INJECTIVE ENDOMORPHISMS OF REAL ALGEBRAIC SETS ARE SURJECTIVE

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Abstract . — Let X be a real algebraic set in $\mathbb{R}^n$. Let $f : X \to X$ be an injective regular (more generally continuous with algebraic graph) map. Then f is surjective.

Introduction

We consider mappings from an algebraic set X to itself. The result that, for regular maps, injectivity implies surjectivity is known in the complex case (more generally over algebraically closed field of char $= 0$) as Ax's theorem [A]. In the real case this result was proved by Białynicki-Birula and Rosenlicht [BR] for $X = \mathbb{R}^n$, later by A. Borel [B] for X smooth (see also [BCR]). A. Tognoli ([T1], [T2], [T3]) proposed a proof in the case where $X \subset \mathbb{R}^n$ may be singular, but his proof is incomplete. We discuss in the end of the paper our main objections to his proof. The goal of this paper is to prove that if $X \subset \mathbb{R}^n$ is algebraic, $f : X \to X$ is an injective continuous mapping such that the graph of f is algebraic (in particular f may be regular), then f is surjective. The proof of the result, in somewhat more general setting, is given in the second section. In a forthcoming paper we shall prove a stronger result, that under the above assumptions f is homeomorphic. The proof of this stronger fact uses the resolution of singularities.

In the first section we recall the notion of arcwise symmetric set, which is our main tool. This notion, introduced in [K], enables us to define a "rigid component" of any semi-algebraic set. In the case of algebraic set arcwise symmetric components are finer than analytic components. We consider also germs of arcwise symmetric sets. We prove (Lemma 1.7) that the images by injective mappings of these germs are again of the same type. This will be helpful when proving the openness of f. Another important fact is that every arcwise symmetric set has a nonzero canonical fundamental class mod 2 (Proposition 1.6).

The crucial ingredient of the proof is the notion of absolutely irreducible germs (Definition 1.9), where we consider germs of horned neighborhoods around analytic arcs.

The main idea of the proof is the same as that of A. Borel in the case where X is smooth ([B], [BCR]). We sketch now his argument.

Let $f : X \to X$ be an injective regular (or more generally continuous with algebraic graph) mapping. Suppose that $Y = X \setminus f(X) \neq \emptyset$, then:

1) Y is semialgebraic and closed (by the invariance of domain);

2) the fundamental class of Y (denoted by $[Y]$) is a non-zero element of $H_d^{BM}(Y, \mathbb{Z}/2)$, where $d = \dim Y$.

The homology we consider is formally the Borel-Moore homology. In in the semialgebraic case, however, it can be considerably simplified by using the semialgebraic Alexandrov compactification and triangulation (cf. [BCR], chap.11).

Denote $f^{k+1} = f \circ f^k$, $f^0 = \mathrm{id}_X$, and $X \setminus f^k(X) = Y_k$. Observe that $[Y] = [Y_1], [Y_2], \ldots, [Y_k]$ are linearly independent in $H_d^{BM}(Y_k, \mathbb{Z}/2)$, since $[Y_\ell] - [Y_{\ell-1}] = [Y_\ell \setminus Y_{\ell-1}]$, $\ell = 1, \ldots, k$ are linearly independent. Hence

$$\dim H_d^{BM}(Y_k, \mathbb{Z}/2) \geq k\,.$$

But from the long exact sequence (see [BCR], chap.11)

$$\longrightarrow H_{d+1}^{BM}\big(f^k(X), \mathbb{Z}/2\big) \longrightarrow H_d^{BM}(Y_k, \mathbb{Z}/2) \longrightarrow H_d^{BM}(X, \mathbb{Z}/2) \longrightarrow$$

and the fact that $f^k : X \to f^k(X)$ is a homeomorphism, we obtain for each $k \in \mathbb{N}$

$$\dim H_d^{BM}(Y_k, \mathbb{Z}/2) \leq \dim H_d^{BM}(X, \mathbb{Z}/2) + \dim H_{d+1}^{BM}(X, \mathbb{Z}/2) = const < \infty$$

which is a contradiction.

In the case where X is singular the problem is with the openess of f.

We shall outline now the main idea of our proof, we combine the idea of A. Borel with the theory of arcwise symmetric sets. We used already in [KR] some properties of these sets to prove that injectivity implies surjectivity of some semialgebraic transformations of $\mathbb{R}^n$. Let $d = \dim X$. Let $\sigma : \widehat{K} \to \widehat{X}$ be a triangulation of $\widehat{X}$ the algebraic Alexandrov compactification of X, where $\widehat{K}$ is a simplicial complex (realized in some $\mathbb{R}^N$), σ is a homeomorphism with semialgebraic graph. Denote

$$K^t = \left\{ p \in \widehat{K} : p \in \sigma^{-1}(X) : \widehat{K} \text{ is a p.l. manifold at } p \right\},$$

and $X^t = \sigma(K^t)$. By the theorem of Shiota and Yokoi [SY] on uniqueness of p.l. structure on semialgebraic set, the set X^t does not depend on triangulation. Clearly X^t is open in X, it is a semialgebraic p.l. manifold and $\dim(X \setminus X^t) < d$.

We proceed by induction on $d = \dim X$.

1) The case $d = 0$ is obvious, since in this case X is finite.

2) It is enough to prove that

$$(*) \qquad f(X^t) = X^t\,.$$

Indeed, by injectivity $f(X \setminus X^t) \subset X \setminus X^t$, hence $f(W) \subset W$, where W is the Zariski closure of $X \setminus X^t$. By induction $f(W) = W$, since $\dim W < d$. Thus $f(X) = X$. To be precise, in the case of mapping with algebraic graph the set W may be smaller than the Zariski closure (c.f. Lemma 2.1 and Example 4.1).

So we are left with proving $(*)$. First we find a closed set Z such that $f(Z) = Z$ and $f(X^t \setminus Z) \subset X^t \setminus Z$. We cannot apply directly the Borel argument, because $X^t \setminus Z$ is not algebraic, even not a difference of two algebraic set. The main difficulty is to prove that if the set $(X^t \setminus Z) \setminus f(X^t \setminus Z)$ is non empty, then it has a nonzero fundamental class.

Through-out the paper we shall not distinguish between a mapping and its graph.

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1. Arcwise symmetric sets

We shall often use the following consequences of the classical Puiseux theorem on parametrisation of 1-dimensional analytic sets.

1.0. Lemma.

a) Let $\eta : (-1, 0) \to \mathbb{R}^n$ be a continuous, bounded arc with semialgebraic (more generally subanalytic) graph. Then, for some $k \in \mathbb{Q}$ and $\varepsilon > 0$, the function $t \to \eta(t^k)$ admits analytic injective extension to $(-\varepsilon, \varepsilon)$.

b) If $\gamma : (-1, 1) \to \mathbb{R}^n$ is an analytic injective mapping, then for each $0 < \varepsilon < 1$, the set $\gamma(-\varepsilon, \varepsilon)$ is semianalytic, locally analytic and locally irreducible.

We recall now a basic definition from [K].

1.1. Definition. — *Let M be an analytic manifold and V a subset of M. We say that $E \subset V$ is arcwise symmetric in V, if one of two equivalent conditions holds:*

i) $$\operatorname{Int} \gamma^{-1}(E) \neq \emptyset \Longrightarrow \gamma(-1, 1) \subset E\,,$$

for every analytic arc $\gamma : (-1, 1) \longrightarrow V \subset M$;

ii) $$\gamma(-1, 0) \subset E \Longrightarrow \gamma(-1, 1) \subset E\,,$$

for every analytic arc $\gamma : (-1,1) \longrightarrow V \subset M$.

In the sequel $M = \mathbb{R}^n$, E and V will be semialgebraic in $\mathbb{R}^n$. We denote by $\mathcal{AR}(V)$ the family of all semialgebraic and arcwise symmetric subsets of V.

Remark. — It follows easily from Lemma 1.0 that in the above definition it is enough to consider only injective arcs (cf. [K]).

1.2. Proposition. — *For a given semialgebraic V there exists a unique noetherian topology on V for which $\mathcal{AR}(V)$ is the family of closed sets. By abuse of language we denote this topology by $\mathcal{AR}(V)$.*

Proof. — The argument is the same as in the proof of Theorem 1.4 in [K]. The main point is that every semialgebraic set is a finite union of connected, analytic manifolds and the following observation: if V is a connected analytic submanifold and $E \in \mathcal{AR}(V)$, $E \neq V$, then $\dim E < \dim V$.

Remark. — For $E \subset M$ and $x \in M$, we put

$$\dim_x E = \max\left\{ \dim \Gamma : \Gamma \text{ is a } C^1 \text{ submanifold}, \Gamma \subset E,\ x \in \overline{\Gamma} \right\},$$

$$\dim E = \max\left\{ \dim_x E : x \in \overline{E} \right\}.$$

Recall ([H], chap. 1) that any closed set in a noetherian topology admits a unique decomposition into irreducible components. So, by Proposition 1.2, every $Y \in \mathcal{AR}(V)$ has a unique decomposition

$$Y = Y_1 \cup \cdots \cup Y_k,$$

where each $Y_i \in \mathcal{AR}(V)$ is irreducible; $Y_i \not\subset Y_j$ for $i \neq j$.

1.3. Remark. — Clearly the $\mathcal{AR}$ topology is finer than (or equal to) the Zariski topology. It follows easily from Lemma 1.0 that, if V is open, then any 1-dimensional $Y \in \mathcal{AR}(V)$ is analytic in V. If $\dim V \geq 3$, then the $\mathcal{AR}$ topology is actually finer than the analytic Zariski topology (cf. [K]). Recall that the Zariski closure of semialgebraic set preserves dimension, hence for every semialgebraic $A \subset V$ we have

$$\dim \overline{A}^{\mathcal{AR}} = \dim A.$$

If V is open (in particular $V = \mathbb{R}^n$), then every $E \in \mathcal{AR}(V)$ is closed (in V) in the strong topology. This is an immediate consequence of the curve selection lemma (cf. [K]).

1.4. Remark. — Let Γ be a semialgebraic, connected analytic submanifold of $\mathbb{R}^n$. Then for any semialgebraic V, such that $\Gamma \subset V$, the closure $\overline{\Gamma}^{\mathcal{AR}}$ is $\mathcal{AR}$-irreducible in V, because, Γ being connected analytic manifold, is $\mathcal{AR}$-irreducible in Γ.

Similarly we have decompositions of germs of arcwise symmetric sets. Let $a \in \overline{E}$, $E \in \mathcal{AR}(V)$. We shall say that the germ E_a is $\mathcal{AR}(V)$*-irreducible*, if there exists $\{U_n\}_{n\in\mathbb{N}}$, a base of neighborhoods of a (in $\mathbb{R}^n$), such that $E \cap U_n$ is $\mathcal{AR}(V \cap U_n)$-irreducible, for each $n \in \mathbb{N}$. We say that E_a is $\mathcal{AR}$*-irreducible* if E_a is $\mathcal{AR}(V)$-irreducible for some neighborhood V of a.

Let $E \in \mathcal{AR}(V)$ and $a \in \overline{E}$. To obtain the decomposition of E_a into $\mathcal{AR}(V)$-irreducible germs we proceed as follows: we find a finite partition $E = \bigcup_{i\in I} \Gamma_i$, and $\{U_n\}_{n\in\mathbb{N}}$ a base of neighborhoods of a (in $\mathbb{R}^n$), such that each $\Gamma_i \cap U_n$ is a connected analytic submanifold. Such a partition can be easily obtained from the local triviality theorem (cf. [BCR], chap. 9). Let

$$E \cap U_n = E_1^n \cup \cdots E_{k(n)}^n$$

be the decomposition into $\mathcal{AR}(V \cap U_n)$-irreducible components. Then, by Remark 1.4, for each $i \in I$ we can find $\sigma(i) \in \{1, \ldots, k(n)\}$ such that $\Gamma_i \cap U_n \subset E_{\sigma(i)}^n$. Since the decomposition is irredundant, the mapping σ must be surjective. So $k(n) \leq \#I$, for every $n \in \mathbb{N}$. This implies that sequence $k(n)$ stabilizes for n large enough and we obtain easily the unique decomposition of the germ E_a.

1.5. Proposition. — *Let* $E, F \in \mathcal{AR}(V)$*. If* E *is* $\mathcal{AR}$*-irreducible in* V *and* $F \subsetneq E$*, then* $\dim F < \dim E$*.*

Proof. — Suppose, contrary to our claim, that $\dim F = \dim E$. Put $A = E \setminus F \neq \emptyset$. If $\dim A < d = \dim E$, than by Remark 1.3, we have $\dim \overline{A}^{\mathcal{AR}} < d$, hence $F \not\subset \overline{A}^{\mathcal{AR}}$. So we have obtained a decomposition $E = F \cup \overline{A}^{\mathcal{AR}}$, which contradicts the irreducibility of E.

Assume now that $\dim A = d$. Denote $B = \overline{A} \setminus A$ and $B_1 = E \cap \overline{B}^{\mathcal{AR}}$, where $\overline{A}$ is the closure in V in the strong topology. Note that $\dim B_1 < d$. We claim that $C = A \cup B_1$ is arcwise symmetric in V. Let $\gamma : (-1, 1) \to V$ be an analytic arc such that $\operatorname{Int} \gamma^{-1}(C) \neq \emptyset$, we should check that $\gamma(-1, 1) \subset C$. Since $E, B_1 \in \mathcal{AR}(V)$, we have $\gamma(-1, 1) \subset E$ and we may suppose that $\operatorname{Int} \gamma^{-1}(A) \neq \emptyset$. By arcwise symmetry of $F = E \setminus A$, it is impossible that $\operatorname{Int} \gamma^{-1}(F) \neq \emptyset$, so

$$\gamma(-1, 1) \subset \overline{A}\,,$$

because $\gamma^{-1}(A)$ is dense in $(-1, 1)$. Thus

$$\gamma(-1, 1) \subset \overline{A} \cap E \subset A \cup B_1 = C$$

and the claim is proved. Note that $F \not\subset C$, since $\dim(F \setminus C) = d$ and obviously $C \not\subset F$. Hence we have obtained a decomposition $E = C \cup F$ which contradicts the irreducibility of E. The proposition follows.

Remark. — In the case where $V = \mathbb{R}^n$ the above proposition was proved in [K] using, in an essential way, the resolution of singularities.

1.6. PROPOSITION. — *Let V be an open semialgebraic subset of $\mathbb{R}^n$, $Y \in \mathcal{AR}(V)$, $d = \dim Y$. Then $[Y]$ is a nonzero element of $H_d^{BM}(Y, \mathbb{Z}/2)$, where $[Y]$ is the class of $\overline{\mathrm{Reg}_d\, Y}$, $\mathrm{Reg}_d\, Y$ is the set of all smooth point of V of dimension d.*

Proof. — The argument is the same as in the real algebraic case (cf. [BCR], chap. 11). The main point is that if we have a triangulation of the semialgebraic Alexandrov compactification of Y, then for each $(d-1)$-simplex the number of adjacent d-simplices is even. This can be proved as follows: taking a suitable section, transversal to a $(d-1)$-simplex we are in the case where $\dim Y = 1$, so Y is analytic, by Lemma 1.0. Now our assertion is obvious, since every analytic set of dimension 1 has an even number of halfbranches at each of its points. Clearly, by the choice of the section, halfbranches correspond to d-simplicies.

1.7. LEMMA. — *Let X be arcwise symmetric in an open subset U of $\mathbb{R}^m$. Let $f : X \to \mathbb{R}^n$ be an injective regular (or more generally continuous with the graph arcwise symmetric in $U \times \mathbb{R}^n$) mapping. Then for each $a \in X$ there exists an open neighborhood U_a, such that the germ $(f(U_a \cap X))_{f(a)}$ is arcwise symmetric in $\mathbb{R}^n$.*

Proof. — Let $\overline{B}(a,r) \subset U$. Then, by Remark 1.3, the set

$$S = \{x \in X : \|x-a\| = r\}$$

is compact. By the continuity and injectivity of f there is a $\delta > 0$ such that $\mathrm{dist}(f(a), f(S)) > \delta$. Put $U_a = B(a,r)$, $V = B(f(a),\delta)$. Then $f(U_a \cap X)$ is arcwise symmetric in V. Indeed, let $\gamma : (-1,1) \to V$ be an analytic, injective arc such that $\gamma(-1,0) \subset f(U_a \cap X)$. By Lemma 1.0 we may suppose that $t \to f^{-1} \circ \gamma(t) = \eta(t)$, $t \in (-1,0)$ extends to an analytic mapping $\eta : (-1,\varepsilon) \to U_a$, for some $\varepsilon > 0$. But X is arcwise symmetric in U_a, hence $\eta(-1,\varepsilon) \subset X$. Note that both γ and $f \circ \eta$ are injective. Since their images contain $\gamma(-1,0)$, we get, by Lemma 1.0 b), that the germs (at $\gamma(0)$) of the images of γ and $f \circ \eta$ are equal. Thus $\gamma(-1,\varepsilon) \subset f(U_a \cap X)$. By the standard "sup" argument this inclusion holds for $\varepsilon = 1$.

1.8. LEMMA. — *Let X be algebraic (or more generally $X \in \mathcal{AR}(U)$, where U is open). Then for every $a \in X^t$ the germ $(X^t)_a = X_a$ is $\mathcal{AR}$-irreducible.*

Proof. — Recall that X^t is the set of points $a \in X$ such that X_a is the germ of a p.l. manifold of dimension $d = \dim X$. Clearly X^t is open in X. We proceed by induction on $\dim X = d$.

1) The case $d = 1$ is obvious.

2) Assume $d \geq 2$. The link of X^t (which is the same as that of X) at a is a p.l. manifold. We denote it by $\ell k(X^t, a)$. This link can be realized as $X^t \cap S(a, \varepsilon)$ for $\varepsilon > 0$ small, where $S(a, \varepsilon) = \{x \in \mathbb{R}^n : \|x - a\| = \varepsilon\}$. Assume, contrary to our claim, that the germ $(X^t)_a$ is reducible *i.e.* there exists an arbitrarily small neighborhood Ω of a (in $\mathbb{R}^n$) such that

$$X^t \cap \Omega = X_1 \cup X_2 \,,$$

where $X_i \in \mathcal{AR}(\Omega)$, $X_i \not\subset X_j$, for $i \neq j$. We can assume that $a \in X_1 \cap X_2$. Take $\varepsilon > 0$ small enough and put

$$\ell(X_i) = \ell k(X_i, a) = X_i \cap S(a, \varepsilon), \ \ i = 1, 2 \,.$$

Observe that, by Remark 1.4, the sets $\ell(X_i)$ are closed $\mathbb{R}^n$. Since $\ell k(X^t, a)$ is homoeomorphic to $(d-1)-$sphere, it must be connected ($d \geq 2$). Therefore, there exists a point $b \in \ell(X_1) \cap \ell(X_2)$ such that for any neighborhood V of b, in $S(a, \varepsilon)$, there is no inclusion relation between $V \cap \ell(X_1)$ and $V \cap \ell(X_2)$. This proves that the germ of $\ell k(X^t, a)$ is reducible at b, which is a contradiction since, by induction, $X^t \cap S(a, \varepsilon)$ is $\mathcal{AR}$-irreducible at each of its points.

1.9. Definition. — *Let Γ be an analytic subset of an open set in $\mathbb{R}^n$. Let $a \in \Gamma$. We say that G is a horned neighborhood of the point a along Γ (we write for short $G \in \mathcal{H}(\Gamma, a)$) if: G is semialgebraic, $G \setminus \{a\}$ is open in $\mathbb{R}^n$ and $\Gamma \cap \Omega \subset G \cap \Omega$, for some Ω neighborhood (in $\mathbb{R}^n$) of a.*

Let $X \in AR(U)$, where U is an open subset of $\mathbb{R}^n$. Let $a \in X$, and $\Gamma \subset X$ be an analytic curve such that the germ Γ_a is irreducible. We say that X is $\mathcal{AR}$-irreducible along Γ at a, if for every $G \in \mathcal{H}(\Gamma, a)$ there exists $H \in \mathcal{H}(\Gamma, a)$ such that $H \subset G$ and $X \cap H$ is $\mathcal{AR}$-irreducible in H.

Finally, we say that the germ X_a is absolutely $\mathcal{AR}$-irreducible if X_a is $\mathcal{AR}$-irreducible and X is $\mathcal{AR}$-irreducible along Γ at a, for every analytic curve $\Gamma \subset X$, Γ_a irreducible.

Note that if X is smooth at a (more precisely, if X_a is the germ of an analytic submanifold), then the germ X_a is absolutely $\mathcal{AR}$-irreducible. Indeed, let Γ be an irreducible germ of curve, $a \in \Gamma \subset X$. We can resolve the singularity of Γ be a finite sequence of punctual blowing-ups. This mapping has a property of lifting of all (nonconstant) analytic arcs, so we can assume that Γ is smooth at a and now it is easily seen that X is $\mathcal{AR}$-irreducible along Γ at a.

Example. — Consider the germ at the origin of the algebraic set $X = \{z^4 - xyz^2 + x^3 = 0\}$. Notice that X can be desingularized by blowing up the y-axis. Let Q be any cube centered at the origin, observe that the strict transform of $Q \cap X$ is connected. Hence the germ X_0 is $\mathcal{AR}$-irreducible (a similar argument is used in [K], theorem 2.6). On the other hand, if G is a "sufficiently sharp" horned neighborhood of the origin along the y-axis then the proper inverse image of $X \cap G$ has two $\mathcal{AR}$-components; one along y axis, the second along the parabola $y = x^2, z = 0$. This proves that X_0 is not $\mathcal{AR}$-irreducible along y-axis, so the germ X_0 is not absolutely $\mathcal{AR}$-irreducible.

2. Proof of the theorem

The theorem for continuous mapping with algebraic graph will be proved by induction on $\dim X$. For the inductive step we shall require the following stronger result.

THEOREM. — *Let X be a semialgebraic arcwise symmetric subset of $\mathbb{R}^n$ and let $f : X \to X$ be a continuous injective semialgebraic mapping. Suppose that there exists an algebraic compactification (W, i) of $\mathbb{R}^n \times \mathbb{R}^n$ and suppose E is arcwise symmetric in W, such that $i(f) = E \cap i(\mathbb{R}^n \times \mathbb{R}^n)$. Then f is surjective.*

By an algebraic compactification of $\mathbb{R}^N$ we mean a pair (W, i), where W is a compact real algebraic variety, $i : \mathbb{R}^N \to W$ is a biregular isomorphism of $\mathbb{R}^N$ onto $i(\mathbb{R}^N)$, moreover the image $i(\mathbb{R}^N)$ is Zariski dense and open in W.

Remark. — Observe that the theorem applies to mappings with algebraic graph; it suffices to take the projective closure of $\mathbb{R}^n \times \mathbb{R}^n$.

Let us fix (W, i), an algebraic compactification of $\mathbb{R}^n \times \mathbb{R}^n$. We shall introduce another noetherian topology $\mathcal{AC}$ which lies between $\mathcal{AR}$-topology and Zariski topology. We say that $A \subset \mathbb{R}^n \times \mathbb{R}^n$ is $\mathcal{AC}$ closed in $\mathbb{R}^n \times \mathbb{R}^n$ (we write $A \in \mathcal{AC}(\mathbb{R}^n \times \mathbb{R}^n)$ if $A = i^{-1}(B)$ for some B which is arcwise symmetric in W. It can be easily proved that the definition is independent of the compactification (W, i).

An advantage of the $\mathcal{AC}$ topology is the following:

2.0. LEMMA. — *Let X be a semialgebraic subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^k$ be a continuous mapping such that its graph is $\mathcal{AC}$ closed. Suppose that $f(A) \subset B$, then $f(\overline{A}^{\mathcal{AC}}) \subset \overline{B}^{\mathcal{AC}}$.*

Proof. — It suffices to show that, if B is $\mathcal{AC}$ closed then $f^{-1}(B)$ is also $\mathcal{AC}$ closed. Let $i = (\alpha \times \beta) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{P}^n \times \mathbb{P}^k = W$ be the canonical multiprojective embedding. Denote by p (resp. q) the projection of W on the first (resp. second) factor. Let

$\gamma : (-1,1) \to \mathbb{P}^n$ be an injective, analytic arc. Assume that $\gamma(-1,0) \subset \alpha(f^{-1}(B))$. Since $f^{-1}(B)$ is closed and $\mathbb{P}^n \setminus \alpha(\mathbb{R}^n)$ is algebraic (hence arcwise symmetric) it is enough to prove that $\gamma(0,\varepsilon) \subset \alpha(f^{-1}(B))$ for some $\varepsilon > 0$. For $t \in (-1,0)$, put

$$\tilde{\gamma}(t) = i(\alpha^{-1}(\gamma(t)), f \circ \alpha^{-1}\gamma(t)).$$

Observe that the graph of $\tilde{\gamma}$ is subanalytic in $\mathbb{R} \times W$, because i is biregular and $i(f)$ is semialgebraic. We now apply Lemma 1.0. Using a suitable reparametrisation of γ, if necessary, we may extend analytically $\tilde{\gamma}$ to $(-1,\varepsilon)$. Clearly $\tilde{\gamma}(0,\varepsilon) \subset i(f)$, since the graph of f is $\mathcal{AC}$ closed. Note that, by analyticity, $\gamma = p \circ \tilde{\gamma}$ for $t \in (-1,\epsilon)$. So $\beta \circ f \circ \alpha^{-1} \circ \gamma = q \circ \tilde{\gamma}$ on $(-1,0) \cup (0,\epsilon)$. Since B is $\mathcal{AC}$ closed and $q \circ \tilde{\gamma}$ is analytic, our assumption yields

$$f \circ \alpha^{-1} \circ \gamma(0,\varepsilon) = \beta^{-1} \circ q \circ \tilde{\gamma}(0,\varepsilon) \subset B\,.$$

Hence the lemma follows.

Remark. — Notice that, by Example 4.1, the lemma fails for f continuous with algebraic graph if the $\mathcal{AC}$ closure is replaced by the Zariski closure.

We fix now $f : X \to X$ satisfying assumptions of the theorem.

2.1. Lemma. — *Under the induction hypothesis there exists an $\mathcal{AC}$ closed set $Z_1 \subset X$ such that $f(Z_1) = Z_1$ and $f(X^t \setminus Z_1) \subset X^t \setminus Z_1$.*

Proof. — Recall that by Reg X we denote the set of all regular points of X. Since our mapping f has generically rank $= \dim X$ we obtain that, $x \in \overline{\operatorname{Reg} X}$ implies $f(x) \in \overline{\operatorname{Reg} X}$.

2.1.1. Sublemma. — *If $x \in \overline{\operatorname{Reg} X}$ and $f(x) \in X^t$, then $x \in X^t$.*

Assume that $x \in \overline{\operatorname{Reg} X} \setminus X^t$ and $f(x) \in X^t$. Then, by Lemma 1.7, there exists U a neighborhood of x such that the germ $f(X \cap U)_{f(x)}$ is arcwise symmetric. Clearly this germ is of dimension $d = \dim X$. So, by Lemma 1.5 and 1.8, we have an equality of germs

$$f(X \cap U)_{f(x)} = X_{f(x)}\,.$$

This proves that f is open at x. We want to show now that this is the case for any z close to x. To repeat the above argument it suffices to check that $\dim_z X = d$. It follows from the proof of Lemma 1.7 that the mapping $f : X \cap U \to f(X \cap U)$ is proper. Let $z \in X \cap U$ be such that $f(z) \in X^t \cap \operatorname{Int} f(X \cap U)$. By properness and injectivity, $\dim_z X = \dim_{f(z)} f(X \cap U) = d$. Thus there exist a neighborhood U' of x and a neighborhood V' of $f(x)$, such that

$$f|_{X \cap U'} : X \cap U' \longrightarrow X \cap V'$$

is a semialgebraic homeomorphism. Hence $x \in X^t$, which is a contradiction. The sublemma follows.

We have proved $f(\overline{\operatorname{Reg} X} \setminus X^t) \subset \overline{\operatorname{Reg} X} \setminus X^t$. Let us put

$$Z_1 = \overline{\overline{\operatorname{Reg} X} \setminus X^t}^{\mathcal{AC}} .$$

By Lemma 2.0 we have $f(Z_1) \subset Z_1$. Clearly $\dim Z_1 < d = \dim X$. So, by the induction hypothesis, $f(Z_1) = Z_1$ (note that the graph of f restricted to Z_1 is $\mathcal{AC}$-closed). Obviously $f(\overline{\operatorname{Reg} X}) \subset \overline{\operatorname{Reg} X}$. Thus $f(X^t \setminus Z_1) \subset X^t \setminus Z_1$, by injectivity.

2.2. *Remark.* — We also have proved that $f : X^t \setminus Z_1 \to X$ is open.

Denote $X^{at} = \{x \in X^t \setminus Z_1 : X_x = (X^t)_x$ is absolutely $\mathcal{AR}$-irreducible$\}$, $X_2 = X^t \setminus X^{at}$ and finally $Z_2 = \overline{X_2}^{\mathcal{AC}}$. Clearly X_2 is included in the singular part of X, so $\dim Z_2 < d = \dim X$. I don't know any example of X where $Z_2 \neq \emptyset$.

2.3. Lemma. — *$f(X_2) \subset X_2$. Hence, by Lemma 2.0, $f(Z_2) \subset Z_2$ and by induction $f(Z_2) = Z_2$.*

Proof. — Assume that $x \in X_2$ and $f(x) \in X^{at}$. Then, by Definition 1.9, there exist an analytic curve Γ irreducible at x and $G \in \mathcal{H}(\Gamma, x)$ such that for every $H \in \mathcal{H}(\Gamma, x)$, $H \subset G$ we have

$$X \cap H = E_1 \cup E_2 ,$$

with $E_i \in \mathcal{AR}(X \cap H)$, $E_i \not\subset E_j$, for $i \neq j$. Clearly $\widetilde{G} = f(G) \in \mathcal{H}(f(\Gamma), f(x))$, because $f|_{X^t \setminus Z_1}$ is open and $f(\Gamma)$ is an analytic curve irreducible at $f(x)$, by injectivity of f and Lemma 1.0 b). For any $\widetilde{H} \in \mathcal{H}(f(\Gamma), f(x))$, $\widetilde{H} \subset \widetilde{G}$ denote $H = f^{-1}(\widetilde{H})$. To get a contradiction it is enough to show that $\widetilde{E_i} = f(E_i) \in \mathcal{AR}(\widetilde{H} \cap X)$.

To this end let us suppose that $\tilde{\gamma}(-1,0) \subset \widetilde{E_i}$, where $\tilde{\gamma} : (-1,1) \to \widetilde{H} \cap X$ is an analytic arc. By Lemma 1.0 a), by changing parametrisation of $\tilde{\gamma}$, if necessary, we may suppose that $\gamma = f^{-1} \circ \tilde{\gamma}$ is analytic in $(-1,1)$. Clearly $\gamma(-1,1) \subset H \cap X$ and $\gamma(-1,0) \subset E_i$, so by the arcwise symmetry of E_i we have $\gamma(-1,1) \subset E_i$ and by consequence $\tilde{\gamma}(-1,1) \subset \widetilde{E_i}$. The lemma follows.

Proof of the theorem. — Let us denote $Z = Z_1 \cup Z_2$, $U = \overline{\operatorname{Reg} X} \setminus Z$. By Lemma 2.1 and 2.3 we have $f(Z) = Z$. Thus, by injectivity of f, $f(U) \subset U$. Notice that $U = X_t \setminus Z$ is locally closed, so in particular U is arcwise symmetric in some open subset V of $\mathbb{R}^n$. Assume that $Y = U \setminus f(U) \neq \emptyset$. Note that if $Y \in \mathcal{AR}(U)$, then $Y \in \mathcal{AR}(V)$. Therefore to get a contradiction it suffices, by Borel's argument (cf. Introduction) and Proposition 1.6, to prove that $Y \in \mathcal{AR}(U)$.

Let us fix $\gamma : (-1,1) \to U \subset \mathbb{R}^n$, an analytic injective arc. Suppose that $\gamma(-1,0) \subset Y$, we have to show that $\gamma(-1,1) \subset Y$.

Let $i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{P}^n \times \mathbb{P}^n$ be the canonical multiprojective embedding. Denote by $\pi : \mathbb{P}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the canonical projection. Put

$$F = \overline{i(f)}^{\mathcal{AR}} \cap \mathbb{P}^n \times \mathbb{R}^n .$$

Clearly $F \in \mathcal{AR}(\mathbb{P}^n \times \mathbb{R}^n)$. The assumption that the graph of f is $\mathcal{AC}$-closed means that $F \cap (\mathbb{R}^n \times \mathbb{R}^n) = i(f)$. This yields the following:

Claim. — *If* $\operatorname{Int} \gamma^{-1}(f(X)) \neq \emptyset$, *then* $\overline{\gamma^{-1}(f(X))} = (-1,1)$.

Indeed, assume that $\gamma(-1,0) \subset f(X)$. Let $\tilde{\gamma} : (-1,0) \to F$ denote the unique lifting of γ to F *i.e.* $\gamma = \pi \circ \tilde{\gamma}$. By Lemma 1.0 we may extend $\tilde{\gamma}$ analytically on $(-1,\varepsilon)$, for some $\varepsilon > 0$ (we reparametrise γ if necessary). Note that by the arcwise symmetry of F we have $\tilde{\gamma}(0,\varepsilon) \subset F$. As $(\mathbb{P}^n \setminus \mathbb{R}^n) \times \mathbb{R}^n$ is algebraic (hence arcwise symmetric) we deduce that $\tilde{\gamma}(0,\varepsilon) \subset i(f) = F \cap (\mathbb{R}^n \times \mathbb{R}^n)$. This implies the claim.

The proof of the inclusion $\gamma(-1,1) \subset Y$ falls into two cases.

Case 1 . — $\operatorname{Int} \gamma^{-1}(f(X)) \cap (-1,0) = \emptyset$. It follows from the claim that $\operatorname{Int} \gamma^{-1}(f(X)) = \emptyset$. Thus for some Σ open and dense in $(-1,1)$ we have $\gamma(\Sigma) \subset U \setminus f(X) \subset U \setminus f(U) = Y$. Recall that, by Remark 2.2, the set Y is closed in U. Hence $\gamma(-1,1) \subset Y$.

Case 2 . — $\operatorname{Int} \gamma^{-1}(f(X)) \cap (-1,0) \neq \emptyset$. The claim implies that the lifting $\tilde{\gamma} : (-1,0) \to F$ can be extended to $(-1,\varepsilon)$; moreover $\tilde{\gamma}(t) \in F$, for $t \in (0,\varepsilon)$. Denote $\Gamma = \gamma(-\varepsilon,\varepsilon)$ and $\tilde{\Gamma} = \tilde{\gamma}(-\varepsilon,\varepsilon)$. It follows from the Łojasiewicz inequality (see e.g. [BCR], Chap. 2), that there exists $\tilde{G} \in \mathcal{H}(\tilde{\Gamma}, \tilde{\gamma}(0))$ such that

$$\overline{\tilde{G}} \cap (\mathbb{P}^n \setminus \mathbb{R}^n) \times \mathbb{R}^n = \{\tilde{\gamma}(0)\} .$$

In other words $\overline{\tilde{G}} \cap F = (\overline{\tilde{G}} \cap i(f)) \cup \{\tilde{\gamma}(0)\}$, hence π is injective on $\overline{\tilde{G}} \cap F$. Recall that F is closed, therefore the set $\tilde{S} = \partial\, \tilde{G} \cap F$ is compact. Observe that $\Gamma \cap \pi(\tilde{S}) = \{\tilde{\gamma}(0\}$. So, again by the Łojasiewicz inequality, there exists $G \in \mathcal{H}(\Gamma, \gamma(0))$ such that $G \cap \pi(\tilde{S}) = \{\gamma(0)\}$. Arguing as in the proof of Lemma 1.7 and using the arcwise symmetry of F it is easily seen that

$$E_1 = G \cap \pi(\tilde{G} \cap F)$$

is arcwise symmetric in G.

Recall that we have to prove that $\gamma(0,1) \subset Y$. Assume, contrary to our claim, that $\gamma(0,\varepsilon) \subset f(U)$, for some $\varepsilon > 0$. Then clearly $\dim_{\gamma(t)} E_1 = d$, for $t > 0$ close to 0. So $\dim E_1 = \dim X \cap G$. By the construction of U, $f^{-1}(\gamma(t)) \notin Z_1$, for $t < 0$ close to 0. Therefore $\dim_{\gamma(t)} E_1 < d$, for $t < 0$ close to 0. Thus we have proved that $E_1 \neq X \cap G$. Hence, by Lemma 1.5, $X \cap G$ cannot be $\mathcal{AR}$-irreducible in G. Obviously the above argument can be repeated for any $H \in \mathcal{H}(\Gamma, \gamma(0))$, $H \subset G$, so X^t is not absolutely $\mathcal{AR}$-irreducible at $\gamma(0)$, a contradiction. The theorem follows.

Remark 2.4. — The proof would be simpler if one could prove the following: suppose that W is an algebraic set and $\gamma : (-1,1) \to W$ an analytic arc, then it is impossible that,

1) $\dim_{\gamma(t)} W < d = \dim W$, for $t < 0$ close to 0;

2) W is a p.l.-manifold (of dim $= d$) at $\gamma(t)$, for $t > 0$ close to 0.

However a surprising example of such a situation was given by Akbulut and King [AK]. Their construction (not explicit) is based on the theory of resolutions towers.

3. Remarks on papers [T2], [T3] by A. Tognoli

We shall explain now why the proof of the theorem, for regular mappings, given by Tognoli, is not complete. First we compare language and notations. Note that an algebraic set (more generally an analytic space) X is strongly non arc analytically connected in a point $x \in X$ (in the sense of Tognoli [T2], [T3], Def. 8) if the germ X_x has at least two $\mathcal{AR}$-components of dimension $d = \dim X$. Incidentally, in Definition 8 of [T2], [T3] the inclusions $B' \subset U'$, $B'' \subset U'$ are missing.

One of the main points in Tognoli's proof is "Lemma 3". It is stated in [T2] as follows: "Let $\varphi : X \to Y$ be an injective algebraic map between real algebraic sets of the same dimension. Suppose that X, Y are irreducible and Y is locally irreducible. Let $x \in X_d = \{x \in X : \dim_x X = \dim X\}$ and suppose that φ is not open in x then Y is strongly non arc analytically connected in $\varphi(x)$."

This statement is not correct. An easy counter-example can be constructed as follows: let

$$Y = \{z^2 - x^2(y^2 - x^3) = 0\} \subset \mathbb{R}^3 ,$$
$$X = \{z^2 - (y^2 - x^3) = 0\} \subset \mathbb{R}^3 ,$$

and $\varphi(x, y, z) = (x, y, xz)$ (restricted to X). Clearly φ is the blowing up of the y-axis (in an affine chart), and X is the strict transform of Y. Obviously φ is not open at the origin, since the y-axis is missing in the image.

The statement of "Lemma 3" was corrected in the published version [T3], where the openness it claimed only for the mapping $\varphi : X_d \to Y_d = \{y \in Y : \dim_y Y = \dim Y\}$.

Notice that this result can be also proved using our Lemma 1.7.

The problem is that in [T3] the author uses actually this incorrect version of "Lemma 3" to prove the crucial inclusion (1) on page 731. Precisely; he deduces (line 30, page 731) that the mapping φ (with values in V and not in V_d !) is open at x, from the fact that V is strongly arc analytically connected at $\varphi(x)$.

Also the next argument is not correct. The author deduces from the fact that an algebraic set $W = \varphi\overline{(V \setminus (V'_c \cup V'_d)}^{\mathcal{Z}}$ contains an open non empty subset of V'_d that W contains T an irreducible component of V'_d. Clearly a "stick" in Whitney's (or Cartan's) umbrella is a counter-example to this statement. In conclusion, inclusion (1), which is crucial for the proof, is not proved in [T3].

4. Examples

4.1. Example. — There exists $g : X \to Y$ which is a homeomorphism with algebraic graph such that Y has 2 irreducible components Y_1 and Y_2, but X is irreducible (here irreducibility is in the Zariski sense). We can assume that $\overline{g^{-1}(Y_1)}^{\mathcal{Z}} \subset \overline{g^{-1}(Y_2)}^{\mathcal{Z}} = X$. Put $V = g^{-1}(Y_2)$, observe that $g(V) \subset Y_2$ but $g(\overline{V}^z) = g(X) \not\subset Y_2 = \overline{Y}_2^z$.

Construction. Put

$$X = \{z^2 - x^2(y^2 - x^3) = 0\} \subset \mathbb{R}^3$$
$$Y_1 = \{x^2 + z^2 = 0\}, \ \ Y_2 = \{z^2 - (y^2 - x^3) = 0\}, \ \ Y = Y_1 \cup Y_2$$
$$\varphi(x, y, z) = (x, y, xz) \ \text{(restricted to } Y_2), \ \ \varphi| = \mathrm{id}_{Y_1}.$$

Finally $g = \varphi^{-1}$.

4.2. Example. — There exists $h : X \to Y$ which is continuous bijective with algebraic graph (even regular) but h^{-1} is non continuous. We put

$$Y = \{z^2 - x^2(y^2 - x^3) = 0\}$$
$$X_1 = \{z^2 - (y^2 - x^3) = 0, \ \ t = 0\}$$
$$X_2 = \{x = z = 0, \ \ yt = 1\}, \ X = X_1 \cup X_2$$
$$h(x, y, z, t) = (x, y, xz, t) \ \text{ on } \ X_1$$
$$h(x, y, z, t) = (x, y, z) \ \text{ on } \ X_2.$$

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