CROSSING NUMBER AND LOWER CENTRAL SERIES OF A SURFACE GROUP

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0. — Let S^g be a closed oriented surface, $\Gamma = \pi_1(S)$. For Z a conjugacy class in Γ a crossing number $d(Z)$ is defined as a minimal number of self-intersection points taken over all immersed loops with only double crossings, representing Z. Let $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \cdots$ be the lower central series of Γ ($\Gamma_{i+1} = [\Gamma_i, \Gamma_i]$). In this paper, we prove the following result.

THEOREM. — *For* $Z \in \Gamma_r$, $d(Z) \geq \frac{1}{2} r + \text{const}$.

Strategy of the proof. — The key point in our approach is the use of group cohomology (of finite p-groups), namely the construction of Bogomolov-Barge-Ghys [BG], and the argument of [Re1].

1. — Let p be a prime, A a vector space over $\mathbb{F}_p$ of dimension $2g$. Let $\omega \in \Lambda^2 A$ be a symplectic form on A^*. Consider a universal central extension

$$0 \longrightarrow \Lambda^2 A \longrightarrow B \longrightarrow A \longrightarrow 0\,.$$

Quotioning by $\mathbb{F}_p \cdot \omega$, we arrive to an extension

$$0 \longrightarrow \Lambda^2 A/\mathbb{F}_p \cdot \omega \longrightarrow B' \longrightarrow A \longrightarrow 0\,.$$

The extension class of B' is just an isomorphism $H_2(A, \mathbb{Z}) \simeq \Lambda^2 A$, followed by a projection fo $\Lambda^2 A/\mathbb{F}_p \omega$.

COROLLARY 1.1. — *The image of the natural homomorphism* $H_2(B', \mathbb{Z}) \to H_2(A, \mathbb{Z})$ *is generated by* ω.

Proof is immediate from the LHS spectral sequence.

Now, by a theorem of Hopf, there is a homomorphism φ of a surface group $\pi_1(S^{g''})$ to B', representing an element of $H_2(B', \mathbb{Z})$, whose projection to $H_2(A, \mathbb{Z})$ is ω. By an argument of [Re1] there is another homomorphism, of a group $\pi_1(S^{g'})$ to B' with the same

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property which is essentially injective, that is, no class of a simple essential loop in $S^{g'}$ is taken to one. We claim $g' \geq g$. Indeed, by construction there is a homomorphism of $\pi_1(S^{g'})$ to A, inducing a map in homology whose image contain ω. Let $V = H_1(S^{g'}, \mathbb{F}_p)$, then the image of $H_2(\pi_1(S^{g'}))$ in $\Lambda_2 A$ is contained in the image of $\Lambda^2 V$ under the induced map. But ω is symplectic, so necessarily $g' \geq g$.

On the other hand we could start with a map of $\pi_1(S^g) = \Gamma$ to A, representing ω (recall that $A \simeq \Gamma/[\Gamma,\Gamma]$). We claim that such map lifts to B'. Indeed, it is equivalent to saying that the restriction class of B', pulled back to Γ becomes 0, but this is obvious.

Summing up we arrive to an essentially injective map of $\pi_1(S^g)$ to B'.

COROLLARY 1.2. — *Let $Z \in \Gamma$ be a conjugacy class of a simple separating essential loop, then the image of Z in $\Gamma_1/[\Gamma,\Gamma_1] \simeq \mathbb{Z}^{2g+\binom{2g}{2})}$ is a nonzero primitive vector.*

Proof. — Let $\psi : \Gamma \to B'$ be an essentially injective map. Since $Z \in \Gamma_1$, $\psi(Z)$ lies in $\Lambda^2 A/\mathbb{F}_p \cdot \omega$ and is nonzero. Since p may be any prime, the result follows.

Proof of the Theorem. — Let L be an immersed loop with minimal number $d(Z)$ of crossings, representing Z. Then $L = L_1 \cup L_2$ where L_1 is simple essential and $d(L_2) < d(Z)$. By the proof of the Corollary 1.2 there is a map ψ of Γ to a 2-nilpotent p-group B' such that $\psi([L_1]) \neq 1$. Let $Z \in \Gamma_r$, $r \geq 2$. Then $\psi(Z) = 1$. Let S' be a normal covering of S of index $|B'|$, corresponding to ψ, that is, $\pi_1(S') = \operatorname{Ker}\psi$. Then L lifts as a closed loop L' to S' and $d(L') \leq d(L) - 1$. Moreover, since B' is two-step nilpotent, then if $Z \in \Gamma_r$, $[L] \in \Gamma'_{r-2}$. The theorem follows now by induction.

References

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