SUBVARIETIES OF SHIMURA VARIETES HAVE POSITIVE SIMPLICIAL VOLUME

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1. Gromov's category and algebraic geometry.

Twenty years ago, Gromov suggested to study a category Gr_n where objects were compact oriented n-dimensional manifolds and morphisms were nonzero degree maps. If $M, N \in \mathrm{Gr}_n$ and $\mathrm{Mor}(M, N) \neq \emptyset$, then one says that M dominates N. The image of the fundamental group of M in that of N has finite index, in particular a simply-connected manifold never dominates a $K(\pi, 1)$-manifold. On the contrary, in Gr_3 any 3-manifold is dominated by a hyperbolic manifold([1]). The simplicial volume (or Gromov's invariant) suits nicely this category, as it satisfies a Gromov-Thurston inequality: if $\varphi : M \to N$ a dominating map, then

$$\mathrm{Vol}_s(M) \geq |\deg \varphi| \, \mathrm{Vol}_s N \, .$$

If X, Y are two smooth proper varieties over $\mathrm{Spec}\,\mathbb{C}$ of dimension n, and $\varphi : X \to Y$ a dominating — in the sense of algebraic geometry — morphism, then X, Y are also objects of Gr_{2n} and φ is dominating in the above sense. In particular, $\mathrm{Vol}_s(X) \geq \deg \varphi \cdot \mathrm{Vol}_s Y$. A fundamental question is:

1.1. MAIN PROBLEM. — *Which algebraic varieties are dominated by ones with positive simplicial volume?*

A theorem of Gromov-Thurston asserts that a smooth compact manifold M which admits a metric of negative sectional curvature, has positive simplicial volume. In the other direction, if M has positive simplicial volume, then it is essential, that is, the natural map $M \to K(\pi_1(M), 1)$ induces a nonzero homomorphism of $H_n(\cdot\,, \mathbb{R})$. Besides this, not much is known [Ber].

2. Locally symmetric spaces, Gromov conjecture and a theorem of Savage.

Let G be a semisimple Lie group, Γ a cocompact lattice in G, K a maximal compact group. The following problem has attracted a lot of attention:

Mots-clés : Shimura varieties, simplicial volume.

(1) I would like to thank M. Boileau for this remark.

2.1. Problem. — *Does* $\Gamma \setminus G/K$ *have a positive simplicial volume?*

A conjecture of Gromov [Gr] says that the answer should be yes if G/K is a symmetric space.

Known cases.

1. If G is a product of the symmetry groups of rank one noncompact symmetric spaces $(\mathbb{R}\mathcal{H}^n, \mathbb{C}\mathcal{H}^n, \mathbb{H}\mathcal{H}^n, \mathbb{C}a\mathcal{H}^2)$ then the answer is yes. This because these spaces carry a metric with negative sectional curvature.

2. If $G = \mathrm{SL}(n, \mathbb{R})$ then the answer is yes [Sav](2).

3. On the other hand, if G has infinite center (like $\widetilde{\mathrm{SL}(2,\mathbb{R})}$) then the answer is no. However, some corollaries of the Gromov-Thurston inequality survive [Re1].

3. Statement of the main results.

Theorem A. — *Let* Γ *be a cocompact lattice in* $\mathrm{Sp}(2n, \mathbb{R})$. *Let* $Y = \Gamma\setminus\mathrm{Sp}(2n, \mathbb{R})/U(n)$ *be a Shimura variety. Let* $X \subset Y$ *be a subvariety. Then* $\mathrm{Vol}_s(X) > 0$.

Theorem B. — *Let* $Z \to X$ *be a family of smooth projective curves of genus* $g \geq 2$ *over a smooth proper base, generic in the sense that the induced morphism* $X \to \mathcal{M}_g$ *has generically maximal rank. Then* $\mathrm{Vol}_s(X) > 0$.

Theorem C. — *Gromov conjecture is true for all Hermitian locally symmetric spaces with a classical group of symmetry, that is, of type I, II, III, IV of Siegel's classification.*

Remarks.

1. The definition of the simplicial volume can be carried over to all proper varieties over $\mathbb{C}$, not necessarily smooth, and the Gromov-Thurston inequality is still valid for dominating (in the sense of algebraic geometry) morphisms. In theorem A we don't suppose that X is smooth.

2. It is very temptating to conjecture that in theorem B, $\mathrm{Vol}_s(Z)$ is also positive. The following related conjecture(3) has been made by Dieter Kotschisck: if $\begin{array}{c}\Sigma \to M \\ \downarrow \\ S\end{array}$ is a C^∞-fibration of closed surfaces over a closed surface, both of genus ≥ 2, then $\mathrm{Vol}_s(M) > 0$.

(2) The note of Leutzinger [L] is unfortunally found wrong.

(3) I thank M. Boileau for telling me about this conjecture.

4. Three cohomology classes in $H^2(\mathrm{Sp}^\delta(2n,\mathbb{R}))$.

4.1. THE EXTENSION CLASS. — The universal covering $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ has a center, isomorphic to $\mathbb{Z}$, so we get an extension $1 \to \mathbb{Z} \to \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R}) \to 1$. Correspondingly, we have an extension class (δ stands for discrete)

$$e \in H^2\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{Z}\right) .$$

4.2. THE REGULATOR. — Let $\mathcal{H} = \mathrm{Sp}(2n,\mathbb{R})/U(n)$ be the Siegel upper half-plane. This is a contractible manifold with a canonical $\mathrm{Sp}(2n,\mathbb{R})$-invariant Kähler form ω. By a general theory ([Re 2], section 3) this defines a class $\mathrm{Bor}(\omega) \in H^2(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R})$. Moreover, this class can be represented by a continuous cocycle (a Dupont construction, see [Re 3]). So we may, and will, be thinking of $\mathrm{Bor}(\omega)$ as an element of $H^2_{\mathrm{cont}}(\mathrm{Sp}(2n,\mathbb{R}),\mathbb{R})$.

4.3. THE MASLOV CLASS. — This class appears in many places, see [Ne], [BG] for example. Fix a Lagrangian plane L in $\mathbb{R}^{2n}$. For $g_1, g_2 \in \mathrm{Sp}(2n,\mathbb{R})$ define

$$\mu(g_1,g_2) = i(L, g_1 L, g_1 g_2 L) ,$$

where i is the Maslov index [GS]. This gives a class in the bounded cohomology group: $\mu \in H^2_b\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right)$.

4.4. COMPARISON THEOREM(4). — *The images of the classes e, $\mathrm{Bor}(\omega)$, m in $H^2\left(\mathrm{Sp}^\delta(2n),\mathbb{R}\right)$ coincide up to a nonzero multiplier.*

5. Proof of the comparison theorem. — We first remark that $H^2_{\mathrm{cont}}(\widetilde{\mathrm{Sp}}(2n,\mathbb{R}),\mathbb{R}) = 0$. Indeed, it follows immediately from the Van Est spectral sequence if we account that $H_1(\widetilde{\mathrm{Sp}}(2n,\mathbb{R})^{\mathrm{top}},\mathbb{R}) = 0$ and $H^2(\mathfrak{s}p(2n,\mathbb{R}),\mathbb{R}) = 0$. Let $\mathrm{Bor}^\delta(\omega)$ be the image of $\mathrm{Bor}(\omega)$ in $H^2(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R})$. It follows that the pullback of $\mathrm{Bor}^\delta(\omega)$ to $H^2\left(\widetilde{\mathrm{Sp}}^\delta(2n,\mathbb{R}),\mathbb{R}\right)$ is zero. It now follows from the Lyndon-Serre-Hochschild spectral sequence that $\mathrm{Bor}^\delta(\omega)$ is proportional to e. Indeed, the spectral sequence reduces to a Gysin long exact sequence

$$H^i\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right) \xrightarrow{\cup e} H^{i+2}\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right) \longrightarrow H^{i+2}\left(\widetilde{\mathrm{Sp}}^\delta(2n,\mathbb{R}),\mathbb{R}\right) \longrightarrow H^{i+1}\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right) \longrightarrow \cdots$$

so the kernel of the pullback map $H^2\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right) \to H^2\left(\widetilde{\mathrm{Sp}}^\delta(2n,\mathbb{R}),\mathbb{R}\right)$ is generated by e. Now, the image of m in $H^2\left(\mathrm{Sp}^\delta(2n,\mathbb{R}),\mathbb{R}\right)$ is proportional to e by [BG]. The theorem will follow if we prove that $\mathrm{Bor}^\delta(\omega) \neq 0$. This is implied by existence of cocompact lattices and the fact that ω is a Kähler form, see the next section.

(4) Inspired by a conversation with M.S. Narashimhan.

6. Proof of the main results.

Proof of Theorem A. — If ω' is a restriction of the Kähler form ω of $Y = \Gamma \setminus \mathrm{Sp}(2n,\mathbb{R})/U(n)$ on X then $\int_X \omega'^m > 0$, where $m = \dim_{\mathbb{C}} X$. On the other hand the inclusion $X \subset Y$ induces a class $[X]$ in $H_{2m}\left(\mathrm{Sp}^{\delta}(2n,\mathbb{R}),\mathbb{Z}\right)$ and by definition of $\mathrm{Bor}(\omega)$ [Re 2] one has

$$\int_X \omega'^m = \left((\mathrm{Bor}^{\delta}(\omega))^m,[X]\right) .$$

It follows that $\mathrm{Bor}^{\delta}(\omega) \neq 0$ (which finishes the proof of the Comparison Theorem). Moreover, by the Comparison Theorem, $\mathrm{Bor}^{\delta}(\omega) = C\cdot\mu$ where $c \neq 0$. But μ is a bounded cohomology class, so $|(\mu^m,[X])| \leq \|\mu^m\|_{\ell^0}\cdot \|[X]\|_{\ell^1}$ by definition. So $\|[X]\|_{\ell^1} \neq 0$.

First proof of Theorem B. — Let Map_g be the mapping class group, $\varphi : \mathrm{Map}_g \to \mathrm{Sp}(2g,\mathbb{Z})$ the canonical homomorphism. It is well-known that $H^2(\mathrm{Map}_g,\mathbb{R}) = \mathbb{R}$ and φ^* is an isomorphism on H^2 level.

Let Ω be the Weil-Peterson Kähler form in the Teichmüller space T_g, invariant under Map_g. A fibration by smooth topological closed surfaces $Z \to X$ defines a flat bundle over $\mathcal{F}$, X with T_g as a fiber [Mor 1]. If this is a holomorphic family, then $\mathcal{F}$ has a holomorphic section, say S.

Next, the general theory of regulators gives a class $\mathrm{Bor}(\Omega) \in H^2(\mathrm{Map}_g,\mathbb{R})$, defined by Ω. This class necessarily is proportional to a pullback of $\mu \in H^2\left(\mathrm{Sp}(2g,\mathbb{R}),\mathbb{R}\right)$ therefore is bounded. On the other hand, the value of $\mathrm{Bor}^m(\Omega)$ on $[X]$ is

$$(\mathrm{Bor}^m(\Omega),[X]) = \int_X (S^*\Omega)^m > 0 ,$$

since S is generic. It follows that $\mathrm{Vol}_s(X) = \|[X]\|_{\ell^1} > 0$.

Second proof of Theorem B. — Let Homeo_s be a group of quasi-symmetric homeomorphisms of a circle. There is an Euler class $e \in H^2_{\mathrm{cont}}(\mathrm{Homeo}_s(S^1))$. It is bounded, in fact given by an explicit cocycle valued in $\{0,\pm 1\}$. There is a homomorphism $\mathrm{Map}_g \to \mathrm{Homeo}_s(S^1)$ so that the pull-back of the Euler class is a generator of $H^2(\mathrm{Map}_g,\mathbb{R})$ ([Mor 2]). Then the proof goes as above.

Remark. — If the fibers of $Z \to X$ are not generically hyperelliptic, then theorem B follows directly from theorem A.

Proof of Theorem C. — Let G/K be a classical hermitian symmetric space. By a theorem of Satake [Sa] there exists an equivariant holomorphic embedding $G/K \to \mathrm{Sp}(2N,\mathbb{R})/U(N)$ for some N, induced by a homomorphism $\psi : G \to \mathrm{Sp}(2N,\mathbb{R})$. The Kähler form of $\mathrm{Sp}(2N,\mathbb{R})/U(N)$ restricts to a G-invariant Kähler form ω on G/K and the latter gives a regulator $\mathrm{Bor}(\omega) \in H^2_{\mathrm{cont}}(G,\mathbb{R})$. On the other hand, the Maslov class of

$\mathrm{Sp}(2N,\mathbb{R})$ restricts to a bounded class m in $H_b^2(G^\delta,\mathbb{R})$. By functoriality, $\mathrm{Bor}(\omega)$ and μ are proportional as classes in $H^2(G^\delta,\mathbb{R})$. Then the proof goes exactly as above.

Remarks.

1. This argument cannot be applied for exceptional domains.

2. We prove in [Re4] that for any compact Kähler surface X whose fundamental group admits a homomorphism into a lattice in $\mathrm{Sp}(4)$ or $\mathrm{SO}(2,n)$, on n odd, such that the corresponding linear representation is rigid, the simplicial volume of X is positive.

3. Kodaira surfaces (families of smooth projective curves over a smooth projective curve) of fiber rank at least three are uniformized by a bounded (non-symmetric) domain in $\mathbb{C}^2$. It is possibly true that the Bergman metric of the domain induces a bounded class in the second cohomology of their fundamental group.

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