+\varepsilon)p, p) \to 0$$

and take the direct images to X by the Leray spectral sequence of the fibration $X_2 \to X$. As $\mathcal{O}_{X_2}(2+\varepsilon,1)$ is relatively ample, the higher R^q sheaves are zero and we see immediately that

$$h^2(Z, \mathcal{O}_{X_2}((2+\varepsilon)p, p)_{|Z}) \le h^2(X, (\pi_{2,0})_\star \mathcal{O}_{X_2}((2+\varepsilon)p, p)) \le h^2(X, E_{2,\nu} T_X^\star)$$

with $\nu = (3+\varepsilon)p$. By Bogomolov's vanishing theorem, the latter group is zero. Thus, we obtain that $\mathcal{O}_{X_2}(2+\varepsilon,1)_{|Z}$ is big, and this implies that $\mathcal{O}_{X_2}(1)_{|Z}$ is also big because we have a sheaf injection

$$\mathcal{O}_{X_2}(2+\varepsilon,1) = \mathcal{O}_{X_2}(3+\varepsilon) \otimes \mathcal{O}(-(2+\varepsilon)D_2) \hookrightarrow \mathcal{O}_{X_2}(3+\varepsilon)$$

(if necessary, pass to suitable tensor multiples to avoid denominators). □

3.5. Corollary. — *Let X be a surface of general type such that $c_1^2 - \frac{9}{7} c_2 > 0$. Then the irreducible components of the Green-Griffiths locus $B_2 \subset X_2$ are of dimension 2 at most, except for the divisor $D_2 \subset X_2$.*

This corollary is not really convincing, since we already have sections in $H^0(X, S^m T_X^\star \otimes \mathcal{O}(-A))$ under the weaker condition $c_1^2 - c_2 > 0$ (a condition which is anyhow too restrictive to encompass the case of surfaces in $\mathbb{P}^3$). Fortunately, under the additional assumption that the surface has Picard group $\mathbb{Z}$, one can get a more precise inequality than the stability inequality, and that inequality turns out to be sufficient to treat the case of generic surfaces of sufficiently high degree in $\mathbb{P}^3$.

4. Proof of the Main Theorem

Since X is of general type and $\mathrm{Pic}(X) = \mathbb{Z}$, the canonical bundle K_X is ample, and we have $c_1^2 > 0$. Condition (b) is stronger than the condition $13\,c_1^2 - 9\,c_2 > 0$, thus by Theorem 3.2 there are non trivial sections in

$$H^0(X_2, \mathcal{O}_{X_2}(m) \otimes \mathcal{O}(-\pi_{2,0}^\star K_X)) \simeq H^0(X, E_{2,m} T_X^\star \otimes \mathcal{O}(-K_X))$$

for m large enough. Pick such a section σ and let Z_σ be its zero divisor,

$$Z_\sigma = m\,u_2 - \pi_{2,0}^\star K_X \qquad \text{in } \mathrm{Pic}(X).$$

Let $Z_\sigma = \sum p_j Z_j$ be the decomposition of Z_σ in irreducible components. From the equality $\mathrm{Pic}(X_2) = \mathrm{Pic}(X) \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2$ and the assumption $\mathrm{Pic}(X) \simeq \mathbb{Z}$, we find

$$Z_j \sim a_{1,j}u_1 + a_{2,j}u_2 - t_j\pi_{2,0}^\star K_X,$$

for suitable integers $a_{1,j},\ a_{2,j} \in \mathbb{Z}$ and rational numbers $t_j \in \mathbb{Q}$. By Lemma 3.3, we either have $Z_j = D_2$ and $(a_{1,j}, a_{2,j}, t_j) = (-1, 1, 0)$, or $Z_j \neq D_2$ and $a_{1,j} \geq 2a_{2,j} \geq 0$. In the latter case either $m_j := a_{1,j}+a_{2,j} > 0$ or $(a_{1,j}, a_{2,j}) = (0, 0)$ and $t_j < 0$. As $\sum p_j t_j = 1$, it follows that there exists an index j such that $t_j > 0$, and thus $a_{1,j} \geq 2a_{2,j} \geq 0$, $m_j = a_{1,j} + a_{2,j} > 0$. Therefore, we get a section

$$\sigma_j \in H^0\big(X_2, \mathcal{O}_{X_2}(m_j) \otimes \pi_{2,0}^\star \mathcal{O}(-t_j K_X)\big)$$

whose divisor is $Z_j + a_{1,j}D_2$. From this, we infer the following easy lemma.

4.1. Lemma. — *Let m the smallest positive integer such that there is a nonzero section*

$$\sigma \in H^0\big(X_2, \mathcal{O}_{X_2}(m) \otimes \pi_{2,0}^\star \mathcal{O}(-tK_X)\big),$$

with $t \in \mathbb{Q}$, $t > 0$. If there are several possible values of t for that integer m, take the largest possible value t. Then the zero divisor of σ has the form $Z_\sigma = Z + a_1 D_2$ where Z is irreducible,

$$Z \sim a_1u_1 + a_2u_2 - t\,\pi_{2,0}^\star K_X, \qquad a_1 \geq 2a_2 > 0, \quad a_1 + a_2 = m,$$

and we have either

$$m = 3, \quad 0 < t \leq 3/4, \quad \text{or} \quad m \geq 4, \quad t \leq 1.$$

Indeed, by repeating the above arguments, it is clear that Z_σ must be of the form $Z + a_1D_2$ described in the lemma, otherwise Z_σ could be split in smaller components Z_j such that one of them at least satisfies $m_j < m$ or $m_j = m$ and $t_j > t$, contradiction. Observe that σ can be considered as a global holomorphic section of the bundle $E_{2,m}T_X^\star \otimes \mathcal{O}(-tK_X)$. The assumption (c) $H^0(X, S^mT_X^\star) = 0$ implies $a_2 > 0$, since $(\pi_{2,0})_\star\big(\mathcal{O}_{X_2}(a_1, 0) \otimes \pi_{2,0}^\star\mathcal{O}(-tK_X)\big) = S^{a_1}T_X^\star \otimes \mathcal{O}(-tK_X)$ has no non trivial sections. In particular $m = a_1 + a_2 \geq 3$. If $m = 3$, the inequality $t \leq 3/4$ follows from assumption (d). Now, assume $m \geq 4$ and $t > 1$. By the filtration of $E_{2,m}T_X^\star$, we have a short exact sequence

$$0 \to S^mT_X^\star \to E_{2,m}T_X^\star \to E_{2,m-3}T_X^\star \otimes \mathcal{O}(K_X) \to 0.$$

Multiply all terms by $\mathcal{O}(-tK_X)$ and consider the associated sequence in cohomology. By assumption (c) again, the first H^0 group vanishes and we get an injection

$$H^0\big(X, E_{2,m}T_X^\star \otimes \mathcal{O}(-tK_X)\big) \hookrightarrow H^0\big(X, E_{2,m-3}T_X^\star \otimes \mathcal{O}(-(t-1)K_X)\big).$$

As $t - 1 > 0$, this contradicts the minimality of m and the lemma is proved. □

Take σ and Z as in Lemma 4.1. Formula (†††) of section 3 gives

$$(2u_1 + u_2)^3 \cdot Z = m(13\,c_1^2 - 9\,c_2) - 12t\,c_1^2,$$

so we obtain

$$(2u_1 + u_2)^3 \cdot Z \geq 3(10c_1^2 - 9c_2) \qquad \text{for } m = 3,\ \ t \leq 3/4,$$
$$(2u_1 + u_2)^3 \cdot Z \geq 4(10c_1^2 - 9c_2) \qquad \text{for } m \geq 4,\ \ t \leq 1.$$

As in the proof of Proposition 3.4, we conclude that the restriction $\mathcal{O}_{X_2}(1)_{|Z}$ is big. Consequently, by Theorem 1.2 (or rather, by the proof of Theorem 1.2, see [Dem95]), every non constant entire curve $f : \mathbb{C} \to X$ is such that $f_{[2]}(\mathbb{C})$ is contained in the base locus of $\mathcal{O}_{X_2}(l) \otimes \pi_{2,0}^\star \mathcal{O}(-A)_{|Z}$ for l large. This base locus is at most 2-dimensional, and projects onto a proper algebraic subvariety Y of X_1. Therefore $f_{[1]}(\mathbb{C})$ is contained in Y, and the Main Theorem is proved. □

5. Non existence of global 2-jet differentials of degree 3

This section is devoted to proving properties (c) and (d) in the Proposition of the introduction. Property (c) is in fact well-known (see e.g. Sakai [Sa78]). We just reprove it briefly for the reader's convenience.

5.1. Lemma. — *Let X be a non singular surface in $\mathbb{P}^3$ and m a positive integer. Then*

(a) $H^0(X, S^m T_X^\star \otimes \mathcal{O}(k)) = 0$ *for all $k \in \mathbb{Z}$, $k \leq m$.*

(b) $H^0(X, S^m T_X^\star \otimes \mathcal{O}(k)) \simeq H^0(\mathbb{P}^3, S^m T_{\mathbb{P}^3}^\star \otimes \mathcal{O}(k))$ *for all $k \leq m + d - 1$.*

Proof. — The Euler exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus 4} \longrightarrow T_{\mathbb{P}^3} \longrightarrow 0$$

gives an exact sequence

$$0 \to S^m T_{\mathbb{P}^3}^\star \otimes \mathcal{O}(k) \to S^m(\mathcal{O}^{\oplus 4}) \otimes \mathcal{O}(k-m) \to S^{m-1}(\mathcal{O}^{\oplus 4}) \otimes \mathcal{O}(k-m+1) \to 0.$$

Since $H^q(\mathbb{P}^3, \mathcal{O}(p)) = 0$ for all $q \leq 2$ and all $p < 0$, we easily conclude that $H^q(\mathbb{P}^3, S^m T_{\mathbb{P}^3}^\star \otimes \mathcal{O}(k)) = 0$ for all $q \leq 2$ and $k \leq m$. By the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \to 0,$$

we now find $H^q(X, S^m T_{\mathbb{P}^3|X}^\star \otimes \mathcal{O}(k)) = 0$ for $q \leq 1$ and $k \leq m$, and that $H^0(X, S^m T_{\mathbb{P}^3|X}^\star \otimes \mathcal{O}(k) \simeq H^0(\mathbb{P}^3, S^m T_{\mathbb{P}^3}^\star \otimes \mathcal{O}(k))$ for $k \leq m + d$. Finally, by taking symmetric powers in the dual sequence of

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^3|X} \longrightarrow \mathcal{O}_X(d) \longrightarrow 0,$$

we find a sequence

$$0 \longrightarrow S^{m-1} T_{\mathbb{P}^3|X}^\star \otimes \mathcal{O}_X(-d) \longrightarrow S^m T_{\mathbb{P}^3|X}^\star \longrightarrow S^m T_X^\star \longrightarrow 0,$$

from which it readily follows that $H^0(X, S^m T_X^\star \otimes \mathcal{O}(k)) = 0$ for $k \leq m$, and $H^0(X, S^m T_X^\star \otimes \mathcal{O}(k)) \simeq H^0(\mathbb{P}^3, S^m T_{\mathbb{P}^3}^\star \otimes \mathcal{O}(k))$ for $k \leq m + d - 1$. □

We now prove the generic nonexistence of 2-jet differentials of degree 3 with values in a sufficiently negative line bundle.

5.2. Proposition. — *Let X be a generic surface of degree d in $\mathbb{P}^3$. Then*

$$H^0(X, E_{2,3}T^\star_X \otimes \mathcal{O}(-tK_X)) = 0 \qquad \textit{for all } t \in \mathbb{Q}^\star_+,\ \ t > \frac{1}{2}.$$

[Here $K_X = \mathcal{O}_X(d-4)$, $d \geq 5$, and $t(d-4)$ is supposed to be a positive integer].

Proof. — Consider the exact sequences

$$0 \longrightarrow S^3T^\star_X \longrightarrow E_{2,3}T^\star_X \longrightarrow K_X \longrightarrow 0,$$

$$0 \longrightarrow S^3T^\star_X \otimes \mathcal{O}(\,-\,tK_X) \longrightarrow E_{2,3}T^\star_X \otimes \mathcal{O}(-tK_X) \longrightarrow \mathcal{O}((1-t)K_X) \longrightarrow 0.$$

As $H^0(X, S^3T^\star_X \otimes \mathcal{O}(-tK_X)) = 0$, to any nonzero section

$$\sigma \in H^0(X, E_{2,3}T^\star_X \otimes \mathcal{O}(-tK_X))$$

corresponds a non zero section $\beta \in H^0(X, \mathcal{O}((1-t)K_X))$. In particular, we must have $t \leq 1$. Let us introduce the $\mathcal{O}(-K_X)$-valued meromorphic differential operator

$$W(f) = \beta(f)^{-1}\sigma(f).$$

We can also consider W as a holomorphic differential operator with values in $\mathcal{O}(B - K_X)$, where $B = \mathrm{div}(\beta)$. It turns out that W is a Wronskian operator of the form

$$W(f) = f' \wedge f''_\nabla, \quad f'' = \nabla_{f'}f',$$

where ∇ is a suitable meromorphic connection on T_X with pole divisor $\leq B$. The relevant type of connections we need are the "meromorphic partial projective connections" introduced in [EG96] and [DEG97]. By definition, a meromorphic connection is an operator

$$\nabla_w v = \sum_{1\leq i,k\leq n}\Big(w_i\frac{\partial v_k}{\partial z_i} + \sum_{1\leq j\leq n}\Gamma^k_{ij}w_iv_j\Big)\frac{\partial}{\partial z_k} = d_wv + \Gamma\cdot(w,v),$$

whose Christoffel symbols $\Gamma = (\Gamma^k_{ij})_{1\leq i,j,k\leq n}$ are meromorphic in any complex coordinates system $(z_1,\ldots,z_n)$. A *meromorphic partial projective connection* is a section of the quotient sheaf of the sheaf of meromorphic connections modulo meromorphic zero order operators of the form $\alpha(w)v + \beta(v)w$. The Christoffel symbols are thus supposed to be determined only up to terms of the form

$$\widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij} = \alpha_i\delta_{jk} + \beta_j\delta_{ik}.$$

Adding such terms to ∇ replaces f''_∇ with $f''_\nabla + \alpha(f')f' + \beta(f')f'$, and thus does not change the corresponding Wronskian operator. In dimension 2, a meromorphic connection depends on 8 Christoffel symbols, but a partial projective meromorphic connection depends only on 4 Christoffel symbols. A straightforward computation in coordinates shows that

$$\begin{aligned} f'\wedge f''_\nabla = \Big(&(f'_1f''_2 - f''_1f'_2) - \Gamma^2_{1,1}f'^3_1 + \Gamma^1_{2,2}f'^3_2 \\ &+ (\Gamma^1_{1,1} - \Gamma^2_{1,2} - \Gamma^2_{2,1})f'^2_1f'_2 - (\Gamma^2_{2,2} - \Gamma^1_{1,2} - \Gamma^1_{2,1})f'_1f'^2_2\Big)\frac{\partial}{\partial z_1}\wedge\frac{\partial}{\partial z_2}. \end{aligned}$$

If B is the pole divisor of the coefficients Γ_{ij}^k, then the Wronskian operator $W(f) = f' \wedge f''_\nabla$ takes values in $\mathcal{O}(B) \otimes \mathcal{O}(-K_X)$, thus

$$W \in H^0(X, E_{2,3}T_X^\star \otimes \mathcal{O}(B - K_X)).$$

Conversely, by what we have seen, any such section W defines a unique meromorphic partial projective connection ∇ with pole divisor $\leq B$. From this, we infer

5.3. Lemma. — *Consider nonzero sections*

$$\sigma_i \in H^0(X, E_{2,3}T_X^\star \otimes \mathcal{O}(-t_i K_X)), \qquad i = 1, 2.$$

If $t_1 + t_2 \geq 1$, the associated Wronskian operators $\beta_1^{-1}\sigma_1$, $\beta_2^{-1}\sigma_2$ are equal. In other words, σ_1, σ_2 induce the same meromorphic partial projective connection ∇.

Proof. — Consider $\sigma = \beta_1\sigma_2 - \beta_2\sigma_1$. This is a section in

$$H^0(X, E_{2,3}T_X^\star \otimes \mathcal{O}((1 - t_1 - t_2)K_X))$$

and its image in $H^0(X, \mathcal{O}((2 - t_1 - t_2)K_X))$ is $\beta_1\beta_2 - \beta_2\beta_1 = 0$. Therefore σ lies in $H^0(X, S^3T_X^\star \otimes \mathcal{O}((1 - t_1 - t_2)K_X))$, which vanishes if $t_1 + t_2 \geq 1$. □

Lemma 5.3 implies in particular that there is at most one meromorphic partial projective connection ∇ with pole divisor $B \leq \frac{1}{2}K_X$. If we could produce one example where B is close to $\frac{1}{2}K_X$, then we conclude that B cannot be much smaller than $\frac{1}{2}K_X$. Such an example is easily constructed by means of Nadel's technique [Na89] (see also [EG96], [DEG97] and [SY97]). Assume that X_λ is a smooth member of a linear system of surfaces

$$X_\lambda = \big\{\lambda_0 s_0(z) + \lambda_1 s_1(z) + \lambda_2 s_2(z) + \lambda_3 s_3(z) = 0\big\}$$

where s_0, s_1, s_2, $s_3 \in \mathbb{C}[z_0, z_1, z_2, z_3]$ are homogeneous polynomials of degree d. We solve the linear system

$$\sum_{0\leq k\leq 3} \widetilde{\Gamma}_{ij}^k \frac{\partial s_\ell}{\partial z_k} = \frac{\partial^2 s_\ell}{\partial z_i \partial z_j}, \qquad 0 \leq i, j, \ell \leq 3,$$

and get in this way a homogeneous meromorphic connection of degree -1 on $\mathbb{C}^4$. One can check that this connection descends to a partial projective meromorphic connection ∇ on $\mathbb{P}^3$ such that X_λ is totally geodesic (see [DEG97]). Let us consider the specific example

$$X_a = \big\{z_0^d + z_1^d + z_2^d + z_3^d + a\, z_0^{k_0} z_1^{k_1} z_2^{k_2} z_3^{k_3} = 0\big\},$$

where k_0, k_1, k_2, $k_3 \geq 0$ are integers such that $\sum k_i = d$. We take in this case

$$s_0 = z_0^{k_0}(z_0^{d-k_0} + a\, z_1^{k_1} z_2^{k_2} z_3^{k_3}), \qquad s_i = z_i^d, \quad i = 1, 2, 3.$$

A short computation shows that X_a is non singular if and only if $a^d \neq (-d)^d \prod k_i^{-k_i}$ and that the pole divisor of the connection ∇ is given by

$$B = \big\{z_0 z_1 z_2 z_3 (d\, z_0^{k_1+k_2+k_3} + a k_0 z_1^{k_1} z_2^{k_2} z_3^{k_3}) = 0\big\}.$$

In particular, the ratio

$$\frac{B}{K_{X_a}} = \frac{4+k_1+k_2+k_3}{d-4}$$

can be taken freely in the range $[4/(d-4)\,,\,(d+4)/(d-4)]$. We adjust it to be equal to $1-t_1 := p/(d-4)$ where $p=[(d-3)/2]$ (the smallest integer such that $1-t_1 \geq 1/2$). This is possible if $d \geq 11$. We then see that X_a has no meromorphic partial connection with pole order

$$\frac{B}{K_{X_a}} \leq 1-t_2 := t_1 = 1 - \frac{p}{(d-4)},$$

except possibly ∇ if $t_1 = t_2 = 1/2$. Therefore

$$H^0(X_a, E_{2,3}T^\star_{X_a} \otimes \mathcal{O}(-tK_{X_a})) = 0 \qquad \text{for} \quad t > 1/2.$$

By the Zariski semicontinuity of cohomology, the above vanishing property holds true for a generic surface X. Proposition 5.2 is proved. □

6. McQuillan's work on algebraic foliations

Recently, using Miyaoka's semi-positivity result for cotangent bundles of non uniruled projective varieties [Mi87] and a dynamic diophantine approximation, McQuillan [McQ97] derived strong Nevanlinna Second Main Theorems for holomorphic mappings $f:\mathbb{C}\to X$ tangent to the leaves of an algebraic foliation.

6.1. Theorem (McQuillan). — *Every parabolic leaf of an algebraic (multi-)foliation on a surface X of general type is algebraically degenerate.*

The assumption $c_1^2 > c_2$ guarantees the existence of an algebraic multi-foliation such that every $f:\mathbb{C}\to X$ is contained in one of the leaves. Thus McQuillan's theorem implies

6.2. Corollary (McQuillan). — *If X is a surface of general type with $c_1^2 > c_2$, then all entire curves of X are algebraically degenerate.*

It turns out that McQuillan's proof is rather involved and goes far beyond the methods presented here (see also M. Brunella [Bru98] for an enlightening presentation of McQuillan's main ideas). Since we do not need the full force of McQuillan's results, we present here special cases of our 1-jet and 2-jet techniques, which should in principle be quite sufficient to deal with our application (modulo a formal computational check which will not be handled here).

6.3. Proposition. — *Let X be a minimal surface of general type, equipped with an algebraic multi-foliation $\mathcal{F} \subset S^m T^\star_X$. Assume that*

$$m(c_1^2 - c_2) + c_1 \cdot c_1(\mathcal{F}) > 0.$$

Then there is a curve Γ in X such that all parabolic leaves of $\mathcal{F}$ are contained in Γ.

Proof. — Notice that every rank 1 torsion free sheaf on a surface is locally free. The inclusion morphism of $\mathcal{F}$ in $S^m T_X^\star$, viewed as a section of $S^m T_X^\star \otimes \mathcal{F}^{-1}$, defines a section of $\mathcal{O}_{X_1}(m) \otimes \pi^\star \mathcal{F}^{-1}$ whose zero divisor $Z \subset X_1 = P(T_X)$ is precisely the divisor associated with the foliation (as explained in the introduction). Therefore $Z = mu - \pi^\star \mathcal{F}$ in $\mathrm{Pic}(X_1)$, and our calculations of section 2 (see (†) and (††)) imply that $\mathcal{O}_{X_1}(1)_{|Z}$ is big as soon as

$$(u_{|Z})^2 = m(c_1^2 - c_2) + c_1 \cdot \mathcal{F} > 0, \qquad (u_{|Z}) \cdot (-c_1) = mc_1^2 + c_1 \cdot \mathcal{F} > 0.$$

However, as X is minimal, we have $c_2 \geq 0$, and Proposition 6.3 follows. □

Again, the above 1-jet result is not sufficient to cover the case of surfaces in $\mathbb{P}^3$, so we have to deal with a 2-jet version instead. Let $Z \subset X_1 = P(T_X)$ be the divisor associated with the given foliation $\mathcal{F}$, and $\sigma \in H^0(X_1, \mathcal{O}_{X_1}(m) \otimes \pi^\star \mathcal{F}^{-1}_{|Z})$ the corresponding section. We let $\mathcal{T}_Z$ be the tangent sheaf to Z, i.e. the rank 2 sheaf $\mathcal{T}_Z$ defined by the exact sequence

$$0 \longrightarrow \mathcal{T}_Z \longrightarrow T_{X_1|Z} \xrightarrow{d\sigma} \mathcal{O}_{X_1}(m) \otimes \pi^\star \mathcal{F}^{-1}_{|Z} \longrightarrow 0.$$

If we define $\mathcal{S} = \mathcal{T}_Z \cap \mathcal{O}(V_1)$ sheaf-theoretically, we find an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow V_{1|Z} \xrightarrow{d\sigma} \mathcal{O}_{X_1}(m)_{|Z} \otimes \pi^\star \mathcal{F}^{-1}_{|Z},$$

where $\mathcal{S}$ is an invertible subsheaf, and a dual exact sequence

$$0 \longrightarrow \mathcal{O}_{X_1}(-m)_{|Z} \otimes \pi^\star \mathcal{F}_{|Z} \longrightarrow V^\star_{1|Z} \longrightarrow \mathcal{S}^\star.$$

We can then lift Z into a surface $\widetilde{Z} \subset X_2$, in such a way that the projection map $\pi_{2,1} : \widetilde{Z} \to Z$ is a modification; at a generic point $x \in Z$, the point of $\widetilde{Z}$ lying above x is taken to be $(x, [\mathcal{S}_x]) \in X_2$. Our goal is to compute the cohomology class of the 2-cycle $\widetilde{Z}$ in $H^\bullet(X_2)$. One of the difficulties is that the cokernel of the map

$$d\sigma_{|V_{1|Z}} : V_{1|Z} \to \mathcal{O}_{X_1}(m)_{|Z} \otimes \pi^\star \mathcal{F}^{-1}_{|Z}$$

may have torsion along a 1-cycle $G_1 \subset Z$, i.e., there is a factorization

$$d\sigma_{|V_{1|Z}} : V_{1|Z} \to \mathcal{O}_{X_1}(m)_{|Z} \otimes \pi^\star \mathcal{F}^{-1}_{|Z} \otimes \mathcal{O}_Z(-G_1) \to \mathcal{O}_{X_1}(m)_{|Z} \otimes \pi^\star \mathcal{F}^{-1}_{|Z}$$

such that the cokernel of the first arrow has 0-dimensional support (of course, G_1 need not be reduced). If the foliation is generic, however, the cokernel of $d\sigma_{|V_{1|Z}}$ will have no torsion in codimension 1, and $d\sigma$ then induces a section of

$$\mathcal{O}_{X_2}(1) \otimes \pi^\star_{2,1}\mathcal{O}_{X_1}(m) \otimes \pi^\star_{2,0}\mathcal{F}^{-1}_{|\pi^{-1}_{2,1}(Z)} \sim (u_2 + m\,u_1 - \mathcal{F})_{|\pi^{-1}_{2,1}(Z)}$$

whose zero locus is $\widetilde{Z}$. As $Z \sim mu_1 - \mathcal{F}$, the cohomology class of $\widetilde{Z}$ in $H^4(X_2)$ is given by

$$\begin{aligned}\{\widetilde{Z}\} &= (mu_1 - \mathcal{F}) \cdot (u_2 + mu_1 - \mathcal{F}) \\ &= m^2 u_1^2 + m\,u_1 \cdot u_2 - 2m\,u_1 \cdot \mathcal{F} - u_2 \cdot \mathcal{F} + \mathcal{F}^2.\end{aligned}$$

A short Chern class computation yields

$$(2u_1 + u_2)^2 \cdot \widetilde{Z} = m^2(4c_1^2 - 3c_2) + m(5c_1^2 - 3c_2) + (8m + 4)c_1 \cdot \mathcal{F} + 3\,\mathcal{F}^2.$$

If the 1-cycle G_1 is non zero, our numerical formula for $\widetilde{Z}$ becomes

$$\{\widetilde{Z}\} = (mu_1 - \mathcal{F}) \cdot (u_2 + mu_1 - \mathcal{F}) - \pi_{2,1}^\star\{G_1\}.$$

On the other hand, we find

$$(2u_1 + u_2)^2 \cdot \pi_{2,1}^\star\{G_1\} = (3u_1 - c_1) \cdot G_1.$$

The general formula for $(2u_1 + u_2)^2 \cdot \widetilde{Z}$ is thus

$$(2u_1+u_2)^2 \cdot \widetilde{Z} = m^2(4c_1^2 - 3c_2) + m(5c_1^2 - 3c_2) + (8m+4)c_1 \cdot \mathcal{F} + 3\,\mathcal{F}^2 - (3u_1 - c_1) \cdot G_1.$$

By using obvious exact sequences, $H^2(\widetilde{Z}, m\mathcal{O}_{X_2}(2,1)_{|\widetilde{Z}})$ is a quotient of

$$H^2(\pi_{2,1}^{-1}(Z), m\mathcal{O}_{X_2}(2,1)_{|\pi_{2,1}^{-1}(Z)}),$$

which is itself controlled by $H^2(X_2, m\mathcal{O}_{X_2}(2,1))$, $H^3(X_2, m\mathcal{O}_{X_2}(2,1) \otimes \mathcal{O}(-Z))$. A direct image argument shows that the latter groups are controlled by groups of the form $H^2(X, E_{2,3m}T_X^\star \otimes L)$, with suitable line bundles L. As in the proof of Theorem 3.4 (possibly after changing $\mathcal{O}_{X_2}(2,1)$ into $\mathcal{O}_{X_2}(2+\varepsilon,1)$ in the above arguments), one can check that the latter H^2 groups vanish. The positivity of $(2u_1 + u_2)^2 \cdot \widetilde{Z}$ thus implies that $\mathcal{O}_{X_2}(2,1)_{|\widetilde{Z}}$ is big, and therefore all parabolic leaves of the (multi)-foliation $\mathcal{F}$ are algebraically degenerate. We thus obtain:

6.4. Proposition. — *Let X be a surface of general type, equipped with a multi-foliation $\mathcal{F} \subset S^mT_X^\star$, and let $\sigma \in H^0(X_1, \mathcal{O}_{X_1}(m) \otimes \pi_{1,0}^\star\mathcal{F})$ be the associated canonical section. Finally, let G_1 be the divisorial part of the subscheme defined by* $\operatorname{coker}(d\sigma_{|V_{1|Z}})$. *Then, under the assumption*

$$m^2(4c_1^2 - 3c_2) + m(5c_1^2 - 3c_2) + (8m+4)c_1 \cdot \mathcal{F} + 3\,\mathcal{F}^2 - (3u_1 - c_1) \cdot G_1 > 0,$$

all parabolic leaves of $\mathcal{F}$ are algebraically degenerate.

6.5. Corollary. — *Let $X \subset \mathbb{P}^3$ be a surface of degree $d \geq 18$ with* $\operatorname{Pic}(X) = \mathbb{Z}$, *and let $\mathcal{F} \subset S^mT_X^\star$ be a generic multi-foliation, in the sense that the 1-cycle G_1 defined above is zero. Then all parabolic leaves of $\mathcal{F}$ are algebraically degenerate and contained in a fixed 1-dimensional algebraic subset $Y \subset X$.*

Proof. — Note that the line subbundle $\mathcal{F} \subset S^mT_X^\star$ must be negative (otherwise $\mathcal{F}$ would yield a non trivial section of $S^mT_X^\star$), hence $c_1 \cdot \mathcal{F} > 0$, $\mathcal{F}^2 > 0$, and likewise we have

$$4\,c_1^2 - 3\,c_2 = d(d^2 - 20\,d + 46) > 0, \qquad 5\,c_1^2 - 3\,c_2 = d(2\,d^2 - 28\,d + 62) > 0$$

for $d \geq 18$. Thus, we get the conclusion if $G_1 = 0$ (but a rather large additional contribution of G_1 would still be allowable; we do not know how much of it can actually occur). □

7. Proof of the Corollaries

Proof of Corollary 1.

Recall that by the Noether-Lefschetz theorem, a very generic surface X in $\mathbb{P}^3$ is such that $\mathrm{Pic}(X) = \mathbb{Z}$, with generator $\mathcal{O}_X(1)$. On the other hand, improving a result of H. Clemens ([Cl86] and [CKM88]), G. Xu [Xu94] has shown that the genus of every curve contained in a very generic surface of degree $d \geq 5$ satisfies the bound $g \geq d(d-3)/2-2$ (this bound is sharp). In particular, such a surface does not contain rational or elliptic curves. Now take a very generic surface in $\mathbb{P}^3$ of degree $d \geq 42$, which has no rational or elliptic curves, and such that the conclusions of the Main Theorem apply, i.e. every nonconstant entire curve $f : \mathbb{C} \to X$ is such that $f_{[1]}(\mathbb{C})$ lies in the leaf of an algebraic foliation on a surface $Z \subset X_1$. Then, by McQuillan's result, f must be algebraically degenerate. The closure $\Gamma = \overline{f(\mathbb{C})}$ would then be an algebraic curve of genus 0 or 1, contradiction.

7.1. Remark. — If one would like to avoid any appeal to McQuillan's deep result, it would remain to check on an example that the multi-foliation defined by Z satisfies the sufficient condition described in Proposition 6.4. This might require for instance a computer check, and is likely to hold without much restriction.

7.2. Remark. — It is extremely likely that Corollary 1 holds true for generic surfaces and not only for very generic ones. In fact, since we have a smooth family of non singular surfaces $\mathcal{X} \to M_d \subset \mathbb{P}^{N_d}$ in each degree d, the Riemann-Roch calculations of sections 3, 4 hold true in the relative situation, and thus produce an algebraic family of divisors $\mathcal{Z}_t \subset (\mathcal{X}_t)_2$ on some Zariski open subset $M'_d \subset M$, $t \in M'_d$. By shrinking M'_d, we can assume that all $\mathcal{Z}_t$ are irreducible, and that we have a flat family $\mathcal{Z} \to M'_d$. By relative Riemann-Roch again, we get a family of divisors $\mathcal{Y}_t \subset \mathcal{Z}_t$, and thus a family of foliations $\mathcal{F}_t$ on the 1-jet bundles $(\mathcal{X}_t)_1$. Finally, if Proposition 6.4 can be applied to these foliations (and we strongly expect that this is indeed the case), we get an algebraic family of curves $\Gamma_t \subset \mathcal{X}_t$ such that all holomorphic maps $f : \mathbb{C} \to \mathcal{X}_t$ are contained in Γ_t. As the degree is bounded, a trivial Hilbert scheme argument implies that the set of t's for which one of the components of Γ_t is rational or elliptic is closed algebraic and nowhere dense. Our claim follows. □

Proof of Corollary 2.

Let $C = \sigma^{-1}(0)$ be a non singular curve of degree d in $\mathbb{P}^2$. Consider the cyclic covering $X_C = \{z_3^d = \sigma(z_0, z_1, z_2)\} \to \mathbb{P}^2$ of degree d, ramified along C. Then X_C is a non singular surface in $\mathbb{P}^3$, and as $\mathbb{C}$ is simply connected, every holomorphic map $f : \mathbb{C} \to \mathbb{P}^2 \smallsetminus C$ can be lifted to X_C. The first step is to check that $\mathrm{Pic}(X_C) = \mathbb{Z}$ for generic C. In fact, one has the following well-known result of Hartshorne [Ha75] (we think Ch. Peters for pointing out the reference): if W is a 3-dimensional projective manifold, and $(X_t)_{t\in\mathbb{P}^1}$ a Lefschetz pencil of surfaces $X_t \subset W$ (i.e. a 1-dimensional linear system of hypersurfaces whose general member is smooth, and whose special members have at most one conical double point) such that $H^0(X_t, K_{X_t}) \neq 0$ for generic t, then $\mathrm{Pic}(X_t) \simeq \mathrm{Pic}(W)$ for generic t. In particular, as every Lefschetz pencil of curves in $\mathbb{P}^2$ can be lifted to a Lefschetz pencil of surfaces in $\mathbb{P}^3$, we conclude that if C is sufficiently generic we indeed have $\mathrm{Pic}(X_C) = \mathbb{Z}$. Similarly, the non existence theorem proved in section 6 holds true for at least one X_C, for example

$$C = \{z_0^d + z_1^d + z_2^d + a\, z_0^{k_0} z_1^{k_1} z_2^{k_2} = 0\},$$
$$X_C = \{z_0^d + z_1^d + z_2^d + z_3^d + a\, z_0^{k_0} z_1^{k_1} z_2^{k_2} = 0\}.$$

We then conclude as above that X_C is hyperbolic for generic C. This implies in particular that $\mathbb{P}^2 \smallsetminus C$ is hyperbolic and hyperbolically embedded in $\mathbb{P}^2$ (see Green [Gr77]). □

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