MULTIPLE FLAG VARIETIES OF FINITE TYPE

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ABSTRACT. We classify all tuples of flag varieties with finitely many orbits under the diagonal action of the general linear group. We also classify the orbits in each case and construct explicit representatives.

1. INTRODUCTION

The Schubert calculus for a reductive group G can be viewed as the study of G-orbits and their closures in the product of two flag varieties $G/P \times G/Q$, where P and Q are parabolic subgroups in G. The number of such G-orbits is always finite. This fundamental fact makes the Schubert calculus especially interesting from the combinatorial standpoint. For example, if B is a Borel subgroup in G then G-orbits in $G/B \times G/B$ are parametrized by the Weyl group W, in view of the Bruhat decomposition $G = \bigcup_{w \in W} BwB$.

In a natural attempt to generalize the Schubert calculus, one can ask for which tuples of parabolic subgroups $(P_1, \ldots, P_k)$ the group G has finitely many orbits when acting diagonally in the product of several flag varieties $G/P_1 \times \cdots \times G/P_k$. Note that if $P_1 = B$ is a Borel subgroup, then $G/P_1 \times \cdots \times G/P_k$ has finitely many G-orbits exactly if $G/P_2 \times \cdots \times G/P_k$ is a spherical variety. Thus, our problem includes that of classifying the multiple flag varieties of spherical type.

To the best of our knowledge, the problem of classifying all finite-orbit tuples for an arbitrary G is still open. In the special case when $k = 3$, $P_1 = B$, and P_2 and P_3 are maximal parabolic subgroups, such a classification was given in [6].

Here we present a complete solution of the classification problem for $G = GL_n$. We also classify the orbits in each case and construct their explicit representatives. Precise formulations of the main results will be given in the next section; the proofs are given in Sections 3 and 4. In Section 4.3 we also discuss some partial results on the generalized Bruhat order given by adjacency of orbits.

We use results and ideas from the theory of quiver representations. In fact, our classification (Theorem 2.2 below) is very close to (but distinct from) the classification of all quiver representations of finite type due to V. Kac [5].

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2. Main results

2.1. Classification theorem. Let $\mathbf{a} = (a_1, \dots, a_p)$ be a nonnegative integer vector with the sum of components equal to n. We call such a vector a *composition* of n, and $a_1, \dots, a_p$ the *parts* of $\mathbf{a}$. For a vector space V of dimension n over an algebraically closed field, we denote by $\mathrm{Fl}_{\mathbf{a}}(V)$ the variety of flags $A = (0 = A_0 \subset A_1 \subset \cdots \subset A_p = V)$ of vector subspaces in V such that

$$\dim(A_i/A_{i-1}) = a_i \quad (i = 1, \dots, p) .$$

A tuple of compositions $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ of the same number n is said to be of *finite type* if the group $GL(V)$ (acting diagonally) has finitely many orbits in the *multiple flag variety* $\mathrm{Fl}_{\mathbf{a}_1}(V) \times \cdots \times \mathrm{Fl}_{\mathbf{a}_k}(V)$. We say that a composition is *trivial* if it has only one non-zero part n. Then the corresponding flag variety consists of a single point, so adding any number of trivial compositions to a tuple gives essentially the same multiple flag varieity, and does not affect the finite type property.

Theorem 2.1. *If a tuple of non-trivial compositions $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ is of finite type then $k \leq 3$.*

In other words, a multiple flag variety of finite type cannot have more than 3 non-trivial factors. By Theorem 2.1, any tuple of compositions of finite type can be made into a triple by adding or removing trivial compositions. Thus we only need to classify *triples* of finite type. We will write $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ instead of $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, and we denote by p, q and r the numbers of non-zero parts in $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, respectively. Henceforth we assume without loss of generality that $p \leq q \leq r$. We also denote by $\min(\mathbf{a})$ the minimum of the non-zero parts of a composition $\mathbf{a}$. Now everything is ready for the formulation of our first main theorem.

Theorem 2.2. *A triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of compositions is of finite type if and only if it belongs to one of the following classes:*

$(A_{q,r})$ $(p, q, r) = (1, q, r),\ 1 \leq q \leq r.$

(D_{r+2}) $(p, q, r) = (2, 2, r),\ 2 \leq r.$

(E_6) $(p, q, r) = (2, 3, 3).$

(E_7) $(p, q, r) = (2, 3, 4).$

(E_8) $(p, q, r) = (2, 3, 5).$

$(E^{(a)}_{r+3})$ $(p, q, r) = (2, 3, r),\ 3 \leq r,\ \min(\mathbf{a}) = 2.$

$(E^{(b)}_{r+3})$ $(p, q, r) = (2, 3, r),\ 3 \leq r,\ \min(\mathbf{b}) = 1.$

$(S_{q,r})$ $(p, q, r) = (2, q, r),\ 2 \leq q \leq r,\ \min(\mathbf{a}) = 1.$

The types in Theorem 2.2 have some obvious overlaps. The class $(A_{q,r})$ covers all multiple flag varieties with less than three non-trivial factors. The type $(S_{q,r})$ appeared in [4]. For relations with the classification of quiver representations of finite type due to V. Kac [5], see Remark 3.4.

Note that, for each of the first five types in Theorem 2.2, there are no restrictions on the dimensions of subspaces in the corresponding flag varieties; only the last three types $(E^{(a)})$, $(E^{(b)})$ and (S) involve such restrictions. The first five types are naturally related to Dynkin graphs without multiple edges (as suggested by their names). Let $T = T_{p,q,r}$ denote the graph with $p+q+r-2$ vertices that consists of 3 chains with p, q, and r vertices, joined together at a common endpoint. We see that the cases in our classification with no restrictions on dimensions are precisely those for which T is one of the Dynkin graphs A_n, D_n, E_6, E_7, or E_8. This is of course no coincidence: we will see that this part of our classification is equivalent to Gabriel's classification of quivers of finite type (and follows from the Cartan-Killing clasification of graphs that give rise to positive-definite quadratic forms).

2.2. **Classification of orbits.** Now we describe a combinatorial parametrization of the set of $GL(V)$-orbits in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ for any triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of compositions of finite type. For a composition $\mathbf{a} = (a_1, \dots, a_p)$, we write

$$|\mathbf{a}| = a_1 + \cdots + a_p \quad , \quad ||\mathbf{a}||^2 = a_1^2 + \cdots + a_p^2 \ ;$$

the number p of parts of $\mathbf{a}$ will be denoted $\ell(\mathbf{a})$ and called the *length* of $\mathbf{a}$.

For any positive integers p, q, and r, let $\Lambda_{p,q,r}$ denote the additive semigroup of all triples of compositions $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ such that $(\ell(\mathbf{a}), \ell(\mathbf{b}), \ell(\mathbf{c})) = (p, q, r)$, and $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. For every $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$, we set

$$Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \dim GL(V) - \dim \mathrm{Fl}_{\mathbf{a}}(V) - \dim \mathrm{Fl}_{\mathbf{b}}(V) - \dim \mathrm{Fl}_{\mathbf{c}}(V) \ ,$$

where V is a vector space of dimension $n = |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. An easy calculation shows that

$$Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{2}(||\mathbf{a}||^2 + ||\mathbf{b}||^2 + ||\mathbf{c}||^2 - n^2) \ ; \tag{2.1}$$

the function Q is called the *Tits quadratic form.*

Let $\Pi_{p,q,r}$ denote the set of all triples $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ of finite type such that $Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 1$.

Theorem 2.3. *Let $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ be a triple of compositions of finite type. Then $GL(V)$-orbits in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ are in a natural bijection with families of nonnegative integers $M = (m_{\mathbf{d}})$ indexed by $\mathbf{d} \in \Pi_{p,q,r}$ such that, in the semigroup $\Lambda_{p,q,r}$,*

$$\sum_{\mathbf{d} \in \Pi_{p,q,r}} m_{\mathbf{d}} \mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}).$$

The set $\Pi_{p,q,r}$ can be explicitly described as follows. For a composition $\mathbf{a}$, we denote by $\mathbf{a}^+$ the partition obtained from $\mathbf{a}$ by removing all zero parts and rearranging the non-zero parts in weakly decreasing order. (For example, if $\mathbf{a} = (0, 2, 1, 0, 3, 2)$ then $\mathbf{a}^+ = (3, 2, 2, 1)$.) We denote $(a^p) = \underbrace{(a, \dots, a)}_{p \text{ parts}}$.

Theorem 2.4. *A triple $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ belongs to $\Pi_{p,q,r}$ if and only if the (unordered) triple of partitions $\{\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+\}$ is one of the following:*

$$\{(1), (1), (1)\}, \quad \{(3^2), (2^3), (2, 1, 1, 1, 1)\}, \quad \{(4, 2), (2^3), (1^6)\},$$
$$\{(m+1, m), (m, m, 1), (1^{2m+1})\} \ (m \geq 2), \quad \{(m, m), (m, m-1, 1), (1^{2m})\} \ (m \geq 2),$$
$$\{(n-1, 1), (1^n), (1^n)\} \ (n \geq 2).$$

To describe the bijection in Theorem 2.3, we introduce the following additive category $\mathcal{F}_{p,q,r}$. The *objects* of $\mathcal{F}_{p,q,r}$ are families $(V; A, B, C)$, where V is a finite-dimensional vector space, and (A, B, C) is a triple of flags in V belonging to $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ for some $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$. The triple $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is called the *dimension vector* of $(V; A, B, C)$. A *morphism* from $(V; A, B, C)$ to $(V'; A', B', C')$ in $\mathcal{F}_{p,q,r}$ is a linear map $f : V \to V'$ such that $f(A_i) \subset A'_i$, $f(B_i) \subset B'_i$, and $f(C_i) \subset C'_i$ for all i. Direct sum of objects is taken componentwise on each member of each flag in the objects.

Comparing definitions, we see that isomorphism classes of objects in $\mathcal{F}_{p,q,r}$ with a given dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ are naturally identified with $GL(V)$-orbits in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$. The advantage of dealing with $\mathcal{F}_{p,q,r}$ is that this category admits direct sums, and so every object $(V; A, B, C)$ of $\mathcal{F}_{p,q,r}$ can be decomposed into a direct sum of indecomposable objects. By the Krull-Schmidt theorem (see §3.1), such a decomposition is unique up to an automorphism of $(V; A, B, C)$. So the isomorphism class of an object is determined by the multiplicities of all non-isomorphic indecomposable objects in its decomposition. Theorem 2.3 now becomes a consequence of the following.

Theorem 2.5. *For every $\mathbf{d} \in \Pi_{p,q,r}$, there exists a unique isomorphism class $I_{\mathbf{d}}$ of indecomposable objects in $\mathcal{F}_{p,q,r}$ with the dimension vector $\mathbf{d}$. For every triple $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ of finite type, any object in $\mathcal{F}_{p,q,r}$ with the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ decomposes (uniquely) into a direct sum of objects $I_{\mathbf{d}}$.*

Corollary 2.6. *The bijection in Theorem 2.3 sends a family $M = (m_{\mathbf{d}})$ $(\mathbf{d} \in \Pi_{p,q,r})$ to the $GL(V)$-orbit Ω_M in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ corresponding to the isomorphism class $\bigoplus_{\mathbf{d} \in \Pi_{p,q,r}} m_{\mathbf{d}} I_{\mathbf{d}}$ of objects in $\mathcal{F}_{p,q,r}$.*

2.3. **Representatives of orbits.** By Corollary 2.6, in order to give an explicit representative of each $GL(V)$-orbit in a multiple flag variety of finite type, it is enough to exhibit a triple of flags that represents every indecomposable object $I_{\mathbf{d}}$ in Theorem 2.5. All possible dimension vectors $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Pi_{p,q,r}$ are described in Theorem 2.4. Note that vanishing of some part a_i in a composition $\mathbf{a}$ means that in any flag $A \in \mathrm{Fl}_{\mathbf{a}}(V)$ the subspace A_i coincides with A_{i-1}. Thus in constructing a representative for $I_{\mathbf{d}}$, we can assume without loss of generality that none of the compositions $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ have zero parts. So it is enough to treat all the dimension vectors $\mathbf{d}$ obtained from the triples $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+)$ in Theorem 2.4 by some permutations of the parts. In particular, we assume that $p = \ell(\mathbf{a}) \leq 2$; thus a flag $A \in \mathrm{Fl}_{\mathbf{a}}(V)$ is determined by one vector subspace A_1 of dimension a_1 in V. Under these assumptions, we will show that a triple of flags $(V; A, B, C)$ representing $I_{\mathbf{d}}$ can be presented in a standard form according to the following definition.

Definition 2.7. An object $(V; A, B, C)$ in $\mathcal{F}_{p,q,r}$ is presented in *standard form* if V is given a basis $e_1, \ldots, e_n$ with the following properties:

(1) Each subspace B_i of the flag B has a basis consisting of the first $b_1 + \cdots + b_i$ standard basis vectors $e_1, e_2, \ldots$, while each C_i has a basis consisting of the last $c_1 + \cdots + c_i$ basis vectors $e_n, e_{n-1}, \ldots$.

(2) $p \leq 2$, and the vector subspace $A_1 \subset V$ has basis vectors $\sum_{l \in S_1} e_l, \ldots, \sum_{l \in S_{a_1}} e_l$ for some subsets $S_1, \ldots, S_{a_1} \subset \{1, \ldots, n\}$.

(3) The subsets S_k satisfy:

$$\left| \bigcup_{k \neq k'} (S_k \cap S_{k'}) \right| \leq 2 \ .$$

Theorem 2.8. *Let $\mathbf{d} \in \Pi_{p,q,r}$ be a triple of compositions obtained from some triple $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+)$ in Theorem 2.4 by permutations of the parts. Then the corresponding* *indecomposable object $I_{\mathbf{d}}$ in $\mathcal{F}_{p,q,r}$ can be presented in standard form with the collection of subsets $S_1, \dots, S_{a_1}$ chosen as follows:*

$\mathbf{d} = ((1),(1),(1)):\quad S_1 = \{1\}.$

$\mathbf{d} = ((4,2),(2^3),(1^6)):\quad S_1 = \{1,5\},\ S_2 = \{2,3,\},\ S_3 = \{2,5,6\},\ S_4 = \{4,5\}.$

$\mathbf{d} = ((3^2),(2^3),(2,1,1,1,1)):\quad S_1 = \{1,2,3\},\ S_2 = \{1,6\},\ S_3 = \{2,4,5\}.$

$\mathbf{d} = ((3^2),(2^3),(1,1,2,1,1)):\quad S_1 = \{1,5,6\},\ S_2 = \{2,3,6\},\ S_3 = \{4,5\}.$

$\mathbf{d} = ((3^2),(2^3),(1,1,1,1,2)):\quad S_1 = \{1,4,6\},\ S_2 = \{2,4,5\},\ S_3 = \{2,3\}.$

$\mathbf{d} = ((3^2),(2^3),(1,2,1,1,1)):\quad S_1 = \{1,2,4,6\},\ S_2 = \{1,3\},\ S_3 = \{1,5\}.$

$\mathbf{d} = ((3^2),(2^3),(1,1,1,2,1)):\quad S_1 = \{1,2,4,6\},\ S_2 = \{1,3\},\ S_3 = \{1,5\}.$

$\mathbf{d} = ((2,4),(2^3),(1^6)):\quad S_1 = \{1,2,3,6\},\ S_2 = \{1,4,5\}.$

$\mathbf{d} = ((m,m+1),(1,m,m),(1^{2m+1})):$
$S_k = \{1,k+1,2m+2-k\}\quad (1 \le k \le m).$

$\mathbf{d} = ((m+1,m),(1,m,m),(1^{2m+1})):$
$S_1 = \{1,2\},\ S_k = \{1,k+1,2m+3-k\}\quad (2 \le k \le m),\quad S_{m+1} = \{1,m+2\}.$

$\mathbf{d} = ((m,m+1),(m,1,m),(1^{2m+1})):$
$S_k = \{k,m+1,2m+2-k\}\quad (1 \le k \le m).$

$\mathbf{d} = ((m+1,m),(m,1,m),(1^{2m+1})):$
$S_1 = \{1,m+1\},\ S_k = \{k,m+1,2m+3-k\}\ (2 \le k \le m),\ S_{m+1} = \{m+1,m+2\}.$

$\mathbf{d} = ((m,m+1),(m,m,1),(1^{2m+1})):$
$S_k = \{k,2m+1-k,2m+1\}\quad (1 \le k \le m).$

$\mathbf{d} = ((m+1,m),(m,m,1),(1^{2m+1})):\quad S_1 = \{1,2m+1\},$
$S_k = \{k,2m+2-k,2m+1\}\quad (2 \le k \le m),\ S_{m+1} = \{m+1,2m+1\}.$

$\mathbf{d} = ((m,m),(1,m-1,m),(1^{2m})):$
$S_k = \{1,k+1,2m+1-k\}\quad (1 \le k \le m-1),\ S_m = \{1,m+1\}.$

$\mathbf{d} = ((m,m),(1,m,m-1),(1^{2m})):$
$S_1 = \{1,2\},\ S_k = \{1,k+1,2m+2-k\}\quad (2 \le k \le m).$

$\mathbf{d} = ((m,m),(m-1,1,m),(1^{2m})):$
$S_k = \{k,m,2m+1-k\}\quad (1 \le k \le m-1),\ S_m = \{m,m+1\}.$

$\mathbf{d} = ((m,m),(m,1,m-1),(1^{2m})):$
$S_1 = \{1,m+1\},\ S_k = \{k,m+1,2m+2-k\}\quad (2 \le k \le m).$

$\mathbf{d} = ((m,m),(m-1,m,1),(1^{2m})):$
$S_k = \{k,2m-k,2m\}\quad (1 \le k \le m-1),\ S_m = \{m,2m\}.$

$\mathbf{d} = ((m,m),(m,m-1,1),(1^{2m})):$
$S_1 = \{1,2m\},\ S_k = \{k,2m+1-k,2m\}\quad (2 \le k \le m).$

$\mathbf{d} = ((n-1,1),(1^n),(1^n)):\quad S_k = \{1,k+1\}\quad (1 \le k \le n-1).$

$\mathbf{d} = ((1,n-1),(1^n),(1^n)):\quad S_1 = \{1,2,\dots,n\}.$

For any composition $\mathbf{a}$, let $\mathbf{a}_{\text{red}}$ denote the composition obtained from $\mathbf{a}$ by removing all zero parts and keeping the non-zero parts in the same order. For a dimension vector $\mathbf{d} = (\mathbf{a},\mathbf{b},\mathbf{c}) \in \Lambda_{p,q,r}$, we set $\mathbf{d}_{\text{red}} = (\mathbf{a}_{\text{red}},\mathbf{b}_{\text{red}},\mathbf{c}_{\text{red}})$ and call $\mathbf{d}_{\text{red}}$ the *reduced dimension vector* of $\mathbf{d}$. Thus $\mathbf{d} \in \Pi_{p,q,r}$ if and only if $\mathbf{d}_{\text{red}}$ is one of the triples in Theorem 2.8.

Example 2.9. Type $(A_{q,r})$: two partial flags. Let $\mathbf{b} = (b_1, \ldots, b_q)$ and $\mathbf{c} = (c_1, \ldots, c_r)$ be two compositions of n. We can identify a pair of partial flags $(B, C) \in \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ with the object $(V; A, B, C)$ in the category $\mathcal{F}_{1,q,r}$, where A is the trivial flag $(0 = A_0 \subset A_1 = V)$. An indecomposable summand of $(V; A, B, C)$ can only have the reduced dimension vector $((1),(1),(1))$. The indecomposable objects with this reduced dimension vector are of the form $I_{ij} = (V'; A', B', C')$ where $1 \le i \le q$ and $1 \le j \le r$: here $\dim V' = 1$, A' is the trivial flag in V', $B' = (0 = B'_0 = \cdots = B'_{i-1} \subset B'_i = \cdots = B'_q = V')$, and $C' = (0 = C'_0 = \cdots = C'_{j-1} \subset C'_j = \cdots = C'_r = V')$.

It follows that $GL(V)$-orbits in $\mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ are parametrized by $q \times r$ nonnegative integer matrices $M = (m_{ij})$ with row sums $b_1, \ldots, b_q$ and column sums $c_1, \ldots, c_r$; the orbit Ω_M corresponds to a direct sum $\bigoplus_{i,j} m_{ij} I_{ij}$. (In particular, if B and C are complete flags, we obtain the usual parametrization of orbits by permutation matrices.) A representative of Ω_M can be given as follows: V has a basis $\{e_{ijk} : 1 \le i \le q,\ 1 \le j \le r,\ 1 \le k \le m_{ij}\}$, each B_i is spanned by the $e_{i'j'k'}$ with $i' \le i$, and each C_j is spanned by the $e_{i'j'k'}$ with $j' \le j$.

Example 2.10. Type $(S_{q,r})$: two partial flags and a line. As in Example 2.9, let $\mathbf{b}$ and $\mathbf{c}$ be any two compositions of n, but now let us take $\mathbf{a} = (1, n-1)$. Let $(V; A, B, C)$ be a triple of flags of type $(S_{q,r})$ with the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{2,q,r}$. By inspection of the cases in Theorem 2.8, we see that an indecomposable summand of $(V; A, B, C)$ can only have the reduced dimension vector $((1),(1),(1))$ or $((1, t-1), (1^t), (1^t))$ for some $t = 2\ \ldots, n$. The corresponding indecomposable objects are of the following form. First each I_{ij} in the previous example can be also considered as an indecomposable object in the present situation: we take V', B', and C' as above, and define the flag A' as $(0 = A'_0 = A'_1 \subset A'_2 = V')$; by abuse of notation, we denote this indecomposable object in $\mathcal{F}_{2,q,r}$ by the same symbol I_{ij}.

Besides these indecomposables, the object $(V; A, B, C)$ must have precisely one indecomposable summand $(V'; A', B', C')$ with $\dim(A'_1) = 1$ (since $\dim(A_1) = 1$). Such indecomposables are indexed by non-empty sets $\Delta = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ with $1 \le i_1 < \ldots < i_t \le q$ and $r \ge j_1 > \ldots > j_t \ge 1$ (such a Δ can be pictured as the outer corners of a Young diagram contained in a $q \times r$ rectangle). The indecomposable object I_Δ is represented by the following triple of flags $(V'; A', B', C')$: the space V' has basis $e_1, \ldots, e_t$, the subspace A'_1 is spanned by $e_1 + \ldots + e_t$, each B'_i is spanned by the e_l with $i_l \le i$, and each C'_j is spanned by the e_l with $j_l \le j$.

We see that $GL(V)$-orbits in $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ correspond to objects $I_\Delta \oplus \bigoplus m'_{ij} I_{ij}$ in $\mathcal{F}_{2,q,r}$ with the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. It is convenient to use the numbers $m_{ij} = m'_{ij} + 1$ for $(i,j) \in \Delta$ and $m_{ij} = m'_{ij}$ otherwise. In this notation, the orbits are parametrized by pairs $(\Delta, M = (m_{ij}))$ where Δ is any set as above, and M is a $q \times r$ nonnegative integer matrix with row sums $b_1, \ldots, b_q$ and column sums $c_1, \ldots, c_r$ that also satisfies $m_{ij} > 0$ for all $(i,j) \in \Delta$. A representative of the orbit $\Omega_{\Delta,M}$ is $(V; A, B, C)$ where $(V; B, C)$ is the representative of Ω_M constructed in the previous example, and A_1 is spanned by the vector $\sum_{(i,j)\in\Delta} e_{ij1}$.

3. Proofs of Theorems 2.1 — 2.5

For a k-tuple of positive integers $(p_1, \ldots, p_k)$, we define the graph $T_{p_1,\ldots,p_k}$, the semigroup $\Lambda_{p_1,\ldots,p_k}$, the Tits quadratic form $Q(\mathbf{d})$ on $\Lambda_{p_1,\ldots,p_k}$, and the additive category $\mathcal{F}_{p_1,\ldots,p_k}$ analogously to their counterparts for $k = 3$. Also let $\Pi_{p_1,\ldots,p_k}$

be the set of k-tuples of compositions of finite type $\mathbf{d} \in \Lambda_{p_1,\dots,p_k}$ with $Q(\mathbf{d}) = 1$. When there is no risk of ambiguity, we drop the subscripts $(p_1, \dots, p_k)$ and write Λ, $\mathcal{F}$, Π, etc.

3.1. **Proof of Theorem 2.5.** The only "non-elementary" part of our argument is the following proposition.

Proposition 3.1. *Suppose $\mathbf{d} \in \Lambda$ is the dimension vector of an indecomposable object of $\mathcal{F}$. Then $Q(\mathbf{d}) \leq 1$. Furthermore:*
(i) if $Q(\mathbf{d}) = 1$ then there is a unique isomorphism class $I_{\mathbf{d}}$ of indecomposable objects with the dimension vector $\mathbf{d}$;
(ii) if $Q(\mathbf{d}) \leq 0$ then there are infinitely many such isomorphism classes.

Proof. Consider $T = T_{p_1,\dots,p_k}$ as a directed graph with all edges pointing toward the central vertex where all the chains are joined. Let $\mathcal{C}$ be the category of quiver representations of T: recall that such a representation is specified by attaching a finite-dimensional vector space to every vertex of T, and a linear map between the corresponding spaces to every arrow (directed edge) of T. There is an obvious functor from $\mathcal{F}$ to $\mathcal{C}$: the quiver representation corresponding to a tuple of flags associates the flag subspaces to vertices of T, and their embeddings to arrows. The image of this functor lies in the subcategory $\mathcal{I}$ of $\mathcal{C}$ consisting of quiver representations with all arrows represented by injective linear maps. In fact, our functor allows us to identify the isomorphism classes of indecomposable objects in $\mathcal{F}$ with those in $\mathcal{I}$. Note that $\mathcal{I}$ is a full additive subcategory of $\mathcal{C}$, and that indecomposables of $\mathcal{I}$ are also indecomposables of $\mathcal{C}$, since an injective linear map can never be a direct sum of non-injective maps.

In view of this translation, our proposition follows from general results due to V. Kac ([5, Theorem 1]) which provide a description of dimension vectors for indecomposable quiver representations of an arbitrary finite directed graph. Recall that these dimension vectors are in a natural bijection with positive roots (real and imaginary) of the simply-laced Kac-Moody Lie algebra corresponding to the graph. If $\mathbf{d}$ is the dimension vector corresponding to a positive root α then Kac proves the following:

(1) If α is a real root then $Q(\mathbf{d}) = 1$ and there is a unique isomorphism class of quiver representations with the dimension vector $\mathbf{d}$.

(2) If α is an imaginary root then $Q(\mathbf{d}) \leq 0$ and there are infinitely many isomorphism classes of quiver representations with the dimension vector $\mathbf{d}$.

These translate directly into the statements of our proposition. □

Note that $\mathcal{F}$ is not an abelian category (since it does not always admit quotients). Nevertheless, since indecomposables of $\mathcal{F}$ are also indecomposables of the abelian category $\mathcal{C}$ of quiver representations, the Krull-Schmidt Theorem (as in [1, Ch 2, Theorem 2.2]) applies also to the subcategory $\mathcal{F}$. That is, each object of $\mathcal{F}$ has a unique splitting into indecomposables.

In general, the condition that $\mathbf{d} \in \Lambda$ has $Q(\mathbf{d}) = 1$ does not imply the existence of an indecomposable object $I_{\mathbf{d}}$ in $\mathcal{F}$ with the dimension vector $\mathbf{d}$. However, we will show that if $\mathbf{d}$ is of finite type (i.e, if $\mathbf{d} \in \Pi$) then $I_{\mathbf{d}}$ exists and has the following important additional property. For any two (isomorphism classes of) objects F and F' in $\mathcal{F}$, let us denote

$$\langle F', F \rangle = \dim \operatorname{Hom}_{\mathcal{F}}(F', F) \ . \tag{3.1}$$

We say that $F \in \mathcal{F}$ is a *Schur indecomposable* if $\langle F, F\rangle = 1$ (which clearly implies that F is indeed an indecomposable object in $\mathcal{F}$).

Proposition 3.2. *If* $\mathbf{d} \in \Pi$ *then there exists a Schur indecomposable* $I_{\mathbf{d}}$ *with the dimension vector* $\mathbf{d}$.

Proof. Since $\mathbf{d}$ is of finite type, the corresponding multiple flag variety $\mathrm{Fl}_{\mathbf{d}}(V) = \mathrm{Fl}_{\mathbf{a}_1}(V) \times \cdots \times \mathrm{Fl}_{\mathbf{a}_k}(V)$ has a (dense) Zariski open orbit Ω. Let $I_{\mathbf{d}}$ be the corresponding isomorphism class in $\mathcal{F}$, and let F be any representative of $I_{\mathbf{d}}$; by abuse of notation, we can think of F as a point in Ω. Then we have

$$\langle I_{\mathbf{d}}, I_{\mathbf{d}}\rangle = \dim \mathrm{Stab}_{GL(V)}(F) = \dim GL(V) - \dim \mathrm{Fl}_{\mathbf{d}}(V) = Q(\mathbf{d}) = 1\ .$$

Therefore, $I_{\mathbf{d}}$ is a Schur indecomposable, as desired. □

By Propositions 3.1 and 3.2, for every $\mathbf{d} \in \Pi$, there exists a unique isomorphism class $I_{\mathbf{d}}$ of indecomposable objects in $\mathcal{F}$ with the dimension vector $\mathbf{d}$. Now the proof of Theorem 2.5 (and hence that of Theorem 2.3 and Corollary 2.6) can be concluded as follows.

We say that a non-zero $\mathbf{d}' \in \Lambda$ is a *summand* of $\mathbf{d} \in \Lambda$ if $\mathbf{d} - \mathbf{d}' \in \Lambda$. It follows from the Krull-Schmidt theorem that if $\mathbf{d}$ is of finite type then every summand of $\mathbf{d}$ is also of finite type. Thus every object in $\mathcal{F}$ whose dimension vector is of finite type decomposes (uniquely) into a direct sum of objects $I_{\mathbf{d}}$ for $\mathbf{d} \in \Pi$, and we are done. □

3.2. **Proofs of Theorems 2.1, 2.2, and 2.4.** The following criterion reduces the classification of tuples of compositions of finite type to an "elementary" problem about the Tits form.

Proposition 3.3. *A tuple of compositions* $\mathbf{d} \in \Lambda$ *is of finite type if and only if* $Q(\mathbf{d}') \geq 1$ *for any summand* $\mathbf{d}'$ *of* $\mathbf{d}$.

Proof. Let $\mathrm{Fl}_{\mathbf{d}}(V)$ be the multiple flag variety corresponding to $\mathbf{d} \in \Lambda$. First suppose $\mathbf{d}$ is of finite type. Since the one-dimensional subgroup of scalar matrices in $GL(V)$ acts trivially on $\mathrm{Fl}_{\mathbf{d}}(V)$, we must have $\dim GL(V) - 1 \geq \dim \mathrm{Fl}_{\mathbf{d}}(V)$, i.e., $Q(\mathbf{d}) \geq 1$. We have already noticed that any summand $\mathbf{d}'$ of $\mathbf{d}$ must also be of finite type, hence we must have $Q(\mathbf{d}') \geq 1$.

Conversely, suppose $Q(\mathbf{d}') \geq 1$ for any summand $\mathbf{d}'$ of $\mathbf{d}$. In view of Proposition 3.1, this implies that every indecomposable summand of an object in $\mathcal{F}$ with the dimension vector $\mathbf{d}$ is uniquely determined by its dimension vector. Therefore, the isomorphism classes of objects with the dimension vector $\mathbf{d}$ are in a bijection with partitions of $\mathbf{d}$ into the sum of dimension vectors of indecomposables. Since there are finitely many such partititons, d must be of finite type, and we are done. □

Remark 3.4. The classification of finite types in the rest of this section closely follows the Cartan-Killing classification, which in our situation amounts to finding all graphs $T_{p_1,\dots,p_k}$ with the positive-definite Tits form.

The criterion in Proposition 3.3 is almost identical to that of V. Kac [5, Proposition 2.4], for finite-type quiver varieties: the quiver variety with a given dimension vector $\mathbf{d}$ has finitely many orbits exactly if $Q(\mathbf{d}') \geq 1$ for all *quiver* summands $\mathbf{d}'$ of $\mathbf{d}$. (A quiver summand need not have positive jumps in dimension along each flag; only the dimension of each space must be positive.)

To illustrate the difference between the two criteria, consider the triple of compositions $\mathbf{d} = ((3,1),(1^4),(1^4))$ of finite type $S_{4,4}$. The corresponding space of quiver representations has infinitely many orbits because it has the quiver summand $\mathbf{d}' = ((2,2),(1^4),(1^4))$ with $Q(\mathbf{d}') = 0$; in terms of dimensions rather than dimension jumps, $\substack{1234321\\3} = \substack{1234321\\2} + \substack{0000000\\1}$. In fact, the infinitely many generic objects of dimension $\mathbf{d}'$ are all subobjects of the unique generic indecomposable $I_{\mathbf{d}}$, but there are no quotient objects in the flag category $\mathcal{F}$, only in the quiver category $\mathcal{C}$. (We thank C. Ringel for this example.)

Proof of Theorem 2.1. Clearly, if $\mathbf{d}$ is a k-tuple of compositions of finite type then any subtuple of $\mathbf{d}$ is also of finite type. Thus it suffices to show that a quadruple $\mathbf{d}$ of non-trivial compositions cannot be of finite type. But any such quadruple has a summand $\mathbf{d}'$ with the reduced dimension vector $((1^2),(1^2),(1^2),(1^2))$. An easy calculation shows that $Q(\mathbf{d}') = 0$, so by Proposition 3.3, $\mathbf{d}$ cannot be of finite type. □

Next, beginning the proof of Theorem 2.2, we eliminate those dimension vectors with a summand corresponding to the minimal imaginary root of an affine root system. Let $N_{p,q,r}$ be the set of all $\mathbf{d}' = (\mathbf{a}', \mathbf{b}', \mathbf{c}') \in \Lambda_{p,q,r}$ such that $\{\mathbf{a}'_{\mathrm{red}}, \mathbf{b}'_{\mathrm{red}}, \mathbf{c}'_{\mathrm{red}}\}$ is one of the following three triples:

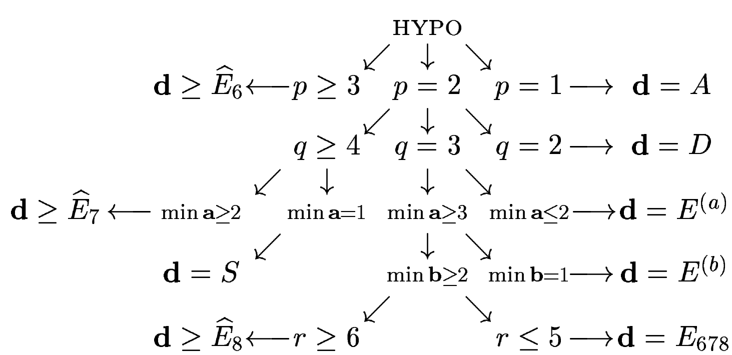
$$\{(1^3),(1^3),(1^3)\}, \quad \{(2^2),(1^4),(1^4)\}, \quad \{(3^2),(2^3),(1^6)\} \ . \tag{3.2}$$

Note that these dimension vectors are associated via the Kac correspondence with minimal imaginary roots for the affine Lie algebras $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$, respectively (cf. the proof of Proposition 3.1). (Note also that the quadruple $((1^2),(1^2),(1^2),(1^2))$ that appeared in the proof of Theorem 2.1 corresponds to the minimal imaginary root for $\widehat{D}_4$.) Using formula (2.1), we find $Q(\mathbf{d}') = 0$ for any $\mathbf{d}' \in N_{p,q,r}$.

Without loss of generality, we can assume that a triple $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ is reduced, i.e., all the compositions $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ have all parts non-zero. Thus, $\mathbf{d} \in \Lambda_{p,q,r}$, where p, q, and r have the same meaning as in Theorem 2.2.

Lemma 3.5. *Let $\mathbf{d} \in \Lambda_{p,q,r}$ be a reduced triple of compositions. Then exactly one of the following holds:*
(i) $\mathbf{d}$ belongs to one of the types A—S in Theorem 2.2.
(ii) The triple $\mathbf{d}$ has some $\mathbf{d}' \in N_{p,q,r}$ as a summand.

Proof. The proof is very similar to the usual classification of Dynkin diagrams. We present it in the schematic form of a tree of implications:



The root of the tree is our

HYPOTHESIS: $\mathbf{d} \in \Lambda_{p,q,r}$ is a reduced triple of compositions, and $1 \le p \le q \le r$.

The arrows coming from a statement point to all possible cases resulting from the statement. We employ the abuse of notation $\mathbf{d} = A$, $\mathbf{d} = D$, etc to indicate that $\mathbf{d}$ belongs to the corresponding type in Theorem 2.2. Similarly we write $\mathbf{d} \geq \widehat{E}_6$, etc to indicate that $\mathbf{d}$ has a summand corresponding to the given affine type. The lemma follows because every case ends in (i) or (ii), and these conditions are clearly disjoint. □

Combining Lemma 3.5 with Proposition 3.3, we prove one direction of Theorem 2.2: if $\mathbf{d}$ is of finite type then it necessarily belongs to one of the types A—S. It remains to show that each of the conditions A—S is *sufficient* for $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ to be of finite type. The following lemma is an immediate consequence of Lemma 3.5.

Lemma 3.6. *If $\mathbf{d} \in \Lambda_{p,q,r}$ is of one of the types A—S then the same is true for any summand of $\mathbf{d}$.*

The remaining part of Theorem 2.2 now follows by combining Proposition 3.3 with Lemma 3.6 and the following.

Lemma 3.7. *Suppose $\mathbf{d} \in \Lambda_{p,q,r}$ is of one of the types A—S. Then $Q(\mathbf{d}) \geq 1$.*

Thus it only remains to prove Lemma 3.7, which we deduce from formula (2.1) and the following elementary estimates.

Lemma 3.8. *Let $\mathbf{b}$ be a reduced composition with $|\mathbf{b}| = n$ and $\ell(\mathbf{b}) = q$. Then*
(1) $||\mathbf{b}||^2 \geq n$, with equality precisely when $\mathbf{b} = (1^n)$;
(2) if $q = 3$ then $||\mathbf{b}||^2 \geq 3(n-2)$, with equality precisely when $\max(b_1, b_2, b_3) \leq 2$;
(3) if $q = 2$, and $n = 2m$ is even then $||\mathbf{b}||^2 \geq 2m^2$, with equality precisely when $\mathbf{b} = (m, m)$;
(4) if $q = 2$, and $n = 2m+1$ is odd then $||\mathbf{b}||^2 \geq 2m^2 + 2m + 1$, with equality precisely when $\mathbf{b}^+ = (m+1, m)$.

Proof. Easy. For example, part (2) is a consequence of the identity:

$$||\mathbf{b}||^2 - 3(n-2) = \sum_{i=1}^{3} (b_i^2 - 3b_i + 2) = \sum_{i=1}^{3} (b_i - 1)(b_i - 2) \ .$$

The other parts are even simpler. □

Proof of Lemma 3.7. Suppose $\mathbf{d} = (\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ is of one of the types A—S. Consider the following cases.

Case 1. Suppose $\mathbf{d}$ is of one of the types A, D, E_6, E_7, or E_8. Then the form Q is positive definite (this is the Cartan-Killing classification), so $Q(\mathbf{d}) \geq 1$. Furthermore, the equality $Q(\mathbf{d}) = 1$ occurs precisely when $\mathbf{d}$ corresponds to a positive root of the associated simple Lie algebra (cf. the proof of Proposition 3.1).

Case 2. Suppose $\mathbf{d}$ is of type $E^{(a)}$. Now the desired inequality $Q(\mathbf{d}) \geq 1$ follows from the equality

$$||\mathbf{a}||^2 - n^2 = 2^2 + (n-2)^2 - n^2 = 8 - 4n$$

and the inequalities $||\mathbf{b}||^2 \geq 3(n-2)$ and $||\mathbf{c}||^2 \geq n$ (Lemma 3.8, parts (1), (2)). The equality $Q(\mathbf{d}) = 1$ occurs precisely when $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+) = ((2,2), (2,1,1), (1^4))$, $((3,2), (2,2,1), (1^5))$, or $((4,2), (2^3), (1^6))$.

Case 3. Suppose $\mathbf{d}$ is of type $E^{(b)}$. Let $\mathbf{b}'$ be the composition obtained from $\mathbf{b}$ by removing a part equal to 1, so that we have $|\mathbf{b}'| = n-1$, $\ell(\mathbf{b}') = 2$, and $||\mathbf{b}||^2 = ||\mathbf{b}'||^2 + 1$. If $n = 2m$ is even then $Q(\mathbf{d}) \geq 1$ follows from the inequalities

$$||\mathbf{a}||^2 \geq 2m^2, \ ||\mathbf{b}'||^2 \geq 2m^2 - 2m + 1, \ ||\mathbf{c}||^2 \geq 2m$$

(Lemma 3.8, parts (1), (3) and (4)). The equality $Q(\mathbf{d}) = 1$ occurs precisely when $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+) = ((m, m), (m, m-1, 1), (1^{2m}))$.

If $n = 2m + 1$ is odd then $Q(\mathbf{d}) \geq 1$ follows from the inequalities

$$||\mathbf{a}||^2 \geq 2m^2 + 2m + 1,\ ||\mathbf{b}'||^2 \geq 2m^2,\ ||\mathbf{c}||^2 \geq 2m$$

(Lemma 3.8, parts (1), (3) and (4)). The equality $Q(\mathbf{d}) = 1$ occurs precisely when $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+) = ((m+1, m), (m, m, 1), (1^{2m+1}))$.

Case 4. Suppose $\mathbf{d}$ is of type S. Then $Q(\mathbf{d}) \geq 1$ follows from

$$||\mathbf{a}||^2 - n^2 = 1 + (n-1)^2 - n^2 = 2 - 2n$$

and $||\mathbf{b}||^2 \geq n,\quad ||\mathbf{c}||^2 \geq n$ (Lemma 3.8, part (1)). The equality $Q(\mathbf{d}) = 1$ occurs precisely when $(\mathbf{a}^+, \mathbf{b}^+, \mathbf{c}^+) = ((n-1, 1), (1^n), (1^n))$.

This completes the proofs of Lemma 3.7 and Theorem 2.2. □

As a by-product of the above argument (the examination of the equality $Q(\mathbf{d}) = 1$), we immediately obtain Theorem 2.4.

4. Orbit representatives and generalized Bruhat order

4.1. **Morphisms of standard triples of flags.** Let $F = (V; A, B, C)$ be an arbitrary object in the category $\mathcal{F}_{p,q,r}$, and let $F' = (V'; A', B', C')$ be an object presented in a standard form according to Definition 2.7 (in particular, we have $p \leq 2$). As a preparation for the proof of Theorem 2.8, we will give an explicit formula for $\langle F', F \rangle$ (cf. formula (3.1)).

Note that every pair of flags (B', C') in an n-dimensional vector space V' satisfying condition (1) in Definition 2.7 can be encoded by a family

$$\Delta = ((i_1, j_1), \ldots, (i_n, j_n))$$

of pairs of indices such that $1 \leq i_1 \leq \cdots \leq i_n \leq q$ and $r \geq j_1 \geq \cdots \geq j_n \geq 1$. Given such a family Δ, each subspace B'_i (resp. C'_j) is spanned by the standard basis vectors e_l such that $i_l \leq i$ (resp. $j_l \leq j$). In the terminology of Example 2.9, the triple $(V'; B', C') \in \mathcal{F}_{q,r}$ is a direct sum $\bigoplus_{l=1}^n I_{i_l, j_l}$.

We see that $F' = (V'; A', B', C')$ is completely determined by the following combinatorial data: a family Δ and a collection of subsets $S_1, \ldots, S_{a_1}$ in $\{1, \ldots, n\}$ satisfying condition (3) in Definition 2.7:

$$\left| \bigcup_{k \neq k'} (S_k \cap S_{k'}) \right| \leq 2\ . \tag{4.1}$$

If $n = \dim V' \geq 2$ (there is not much loss in generality in this assumption) then (4.1) is equivalent to the following: there exist two distinct indices μ and ν in $\{1, \ldots, n\}$ such that the sets $S_1 \setminus \{\mu, \nu\}, \ldots, S_{a_1} \setminus \{\mu, \nu\}$ are pairwise disjoint.

Returning to an object $F = (V; A, B, C)$, we set, for $l = 1, \ldots, n$,

$$D_l = B_{i_l} \cap C_{j_l}\ .$$

Let $U^{(\mu)}, U^{(\nu)}$ and $U^{(\mu\nu)}$ denote the following three subspaces in V/A_1:

$$U^{(\mu)} = \frac{(A_1 + D_\mu) \cap \bigcap_{\mu \in S_k, \nu \notin S_k} (A_1 + \sum_{l \in S_k \setminus \{\mu\}} D_l)}{A_1}\ ,$$

$$U^{(\nu)} = \frac{(A_1 + D_\nu) \cap \bigcap_{\mu \notin S_k, \nu \in S_k} (A_1 + \sum_{l \in S_k \setminus \{\nu\}} D_l)}{A_1}\ ,$$

$$U^{(\mu\nu)} = \frac{\bigcap_{\mu,\nu \in S_k} (A_1 + \sum_{l \in S_k \setminus \{\mu,\nu\}} D_l)}{A_1} .$$

(Here the intersection (resp. sum) taken over the empty set of indices is understood as the ambient space V/A_1 (resp. as the zero space).) One final piece of notation: for a family of subspaces $(U_1, \dots, U_s)$ in some finite dimensional vector space U, we define

$$\mathrm{dd}(U_1, \dots, U_s) = \sum_{i=1}^{s} \dim\,(U_i) - \dim\,(\sum_{i=1}^{s} U_i) . \tag{4.2}$$

(Here "dd" is an abbreviation for *dependency degree*; it measures the difference between the sum of subspaces and their direct sum.)

Proposition 4.1. *In the above notation, we have*

$$\begin{aligned} \langle F'F \rangle &= \mathrm{dd}\,(A, D_\mu) + \mathrm{dd}\,(A, D_\nu) \\ &+ \textstyle\sum_{k=1}^{a_1} \mathrm{dd}\,(A, D_l : l \in S_k \setminus \{\mu,\nu\}) + \mathrm{dd}\,(U^{(\mu)}, U^{(\nu)}, U^{(\mu\nu)}) . \end{aligned} \tag{4.3}$$

Proof. A morphism from F' to F is a linear map from V' to V, and so is determined by the images of the basis vectors $e_1, \dots, e_n$; let us denote these images by $v_1, \dots, v_n$. By the definition, the vectors v_l must satisfy the following conditions:

$$v_l \in D_l \ (1 \le l \le n) , \ \sum_{l \in S_k} v_l \in A_1 \ (1 \le k \le a_1) . \tag{4.4}$$

Thus $\langle F', F \rangle$ is equal to the dimension of the subspace $U \subset V^n$ formed by n-tuples $(v_1, \dots, v_n)$ satisfying (4.4). Let $v \mapsto \overline{v}$ denote the natural projection $V \to V/A_1$. Clearly, $U = \mathrm{Ker}\,(\varphi)$, where $\varphi : \bigoplus_{l=1}^n D_l \to (V/A_1)^{a_1}$ is the linear map given by

$$(v_1, \dots, v_n) \mapsto (\sum_{l \in S_1} \overline{v_l}, \dots, \sum_{l \in S_{a_1}} \overline{v_l}) .$$

Thus we have

$$\langle F', F \rangle = \sum_{l=1}^{n} \dim(D_l) - \mathrm{rk}\,(\varphi) . \tag{4.5}$$

Let us denote

$$W^{(\mu)} = (A_1 + D_\mu)/A_1, \quad W^{(\nu)} = (A_1 + D_\nu)/A_1 ,$$
$$W_k = (A_1 + \sum_{l \in S_k \setminus \{\mu,\nu\}} D_l)/A_1 \quad (1 \le k \le a_1) ,$$

and consider the subspace

$$W = W^{(\mu)} \oplus W^{(\nu)} \oplus \bigoplus_{k=1}^{a_1} W_k \subset (V/A_1)^{a_1+2} .$$

Then φ can be factored as $\varphi = \varphi_1 \circ \varphi_2$, where the linear maps $\varphi_1 : W \to (V/A_1)^{a_1}$ and $\varphi_2 : \bigoplus_{l=1}^n D_l \to W$ are given by

$$\varphi_1 : (w^{(\mu)}, w^{(\nu)}, w_1, \dots, w_{a_1}) \mapsto$$
$$(\chi_1(\mu) w^{(\mu)} + \chi_1(\nu) w^{(\nu)} + w_1, \dots, \chi_{a_1}(\mu) w^{(\mu)} + \chi_{a_1}(\nu) w^{(\nu)} + w_{a_1}) ;$$

$$\varphi_2 : (v_1, \dots, v_n) \mapsto$$

$$(\overline{v_\mu}, \overline{v_\nu}, \sum_{l \in S_1 \setminus \{\mu,\nu\}} \overline{v_l}, \ldots, \sum_{l \in S_{a_1} \setminus \{\mu,\nu\}} \overline{v_l}) \ .$$

(Here χ_k stands for the indicator function of the set S_k, i.e., $\chi_k(l) = 1$ if $l \in S_k$, otherwise $\chi_k(l) = 0$.) Since the sets $S_1 \setminus \{\mu, \nu\}, \ldots, S_{a_1} \setminus \{\mu, \nu\}$ are pairwise disjoint, the map φ_2 is surjective. It follows that

$$\begin{aligned} \text{rk}\ (\varphi) &= \text{rk}\ (\varphi_1) = \dim(W) - \dim(\text{Ker}\ (\varphi_1)) \\ &= \dim(W^{(\mu)}) + \dim(W^{(\nu)}) + \textstyle\sum_{k=1}^{a_1} \dim(W_k) - \dim(\text{Ker}\ (\varphi_1)) \ . \end{aligned} \tag{4.6}$$

It remains to compute $\dim(\text{Ker}\ (\varphi_1))$. The definition of φ_1 implies that the projection $(w^{(\mu)}, w^{(\nu)}, w_1, \ldots, w_{a_1}) \mapsto (w^{(\mu)}, w^{(\nu)})$ restricts to an isomorphism between Ker (φ_1) and the space of pairs $(w^{(\mu)}, w^{(\nu)})$ such that $w^{(\mu)} \in U^{(\mu)}$, $w^{(\nu)} \in U^{(\nu)}$, and $w^{(\mu)} + w^{(\nu)} \in U^{(\mu\nu)}$. It follows that

$$\dim(\text{Ker}\ (\varphi_1)) = \text{dd}(U^{(\mu)}, U^{(\nu)}, U^{(\mu\nu)}) \ . \tag{4.7}$$

Combining (4.5), (4.6), and (4.7) and simplifying, we obtain (4.3). □

Note that if (4.1) becomes an equality then the indices μ and ν are determined by

$$\bigcup_{k \neq k'} (S_k \cap S_{k'}) = \{\mu, \nu\} \ .$$

However if the inequality in (4.1) is strict then there is no distinguished pair of indices, and (4.3) can be simplified as follows.

Corollary 4.2. *In the same notation as in Proposition 4.1, if* $\bigcup_{k \neq k'} (S_k \cap S_{k'}) = \{\mu\}$ *for some index* μ *then*

$$\begin{aligned} \langle F', F \rangle &= \text{dd}\ (A_1, D_\mu) + \textstyle\sum_{k=1}^{a_1} \text{dd}(A_1, D_l : l \in S_k \setminus \{\mu\}) \\ &+ \dim\ (A_1 + D_\mu) \cap \textstyle\bigcap_{k: \mu \in S_k} (A_1 + \sum_{l \in S_k \setminus \{\mu\}} D_l) - \dim\ A_1 \ . \end{aligned} \tag{4.8}$$

Furthermore, if all S_k *are pairwise disjoint then*

$$\langle F', F \rangle = \textstyle\sum_{k=1}^{a_1} \text{dd}(A_1, D_l : l \in S_k) \ . \tag{4.9}$$

4.2. **Proof of Theorem 2.8.** Let $F \in \mathcal{F}_{p,q,r}$ be one of the triples of flags in Theorem 2.8; thus $p \leq 2$, and F is presented in a standard form according to Definition 2.7. It suffices to show that F is a Schur indecomposable, i.e., that $\langle F, F \rangle = 1$ (cf. (3.1) and Proposition 3.2).

In the first and last case on the list (i.e., for $\mathbf{d} = ((1), (1), (1))$ or $\mathbf{d} = ((1, n-1), (1^n), (1^n)))$, the equality $\langle F, F \rangle = 1$ follows at once from (4.9).

In each of the first four 6-dimensional cases on the list, the desired equality $\langle F, F \rangle = 1$ is a direct consequence of (4.3). In each of these cases it is also easy to check by an independent calculation that every morphism from F to itself is scalar. For instance, let us do this for $\mathbf{d} = ((4, 2), (2^3), (1^6))$. Let (x_{ij}) be a 6×6 matrix that represents a morphism $\varphi : V \to V$ in the standard basis $e_1, \ldots, e_6$. The condition that φ preserves the flags B and C means that the only non-zero matrix entries can be $x_{11}, x_{21}, x_{22}, x_{33}, x_{43}, x_{44}, x_{55}, x_{65}$, and x_{66}. Thus we have $\varphi(e_2 + e_3) = x_{22}e_2 + x_{33}e_3 + x_{43}e_4$; the condition that this vector lies in A_1 implies that $x_{22} = x_{33}$ and $x_{43} = 0$. Similarly, the condition that $\varphi(e_4 + e_5) \in A_1$ implies that $x_{44} = x_{55}$ and $x_{65} = 0$. Finally, the two remaining conditions that $\varphi(e_1 + e_5)$ and $\varphi(e_2 + e_5 + e_6)$ lie in A_1 imply that $x_{11} = x_{55}$, $x_{21} = 0$, and $x_{22} = x_{55} = x_{66}$. Combining all these equalities, we see that φ is scalar, as desired.

For the rest of the list, the equality $\langle F,F\rangle = 1$ can be checked case by case with the help of (4.8). To simplify this procedure, we observe that all these cases satisfy the following strengthened form of condition (3) in Definition 2.7:

(3′) The intersection $\bigcap_{k=1}^{a_1} S_k$ consists of one index μ, and the subsets $S'_k = S_k \setminus \{\mu\}$ are nonempty, pairwise disjoint, and have $\{1,\dots,n\}\setminus\{\mu\}$ as their union.

Assuming (3′), we will give combinatorial conditions on subsets S_k that are necessary and sufficient for the corresponding object F to be Schur indecomposable. This requires some terminology.

Let $F = (V; A, B, C)$ be a triple of flags with the dimension vector $(\mathbf{a},\mathbf{b},\mathbf{c})$; let $n = \dim V$. We associate to $\mathbf{b}$ the subdivision of $[1,n] = \{1,\dots,n\}$ into consecutive blocks $[1,b_1], [b_1+1, b_1+b_2], \dots$ of sizes $b_1,\dots,b_q$. The blocks of this subdivision will be called $\mathbf{b}$-blocks. We define the $\mathbf{c}$-blocks in a similar way, with the only difference that they go opposite way (so that the first $\mathbf{c}$-block is $[n-c_1+1, n]$). We say that an index $l \in [1,n]$ is $\mathbf{b}$-*separated* (resp. $\mathbf{c}$-*separated*) from a subset $S \subset [1,n]$ if no element of S smaller (resp. larger) than l lies in the same $\mathbf{b}$-block (resp. $\mathbf{c}$-block) with l. If l is both $\mathbf{b}$-separated and $\mathbf{c}$-separated from S, we say that l is $\mathbf{bc}$-*separated* from S. Now suppose that F is presented in a standard form satisfying (3′), and denote $S'_0 = \{\mu\}$.

Proposition 4.3. *Suppose $F = (V; A, B, C) \in \mathcal{F}_{p,q,r}$ has the dimension vector $(\mathbf{a},\mathbf{b},\mathbf{c})$ and is presented in a standard form satisfying (3′). Then F is a Schur indecomposable if and only if the subsets S'_k for $k = 0,\dots,a_1$ satisfy the following conditions:*
(1) No two elements of the same S'_k lie in the same $\mathbf{b}$-block or in the same $\mathbf{c}$-block.
(2) For every two distinct indices j and k, each $l \in S'_j$ is either $\mathbf{b}$-separated or $\mathbf{c}$-separated from S'_k.
(3) In the situation of (2), S'_j contains an index $\mathbf{bc}$-separated from S'_k.
(4) For every j, any two elements of S'_j are equivalent to each other with respect to the equivalence relation generated by the following: $l \sim l'$ if, for some $k \neq j$, both l and l' are $\mathbf{bc}$-separated from S'_k.

Proof. We compute $\langle F,F\rangle$ using (4.8). Note that under the condition (3′), formula (4.8) further simplifies as follows:

$$\langle F,F\rangle = \sum_{k=0}^{a_1} \mathrm{dd}(A_1, D_l : l \in S'_k) + \dim\Big(\bigcap_{k=0}^{a_1}(A_1 + \sum_{l\in S'_k} D_l)/A_1\Big)\ . \tag{4.10}$$

Tracing the definitions, we observe that each subspace $D_l = B_{i_l} \cap C_{j_l}$ is spanned by all the basis vectors $e_{l'}$ such that l' is *not* $\mathbf{bc}$-separated from $\{l\}$. In particular, $e_l \in D_l$. It follows that

$$A_1 + \langle e_\mu\rangle \subset \bigcap_{k=0}^{a_1}(A_1 + \sum_{l\in S'_k} D_l)\ , \tag{4.11}$$

where $\langle e_\mu\rangle$ stands for the one-dimensional subspace spanned by e_μ. Recalling (4.2), we see that the equality $\langle F,F\rangle = 1$ is equivalent to the following conditions:
(i) for each $k = 0,\dots,a_1$, the sum $\sum_{l\in S'_k} D_l$ is direct;
(ii) for each k, we have $A_1 \cap \sum_{l\in S'_k} D_l = 0$;
(iii) the inclusion in (4.11) is an equality.

It is now completely straightforward to show that conditions (i) – (iii) are equivalent to conditions (1) – (4) in our proposition. To be more precise, (i) translates into (1) and (2), (ii) translates into (3), and (iii) into (4). □

Now an easy inspection shows that all the remaining cases in Theorem 2.8 satisfy conditions (1) – (4) in Proposition 4.3 (in most of these cases, the inspection is simplified even more by the following observation: if $\mathbf{c} = (1^n)$ then condition (2) is automatic). This completes the proof of Theorem 2.8. □

It is easy to show that in the four exceptional 6-dimensional cases it is impossible to find subsets $S_1, \ldots, S_{a_1}$ satisfying the conditions in Proposition 4.3 (this check starts with the observation that the case $j = 0$ in condition (3) means that an index μ must be the minimal element of its $\mathbf{b}$-block and the maximal element of its $\mathbf{c}$-block). This justifies our efforts in obtaining (4.3).

Note that it is possible to construct the list of representatives in Theorem 2.8 and obtain an alternative proof by a recursive procedure which is a special case of the mutations of exceptional pairs studied by Rudakov, Schofield, Crawley-Boevey, and Ringel (see [8]). In our situation, this procedure relies on the following simple general proposition.

Proposition 4.4. *Suppose there is a short exact sequence*

$$0 \to F' \to F \to F'' \to 0$$

in $\mathcal{F}$ with the following properties: both F' and F'' are Schur indecomposables, and $\langle F', F'' \rangle = \langle F, F' \rangle \, \langle F'', F \rangle = 0$. Then F is a Schur indecomposable.

It turns out that, for every dimension vector $\mathbf{d}$ in Theorem 2.8, one can construct a short exact sequence as in Proposition 4.4 such that F has dimension vector $\mathbf{d}$, and one of the Schur indecomposables F' and F'' has reduced dimension vector $((1), (1), (1))$. If $\mathbf{d} \neq ((1), (1), (1))$ then another one of F' and F'' is smaller than $\mathbf{d}$ and is also on our list so we can assume that we already know its "nice" presentation; we can then use an explicit form of the short exact sequence to construct a "nice" presentation for F. For instance, if $\mathbf{d} = ((4,3), (3,1,3), (1^7))$ then we can choose the dimension vectors of F' and F'' to be respectively $\mathbf{d}' = ((3,3), (3,1,2), (1,1,0,1,1,1,1))$, and $\mathbf{d}'' = ((1,0), (0,0,1), (0,0,1,0,0,0,0))$. Iterating this procedure, one can construct representatives for all Schur indecomposables on our list (this was in fact our original way to do it).

4.3. **Generalized Bruhat order.** Having determined the orbits in triple flag varieties of finite type, we naturally ask how they fit together. Recall that a parametrization of orbits is given by Theorem 2.3 and Corollary 2.6. We define the partial order (called *degeneration order* or *generalized Bruhat order*) on the set of families $M = (m_{\mathbf{d}})$ ($\mathbf{d} \in \Pi_{p,q,r}$) by setting $M \overset{deg}{\leq} M'$ if Ω_M lies in the Zariski closure of $\Omega_{M'}$.

Recall from Theorem 2.5 that the orbit Ω_M corresponds to the isomorphism class $\bigoplus_{\mathbf{d} \in \Pi_{p,q,r}} m_{\mathbf{d}} I_{\mathbf{d}}$ of objects in the category $\mathcal{F}_{p,q,r}$; denote this isomorphism class by F_M. The following proposition is a special case of a result due to C. Riedtmann (cf. [7, 2, 3]).

Proposition 4.5. *If $M \overset{deg}{\leq} M'$ then $\langle I_{\mathbf{d}}, F_M \rangle \geq \langle I_{\mathbf{d}}, F_{M'} \rangle$ for any $\mathbf{d} \in \Pi_{p,q,r}$.*

It would be interesting to know if the converse statement is also true, i.e., if the degeneration order $M \overset{deg}{\le} M'$ is given by the inequalities $\langle I_{\mathbf{d}}, F_M\rangle \ge \langle I_{\mathbf{d}}, F_{M'}\rangle$ for all $\mathbf{d} \in \Pi_{p,q,r}$. This statement is true when the graph $T_{p,q,r}$ is one of the Dynkin graphs A_n, D_n, E_6, E_7, or E_8; this follows from general results due to K. Bongartz (cf. [2, §4], [3, §5.2]).

Note that Theorem 2.8 and formulas (4.3), (4.9), and (4.10) allow us to compute $\langle I_{\mathbf{d}}, F_M\rangle$ explicitly for all $\mathbf{d} \in \Pi_{p,q,r}$. In particular, it is easy to compute $\langle I_{\mathbf{d}}, I_{\mathbf{d}'}\rangle$ for any two Schur indecomposables of finite type. Knowing these numbers yields an explicit formula for $\langle F_M, F_M\rangle$:

$$\langle F_M, F_M\rangle = \sum_{\mathbf{d},\mathbf{d}' \in \Pi_{p,q,r}} \langle I_{\mathbf{d}}, I_{\mathbf{d}'}\rangle \; m_{\mathbf{d}} m_{\mathbf{d}'} \ . \tag{4.12}$$

One application of this is a formula for the (co)dimension of any orbit Ω_M.

Proposition 4.6. *The codimension of Ω_M in a multiple flag variety $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ of finite type is given by*

$$\mathrm{codim}\ (\Omega_M) = \langle F_M, F_M\rangle - Q(\mathbf{a}, \mathbf{b}, \mathbf{c}) \ . \tag{4.13}$$

Proof. This follows at once from the formula

$$\langle F_M, F_M\rangle = \dim \mathrm{Stab}_{GL(V)}(F) = \dim GL(V) - \dim \Omega_M \ ,$$

where F is any representative of Ω_M (cf. the proof of Proposition 3.2). □

In conclusion, let us illustrate the material in this subsection by two examples that continue Examples 2.9 and 2.10.

Example 4.7. Type $(A_{q,r})$: two partial flags. We will use the notation in Example 2.9, so that Ω_M denotes the orbit in $\mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ corresponding to a $q \times r$ nonnegative integer matrix $M = (m_{ij})$ with row sums $b_1, \ldots, b_q$ and column sums $c_1, \ldots, c_r$. Formula (4.9) specializes to

$$\langle I_{ij}, F\rangle = \dim(B_i \cap C_j)$$

for any pair of flags $F = (V; B, C)$. It follows that

$$\langle I_{ij}, F_M\rangle = \sum_{i'=1}^{i} \sum_{j'=1}^{j} m_{i'j'} \ . \tag{4.14}$$

By Proposition 4.5 and results of Bongartz quoted above, the degeneration order $M \overset{deg}{\le} M'$ is given by the inequalities

$$\sum_{i'=1}^{i} \sum_{j'=1}^{j} m_{i'j'} \ge \sum_{i'=1}^{i} \sum_{j'=1}^{j} m'_{i'j'}$$

for all i and j. (If $\mathbf{b} = \mathbf{c} = (1^n)$, this is a well known description of the Bruhat order on the symmetric group.) Finally, (4.13) after some simplification implies the following formula for the codimension of any orbit Ω_M:

$$\mathrm{codim}\ (\Omega_M) = \sum_{i'<i,\ j'<j} m_{ij} m_{i'j'} \ . \tag{4.15}$$

Example 4.8. Type $(S_{q,r})$: two partial flags and a line. We will use the notation in Example 2.10. In particular, $\Omega_{\Delta,M}$ denotes the orbit in $\mathrm{Fl}_{\mathbf{a}}(V)\times\mathrm{Fl}_{\mathbf{b}}(V)\times\mathrm{Fl}_{\mathbf{c}}(V)$ corresponding to a $q \times r$ nonnegative integer matrix $M = (m_{ij})$ as above and to a non-empty set $\Delta = \{(i_1, j_1), (i_2, j_2), \ldots, (i_t, j_t)\}$ such that $1 \le i_1 < \ldots < i_t \le q$, $r \ge j_1 > \ldots > j_t \ge 1$, and $m_{ij} > 0$ for $(i,j) \in \Delta$. Using (4.9), we obtain

$$\langle I_{ij}, F\rangle = \dim(B_i \cap C_j) \ ,$$

$$\langle I_{\Delta_1}, F\rangle = \dim\ (A_1 \cap \sum_{(i,j)\in\Delta_1} (B_i \cap C_j)) + \mathrm{dd}(B_i \cap C_j : (i,j) \in \Delta_1)$$

for any triple of flags $F = (V; A, B, C)$ (recall that A_1 is the only proper subspace in the flag A, and $\dim A_1 = 1$). These formulas imply that

$$\begin{aligned} &\langle I_{ij}, F_{\Delta,M}\rangle = \textstyle\sum_{i'=1}^{i} \sum_{j'=1}^{j} m_{i'j'} \ ; \\ &\langle I_{\Delta_1}, F_{\Delta,M}\rangle = \delta_{\Delta\le\Delta_1} + \textstyle\sum_{(i,j)\in\Delta_1'} \sum_{i'=1}^{i} \sum_{j'=1}^{j} m_{i'j'} \ , \end{aligned} \tag{4.16}$$

where we use the following notation: $\Delta \le \Delta_1$ means that for any $(i', j') \in \Delta$ there exists $(i,j) \in \Delta_1$ such that $i' \le i$ and $j' \le j$; the δ-symbol has the usual indicator meaning; and if $\Delta_1 = \{(i_1, j_1), \ldots, (i_t, j_t)\}$ with $1 \le i_1 < \ldots < i_t \le q$ and $r \ge j_1 > \ldots > j_t \ge 1$ then Δ_1' denotes the set $\{(i_1, j_2), (i_2, j_3), \ldots, (i_{t-1}, j_t)\}$. Finally, (4.13) after some simplification implies the following formula for the codimension of any orbit $\Omega_{\Delta,M}$:

$$\mathrm{codim}\ (\Omega_{\Delta,M}) = \sum_{i'<i,\ j'<j} m_{ij} m_{i'j'} + \sum_{\{(i,j)\}\not\le\Delta} m_{ij} \ . \tag{4.17}$$

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