ON SERRE DUALITY WITH SUPPORT CONDITIONS AND SEPARATION THEOREMS

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0. Introduction and statement of the results

If X is an n-dimensional complex manifold and E a holomorphic vector bundle over X, then we denote by $C^0_{s,r}(X,E)$ the Fréchet space of continuous E-valued (s,r)-forms on X, by $Z^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-closed forms, and by $E^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-exact forms ($E^0_{s,0}(X,E):=\{0\}$). As usual, the factor space

$$H^{s,r}(X,E):=Z^0_{s,r}(X,E)/E^0_{s,r}(X,E).$$

will be considered as topological vector space endowed with the factor topology. Recall that this topology is separated if and only if $E^0_{s,r}(X,E)$ is closed with respect to the topology of $C^0_{s,r}(X,E)$. If E is the trivial line bundle, then we write also $C^0_{s,r}(X)$ instead of $C^0_{s,r}(X,E)$ etc.

0.1. Definition. Let X be an n-dimensional complex manifold X and let q,q^* be integers with $1\leq q\leq n-1$ and $0\leq q^*\leq n$. X will be called *q-concave-q^*-convex* if X is connected and there exists a real C^2 function ρ on X such that, if $\inf\rho:=\inf_{\zeta\in X}\rho(\zeta)$ and $\sup\rho:=\sup_{\zeta\in X}\rho(\zeta)$, then $\inf\rho<\rho(\zeta)$ for all $\zeta\in X$, the sets $\{\alpha\leq\rho\leq\beta\}$, $\inf\rho<\alpha<\beta<\sup\rho$, are compact, and the following two conditions are fulfilled:

(i) There exists $\alpha\in]\inf\rho,\sup\rho[$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues everywhere on $\{\rho\leq\alpha\}$.

(ii) If $q^*=0$, then, for all $\alpha\in]\inf\rho,\sup\rho[$, the set $\{\rho\geq\alpha\}$ is compact (and hence $\sup\rho=\max\rho$), i.e. X is q-concave in the sense of Andreotti-Grauert. If $1\leq q^*\leq n$, then there exists $\beta\in]\inf\rho,\sup\rho[$ such that the Levi form of ρ has at least $n-q^*+1$ positive eigenvalues everywhere on $\{\rho\geq\beta\}$ (and hence $\sup\rho>\rho(\zeta)$ for all $\zeta\in X$).

The following separation theorem is well known:

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0.2. Theorem. *Let X be an n-dimensional complex manifold which is q-concave-q^*-convex, where $1 \leq q \leq n-1$ and $0 \leq q^* \leq n-q-1$. Then, for any holomorphic vector bundle E over X, $H^{0,n-q}(X,E)$ is separated.*

For $q^* = 0$ this theorem was proved by Andreotti and Vesentini [A-V]. The general case is contained in Theorem 2 of [R] of J.P. Ramis, where the more general situation of sheaves over complex spaces is studied. A simple direct proof of Theorem 0.2 is given in [La-L 1].

Consider the case

$$q^* = n - q, \qquad 1 \leq q \leq n-1.$$

First note that then it may happen that $H^{0,n-q}(X,E)$ is not separated. This follows from an example of Rossi [Ros] and Theorem 23.3 in [H-L] (for the details cp. the introduction of [La-L 1]). The example of Rossi is a 2-dimensional 1-concave-1-convex manifold such that $H^{0,1}(X)$ is not separated.[1]

However, in [H-L] it was proved that if ' the q-convex hole can be repaired', then nevertheless $H^{0,n-q}(X,E)$ is separated. More precisely, the following theorem holds:

0.3. Theorem. (cp. Theorem 19.1' in [H-L]) *Let X be an n-dimensional complex manifold which is q-concave-$(n-q)$-convex, $1 \leq q \leq n-1$, such that additional the following condition is fulfilled:*

(A) There exists a complex manifold Y with a relatively compact open subset H such that: X is an open subset of Y, $Y = X \cup H$ and if ρ is as in Definition 0.1, then, for certain γ with $\inf \rho < \gamma < \sup \rho$, $X \cap H = \{\rho < \gamma\}$.

Then, for any holomorphic vector bundle E over Y, $H^{0,n-q}(X,E)$ is separated.

The proof of Theorem 0.3 given in [H-L] is rather long and difficult and uses many estimates for integral operators of the Grauert-Henkin-Lieb type (not only the well known Hölder estimates). In Sect. 4 of the present paper we give a simple proof of Theorem 0.3 using only Andreotti-Grauert finiteness theorems and Serre duality.

In Sect. 5 we prove a finiteness and separation theorem for certain special families of supports (compact with respect to a part of the boundary and arbitrary with respect to the other part).

Then in Sect. 6, using this result from Sect. 5 and the arguments of Sect. 4, we prove that the conclusion of Theorem 0.3 remains valid also without condition (A) if $q < n/2$, i.e. we prove the following

0.4. Theorem. *Let X be an n-dimensional complex manifold, $n \geq 3$, which is q-concave-$(n-q)$-convex, $1 \leq q \leq n-1$. If*

$$q < \frac{n}{2},$$

[1]Note that there is a misprint in the formulation of Theorem 2 in [R] - by this formulation $H^{0,1}(X)$ should be separated also for the Rossi example.

then, for any holomorphic vector bundle E over X, $H^{0,n-q}(X,E)$ is separated.

Note that for $q=1$ the assertion of this theorem follows already from Theorem 0.3, because then, by a theorem of Rossi [Ros], the 1-concave 'hole' can be repaired.

In Sect. 7 we show that the arguments of Sect. 4 can be applied also to the situation considered in [Mi]. We prove the following theorem (see Theorem 7.5):

0.5. Theorem. *Let Y be a compact complex space of dimension n whose singular part S consists of a finite number of points. Set $X = Y \setminus S$, let some subset S_0 of S be fixed and denote by Φ the family of all closed subsets C of X such that $Y \setminus C$ is a neighborhood of S_0.*

Then, for any holomorphic vector bundle E over X, $H^{0,n-1}_{\Phi}(X,E)$ is separated.

In [Mi] this result was proved under the additional hypothesis that the bundle $K_X^{-1} \otimes E$ is extendable to S_0 as a holomorphic vector bundle. Note also that in the present paper the more general situation is admitted when the manifold X has arbitrary 1-convex 'holes' which, possibly, cannot be filled in by complex spaces.

The basic tool of the present paper is Serre duality on *non-compact* complex manifolds, where we are interested not only in the standard situation - the relation between the usual cohomology and the cohomology with compact supports, but also in the case of more general support conditions. Although many seems to be known in this direction [S,A-K,C-S], we could not find in the literature correct proofs for everything what we need.

Therefore we begin this paper (Sects. 1-3) with a study of Serre duality for certain special families of supports (which we call *nice*), repeating also well known things, for the sake of completeness.

Finally, we have to confess that in our preprint [La-L 3] (a first version of the present paper), there is a mistake in the general part on Serre duality (the proof of Lemma 2.8 is false). Note however that the results of Sects. 3 - 7 of this preprint are correct. They are proved in the present paper (for Sects. 3 - 6) and [La-L 2] (for Sect. 7).

1. Families of supports and algebraic relations between corresponding Dolbeault groups

In this section X is an n-dimensional complex manifold countable at infinity and E is a holomorphic vector bundle over X.

1.1. Notations. Throughout this paper we use the following notations:

If $Y \subseteq X$, then we denote by $C^0_{s,r}(Y;X,E)$ the subspace of all $f \in C^0_{s,r}(X,E)$ with $\operatorname{supp} f \subseteq Y$, and we set $Z^0_{s,r}(Y;X,E) = Z^0_{s,r}(X,E) \cap C^0_{s,r}(Y;X,E)$.

If Φ is a family of subsets of X, then:

- $C^0_{s,r}(\Phi;X,E)$ is the space of all $f \in C^0_{s,r}(X,E)$ with $\operatorname{supp} f \in \Phi$,
- $Z^0_{s,r}(\Phi;X,E) := Z^0_{s,r}(X,E) \cap C^0_{s,r}(\Phi;X,E)$,

- $E^0_{s,r}(\Phi; X, E) := Z^0_{s,r}(\Phi; X, E) \cap \overline{\partial} C^0_{s,r-1}(\Phi; X, E)$ if $r \geq 1$,
- $E^0_{s,r}(\Phi; X, E) := \{0\}$ if $r = 0$,
- $H^{s,r}_\Phi(X, E) := Z^0_{s,r}(\Phi; X, E)/E^0_{s,r}(\Phi; X, E)$.

Note that $H^{s,r}_\Phi(X, E) = H^{s,r}(X, E)$ if Φ consists of all closed subsets of X. As usual, we write

$$H^{s,r}_c(X, E) := H^{s,r}_\Phi(X, E)$$

if Φ consists of the compact subsets of X.

If $Y' \subseteq Y \subseteq X$ and $\Phi' \subseteq \Phi$ are families of subsets of X then we use the abbreviations

$$E^0_{s,r}(Y \to Y'; X, E) := Z^0_{s,r}(Y'; X, E) \cap \overline{\partial} C^0_{s,r-1}(Y; X, E),$$
$$E^0_{s,r}(\Phi \to \Phi'; X, E) := Z^0_{s,r}(\Phi'; X, E) \cap \overline{\partial} C^0_{s,r-1}(\Phi; X, E),$$
$$E^0_{s,r}(\Phi \to Y; X, E) := Z^0_{s,r}(Y; X, E) \cap \overline{\partial} C^0_{s,r-1}(\Phi; X, E).$$

1.2. Families of supports. By a *family of supports in* X we mean a collection Φ of closed subsets of X such that the following conditions are fulfilled (cf. [S]):

(S_1) if $C \in \Phi$, then each closed subset of C belongs to Φ;

(S_2) if $C_1, C_2 \in \Phi$, then $C_1 \cup C_2 \in \Phi$;

(S_3) for each $C \in X$ there exists an open neighborhood U of C with $\overline{U} \in \Phi$.

Note that, for each family of supports Φ in X, the union $\bigcup \Phi$ is open (possibly $\neq X$), and Φ contains all compact subsets of $\bigcup \Phi$.

1.3. The family $\Phi * \Phi'$. If Φ is a family of supports in X and $\Phi' \subseteq \Phi$ is a subfamily which is also a family of supports in X, then we denote by $\Phi * \Phi'$ the family of open sets $U \subseteq X$ such that $C \setminus U \in \Phi'$ for all $C \in \Phi$.

It is easy to see that every finite intersection of sets of $\Phi * \Phi'$ is in $\Phi * \Phi'$, and, unless $\Phi = \Phi'$, the empty set never belongs to $\Phi * \Phi'$. Furthermore, it is clear that $X \setminus C' \in \Phi * \Phi'$ if $C' \in \Phi'$. However it is not true in general that $X \setminus U \in \Phi'$ if $U \in \Phi * \Phi'$ (cf. Example II in Sect. 1.4 below).

1.4. Complete subfamilies. Let $\Phi' \subseteq \Phi$ be two families of supports in X. Then the follwing conditions are equivalent:

(i) There exists $C_0 \in \Phi$ with $\overline{C \setminus C_0} \in \Phi'$ for all $C \in \Phi$.

(ii) There exists $U \in \Phi * \Phi'$ with $\overline{U} \in \Phi$.

(iii) For each $U \in \Phi * \Phi'$ there exists $V \in \Phi * \Phi'$ such that $\overline{V} \subseteq U$ and $\overline{V} \in \Phi$.

If these equivalent conditions are fulfilled, then Φ' will be called *complete* in Φ.

Proof of the equivalence. (ii) $\Longrightarrow$ (i): Assume U is as in (ii). Set $C_0 = \overline{U}$ and consider $C \in \Phi$. Then, by definition of $\Phi * \Phi'$, $C \setminus U \in \Phi'$, and hence

$$C \setminus C_0 = C \setminus \overline{U} \subseteq C \setminus U \in \Phi'.$$

(i) $\Longrightarrow$ (ii): Let C_0 be as in (i). By condition (S_3) in the definition of a family of supports, there is an open neighborhood U of C_0 with $\overline{U} \in \Phi$. Then $C \setminus U \subseteq \overline{C \setminus C_0} \in \Phi'$ for all $C \in \Phi$. Hence $U \in \Phi * \Phi'$.

(ii) $\Longrightarrow$ (iii): Let $U \in \Phi * \Phi'$ be given. By (ii), we have $U_0 \in \Phi * \Phi'$ with $\overline{U_0} \in \Phi$. Set $W = U \cap U_0$. Then $W \in \Phi * \Phi'$ and $\overline{W} \in \Phi$. Hence the boundary $\partial W = \overline{W} \setminus W$ belongs to Φ'. Take an open neighborhood W' of ∂W with $\overline{W'} \in \Phi'$. Then $X \setminus \overline{W'} \in \Phi * \Phi'$ and hence

$$V := W \setminus \overline{W'} = W \cap (X \setminus \overline{W'}) \in \Phi * \Phi'.$$

Moreover it is clear that $\overline{V} \subseteq W \subseteq U$. Since $\overline{W} \in \Phi$ and $\overline{V} \subseteq \overline{W}$, we have also that $\overline{V} \in \Phi$.

(iii) $\Longrightarrow$ (ii) is trivial. □

It is easy to see that in each of the following examples, $\Phi' \subseteq \Phi$ are families of supports in X, where Φ' is complete in Φ.

Example I: Let Φ be the family of all closed subsets of X, and Φ' the family of the compact subsets of X. Then $\Phi * \Phi'$ consists of all complements of compact sets.

Example II: Let K be a fixed compact subset of X, Φ the family of the compact subsets of X, and Φ' the family of all $C \in \Phi$ with $K \cap C = \emptyset$. Then $\Phi * \Phi'$ is the family of neighborhoods of K.

Example III (cf. [M]): Let $X = \tilde{X} \setminus S$ where $\tilde{X}$ is a compact complex space whose singular points are isolated and S is the set of all singular points of $\tilde{X}$. Assume that S is divided into two non-empty subsets S_1 and S_2. Let Φ be the family of all closed subsets of X, and Φ' the family of all $C' \in \Phi$ such that $C' \cap U = \emptyset$ for some neighborhood U of S_1 in $\tilde{X}$. Then $\Phi * \Phi'$ is the family of open subsets U of X which are of the form $U = \tilde{U} \setminus S_1$ where $\tilde{U}$ is a neighborhood of S_1 in $\tilde{X}$.

Example IV: Let X be an open subset of $\mathbb{C}^n$, K a closed subset of the boundary of X in $\mathbb{C}^n$, Φ the family of all subsets of X which are closed in X, and Φ' the family of all $C' \in \Phi$ such that $C' \cap U = \emptyset$ for some $\mathbb{C}^n$-open neighborhood U of K. Then $\Phi * \Phi'$ consits of all sets of the form $U \cap X$ where U ranges over the $\mathbb{C}^n$-open neighborhoods of K.

Now let $\Phi' \subseteq \Phi$ be two families of supports in X. Then we use the following notations:

Two forms $f \in C^0_{s,r}(U,E)$, $g \in C^0_{s,r}(V,E)$ where $U,V \in \Phi * \Phi'$, will be called *equivalent* if there is an open $W \subseteq U \cap V$ with $f\big|_W = g\big|_W$. The corresponding space of equivalence classes of the disjoint union of all $C^0_{s,r}(U,E)$, $U \in \Phi * \Phi'$, will be denoted by $C^0_{s,r}(\Phi*\Phi',E)$. $Z^0_{s,r}(\Phi*\Phi',E)$ denotes the subspace of $C^0_{s,r}(\Phi*\Phi',E)$ defined by $\overline{\partial}$-closed forms, and $E^0_{s,r}(\Phi*\Phi',E)$ denotes the subspace of $Z^0_{s,r}(\Phi*\Phi',E)$ defined by $\overline{\partial}$-exact forms.

We set

$$H^{s,r}(\Phi * \Phi', E) = Z^0_{s,r}(\Phi * \Phi', E)/E^0_{s,r}(\Phi * \Phi', E).$$

Furthermore, we denote by $Z^0_{s,r}(\Phi; X, E)\big|\Phi * \Phi'$ the image of $Z^0_{s,r}(\Phi; X, E)$ in

$Z^0_{s,r}(\Phi * \Phi', E)$ under the restriction map, and set

$$\hat{H}^{s,r}(\Phi * \Phi', E) = Z^0_{s,r}(\Phi * \Phi', E)/\big[Z^0_{s,r}(\Phi; X, E)\big|\Phi * \Phi'\big].$$

1.5. Lemma. *If $\Phi' \subseteq \Phi$ are two families of supports in X such that Φ' is complete in Φ, then:*

(i) For all s, r with $0 \leq s, r \leq n$, we have the relation

$$E^0_{s,r}(\Phi * \Phi', E) \subseteq Z^0_{s,r}(\Phi; X, E)\big|\Phi * \Phi'. \tag{1.1}$$

and therefore the inequality

$$\dim \hat{H}^{s,r}(\Phi * \Phi', E) \leq \dim H^{s,r}(\Phi * \Phi', E). \tag{1.2}$$

(ii) For all s, r with $0 \leq s \leq n$ and $1 \leq r \leq n$, we have a natural isomorphism

$$\hat{\delta} : E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E) \longrightarrow \hat{H}^{s,r-1}(\Phi * \Phi', E), \tag{1.3}$$

and hence the equality

$$\dim \big[E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E)\big] = \dim \hat{H}^{s,r-1}(\Phi * \Phi', E). \tag{1.4}$$

(iii) For all s, r with $0 \leq s \leq n$ and $1 \leq r \leq n$, we have a natural linear epimorphism

$$\delta : E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E) \longrightarrow H^{s,r-1}(\Phi * \Phi', E), \tag{1.5}$$

and hence the inequality

$$\dim \big[E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E)\big] \leq \dim H^{s,r-1}(\Phi * \Phi', E). \tag{1.6}$$

Proof. (i): Let $f \in E^0_{s,r}(\Phi * \Phi', E)$ be given. If $r = 0$, then $f = 0$ and hence $f \in Z^0_{s,r}(\Phi; X, E)\big|\Phi*\Phi'$. If $r \geq 1$, then there exists $U \in \Phi*\Phi'$ and $\varphi \in C^0_{s,r-1}(U, E)$ with continuous $\overline{\partial}\varphi$ such that f is defined by $\overline{\partial}\varphi$. By condition (iii) in 1.4, after shrinking U, we may assume that $\overline{U} \in \Phi$, and, by the same argument, we can find $V \in \Phi * \Phi'$ with $\overline{V} \subseteq U$. Take a real C^∞-function χ on X with $\operatorname{supp}\chi \subseteq U$ and $\chi \equiv 1$ on V. Let $\psi \in Z^0_{s,r}(X, E)$ be the form defined by

$$\psi = \overline{\partial}(\chi\varphi) = \overline{\partial}\chi \wedge \varphi + \chi\overline{\partial}\varphi$$

on U and $\psi \equiv 0$ outside U. Since $\overline{U} \in \Phi$, then $\psi \in Z^0_{s,r}(\Phi; X, E)$. Since $\psi = \overline{\partial}\varphi$ on V and therefore the germ f is defined by $\psi\big|V$, this implies that $f \in Z^0_{s,r}(\Phi; X, E)\big|\Phi*\Phi'$.

(ii): Let $f \in E^0_{s,r}(\Phi \to \Phi'; X, E)$. Take $u \in C^{s,r-1}(\Phi; X, E)$ with $\overline{\partial}u = f$. Then $u|_{(X\setminus \operatorname{supp} f)} \in Z^0_{s,r}(X \setminus \operatorname{supp} f, E)$. Therefore, since $X \setminus \operatorname{supp} f \in \Phi * \Phi'$, $u|_{(X\setminus \operatorname{supp} f)}$

defines an element in $\hat{H}^{s,r-1}(\Phi * \Phi', E)$. Denote this element by $\hat{\delta} f$. This element does not depend on the choice of u, for if $\tilde{u} \in C^{s,r-1}(\Phi; X, E)$ is another form with $\overline{\partial}\tilde{u} = f$, then $u - \tilde{u} \in Z^0_{s,r-1}(\Phi; X, E)$. Hence a linear map

$$\hat{\delta} : E^0_{s,r}(\Phi \to \Phi'; X, E) \longrightarrow \hat{H}^{s,r-1}(\Phi * \Phi', E)$$

is well defined. It remains to show that $\hat{\delta}$ is surjective and

$$\ker \hat{\delta} = E^0_{s,r}(\Phi'; X, E). \tag{1.7}$$

Proof of the surjectivity: Let $F \in \hat{H}^{s,r-1}(\Phi * \Phi', E)$ be given. Take $U \in \Phi * \Phi'$ and $f \in Z^0_{s,r-1}(U, E)$ such that F is defined by f. By condition (iii) in 1.4, we can find open sets $V, W \in \Phi * \Phi'$ such that $\overline{V} \subseteq W$, $\overline{W} \subseteq U$ and $\overline{W} \in \Phi$. Take a real C^∞-function χ on X with $\operatorname{supp} \chi \subseteq W$ and $\chi \equiv 1$ on $\overline{V}$ and let g be the form on X defined by $g = \overline{\partial}(\chi f)$ on W and by zero outside W. Since $\overline{W} \in \Phi$, then $g \in E^0_{s,r}(\Phi; X, E)$. Since $g \equiv 0$ outside $\overline{W} \setminus V$ and $\overline{W} \setminus V \in \Phi'$, we see that even $g \in E^0_{s,r}(\Phi \to \Phi'; X, E)$. As $(\chi f)|_V = f|_V$ defines F, we see that $\hat{\delta} g = F$.

Proof of (1.7): First let $f \in E^0_{s,r}(\Phi'; X, E)$ be given. Then there exists $u \in Z^0_{s,r-1}(X, E)$ with $\overline{\partial} u = f$ and $\operatorname{supp} u \in \Phi'$. Since $(X \setminus \operatorname{supp} u) \in \Phi * \Phi'$, then, by definition of $\hat{\delta}$, $\hat{\delta} f$ is defined by the form $u|_{(X \setminus \operatorname{supp} u)}$ which is zero.

Now let $f \in \ker \hat{\delta}$ be given, i.e. $f = \overline{\partial} u$ where u is a form from $C^0_{s,r-1}(\Phi; X, E)$ such that, for certain $v \in Z^0_{s,r-1}(\Phi; X, E)$ and some $U \in \Phi * \Phi'$, $u = v$ on U. Then $\operatorname{supp}(u - v) \in \Phi'$ and $\overline{\partial}(u - v) = f$, i.e. $f \in E^0_{s,r}(\Phi'; X, E)$.

(iii) follows from (i) and (ii). □

1.6. Corollary. *If $\Phi' \subseteq \Phi$ are two families of supports in X such that Φ' is complete in Φ, then*

$$\dim H^{s,r}_{\Phi'}(X, E) \le \dim H^{s,r}_{\Phi}(X, E) + \dim H^{s,r-1}(\Phi * \Phi', E) \tag{1.8}$$

for all s, r with $0 \le s \le n$ and $1 \le r \le n$.

Proof. From

$$E^0_{s,r}(\Phi'; X, E) \subseteq E^0_{s,r}(\Phi \to \Phi'; X, E) \subseteq Z^0_{s,r}(\Phi'; X, E)$$

it follows that

$$\begin{aligned}\dim H^0_{s,r}(\Phi'; X, E) = \dim \big[E^0_{s,r}(\Phi \to \Phi'; X, E) / E^0_{s,r}(\Phi'; X, E)\big] \\ + \dim \big[Z^0_{s,r}(\Phi'; X, E) / E^0_{s,r}(\Phi \to \Phi'; X, E)\big].\end{aligned}$$

In view of Lemma 1.5 (iii) and the obvious inequality

$$\dim Z^0_{s,r}(\Phi'; X, E) / E^0_{s,r}(\Phi \to \Phi'; X, E) \le \dim H^{s,r}_{\Phi}(X, E),$$

this implies (1.8). □

2. Topological preparations

In this section X is an n-dimensional complex manifold countable at infinity.

2.1. The LF-topology of $C^0_{s,r}(\Phi; X, E)$ and $H^{s,r}_\Phi(X, E)$. Let E be a holomorphic vector bundle over X, and Φ a family of supports in X.

As usual, we consider $C^0_{s,r}(X, E)$ as Fréchet space with the topology of uniform convergence on compact subsets of X. If C is a closed subset of X, then $C^0_{s,r}(C; X, E)$ will be also consider as Fréchet space, with the topology induced from $C^0_{s,r}(X, E)$.

The space $C^0_{s,r}(\Phi; X, E)$ will be provided with the inductive limit topology of the Fréchet spaces $C^0_{s,r}(C; X, E)$, $C \in \Phi$, i.e. the finest locally convex topology such that, for each $C \in \Phi$, the natural injection of $C^0_{s,r}(C; X, E)$ in $C^0_{s,r}(\Phi; X, E)$ is continuous.

A family of supports Φ in X will be called *cofinal* (cf. [C-S]) if there exists a sequence $(C_j)_{j\in\mathbb{N}}$ of sets $C_j \in \Phi$ such that each $C \in \Phi$ is contained in certain C_j. In view of condition (S_3) in the definition of a family of supports, then this sequence always can be chosen so that each C_j is contained in the interior of C_{j+1}.

If Φ is a cofinal family of supports in X, then $C^0_{s,r}(\Phi; X, E)$ is an LF-space, i.e. a *countable strict inductive limit of Fréchet spaces* (cf., e.g., Chapter 13 in [T]) - if $(C_j)_{j\in\mathbb{N}}$ is a sequence as in the definition of cofinality such that each C_j is contained in the interior of C_{j+1}, then the sequence of Fréchet spaces $C^0_{s,r}(C_j; X, E)$ may serve as defining sequence.

It is easy to see that all families of supports considered in Examples I - IV of Sect. 1 are cofinal.

We provide $Z^0_{s,r}(\Phi; X, E)$ and $E^0_{s,r}(\Phi; X, E)$ with the topology of $C^0_{s,r}(\Phi; X, E)$, and $H^{s,r}_\Phi(X, E)$ with the corresponding factor toppology. The space of continuous linear forms on $H^{s,r}_\Phi(X, E)$ will be denoted by $\big(H^{s,r}_\Phi(X, E)\big)'$. Recall that the topology of $H^{s,r}_\Phi(X, E)$ is separated if and only if $E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(\Phi; X, E)$.

2.2. The dual family Φ^*. If Φ is a family of supports in X, then we denote by Φ^* the family of all closed subsets C^* of X such that, for all $C \in \Phi$, the intersection $C^* \cap C$ is compact. Φ^* will be called the *dual family of* Φ.

A family of supports Φ in X will be called *reflexive* if $\Phi^{**} = \Phi$.

If Φ is a family of supports in X, then, obviously, conditions (S_1) and (S_2) in the definition of a family of supports are also fulfilled for Φ^*. However, condition (S_3) is not fulfilled in general for Φ^*. We thank Lee Stout for submitting us by e-mail the following counterexample:

Let $\mathbb{R}_+$ be the nonnegative part of the real axis in $\mathbb{C}$. Denote by Φ the family of all closed subsets C of $\mathbb{C}$ for which $C \cap \mathbb{R}_+$ is compact. Then Φ is a family for supports in $\mathbb{C}$, but the dual family Φ^* does not satisfy condition (S_3). ($\mathbb{R}_+ \in \Phi^*$, but there is no neighborhood U of $\mathbb{R}_+$ with $\overline{U} \in \Phi^*$.)

It is easy to see that the family Φ in the example of STOUT is not cofinal. This is consistent with the following lemma.

2.3. Lemma. *If Φ is a cofinal family of supports in X, then Φ^* is a family of supports in X.*

Proof. Let $C^* \in \Phi^*$ be given. We have to find a neighborhood V of C^* such that $\overline{V} \in \Phi^*$. Since Φ is cofinal we have a sequence $C_j \in \Phi$, $j = 1, 2, \ldots$ such that each $C \in \Phi$ is contained in some C_j and if U_j is the interior of C_j, then $C_j \subseteq U_{j+1}$. Set $C_1^* = C^* \cap C_1$ and $C_j^* = C^* \cap (C_j \setminus U_{j-1})$ if $j \geq 2$. Then all C_j^* are compact and $C_j^* \cap C_{j-2} = \emptyset$ if $j \geq 3$. Take for each $j \geq 1$ a relatively compact open set V_j with

$$C_j^* \subseteq V_j \qquad \text{and} \qquad \overline{V}_j \cap C_{j-2} = \emptyset \quad \text{if} \quad j \geq 3$$

Then

$$V := \bigcup_{j=1}^{\infty} V_j.$$

has the required properties. □

It is not true in general that the dual family of a cofinal family of supports is again cofinal. For example, the dual family of the family Φ' in Example IV, Sect. 1.4, is not cofinal.

2.4. Lemma. *For any holomorphic vector bundle E over X and all integers s, r with $0 \leq s, r \leq n$, the following two assertions hold:*

(i) Let C be a closed subset of X, and f a continuous linear functional on $C^0_{s,r}(C; X, E)$. Then there exists a compact set $K \subseteq C$ such that $f \equiv 0$ on $C^0_{s,r}(C \setminus K; X, E)$.

(ii) Let Φ be a family of supports in X with $\bigcup \Phi = X$, and let f be a continuous linear functional on $C^0_{s,r}(\Phi; X, E)$. Denote by f_X the current which is then defined by f on all of X (for $\bigcup \Phi = X$). Then

$$\operatorname{supp} f_X \in \Phi^*.$$

Proof. (i) Take a sequence $(K_j)_{j \in \mathbb{N}}$ of compact subsets of X such that each K_j is contained in the interior of K_{j+1} and $\bigcup_{j \in \mathbb{N}} K_j = X$. We have to prove that $f \equiv 0$ on $C^0_{s,r}(C \setminus K_j; X, E)$ if j is sufficiently large.

Assume the contrary. Then we can find a sequence $(\varphi_j)_{j \in \mathbb{N}}$ of forms $\varphi_j \in C^0_{s,r}(C \setminus K_j; X, E)$ such that $f(\varphi_j) = 1$ for all j. This contradicts the continuity of f, since, by definition of the topology of $C^0_{s,r}(C; X, E)$, the sequence $(\varphi_j)_{j \in \mathbb{N}}$ converges to zero in $C^0_{s,r}(C; X, E)$.

(ii) Let C be any element of Φ, by condition (S_3) in the definition of a family of supports, we can find a neighborhood U of C with $\overline{U} \in \Phi$. Then, by part (i) of the lemma, there is a compact set $K \subseteq \overline{U}$ such that $f \equiv 0$ on $C^0_{s,r}(\overline{U} \setminus K; X, E)$. In particular, then $f_X \equiv 0$ over $U \setminus K$, i.e. $(\operatorname{supp} f_X) \cap (U \setminus K) = \emptyset$. Hence the set $(\operatorname{supp} f_X) \cap C$ is contained in K and therefore compact. □

2.5. Lemma. *Let Φ be a family of supports in X such that $\bigcup \Phi = X$ and Φ^* is also a family of supports. Further, let E be a holomorphic vector bundle over X and E^* the dual of E. Then, for all integers s, r with $0 \le s, r \le n$, there is a natural linear epimorphism*

$$h'_{s,r} : H^{n-s,n-r}_{\Phi^*}(X, E^*) \longrightarrow \big(H^{s,r}_{\Phi}(X, E)\big)',$$

which is an <u>*isomophism*</u> *if and only if*

$$(2.1)\quad E^0_{n-s,n-r}(\Phi^*; X, E^*) = \left\{ f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*) \;\middle|\; \int_X f \wedge g = 0 \quad \textit{for all} \quad g \in Z^0_{s,r}(\Phi; X, E) \right\}.$$

Proof. Since, for $C \in \Phi$ and $C^* \in \Phi^*$, the intersection $C \cap C^*$ is compact, for each $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$, setting

$$f'(g) := \int_X f \wedge g \qquad \text{for} \quad g \in Z^0_{s,r}(\Phi; X, E),$$

we can define a continuous linear functional f' on $Z^0_{s,r}(\Phi; X, E)$, where, by Stokes' theorem, $f'(g) = 0$ if $f \in E^0_{n-s,n-r}(\Phi^*; X, E^*)$ or $g \in E^0_{s,r}(\Phi; X, E)$. Hence in this way we get a linear map from $H^{n-s,n-r}_{\Phi^*}(X, E^*)$ to $\big(H^{s,r}_{\Phi}(X, E)\big)'$ which we denote by $h'_{s,r}$.

Obviously, the injectivity of this map is equivalent to (2.1). Therefore it remains to prove that $h'_{s,r}$ is surjective if condition (2.1) is fulfilled.

Assume (2.1) holds, consider an arbitrary functional $F \in \big(H^{s,r}_{\Phi}(X, E)\big)'$, and let

$$p : Z^0_{s,r}(\Phi; X, E) \longrightarrow H^{s,r}_{\Phi}(X, E)$$

be the canonical projection. Then we have to find $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$ with

$$(2.2)\qquad \int_X f \wedge g = (F \circ p)(g) \qquad \text{for all} \quad g \in Z^0_{s,r}(\Phi; X, E).$$

First, by the Hahn-Banach theorem, we can find a continuous linear functional $\tilde F$ on $C^0_{s,r}(\Phi; X, E)$ with $\tilde F = F \circ p$ on $Z^0_{s,r}(\Phi; X, E)$. Since $\bigcup \Phi = X$, all compact subsets of X belong to Φ. Therefore, the continuity of $\tilde F$ on $C^0_{s,r}(\Phi; X, E)$ in particular means that $\tilde F$ is an E^*-valued current of bidegree $(n-s, n-r)$ on X. Since $\tilde F$ vanishes on $E^0_{s,r}(\Phi; X, E)$, $\tilde F$ is $\overline\partial$-closed, and it follows from Lemma 2.4 (ii) that $\operatorname{supp} \tilde F \in \Phi^*$.

If $r = n$, by regularity of $\overline\partial$, there exists $f \in Z^0_{n-s,0}(\Phi^*; X, E^*)$ (f is even holomorphic) such that

$$\int_X f \wedge g = \tilde F(g) = (F \circ p)(g) \qquad \text{for all} \quad g \in Z^0_{s,n}(\Phi; X, E),$$

i.e. (2.2) is fulfilled.

Now let $r \le n-1$. Since Φ^* is a family of supports, by condition (S_3) in the definition of a family of supports and by regularity of $\overline{\partial}$ (see, e.g., Corollary 2.15 in [H-L]), we can find a current S on X with $\operatorname{supp} S \in \Phi^*$ such that the current $\tilde{F} - \overline{\partial} S$ is defined by a continuous form with support in Φ^*, i.e., we have a form $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$ such that

$$\int_X f \wedge \varphi = (\tilde{F} - \overline{\partial} S)(\varphi) \tag{2.3}$$

for all E-valued $C^\infty_{s,r}$-forms φ with compact support on X. It remains to prove (2.2).

First consider a form $g_\infty \in Z^0_{s,r}(\Phi; X, E)$ which is of class C^∞. Since

$$(\operatorname{supp} S \cup \operatorname{supp} f \cup \operatorname{supp} \tilde{F}) \cap \operatorname{supp} g_\infty$$

is compact, then it follows from (2.3) that

$$\int_X f \wedge g_\infty = (\tilde{F} - \overline{\partial} S)(g_\infty) = \tilde{F}(g_\infty). \tag{2.4}$$

Now let $g \in Z^0_{s,r}(\Phi; X, E)$ be arbitrary. If $r = 0$, then g is holomorphic and (2.2) follows from (2.4). Therefore we may assume that $r \ge 1$. Then, as above, since Φ is a family of supports, by condition (S_3) in the definition of a family of supports and by regularity of $\overline{\partial}$, we can find a C^∞-form $g_\infty \in Z^0_{s,r}(\Phi; X, E)$ and a form $\psi \in C^0_{s,r-1}(\Phi; X, E)$ such that

$$g = g_\infty + \overline{\partial}\psi.$$

It follows from Stokes' theorem and (2.4) that

$$\int_X f \wedge g = \int_X f \wedge g_\infty + \int_X f \wedge \overline{\partial}\psi = \int_X f \wedge g_\infty = \tilde{F}(g_\infty).$$

Since $g - g_\infty = \overline{\partial}\psi \in E^0_{s,r}(\Phi; X, E)$ and therefore $\tilde{F}(g_\infty) = \tilde{F}(g)$, this implies (2.2). $\square$

2.6. Lemma. *Let Φ be a cofinal family of supports in X. Further, let E be a holomorphic vector bundle over X, $C \in \Phi$, $0 \le s \le n$ and $1 \le r \le n$. Suppose there exists a finite dimensional linear subspace F of $C^0_{s,r}(C; X, E)$ such that the linear space*

$$F + E^0_{s,r}(\Phi \to C; X, E)$$

is topologically closed in $C^0_{s,r}(C; X, E)$. Then also $E^0_{s,r}(\Phi \to C; X, E)$ is topologically closed in $C^0_{s,r}(C; X, E)$ and, moreover, there exists $C_0 \in \Phi$ with

$$E^0_{s,r}(\Phi \to C; X, E) = E^0_{s,r}(C_0 \to C; X, E). \tag{2.5}$$

Proof. Since Φ is cofinal, we can find a sequence $C_j \in \Phi$ such that each $C \in \Phi$ is contained in some C_j. Then

$$E^0_{s,r}(\Phi \to C; X, E) \subseteq \bigcup_{j=1}^{\infty} \overline{\partial} C^0_{s,r-1}(C_j; X, E).$$

Since $F + E^0_{s,r}(\Phi \to C; X, E)$ is a Fréchet space, this implies that for certain j_0, the space

$$F + E^0_{s,r}(\Phi \to C; X, E) \cap \overline{\partial} C^0_{s,r-1}(C_{j_0}; X, E) \tag{2.6}$$

is of second Baire categorie in $F + E^0_{s,r}(\Phi \to C; X, E)$. Let D_0 be the linear subspace of all $\varphi \in C^0_{s,r-1}(C_{j_0}; X, E)$ with $\overline{\partial}\varphi \in C^0_{s,r}(C; X, E)$, and let $1_F \oplus \overline{\partial}_0$ be the linear operator with domain of definition $F \oplus D_0$ between the Fréchet spaces $F \oplus C^0_{s,r-1}(C_{j_0}; X, E)$ and $F + E^0_{s,r}(\Phi \to C; X, E)$ defined by $(1_F \oplus \overline{\partial}_0)(f, \varphi) = f + \overline{\partial}\varphi$ for $(f, \varphi) \in F \oplus D_0$. Since this operator is closed, and its image (which is equal to (2.6)) is of second Baire categorie in $F + E^0_{s,r}(\Phi \to C; X, E)$, it follows by the open mapping theorem that this operator is onto. Hence

$$(1_F \oplus \overline{\partial}_0)(F \oplus D_0) = F + E^0_{s,r}(\Phi \to C; X, E) \tag{2.7}$$

and

$$\overline{\partial}(D_0) = (1_F \oplus \overline{\partial}_0)(\{0\} \oplus D_0)$$

is finite codimensional in $F + E^0_{s,r}(\Phi \to C; X, E)$. Since $\overline{\partial}$ (with D_0 as domain of definition) is closed, it follows from the open mapping theorem that $\overline{\partial}(D_0)$ is moreover topologically closed in $F + E^0_{s,r}(\Phi \to C; X, E)$. This implies that $E^0_{s,r}(\Phi \to C; X, E)$ is topologically closed in $F + E^0_{s,r}(\Phi \to C; X, E)$, for $\overline{\partial}(D_0) \subseteq E^0_{s,r}(\Phi \to C; X, E) \subseteq F + E^0_{s,r}(\Phi \to C; X, E)$. (2.5) follows from (2.7) repeating the first part of the proof with $F = \{0\}$. $\square$

3. Serre duality for nice families of supports

In this section X is always a complex manifold of dimension n countable at infinity.

3.1. Compact boundaries and nice families of supports. Γ will be called a *compact boundary in* X if Γ is a smooth compact oriented real hypersurface in X which is the boundary (in the sense of oriented manifolds[2]) of certain open subset of X. This open subset then will be denoted by $Z_+(\Gamma, X)$. Further then we set $Z_-(\Gamma, X) = X \setminus \overline{Z_+(\Gamma, X)}$, and we denote by $-\Gamma$ the oriented manifold which is equal to Γ as a manifold, but carries the opposite orientation. Note that

$$Z_\pm(-\Gamma, X) = Z_\mp(\Gamma, X).$$

[2]We always assume that complex manifolds are oriented by $dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$ if $x_1, y_1, \ldots, x_n, y_n$ are real coordinates such that $x_1 + iy_1, \ldots, x_n + iy_n$ are holomorphic coordinates.

If Γ is a compact boundary in X, then we denote by $\Phi(\Gamma, X))$ the family of all closed subsets C of X such that $C \cap \overline{Z_+(\Gamma, X)}$ is compact. Then $\Phi(\Gamma, X)$ is a reflexive cofinal family of supports in X, where $\Phi(\Gamma, X)^* = \Phi(-\Gamma, X)$.

A family of supports Φ in X will be called *nice* if there exists a compact boundary Γ in X with $\Phi = \Phi(\Gamma, X)$.

3.2. The families Φ^0 and Φ^{*0} and the restrictions $\Phi|_U$, $U \in \Phi^0 \cup \Phi^{*0}$. Let Φ be a nice family of supports in X. Then we denote by Φ^0 the family of all open sets $U \subseteq X$ such that $\overline{U} \in \Phi$ and $\overline{Z_-(\Gamma, X)} \subseteq U$ for certain compact boundary Γ in X with $\Phi = \Phi(\Gamma, X)$. Φ^{*0} $(= (\Phi^*)^0)$ then consists of all open sets $U^* \subseteq X$ such that $\overline{U}^* \in \Phi^*$ and $\overline{Z_+(\Gamma, X)} \subseteq U^*$ for certain compact boundary Γ in X with $\Phi = \Phi(\Gamma, X)$.

If $U \in \Phi^0 \cup \Phi^{*0}$, then we set

$$\Phi|_U = \{C \mid C \in \Phi \quad \text{and} \quad C \subseteq U\} \qquad \text{if} \quad U \in \Phi^0$$

and

$$\Phi|_U = \{C \mid C = C_0 \cap U \quad \text{for some} \quad C_0 \in \Phi\} \qquad \text{if} \quad U \in \Phi^{*0},$$

i.e.

$$\Phi|_U = \Phi(\Gamma, U).$$

if Γ is a compact boundary in X with $\overline{Z_-(\Gamma, X)} \subseteq U$ (if $U \in \Phi^0$) resp. $\overline{Z_+(\Gamma, X)} \subseteq U^*$ (if $U \in \Phi^{*0}$). Note that

$$(\Phi|_U)^* = \Phi^*|_U.$$

By definition, $\Phi|_U$ is a family of supports in U (and not in X). However, if $U \in \Phi^0$, then $\Phi|_U$ may be considered also as a family of supports in X.

3.3. The conditions $Cl_{s,r}(\Phi; X, E)$ and $SCl_{s,r}(\Phi; X, E)$. Let Φ be a nice family of supports in X, E a holomorphic vector bundle over X and $0 \leq s \leq n$, $1 \leq r \leq n$.

We say *condition* $Cl_{s,r}(\Phi; X, E)$ *is fulfilled* if, for all $C \in \Phi$, the space $E^0_{s,r}(\Phi \to C; X, E)$ is topologically closed in $C^0_{s,r}(X, E)$ (with respect to uniform convergence on compact subsets, cf. Sect. 2.1).

A pair (U, V) will be called a $Cl_{s,r}(\Phi; X, E)$*-pair* if $U, V \in \Phi^0$, $U \subseteq V$ and, for certain $C \in \Phi$ with $C \subseteq V$,

$$E^0_{0,r}(\Phi \to \overline{U}; X, E) = E^0_{0,r}(C \to \overline{U}; X, E).$$

It follows from Lemma 2.6 that if condition $Cl_{s,r}(\Phi; X)$ is fulfilled, then, for all $U \in \Phi^0$, there exists $V \in \Phi^0$ such that (U, V) is a $Cl_{s,r}(\Phi; X)$-pair.

We denote by $\Phi^{*0}_{s,r}(E)$ the family of all sets $U^* \in \Phi^{*0}$ such that condition $Cl_{s,r}(\Phi|_{U^*}; U^*, E)$ is fulfilled and, moreover, the following holds:

$$(3.1) \quad \begin{cases} \text{If } \varphi \in Z^0_{n-s,n-r+1}(U^*; X, E^*) \text{ such that } \int_{U^*} \varphi \wedge g = 0 \text{ for all} \\ g \in Z^0_{s,r-1}(\Phi; X, E), \text{ then } \int_{U^*} \varphi \wedge g = 0 \text{ for all } g \in Z^0_{s,r-1}(\Phi|_{U^*}; U^*, E). \end{cases}$$

We say *condition* $SCl_{s,r}(\Phi; X, E)$ (resp. *the strong condition* $Cl_{s,r}(\Phi; X, E)$) *is fulfilled* if condition $Cl_{s,r}(\Phi; X, E)$ is fulfilled and, moreover, for each $C \in \Phi^*$, there exists $C_0 \in \Phi^*$ and a sequence $(U_j^*)_{j\in\mathbb{N}} \subseteq \Phi_{s,r}^{*0}(E)$ such that $C \subseteq U_j^* \subseteq C_0$ and $\overline{U_j^*} \subseteq U_{j+1}^*$ for all $j \in \mathbb{N}$.

Note that if Φ is the family of all compact subsets of X, then in this definition we can always take the whole manifold X for both C_0 and all U_j^*. Hence in this case conditions $SCl_{s,r}(\Phi; X, E)$ and $Cl_{s,r}(\Phi; X, E)$ are equivalent.

3.4. Lemma. *Let Φ be a nice family of supports in X, E a holomorphic vector bundle over X and $0 \le s \le n$, $1 \le r \le n$. Further, let $U \in \Phi^0$ and $U^* \in \Phi_{s,r}^{*0}(E)$. Then, for each $\varphi \in Z_{n-s,n-r+1}^0(U^*; X, E^*)$ with*

$$\int_{U^*} \varphi \wedge g = 0 \qquad \textit{for all} \quad g \in Z_{s,r-1}^0(\Phi; X, E), \tag{3.2}$$

there exists $\psi \in C_{n-s,n-r}^0(U \cap U^*; U, E^*)$ which solves the equation

$$\bar\partial\psi = \varphi \qquad \textit{over} \quad U.$$

Proof. Let $\varphi \in Z_{n-s,n-r+1}^0(U^*; X, E^*)$ with (3.2) be given. Since $U^* \in \Phi_{s,r}^{*0}(E)$ and therefore condition $Cl_{s,r}(\Phi|_{U^*}; U^*, E)$ is fulfilled and since $\overline{U} \cap U^* \in \Phi|_{U^*}$, the space $E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$ is topologically closed in $C_{s,r}^0(U^*, E)$ and, by Lemma 2.6, we can find $U_0 \in \Phi^0$ with

$$E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E) \subseteq \bar\partial C_{s,r-1}^0(\overline{U}_0 \cap U^*; U^*, E). \tag{3.3}$$

Furthermore, since $U^* \in \Phi_{s,r}^{*0}(E)$, it follows from (3.1) and (3.2) that

$$\int_{U^*} \varphi \wedge g = 0 \qquad \text{for all} \quad g \in Z_{s,r-1}^0(\overline{U}_0 \cap U^*; U^*, E). \tag{3.4}$$

By (3.3) and (3.4), a linear functional ψ_1 on $E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$ can be defined as follows: For each $f \in E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$ we take $u \in C_{s,r-1}^0(\overline{U}_0 \cap U^*; U^*, E)$ with $\bar\partial u = f$ and set

$$\psi_1(f) = \int_{U^*} \varphi \wedge u. \tag{3.5}$$

Since $E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$ is a Fréchet space with respect to the topology of $C_{s,r}^0(U^*, E)$, and, by (3.3), $\bar\partial$ is surjective as operator between $C_{s,r-1}^0(\overline{U}_0 \cap U^*; U^*, E)$ and $E_{s,r}^0(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$, it follows from the open mapping theorem (for closed linear surjections between Fréchet spaces) that ψ_1 is continuous with respect to the topology of $C_{s,r}^0(U^*, E)$. Let ψ_2 be a Hahn-Banach extension of ψ_1 to all of $C_{s,r}^0(U^*, E)$. Then ψ_2 is a current over U^* with supp $\psi_2 \subset\subset U^*$ (cf. Lemma 2.4 (i) with $C = X = U^*$) such that, by (3.5), $\bar\partial\psi_2 = \pm\varphi$ on $U \cap U^*$. Let K be a compact subset of U^* which is a neighborhood of supp ψ_2. Since φ is continuous, then it follows by regularity of $\bar\partial$ that there exists a form $\psi \in C_{n-s,n-r}^0(K \cap U; U \cap U^*, E^*)$ with $\bar\partial\psi = \varphi$ on $U \cap U^*$. After extending by zero we may assume that $\psi \in C_{n-s,n-r}^0(K \cap U; U, E^*)$. Since both ψ and φ vanish outside U^*, this implies that the equation $\bar\partial\psi = \varphi$ holds on all of U. $\square$

3.5. Lemma. *Let Φ be a nice family of supports in X, E a holomorphic vector bundle over X, and $0 \le s \le n$, $1 \le r \le n$. Further, let $U, U_0, U_{00} \in \Phi^0$ and $U^*, U_0^* \in \Phi^{*0}_{s,r}(E)$ with $\overline{U} \subseteq U_0$, $\overline{U}_0 \subseteq U_{00}$ and $\overline{U^*} \subseteq U_0^*$ such that $(U \cap U^*, U_0 \cap U^*)$ is a $Cl_{s,r}(\Phi|_{U^*}; U^*, E)$-pair, and let $f \in Z^0_{s,r}(U \cap U_0^*; U_0^*, E)$ which extends continuously to a neighborhood of $\overline{U_0^*}$ such that*

$$\int_{U_{00}\cap U_0^*} f \wedge \psi = 0 \qquad \text{for all} \quad \psi \in Z^0_{n-s,n-r}(U_{00} \cap U_0^*; U_{00}, E^*). \tag{3.6}$$

Then there exists $u \in C^0_{s,r-1}(U_0 \cap U^*; U^*, E)$ which solves the equation

$$\overline{\partial} u = f \qquad \text{over} \quad U^*.$$

Proof. Denote by H the space of all $\varphi \in Z^0_{n-s,n-r+1}(U_0^*; X, E^*)$ such that

$$\int_{U_0^*} \varphi \wedge g = 0 \qquad \text{for all} \quad g \in Z^0_{s,r-1}(\Phi; X, E).$$

Then, by Lemma 3.4,

$$H|_{U_{00}} \subseteq \overline{\partial} C^0_{n-s,n-r}(U_{00} \cap U_0^*; U_{00}, E^*). \tag{3.7}$$

Now let F_1 be the Fréchet space of all $\varphi \in H$ with $\operatorname{supp} \varphi \subseteq \overline{U}^*$ endowed with the topology of $C^0_{n-s,n-r+1}(X, E^*)$, and let $F_2 := C^0_{n-s,n-r}(U_{00} \cap \overline{U}^*_0; U_{00}, E^*)$ which is also a Fréchet space. Then it follows from (3.7) that

$$F_1|_{U_{00}} \subseteq \overline{\partial} F_2. \tag{3.8}$$

By (3.8) and (3.6) a linear functional u_1 on F_1 can be defined as follows: For each $\varphi \in F_1$ we take $\psi \in F_2$ with $\overline{\partial}\psi = \varphi$ on U_{00} and set

$$u_1(\varphi) = \int_{U_{00}\cap U_0^*} f \wedge \psi.$$

Since, by (3.8), $\overline{\partial}$ is surjective as operator between F_1 and F_2 and f extends continuously to a neighborhood of $\overline{U}^*_0$, it follows from the open mapping theorem (for closed linear surjections between Fréchet spaces) that u_1 is continuous with respect to the topology of $C^0_{n-s,n-r+1}(X, E^*)$. Let u_2 be a Hahn-Banach extension of u_1 to all of $C^0_{n-s,n-r+1}(X, E^*)$. Then u_2 is a current with compact support on X. Let α be a smooth $(n-s, n-r)$-form with compact support in U^* and values in E^*. Since $\operatorname{supp} f \subset U \cap U_0^*$ and $\overline{U} \subseteq U_{00}$ and as $\overline{\partial}\alpha \in F_1$ and $\alpha|_{U_{00}} \in F_2$, we have

$$u_2(\overline{\partial}\alpha) = u_1(\overline{\partial}\alpha) = \int_{U_{00}\cap U_0^*} f \wedge \alpha,$$

which means that $\overline{\partial} u_2 = \pm f$ on U^*. Take a compact subset K of X which is a neighborhood of $\operatorname{supp} u_2$. Since f is continuous, then by regularity of $\overline{\partial}$ we can find a form $u_3 \in C^0_{s,r-1}(K \cap U^*; U^*, E)$ with $\overline{\partial} u_3 = f$ on U^*. Since $K \cap U^* \in \Phi|_{U^*}$, this implies that $f \in E^0_{s,r}(\Phi|_{U^*} \to \overline{U} \cap U^*; U^*, E)$. Since $(U \cap U^*, U_0 \cap U^*)$ is a $Cl_{s,r}(\Phi|_{U^*}; U^*, E)$-pair, it follows that the equation $\overline{\partial} u = f$ can be solved on U^* even with some $u \in C^0_{s,r-1}(U_0 \cap U^*; U^*, E)$. □

3.6. Lemma. *Let Φ be a nice family of supports in X, E a holomorphic vector bundle over X, and $0 \le s \le n$, $1 \le r \le n$. Further, let $U, U_0, U_{00} \in \Phi^0$, $U^*, U_0^*, \in \Phi_{s,r}^{*0}(E)$, and $U_{00}^* \in \Phi^{*0}$ with $\overline{U} \subseteq U_0$, $\overline{U}_0 \subseteq U_{00}$, $\overline{U}^* \subseteq U_0^*$, $\overline{U}_0^* \subseteq U_{00}^*$ such that $(U \cap U^*, U_0 \cap U^*)$ is a $Cl_{s,r}(\Phi|_{U^*}; U^*, E)$-pair. Then the closure (with respect to the topology of $C^0_{n-s,n-r}(U, E^*)$) of the image of the restriction map*

$$Z^0_{n-s,n-r}(U_{00} \cap U_{00}^*; U_{00}, E^*) \longrightarrow C^0_{n-s,n-r}(U, E^*)$$

contains the image of the restriction map

$$Z^0_{n-s,n-r}(U_0 \cap U^*; U_0, E^*) \longrightarrow C^0_{n-s,n-r}(U, E^*).$$

Proof. Let F be a continuous linear functional on $C^0_{n-s,n-r}(U, E^*)$ such that

$$F(\psi) = 0 \qquad \text{for all} \quad \psi \in Z^0_{n-s,n-r}(U_{00} \cap U_{00}^*; U_{00}, E^*). \tag{3.10}$$

By the Hahn-Banach theorem we have to prove that then

$$F(\psi) = 0 \qquad \text{for all} \quad \psi \in Z^0_{n-s,n-r}(U_0 \cap U^*; U_0, E^*). \tag{3.11}$$

Since F is continuous on $C^0_{n-s,n-r}(U, E^*)$, it follows from Lemma 2.4 (i) (with $C = X = U$) that F is a current with compact support on U. Extending by zero we may assume that F is defined on X. By (3.10) this current is $\overline{\partial}$-closed on U_{00}^*. Hence, by regularity of $\overline{\partial}$, we can find a form $f \in C^0_{s,r}(U_{00}^*, E)$ whose support is contained in $K \cap U_{00}^*$, where K is some compact subset of U such that

$$\int_U f \wedge \psi = F(\psi) \qquad \text{if} \quad \psi \in Z^0_{n-s,n-r}(U_{00} \cap U_{00}^*; U_{00}, E^*). \tag{3.12}$$

From (3.12) and (3.10) it follows that

$$\int_U f \wedge \psi = F(\psi) = 0 \qquad \text{for all} \quad \psi \in Z^0_{n-s,n-r}(U_{00} \cap U_{00}^*; U_{00}, E^*). \tag{3.13}$$

Since the support of f is contained in $K \cap U_{00}^*$, where K is a compact subset of U, it is moreover clear that the restriction of f to U_0^* belongs to $Z^0_{s,r}(U \cap U_0^*; U_0^*, E)$ and extends continuously to a neighborhood of $\overline{U_0^*}$. Therefore we can apply Lemma 3.5 and get a form $u \in C^0_{s,r-1}(U_0 \cap U^*; U^*, E)$ which solves the equation $\overline{\partial} u = f$ over U^*. Together with (3.12) and Stokes' formula this implies (3.11). □

3.7. Theorem. *Let Φ be a nice family of supports in X, E a holomorphic vector bundle over X, and $0 \le s \le n$, $1 \le r \le n$ such that condition $SCl_{s,r}(\Phi; X, E)$ is fulfilled. Then $E^0_{n-s,n-r+1}(\Phi^*; X, E^*)$ consists of all $\varphi \in Z^0_{n-s,n-r+1}(\Phi^*; X, E^*)$ such that*

$$\int_X \varphi \wedge g = 0 \qquad \textit{for all} \quad g \in Z^0_{s,r-1}(\Phi; X, E). \tag{3.14}$$

In particular, then $H^{n-s,n-r+1}_{\Phi^*}(X,E^*)$ *is separated.*

Proof. That condition (3.14) is necessary, follows from Stokes' formula. Now let $\varphi \in Z^0_{n-s,n-r+1}(\Phi^*;X,E^*)$ with (3.14) be given.

Since $\operatorname{supp}\varphi \in \Phi^*$ and condition $SCl_{s,r}(\Phi;X,E)$ is fulfilled, we can find $C_0 \in \Phi^*$ and a sequence $(U_j^*)_{j\in\mathbb{N}} \subseteq \Phi^{*0}_{s,r}(E)$ such that $\operatorname{supp}\varphi \subseteq U_0^*$ and $\overline{U}_j^* \subseteq U_{j+1}^* \subseteq C_0$ for all j. Since then, for each j, condition $Cl_{s,r}(\Phi|_{U_j^*};U_j^*,E)$ is fulfilled, by means of Lemma 2.6, we can inductively construct a sequence $(U_j)_{j\in\mathbb{N}} \subseteq \Phi^0$ such that $\bigcup_{j\in\mathbb{N}} U_j = X$, and, for all j, $\overline{U}_j \subseteq U_{j+1}$ and $(U_j\cap U_{2j}^*, U_{j+1}\cap U_{2j}^*)$ is a $Cl_{s,r}(\Phi|_{U_{2j}^*};U_{2j}^*,E)$-pair. Further we fix some sequence $(K_j)_{j\in\mathbb{N}}$ of compact sets $K_j \subseteq U_j$ such that $\bigcup_{j\in\mathbb{N}} K_j = X$ and each K_j is contained in the interior of K_{j+1}.

Now it is sufficient to construct a sequence $(\psi_j)_{j\in\mathbb{N}}$ of forms $\psi_j \in C^0_{n-s,n-r}(U_j\cap U_{2j}^*;U_j,E^*)$ such that, for all $j\in\mathbb{N}$,

$$\overline{\partial}\psi_j = \varphi \quad \text{on} \quad U_j \tag{3.15}$$

and moreover

$$\sup_{z\in K_{j-1}} \| \psi_{j-1}(z) - \psi_j(z) \| \leq \frac{1}{2^j} \qquad \text{if} \quad j \geq 1. \tag{3.16}$$

(Here $\|\cdot\|$ denotes some Riemannian norm.) In fact, then the limit $\psi := \lim \psi_j$ solves the equation $\overline{\partial}\psi = \varphi$ on all of X and, since $\operatorname{supp}\psi_j \subseteq C_0$ for all j, this limit belongs to $C^0_{n-s,n-r}(\Phi^*;X,E^*)$.

To construct this sequence we first observe that, since $U_0^* \in \Phi^{*0}_{s,r}$ and $(U_j^*)_{j\in\mathbb{N}} \subseteq \Phi^{*0}_{s,r}(E)$, from Lemma 3.4 we get a sequence $(\tilde{\psi}_j)_{j\in\mathbb{N}}$ of forms $\tilde{\psi}_j \in C^0_{n-s,n-r}(U_j\cap U_0^*;U_j,E^*)$ with

$$\overline{\partial}\tilde{\psi}_j = \varphi \quad \text{on} \quad U_j \qquad \text{for all} \quad j\in\mathbb{N}. \tag{3.17}$$

Proceeding inductively,now we set $\psi_0 = \tilde{\psi}_0$ and assume that for certain $m\in\mathbb{N}$, forms $\psi_j \in C^0_{n-s,n-r}(U_j\cap U_{2j}^*;U_j,E^*)$, $0\leq j\leq m$, are already constructed such that (3.15) and (3.16) hold for $0\leq j\leq m$. Then

$$\tilde{\psi}_{m+1} - \psi_m \in Z^0_{n-s,n-r}(U_m\cap U_{2m}^*;U_m,E)$$

and from Lemma 3.6 we get a form $\omega \in Z^0_{n-s,n-r}(U_{m+1}\cap U_{2(m+1)}^*;U_{m+1},E)$ such that

$$\sup_{z\in K_m} \| \tilde{\psi}_{m+1}(z) - \psi_m(z) - \omega(z) \| \leq \frac{1}{2^{m+1}}.$$

It remains to set $\psi_{m+1} = \tilde{\psi}_{m+1} - \omega$. $\square$

3.8. Theorem. *Let Φ_c be the family of all compact subsets of X, E a holomorphic vector bundle over X, and $0 \le s \le n$, $1 \le r \le n$. Then the following conditions are equivalent:*

(i) condition $Cl_{s,r}(\Phi_c; X, E)$;

(ii) $H_c^{s,r}(X, E)$ is separated;

(iii) $E^0_{s,r}(\Phi_c; X, E)$ consists of all $f \in Z^0_{s,r}(\Phi_c; X, E)$ with

$$\int_X f \wedge \psi = 0 \qquad \text{for all} \quad \psi \in Z^0_{n-s,n-r+1}(X, E).$$

(iv) $H^{n-s,n-r+1}(X, E^*)$ is separated;

(v) $E^0_{n-s,n-r+1}(X, E^*)$ consists of all $\varphi \in Z^0_{n-s,n-r+1}(X, E^*)$ with

$$\int_X \varphi \wedge g = 0 \qquad \text{for all} \quad g \in Z^0_{s,r-1}(\Phi_c; X, E).$$

Proof. Conclusion $(iv) \Rightarrow (iii)$ is the theorem of Serre [S], the conclusions $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(v) \Rightarrow (iv)$ are trivial, and conclusion $(i) \Rightarrow (v)$ follows from Theorem 3.7, since, as observed in Sect. 3.3, conditions $Cl_{s,r}(\Phi_c; X, E)$ and $SCl_{s,r}(\Phi_c; X, E)$ are equivalent. □

4. A simple proof of Theorem 0.3

By Serre duality (Theorem 3.8), Theorem 0.3 is equivalent to the following

4.1. Theorem. *Let the hypotheses of Theorem 0.3 be fulfilled. Then, for any holomorphic vector bundle E over Y, $H_c^{0,q+1}(X, E)$ is separated.*

Proof of Theorem 4.1. Let Φ_Y and Φ_X be the families of compact subsets of Y, resp. X. By Theorem 3.8 it is sufficient to prove that for each $K \in \Phi_X$, the space $E^0_{0,q+1}(\Phi_X \to K; X, E)$ is topologically closed in $C^0_{0,q+1}(K; X, E)$. By Lemma 2.6 this is equivalent to the following statement:

$$(4.1) \qquad \begin{cases} \text{For each } K \in \Phi_X, \text{ there exists a finite dimensional subspace } F_K \\ \text{of } C^0_{0,q+1}(K; X, E) \text{ such that } F_K + E^0_{0,q+1}(\Phi_X \to K; X, E) \\ \text{is topologically closed in } C^0_{0,q+1}(K; X, E). \end{cases}$$

To prove (4.1), we first note that Φ_X is complete in Φ_Y and therefore, by Lemma 1.5,

$$(4.2) \quad \dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)/E^0_{0,q+1}(\Phi_X; Y, E)\right] \le \dim H^{0,q}(\Phi_Y * \Phi_X, E).$$

Let ρ be as in Definition 0.1 and let $\alpha_0 \in]\inf\rho, \sup\rho[$ such that the Levi form of ρ has at least $n - q + 1$ positive eigenvalues on $\{\rho \le \alpha_0\}$. Then the manifolds

$U_\alpha := (Y \setminus X) \cup \{\rho < \alpha\}$, $\inf \rho < \alpha \leq \alpha_0$, are q-convex in the sense of Andreotti-Grauert and hence, by the Andreotti-Grauert finiteness theorem [A-G],

$$\dim H^{0,q}(U_\alpha, E) < \infty, \qquad \text{if} \quad \inf \rho < \alpha \leq \alpha_0.$$

Since $U_\alpha \in \Phi_Y * \Phi_X$ for all $\alpha \in]\inf \rho, \alpha_0]$ and, conversely, for each $U \in \Phi_Y * \Phi_X$, we can find $\alpha \in]\inf \rho, \alpha_0]$ with $U_\alpha \subseteq U$, it follows that

$$\dim H^{0,q}(\Phi_Y * \Phi_X; E) < \infty.$$

In view of (4.3) this implies that

$$\dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)/E^0_{0,q+1}(\Phi_X; Y, E)\right] < \infty,$$

i.e.

$$\dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)/E^0_{0,q+1}(\Phi_X; X, E)\right] < \infty \tag{4.3}$$

where $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ denotes the image of $E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)$ under the restriction map $Y \to X$. Since Y is $(n-q)$-convex, it follows from the Andreotti-Grauert finiteness theorem that

$$\dim H^{n,n-q}(Y, E) < \infty.$$

Hence, by Serre's duality theorem [S], $E^0_{0,q+1}(\Phi_Y; Y, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_Y; Y, E)$. Since the extending-by-zero map

$$C^0_{0,q+1}(\Phi_X; X, E) \longrightarrow C^0_{0,q+1}(\Phi_Y; Y, E)$$

is continuous and $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ is the preimage of $E^0_{0,q+1}(\Phi_Y; Y, E)$ with respect to this map, it follows that $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$. By (4.3) this means in particular that (4.1) holds. □

5. A finiteness and separation theorem with support condition on q-concave-q^*-convex manifolds

In this section X is an n-dimensional complex manifold which is q-concave-q^*-convex where $1 \leq q \leq n-1$ and $1 \leq q^* \leq n$. Further we assume that ρ, $\inf \rho$ and $\sup \rho$ are as in Definition 0.1, and we denote by Φ the family of all closed subsets C of X such that the sets $C \cap \{\rho \leq \alpha\}$, $\inf \rho < \alpha < \sup \rho$, are compact. Then the dual family Φ^* of Φ consists of all closed subsets C of X such that the sets $C \cap \{\rho \geq \alpha\}$, $\inf \rho < \alpha < \sup \rho$, are compact.

It is the aim of the present section to prove the following

5.1. Theorem. *For each holomorphic vector bundle E over X and all s, $0 \leq s \leq n$, the following three statements hold:*

(i) If $\max(q+1, q^*) \leq r \leq n$, *then* $\dim H^{s,r}_{\Phi}(X, E) < \infty$.

(ii) If $0 \leq r \leq \min(n-q, n-q^*+1)$, *then*

$$E^0_{s,r}(\Phi^*; X, E^*) = \tag{5.1}$$

$$\left\{ f \in Z^0_{s,r}(\Phi^*; X, E^*) \;\middle|\; \int_X f \wedge g = 0 \quad \text{for all} \quad g \in Z^0_{n-s,n-r}(\Phi; X, E) \right\}.$$

In particular, then $H^{s,r}_{\Phi^*}(X, E^*)$ *is separated.*

(iii) If even $0 \leq r \leq \min(n-q-1, n-q^*)$, *then* $\dim H^{s,r}_{\Phi^*}(X, E^*) < \infty$.

Note that in the case $q = n-1$ and $q^* = n$, (i) is already proven in [La-L 2]. It is the main step in the proof of the Malgrange vanishing theorem with support conditions, which says that under these assumptions $H^{s,n}_{\Phi}(X, E) = 0$.

The proof of Theorem 5.1 is given at the end of this section. First we need some preparations.

In the remainder of this section, we assume, as possible by the Morse pertubation argument (see, e.g., the theorem on page 43 in [G-P]), that all critical points of ρ are non-degenerate, and we shall use the following notations:

If $\inf \rho < \alpha < \beta < \sup \rho$, then $C^0_{s,r}(\{\rho \geq \alpha\}; \{\rho \leq \beta\}, E)$ is the Banach space of all continuous E-valued forms f on $\{\rho \leq \beta\}$ with $\operatorname{supp} f \subseteq \{\rho \geq \alpha\}$ endowed with the topology of uniform convergence, and by $Z^0_{s,r}(\{\rho \geq \alpha\}; \{\rho \leq \beta\}, E)$ we denote its subspace of $\overline{\partial}$-closed forms endowed with the same topology.

Further we fix some numbers α_0, β_0 with $\inf \rho < \alpha_0 < \beta_0 < \sup \rho$ such that $\{\rho = \beta_0\}$ is smooth, and the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $\{\rho \leq \alpha_0\}$, and at least $n-q^*+1$ positive eigenvalues on $\{\rho \geq \beta_0\}$.

Note that Φ is a nice family of supports in X (cp. Sect. 3.1). In fact, if Γ is the boundary of $\{\rho < \beta_0\}$, then $\Phi = \Phi(\Gamma, X)$.

5.2. Lemma. *If* $\max(q+1, q^*) \leq r \leq n$, *then, for each holomorphic vector bundle E over X, for all s, $0 \leq s \leq n$, and any $\varepsilon > 0$, the space*

$$Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \,\cap\, \overline{\partial}\, C^0_{s,r-1}(\{\rho \geq \alpha_0 - \varepsilon\}; \{\rho \leq \beta_0\}, E) \tag{5.2}$$

is of finite codimension and topologically closed in $Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$.

Proof. If $\inf \rho < \gamma < \beta_0$, then we denote by $C^{1/2}_{s,r-1}(\{\rho \geq \gamma\}; \{\rho \leq \beta_0\}, E)$ the Banach space of forms in $C^0_{s,r-1}(\{\rho \geq \gamma\}; \{\rho \leq \beta_0\}, E)$ which are Hölder continuous with exponent $1/2$ on $\{\gamma \leq \rho \leq \beta_0\}$. We shall prove that even the space

$$C^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \,\cap\, \overline{\partial}\, C^{1/2}_{s,r-1}(\{\rho \geq \alpha_0 - \varepsilon\}; \{\rho \leq \beta_0\}, E)$$

is of finite codimension and topologically closed in $Z^0_{0,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$. By Ascoli's theorem and Fredholm theory, for this it is sufficient to construct continuous linear operator

$$A : Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \longrightarrow C^{1/2}_{s,r-1}(\{\rho \geq \alpha_0 - \varepsilon\}; \{\rho \leq \beta_0\}, E)$$

and

$$K : Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \longrightarrow C^{1/2}_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$$

such that

$$\overline{\partial} Af = f + Kf \qquad \text{on} \quad \{\rho \leq \beta_0\}$$

for all $f \in Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$.

Take $\delta > 0$ so small that $\alpha_0 + \delta < \beta_0$ and the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $\{\rho \leq \alpha_0 + \delta\}$. Since $r \geq q+1$, then by Lemma 1.2 in [La-L 1], there exists a continuous linear operator

$$A_0 : Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \longrightarrow C^{1/2}_{s,r-1}(\{\rho \geq \alpha_0 - \varepsilon\}; \{\rho \leq \beta_0\}, E)$$

such that $\overline{\partial} A_0 f = f$ on $\{\rho \leq \alpha_0 + \delta\}$ for all $f \in Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$. Since $r \geq q^*$ and the boundary $\{\rho = \beta_0\}$ is smooth, we can use the local integral operators of Fischer and Lieb [F-Li] (see also Sects. 7 and 9 in [H-L]) and obtain open sets $U_1, \dots, U_N \subset\subset X$ with

$$\{\alpha_0 + \delta \leq \rho \leq \beta_0\} \subseteq U_1 \cup \ldots \cup U_N \subseteq \{\rho > \alpha_0\}$$

as well as continuous linear operators

$$A_j : Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E) \longrightarrow C^{1/2}_{s,r-1}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$$

such that $\overline{\partial} A_j f = f$ on $U_j \cap \{\rho \leq \beta_0\}$ for all $f \in Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$, $j = 1, \dots, N$. Take real C^∞-functions $\chi_0, \dots, \chi_N$ on X with $\operatorname{supp} \chi_0 \subset\subset \{\rho < \alpha_0 + \delta\}$, $\operatorname{supp} \chi_j \subset\subset U_j$ if $1 \leq j \leq N$, and $\chi_0 + \ldots + \chi_N \equiv 1$ on $\{\alpha_0 - \varepsilon \leq \rho \leq \beta_0\}$. Then the operators

$$A := \sum_{j=0}^{N} \chi_j \qquad \text{and} \qquad K := \sum_{j=0}^{N} \overline{\partial} \chi_j \wedge A_j$$

have the required property. □

Lemma 5.3. *Let* $\max(q+1, q^*) \leq r \leq n$. *Then for each holomorphic vector bundle* E *over* X *and all* s, $0 \leq s \leq n$, *the following two assertions hold:*

(i) If $\beta \in [\beta_0, \sup \rho[$ *and* $\{\rho = \beta\}$ *is smooth, then the space* $Z^0_{s,r-1}(\{\rho \geq \alpha_0\}; X, E)$ *is dense in* $Z^0_{s,r-1}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta\}, E)$ *(with respect to uniform convergence).*

(ii) For each $\varepsilon > 0$, *the space* $E^0_{s,r}(\{\rho \geq \alpha_0 - \varepsilon\} \to \{\rho \geq \alpha_0\}; X, E)$ *is the preimage of the space (5.2) with respect to the restriction map*

$$Z^0_{s,r}(\{\rho \geq \alpha_0\}; X, E) \longrightarrow Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E).$$

Proof. Since by Lemma 5.2, the space (5.2) is topologically closed in $Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$, this can be proved in the same way as Theorems 12.11 and 12.13 (ii) in [H-L]. (The domain D in Theorems 12.11 and 12.13 in [H-L] is assumed to be relatively compact, but in the proof of these theorems only the consequence is used that then $Z^0_{0,r}(\overline{D}, E) \cap \overline{\partial} C^0_{0,r-1}(\overline{D}, E)$ is topologically closed in $Z^0_{0,r}(\overline{D}, E)$.) □

5.4. Proof of Theorem 5.1 (i). By definition of Φ, for each $C \in \Phi$, there exists $\gamma \in]\inf \rho, \sup \rho[$ with $C \subseteq \{\rho \geq \gamma\}$. Therefore it follows from Lemma 1.2 (i) in [La-L 1] that

$$Z^0_{s,r}(\Phi; X, E) = E^0_{s,r}(\Phi; X, E) \; + \; Z^0_{s,r}(\{\rho \geq \alpha_0\}; X, E). \tag{5.3}$$

By Lemma 5.2, the space (5.2) is of finite codimension in $Z^0_{s,r}(\{\rho \geq \alpha_0\}; \{\rho \leq \beta_0\}, E)$. Therefore it follows from Lemma 5.3 (ii) that the space $E^0_{s,r}(\{\rho \geq \alpha_0 - \varepsilon\} \to \{\rho \geq \alpha_0\}; X, E)$ is of finite codimension in $Z^0_{s,r}(\{\rho \geq \alpha_0\}; X, E)$. Since (5.2) is a subspace of $E^0_{s,r}(\Phi; X, E)$, it follows that $Z^0_{s,r}(\{\rho \geq \alpha_0\}; X, E) \cap E^0_{s,r}(\Phi; X, E)$ is of finite codimension in $Z^0_{s,r}(\{\rho \geq \alpha_0\}; X, E)$. In view of (5.3), this implies that $E^0_{s,r}(\Phi; X, E)$ is of finite codimension in $Z^0_{s,r}(\Phi; X, E)$, i.e. $\dim H^{s,r}(\Phi; X, E) < \infty$. □

5.5. Lemma. *If* $\max(q+1, q^*) \leq r \leq n$, *then for each holomorphic vector bundle* E *over* X *and for all* s, $0 \leq s \leq n$, *condition* $SCl_{s,r}(\Phi; X, E)$ *is fulfilled.*

Proof. Consider some $C \in \Phi$. Then it follows from part (i) of Theorem 5.1 (which is already proved) that $E^0_{s,r}(\Phi \to C; X, E)$ is of finite codimension in $Z^0_{s,r}(C; X, E)$. Since $E^0_{s,r}(\Phi \to C; X, E)$ is the image of $\overline{\partial}$ as operator between the LF-space $C^0_{s,r-1}(\Phi; X, E)$ and the Fréchet space $Z^0_{s,r}(C; X, E)$, this implies, by the open mapping theorem, that $E^0_{s,r}(\Phi \to C; X, E)$ is topologically closed in $Z^0_{s,r}(C; X, E)$ and consequently in $C^0_{s,r}(C; X, E)$. This proves that condition $Cl_{s,r}(\Phi; X, E)$ is fulfilled.

Let Σ be the set of all $\beta \in [\beta_0, \sup \rho[$ such that $\{\rho = \beta\}$ is smooth. Then the manifolds $U^*_\beta := \{\rho < \beta\}$, $\beta \in \Sigma$, are also q-concave-q^*-convex, $U^*_\beta \in \Phi^{*0}$ and, applying the above arguments to each U^*_β, we see that, for each $\beta \in \Sigma$, condition $Cl_{s,r}(\Phi|_{U^*_\beta}; U^*_\beta, E)$ is fulfilled. Moreover, it follows from Lemma 5.3 (i) that each U^*_β, $\beta \in \Sigma$, satisfies condition (3.1) in the definition of $\Phi^{*0}_{s,r}(E)$. Hence $U^*_\beta \in \Phi^{*0}_{s,r}(E)$ for all $\beta \in \Sigma$. Since all critical points of ρ are non-degenerate and therefore $[\beta_0, \sup \rho[\setminus \Sigma$ is discrete, this yields that condition $SCl_{s,r}(\Phi; X, E)$ is fulfilled. □

5.6. Proof of Theorem 5.1 (ii) and (iii). The case $r = 0$ is trivial. In fact, since $q^* \geq 1$ and therefore $X \setminus C^* \neq \emptyset$ for all $C^* \in \Phi^*$ it follows from uniqueness of holomorphic functions (X is connected) that $Z^0_{s,r}(\Phi^*; X, E) = \{0\}$ if $r = 0$.

Now let $1 \leq r \leq \min(n - q, n - q^* + 1)$. Then $\max(q+1, q^*) \leq n - r + 1 \leq n$. Therefore, by Lemma 5.5, condition $SCl_{n-s,n-r+1}(\Phi; X, E)$ is fulfilled, and it follows from Theorem 3.7 that (5.1) holds, i.e. (ii) is proved.

From (5.1) and Lemma 2.5 it follows that

$$\dim H^{s,r}_{\Phi^*}(X, E^*) = \dim H^{n-s,n-r}_{\Phi}(X, E).$$

Therefore (iii) follows from (i). □

6. Proof of Theorem 0.4

By Serre duality (Theorem 3.8), Theorem 0.4 is equivalent to the following

6.1. Theorem. *Let X be an n-dimensional complex manifold, $n \geq 3$, which is q-concave-$(n-q)$-convex, $1 \leq q \leq n-1$. If*

$$q < \frac{n}{2},$$

then, for any holomorphic vector bundle E over X and all s with $0 \leq s \leq n$, $H_c^{s,q+1}(X,E)$ is separated.

Proof. Let ρ, $\inf\rho$ and $\sup\rho$ be as in Definition 0.1 and let Φ^* be the family of all closed subsets C of X such that the sets $C \cap \{\rho \geq \alpha\}$, $\inf\rho < \alpha < \sup\rho$, are compact. Denote by Φ_c the family of all compact subsets of X. Then Φ_c is complete in Φ^* and it follows from Lemma 1.3 that

$$\text{(6.1)} \qquad \dim\left[E^0_{s,q+1}(\Phi^* \to \Phi_c; X, E)/E^0_{s,q+1}(\Phi_c; X, E)\right] \leq \dim H^{s,q}(\Phi^* * \Phi_c, E).$$

Take $\alpha_0 \in]\inf\rho, \sup\rho[$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $\{\rho \leq \alpha_0\}$. Then, by the Andreotti-Grauert finiteness theorem,

$$\dim H^{s,r}(\{\rho < \alpha\}, E) < \infty \qquad \text{if} \quad q \leq r \leq n-q-1 \quad \text{and} \quad \inf\rho < \alpha \leq \alpha_0.$$

Since $q < n/2$ and hence $q \leq n-q-1$, it follows that

$$\dim H^{s,q}(\{\rho < \alpha\}, E) < \infty \qquad \text{if} \quad \inf\rho < \alpha \leq \alpha_0.$$

Since, for each $U \in \Phi^* * \Phi_c$, there exists $\alpha \in]\inf\rho, \alpha_0]$ with $\{\rho < \alpha\} \subseteq U$, this implies that

$$\dim H^{s,q}(\Phi^* * \Phi_c, E) < \infty.$$

Hence, by (6.1),

$$\text{(6.2)} \qquad \dim\left[E^0_{s,q+1}(\Phi^* \to \Phi_c; X, E)/E^0_{s,q+1}(\Phi_c; X, E)\right] < \infty.$$

Furthermore, the hypothesis $q < n/2$ implies that $q+1 \leq \min(n-q, n-q^*+1)$ with $q^* := n-q$. Therefore it follows from Theorem 5.1 (ii) that $E^0_{s,q+1}(\Phi^*; X, E)$ is topologically closed in $C^0_{s,q+1}(\Phi^*; X, E)$. Since the embedding map

$$C^0_{s,q+1}(\Phi_c; X, E) \longrightarrow C^0_{s,q+1}(\Phi^*; X, E)$$

is continuous and $E^0_{s,q+1}(\Phi^* \to \Phi_c; X, E)$ is the preimage of $E^0_{s,q+1}(\Phi^*; X, E)$ with respect to this map, it follows that $E^0_{s,q+1}(\Phi^* \to \Phi_c; X, E)$ is topologically closed in $C^0_{s,q+1}(\Phi_c; X, E)$. Hence, by (6.2), there is a finite dimensional linear subspace F of $C^0_{s,q+1}(\Phi_c; X, E)$ such that $F + E^0_{s,q+1}(\Phi_c; X, E)$ is topologically closed in $C^0_{s,q+1}(\Phi_c; X, E)$. By Lemma 2.6 this yields that condition $Cl_{s,q+1}(\Phi_c; X, E)$ is fulfilled. Now the assertion follows from Theorem 3.8. □

7. Proof of Theorem 0.5 (and some generalization)

In this section we assume that X is a connected n-dimensional complex manifold which is q-concave in the sense of Andreotti-Grauert, $1 \le q \le n$, i.e. we have a real C^2 function ρ on X such that

(i) for all $\alpha > \inf \rho$, the set $\{\rho \ge \alpha\}$ is compact;

(ii) outside some compact set, the Levi form of ρ has at least $n-q+1$ positive eigenvalues.

Moreover, as possible by the Morse pertubation argument (see, e.g., the theorem on page 43 in [G-P]), we assume that all critical points of ρ are non-degenerate and hence isolated.

We fix some $\alpha_0 > \inf \rho$ as well as two open subsets Z and Z^* of X such that $d\rho(\zeta) \neq 0$ for $\rho(\zeta) = \alpha_0$,

$$Z^* \cap Z = \emptyset \quad \text{and} \quad Z^* \cup Z = \{\rho < \alpha_0\}.$$

Denote by Φ the family of all closed subsets C of X such that $C \cap \overline{Z}^*$ is compact. Φ is a nice family of supports in X. In fact, with the notation from Sect. 3.1, we have $\Phi = \Phi(\partial Z^*, X) = \Phi(-\partial Z, X)$. The dual family Φ^* of Φ consists of all closed sets $C \subseteq X$ such that $C \cap Z$ is compact.

7.1. Lemma. *If*

$$q < \frac{n}{2} \qquad \textit{and} \qquad q+1 \le r \le n-q,$$

then, for each holomorphic vector bundle E over X and all s, $0 \le s \le n$, condition $Cl_{s,r}(\Phi; X, E)$ is fulfilled.

Proof. Set $Z^*_\alpha = Z^* \cap \{\rho < \alpha\}$ and take α_1 with $\inf \rho < \alpha_1 < \alpha_0$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $Z^*_{\alpha_1}$. Then, by the Andreotti-Grauert finiteness theorem [A-G],

$$\dim H^{s,r}(Z^*_\alpha, E) < \infty \quad \text{if } \inf \rho < \alpha < \alpha_1 \quad \text{and } q \le r \le n-q-1. \tag{7.1}$$

Let Ψ be the family of all closed subsets of X. Then $\Psi * \Phi$ consists of all open sets $U \subseteq X$ such that $\overline{Z}^* \setminus U$ is compact, and Φ is complete in Ψ. Therefore, by Lemma 1.5 (iii),

$$\dim \left[E^0_{s,r}(\Psi \to \Phi; X, E)/E^0_{s,r}(\Phi; X, E)\right] \le \dim H^{s,r-1}(\Psi * \Phi; E)$$

for all $r \ge 1$. Together with (7.1) this implies that

$$\dim \left[E^0_{s,r}(\Psi \to \Phi; X, E)/E^0_{s,r}(\Phi; X, E)\right] < \infty \quad \text{if} \quad q+1 \le r \le n-q. \tag{7.2}$$

By the classical Andreotti-Vesentini theorem, $E^0_{s,r}(X, E)$ is topologically closed in $C^0_{s,r}(X, E)$ if $0 \le r \le n-q$. Since the topology of $C^0_{s,r}(\Phi; X, E)$ is stronger than the topology of $C^0_{s,r}(X, E)$ and $E^0_{s,r}(\Psi \to \Phi; X, E) = E^0_{s,r}(X, E) \cap C^0_{s,r}(\Phi; X, E)$, it follows that $E^0_{s,r}(\Psi \to \Phi; X, E)$ is topologically closed in $C^0_{s,r}(\Phi; X, E)$ if $0 \le r \le n-q$. In view of (7.2) and Lemma 2.6, this completes the proof. □

7.2. Lemma. *Let* $\inf \rho < \alpha < \alpha_0$ *such that the Levi form of* ρ *has at least* $n-q+1$ *positive eigenvalues on* $\{\rho \le \alpha\}$, *and let*

$$q < \frac{n}{2} \qquad \textit{and} \qquad q+1 \le r \le n-q.$$

Then, for each holomorphic vector bundle E *over* X *and all* s, $0 \le s \le n$, *the set*

$$U_\alpha^* := Z^* \cup \{\rho > \alpha\}$$

belongs to $\Phi^{*0}_{s,r}(E)$.

Proof. First note that in Lemma 7.1 instead of the manifold X we may take U_α^* if we replace Z by $Z \cap U_\alpha^*$. Therefore it follows from Lemma 7.1 that condition $Cl_{s,r}(\Phi|_{U_\alpha^*}; U_\alpha^*, E)$ is fulfilled.

Further, since $r-1 < n-q-1$, by Andreotti-Grauert theory (see, e.g., Theorem 15.9 in [H-L]) we have the following statement: For each $g \in Z^0_{s,r-1}(U_\alpha^*, E)$ and all $\varepsilon > 0$, there exists $\tilde{g} \in Z^0_{s,r-1}(X, E)$ such that

$$\tilde{g} = g \qquad \text{on} \qquad Z^* \cup \{\rho > \alpha + \varepsilon\}.$$

This implies that also condition (3.1) in the defintion of $\Phi^{*0}_{s,r}(E)$ is fulfilled for U_α^*. $\square$

7.3. Corollary to Lemma 7.2. *If*

$$q < \frac{n}{2} \qquad \textit{and} \qquad q+1 \le r \le n-q,$$

then, for each holomorphic vector bundle E *over* X *and all* s, $0 \le s \le n$, *condition* $SCl_{s,r}(\Phi; X, E)$ *is fulfilled.*

7.4. Theorem. *(i) If*

$$q < \frac{n}{2} \qquad \textit{and} \qquad q+1 \le r \le n-q,$$

then, for each holomorphic vector bundle E *over* X *and all* s, $0 \le s \le n$, *the space* $E^0_{s,r}(\Phi; X, E)$ *consists of all* $\varphi \in Z^0_{s,r}(\Phi; X, E)$ *such that*

$$\int_X \varphi \wedge g = 0 \qquad \textit{for all} \quad g \in Z^0_{n-s,n-r+1}(\Phi^*; X, E^*).$$

In particular, then $H^{s,r}_\Phi(X, E)$ *is separated.*

(ii) If

$$q < \frac{n-1}{2} \qquad \textit{and} \qquad q+1 \le r \le n-q-1,$$

then moreover, for each holomorphic vector bundle E *over* X *and all* s, $0 \le s \le n$,

$$\dim H^{s,r}_\Phi(X, E) < \infty.$$

Proof of (i). The relation $q+1 \leq r \leq n-q$ is equivalent with $q+1 \leq n-r+1 \leq n-q$. Further, interchanging the roles of Z and Z^*, we see that the statement of Corollary 7.3 holds also with Φ^* instead of Φ. Therefore condition $SCl_{n-s,n-r+1}(\Phi^*; X, E^*)$ is fulfilled and assertion follows from Theorem 3.7. □

Proof of (ii). By the Andreotti-Grauert finiteness theorem [A-G], $\dim H^{s,r}(X,E) < \infty$. In particular, if Ψ denotes the family of all closed subsets of X, then

$$\dim \left[Z^0_{s,r}(\Phi; X, E) / E^0_{s,r}(\Psi \to \Phi; X, E) \right] < \infty.$$

Together with (7.2) (in the proof of Lemma 7.1) this completes the proof. □

7.5. Theorem. *If $q = 1$, then, for each holomorphic vector bundle E over X and all s, $0 \leq s \leq n$, the space $E^0_{s,n-1}(\Phi; X, E)$ consists of all $\varphi \in Z^0_{s,n-1}(\Phi; X, E)$ such that*

$$\int_X \varphi \wedge g = 0 \qquad \text{for all} \quad g \in Z^0_{n-s,1}(\Phi^*; X, E^*).$$

In particular, then $H^{s,n-1}_\Phi(X,E)$ is separated.

Proof. This follows from Theorem 7.4 if $n \geq 3$, and from Theorem 4.2 in [La-L 2] if $n = 2$. □

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