ON THE REPRESENTATIONS OF FUNDAMENTAL GROUPS OF RAMIFIED COVERINGS IN $SL_2(\overline{\mathbb{F}}_q)$

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Recently a spectacular progress in our understanding of a big class of closed three-manifolds has been made by Freedman-Freedman. Namely, in [FF] they proved a theorem, whose refined version [CL] implies:

(*) Let $K \subset M$ be a non-fibered knot in a homology sphere. Let $M_n \to M$ be a ramified covering along K. Then for $M \gg 1$, M_n is Haken.

A purely different analytic approach, also proving the above theorem, has been developed in [Re 1]. Moreover in [Re 1] it is proved that *any* ramified covering (say, with $n = 2$) is Haken after a big Dehn surgery ([Re 1], theorem A6).

Before [FF], essentially the only way to construct incompressible surface was to exhibit an action of π_1 on a tree of $\mathrm{SL}_2(K_v)$ where K_v is a discrete valuation field. Though neither the approach of [FF] and [CL] nor the approach of [Re 1] deal with such actions, one may ask:

(*) Do π_1 of ramified coverings admit a non-rigid representations to $\mathrm{SL}_2(K)$, K a field?

Infinitesimally, the answer is yes, as the our main result asserts:

THEOREM. — *Let M be an irreducible homology sphere with rich fundamental group, e.g. hyperbolic. Let $\rho : \pi_1(M) \to \mathrm{SL}_2(F)$, F a number field, is a rigid representation, and let $\rho_p : \pi_1(M) \to \mathrm{SL}_2(\mathbb{F}_q)$, q a power of a prime p, is a surjective reduction. For K a knot in M, not contained in a ball, let $\bar{\rho}_p : \pi_1(M_p) \to \mathrm{SL}_2(\mathbb{F}_q)$ be a composition homomorphism. Then for all but finitely many p,*

$$H^1\left(\pi_1(M_p), \mathfrak{s}l_2(\mathbb{F}_q)\right) \neq 0\,.$$

Remarks.

1. We refer to [Re 2], [RM] for all notions used in the statement of the theorem.

2. To actually have a representation variety one need to check that all Massey products are zero and that a formal variation of a representation is integrated to an actual variation. We do not know if that is true.

Mots-clés : ramified coverings, action on trace.

Lemma (1.1). — *Let $K \subset M$ be a knot as above, let $T = \partial N(K)$ be a boundary of a tubular neighbourhood of K in M. Let $x, y \in \pi_1(T)$ be a parallel and a meridian (y is not defined uniquely). Let $\pi_1(M \setminus K) \xrightarrow{\varphi} \pi_1(M)$ be a canonical surjection and let $Q = \operatorname{Ker}\varphi$ and let $W = Q/[Q,Q]$ as $\pi_1(M)$-modules. Then $W \simeq \mathbb{Z}[\pi_1(M)/\langle x\rangle]$.*

Proof. — Let V be a $\pi_1(M)$-module. The Meyer-Wietoris sequence for V is (observe that both M and $M \setminus K$ are acyclic)

$$H_2(\pi_1(M), V) \longrightarrow H_1(\mathbb{Z} \oplus \mathbb{Z}, V) \longrightarrow H_1(\pi_1(M \setminus K), V) \oplus H_1(\langle y\rangle, V) \longrightarrow H_1(\pi_1(M), V) \longrightarrow \cdots$$

If we put $V = \mathbb{Z}[\pi_1(M)]$, then $H_1(\pi_1(M \setminus K), V) = W$ by Shapiro's lemma. Moreover, $H_i(\pi_1(M), V) = 0$. Computing $H_1(\mathbb{Z} \oplus \mathbb{Z}, V)$ by the spectral sequence, we arrive immediately to the result.

Now, let M_p be a p-fold ramified covering along K. We have a diagram

$$\begin{array}{ccccc}
1 & & & & \\
\downarrow & & & & \\
\pi_1(M_p \setminus K) & \longrightarrow & \pi_1(M_p) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(M \setminus K) & \longrightarrow & \pi_1(M) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\mathbb{Z}_p & & 1 & &
\end{array}$$

Let R be a kernel of the composed map $\pi_1(M_p \setminus K) \to \pi_1(M)$. We will compute $R/[R,R]$ in two ways. First, we have a sequence

$$1 \longrightarrow R \longrightarrow Q \longrightarrow \mathbb{Z}_p \longrightarrow 1$$

from which we have a short exact sequence

$$0 \longrightarrow H_0(\mathbb{Z}_p, R/[R,R]) \longrightarrow W \xrightarrow{\varepsilon} \mathbb{Z}_p \longrightarrow 0$$

where after identification $W \simeq \mathbb{Z}[\pi_1(M)/\langle x\rangle]$, ε becomes an argumentation map.

Second, we have a short exact sequence

$$1 \longrightarrow Q_p \longrightarrow R \longrightarrow \operatorname{Ker}(\pi_1(M_p) \to \pi_1(M)) \longrightarrow 1$$

where Q_p is the kernel of the map $\pi_1(M_p \setminus K) \longrightarrow \pi_1(M_p)$. This gives

$$H_0\left(\operatorname{Ker}(\pi_1(M_p) \to \pi_1(M)), \mathbb{Z}[\pi_1(M_p)/\langle y^p\rangle]\right) \longrightarrow R/[R,R] \longrightarrow H_1\left(\operatorname{Ker}(\pi_1(M_p) \to \pi_1(M))\right) \longrightarrow 0\,.$$

Let $G = \operatorname{Ker}(\pi_1(M_p) \to \pi_1(M))$. We have

$$H_0(G, \mathbb{Z}[\pi_1(M_p), \langle y^p\rangle]) = \mathbb{Z}[\pi_1(M)/\langle y^p\rangle]$$

so that the sequence above becomes

$$\mathbb{Z}[\pi_1(M)/\langle y^p\rangle] \longrightarrow R[R,R] \longrightarrow H_1(G) \longrightarrow 0$$

(the first map can in principle be not injective because of d_2 in the LHS spectral sequence).

Applying the functor $H_0(\mathbb{Z}_p, \cdot\,)$ we get

$$\mathbb{Z}[\pi_1(M)/\langle y^p\rangle] \longrightarrow H_0(\mathbb{Z}_p, R/[R,R]) \longrightarrow H_0(\mathbb{Z}_p, H_1(G)) \longrightarrow 0,$$

or

$$\mathbb{Z}[\pi_1(M)/\langle y^p\rangle] \longrightarrow \operatorname{Ker}\varepsilon \longrightarrow H_0(\mathbb{Z}_p, H_1(G)) \longrightarrow 0.$$

The first map is the projection $\mathbb{Z}[\pi_1(M)/\langle y^p\rangle) \to \mathbb{Z}[\pi_1(M)/\langle y\rangle]$ followed by the multiplication by p, so $H_0(\mathbb{Z}_p, H_1(G)) = \mathbb{F}_p[\pi_1(M)/\langle y\rangle]$.

Now, let D be any $\mathbb{F}_p[\pi_1(M)]$-module. We have a piece of the spectral sequence.

$$H^0(\pi_1(M), H^1(G,D))$$

$$\Longrightarrow H^{i+j}(\pi_1(M_p), D)$$

$$H^1(\pi_1(M), D) \qquad H^2(\pi_1(M), D) \qquad H^3(\pi_1(M), D)$$

Since D as a G-module is trivial,

$$H^0(\pi_1(M), H^1(G,D)) = \operatorname{Hom}_{\pi_1(M)}(G, D) = \operatorname{Hom}_{\pi_1(M)}(H_1(G), D)\,.$$

Now,

$$\begin{aligned} H^0(\mathbb{Z}_p, \operatorname{Hom}_{\pi_1(M)}(H_1(G), D)) &= \operatorname{Hom}_{\pi_1(M)}(H_0(\mathbb{Z}_p, H_1(G)), D) \\ &= \operatorname{Hom}_{\pi_1(M)}(\mathbb{F}_p[\pi_1(M)/\langle y\rangle], D) \\ &= \operatorname{Inv}_{\langle y\rangle} D\,. \end{aligned}$$

Now, $H^i(\pi_1(M), D) = H_{3-i}(\pi_1(M), D)$ by Poincaré duality. So if $\operatorname{Inv}_{\langle y\rangle} D \neq 0$ and $H_i(\pi_1(M), D) = 0$, $i = 0, 1$, then $H^1(\pi_1(M_p), D) \neq 0$.

Now we let $D = \mathfrak{sl}_2(\mathbb{F}_q)$, an adjoint module. Obviously $\operatorname{Inv}_{\langle y\rangle} D \neq 0$. Next, $H_0(\pi_1(M), D) = 0$ as there is no center in the Lie algebra $\mathfrak{sl}_2(\mathbb{F}_q)$. The only question to check is wether $H_1(\pi_1(M), \mathfrak{sl}_2(\mathbb{F}_q)) = 0$.

LEMMA (2.1). — *Let F be a number field, Γ a finitely generated group, V a finite dimensional over F, $F[\Gamma]$-module. If for infinitely many prime ideals $\mathcal{P}$ in F the reduction module $V_{\mathcal{P}}$ satisfies $H_i(\Gamma, V_{\mathcal{P}}) \neq 0$, then $H_i(\Gamma, V) \neq 0$.*

Remark. — The reduction module is defined for all but finitely many $\mathcal{P}$.

Proof is simple linear algebra and is left to the reader.

Now, $H_1(\pi_1(M), \mathfrak{sl}_2(\mathbb{C}))$ is 0 because ρ is rigid, so $H_1(\pi_1(M), \mathfrak{sl}_2(\mathbb{F}_q)) = 0$ for all but finitely many $\mathcal{P}$. The theorem is proved.

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