RATIONAL SMOOTHNESS AND FIXED POINTS OF TORUS ACTIONS

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ABSTRACT . — We obtain a criterion for rational smoothness of an algebraic variety with a torus action, with applications to orbit closures in flag varieties, and to closures of double classes in regular group completions.

Introduction

For a complex algebraic group acting on a complex flag variety with finitely many orbits, the geometry of orbit closures is of importance in representation theory; the most interesting cases are Schubert varieties (in relation with category $\mathcal{O}$), and orbit closures of symmetric subgroups (in relation with Harish-Chandra modules), see e.g. [Ka]. In particular, it would be useful to characterize rationally smooth points of an orbit closure, i.e., those points where the local cohomology with constant coefficients is the same as for a point of a smooth variety.

Criteria for rational smoothness of Schubert varieties have been obtained by Kazhdan-Lusztig [K-L1], [K-L2] and then by Kumar [Ku], Carrell-Peterson [C] and Arabia [A]. The latter criteria hold, more generally, for varieties where a torus acts with isolated fixed points, such that all weights of the tangent space at such a fixed point are contained in an open half-space and have multiplicity one. But that condition can fail for orbit closures of symmetric subgroups in flag varieties (e.g., for SO_n acting on the flag variety of SL_n).

In the present paper, we obtain a criterion for rational smoothness of varieties with a torus action, which applies to the latter situation. Our main result can be stated as follows, in a somewhat weakened version.

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THEOREM (1.3). — *Let X be a complex algebraic variety with an action of a torus T. Let $x \in X$ be an attractive fixed point of T, that is, all weights of T in the tangent space $T_x X$ are contained in an open half-space. For a subtorus $T' \subset T$, let $X^{T'} \subset X$ be its fixed point set. Then we have*

$$\dim_x(X) \leq \sum_{T'} \dim_x(X^{T'})$$

(sum over all subtori of codimension one), and this sum is finite. Furthermore, X is rationally smooth at x if and only if the following conditions hold:

(i) A punctured neighborhood of x in X is rationally smooth.

(ii) For any subtorus $T' \subset T$ of codimension one, the fixed point subset $X^{T'}$ is rationally smooth at x.

(iii) We have $\dim_x(X) = \sum_{T'} \dim_x(X^{T'})$ (sum over all subtori of codimension one).

Assume moreover that all weights in the tangent space $T_x X$ have multiplicity one. Then the subsets $X^{T'}$ identify with coordinate lines in $T_x X$, and the sum of their dimensions is the number $n(X, x)$ of irreducible T-stable curves through x. So we obtain $\dim_x(X) \leq n(X, x)$, a result of Carrell and Peterson [C] Theorem D. In this case, the fact that equality holds for rationally smooth x follows also from work of Arabia [A].

Consider now a connected semisimple group G, its flag variety $\mathcal{B}(G)$, and a symmetric subgroup $H \subset G$, that is, the fixed point subgroup of an involution θ of G. Let T_H be a maximal torus of H, with centralizer T in G. Then T is a maximal torus of G, stable by θ. The T_H-fixed points in $\mathcal{B}(G)$ are the (finitely many) T-fixed points, and the fixed points of subtori $T' \subset T_H$ of codimension one can be described completely in terms of the action of θ on roots of (G, T) (2.5).

Then our main result leads to an inequality for the dimension of an H-orbit closure $X \subset \mathcal{B}(G)$, with equality if X is rationally smooth at a T_H-fixed point (2.5); this generalizes a result of Springer [Sp2] concerning inner involutions. Actually, much of our analysis extends to any reductive subgroup $H \subset G$ having only finitely many orbits in $\mathcal{B}(G)$ (2.2, 2.3). However, such orbits need not admit an attractive slice (2.3), whereas orbits of a symmetric subgroup do admit such a slice, see [M-S] 6.4.

Another application of our criterion is given in Section 3; it concerns double classes BgB where B is a Borel subgroup of a connected reductive group G, and their closures $\overline{BgB}$ in a smooth $(G \times G)$-equivariant completion of G which is regular in the sense of [B-D-P]. We show in 3.1 that these closures admit attractive slices at all points, and that they are rationally smooth in codimension two. This generalizes a classical result for Schubert varieties [K-L1]. However, closures of double classes are not rationally smooth, apart from very few cases (3.3). The fact that almost all closures of double classes are singular in codimension two was proved in [Br1] by more ad hoc arguments.

Although our results are stated for complex algebraic varieties, our arguments adapt to the case of an algebraically closed field of any characteristic, with rational cohomology replaced by l-adic cohomology. This makes the exposition rather heavy at several places. An appendix collects results on rational smoothness and on torus actions, for which we did not find suitable references.

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1. A criterion for rational smoothness

1.1. Necessary conditions for rational smoothness.

In what follows, we consider complex algebraic varieties, that is, separated reduced schemes of finite type over $\mathbf{C}$. With this convention, varieties need not be irreducible. For such a variety X, we denote by $H^*(X)$ cohomology of X with rational coefficients. For a point $x \in X$, we denote by $H^*_x(X)$ cohomology with support in $\{x\}$, and by $\dim_x(X)$ the dimension of the local ring of X at x.

Definition. — X is *rationally smooth at* x if, for all y in a neighborhood of x in the complex topology, $H^m_y(X) = 0$ for $m \neq 2\dim_x(X)$, and $H^{2\dim_x(X)}_y(X)$ is isomorphic to $\mathbf{Q}$.

If X is rationally smooth at a point x, then it is irreducible at that point (see e.g. Proposition A1). Any smooth point is rationally smooth; moreover, rational smoothness is preserved under quotient by a finite group action (see e.g. Proposition A1). Other examples of rationally smooth varieties are unibranched curves.

We will obtain necessary conditions for rational smoothness of a variety X at a fixed point of an algebraic action of a torus T (that is, T is an algebraic group isomorphic to a product of copies of the multiplicative group $\mathbf{G}_m$). We will always assume that X is covered by open affine T-stable subsets. By [Su], this assumption holds for T-stable subvarieties of normal T-varieties.

Theorem. — *Let T be a torus acting on a variety X with a fixed point x. If X is rationally smooth at x, then, for each subtorus $T' \subset T$, the fixed point set $X^{T'}$ is rationally smooth at x. Moreover, we have*

$$\dim_x(X) - \dim_x(X^T) = \sum_{T'} (\dim_x(X^{T'}) - \dim_x(X^T))$$

(sum over all subtori $T' \subset T$ of codimension one).

Proof. — We use equivariant cohomology (see e.g. [H]) which we briefly review. Let $E_T \to B_T$ be a universal principal bundle for T. Then T acts diagonally on $X \times E_T$ with a quotient denoted by $X \times_T E_T$. Let

$$H_T^*(X) := H^*(X \times_T E_T)$$

be the T-equivariant cohomology ring of X with rational coefficients. The map

$$X \times_T E_T \to E_T/T = B_T$$

is a fibration with fiber X, and B_T is simply connected. Thus, there is a spectral sequence

$$H^p(B_T) \otimes H^q(X) \Rightarrow H_T^{p+q}(X)$$

and $H_T^*(X)$ is a module over $H^*(B_T)$. The latter is the symmetric algebra of the character group of T, where each character has degree 2. The inclusion $i_T : X^T \to X$ induces a $H^*(B_T)$-linear map

$$i_T^* : H_T^*(X) \to H_T^*(X^T) \cong H^*(B_T) \otimes H^*(X^T).$$

By the localization theorem (see [H] or Proposition A5), i_T^* becomes an isomorphism after inverting all non trivial characters of T.

Let $y \in X^T$. Denote by

$$H_{T,y}^*(X) := H_{y \times_T E_T}^*(X \times_T E_T)$$

equivariant cohomology of X with support in $\{y\}$, and consider the map

$$i_{T,y}^* : H_{T,y}^*(X) \to H_{T,y}^*(X^T) = H^*(B_T) \otimes H_y^*(X^T).$$

Applying the localization theorem to X and $X \setminus \{y\}$, we see that $i_{T,y}^*$ is an isomorphism after inverting all non trivial characters. On the other hand, because X is rationally smooth at x, the spectral sequence

$$H^p(B_T) \otimes H_y^q(X) \Rightarrow H_{T,y}^{p+q}(X)$$

degenerates for all y in a neighborhood of x. Thus, $H_{T,y}^*(X)$ is a free $H^*(B_T)$-module generated by an element of degree $2\dim_y(X) = 2\dim_x(X)$. It follows that the space $H_y^*(X^T)$ is one dimensional, and that X^T is rationally smooth at x (e.g. by Proposition A1). Moreover, identifying the $H^*(B_T)$-modules $H_{T,x}^*(X)$ and $H_{T,x}^*(X^T)$ with $H^*(B_T)$, the map $i_{T,x}^*$ becomes multiplication by a homogeneous element $f \in H^*(B_T)$ of degree $2\dim_x(X) - 2\dim_x(X^T)$. By the localization theorem, f is a scalar multiple of a product of characters.

Let χ be a primitive character dividing f, and let T' be the kernel of χ, a subtorus of T of codimension one. Then $i_T : X^T \to X$ factors as $i_{T,T'} : X^T \to X^{T'}$ followed by $i_{T'} : X^{T'} \to X$. By the localization theorem again, the map

$$i_{T',x}^* : H_{T,x}^*(X) \to H_{T,x}^*(X^{T'})$$

becomes an isomorphism after inverting all characters of T which restrict non trivially to T', i.e., which are not multiples of χ. Moreover, $X^{T'}$ is rationally smooth at x by the first step of the proof. Thus, we can identify $H^*_{T,x}(X^{T'})$ with $H^*(B_T)$; then $i_{T',x}$ identifies with multiplication by a product of non multiples of χ.

Choose a subtorus $T'' \subset T$ of dimension one, such that the product map $T' \times T'' \to T$ is an isomorphism. Then the character group of T'' is generated by restriction of χ. Moreover, we can take $E_T = E_{T'} \times E_{T''}$, then $X^{T'} \times_T E_T \cong B_{T'} \times (X^{T'} \times_{T''} E_{T''})$ and $X^T \times_T E_T \cong B_{T'} \times (X^T \times_{T''} E_{T''})$. Thus, we have isomorphisms

$$H^*_{T,x}(X^{T'}) \cong H^*(B_{T'}) \otimes H^*_{T'',x}(X^{T'}),\ H^*_{T,x}(X^T) \cong H^*(B_{T'}) \otimes H^*_{T'',x}(X^T)$$

compatible with $i_{T,T',x}$. Applying the localization theorem to the T''-variety $X^{T'}$, it follows that

$$i_{T,T',x} : H^*_{T,x}(X^{T'}) \to H^*_{T,x}(X^T)$$

is an isomorphism after inverting χ. In other words, $i^*_{T,T',x}$ identifies with multiplication by a power χ^{n_χ}, and f is divisible by χ^{n_χ} but not by $\chi^{n_\chi+1}$. Taking degrees, we obtain $2n_\chi = 2\dim_x(X^{T'}) - 2\dim_x(X^T)$. Now f is a scalar multiple of $\prod_\chi \chi^{n_\chi}$ (product over all primitive characters) and our relation on dimensions follows by taking degrees.

1.2. An inequality for dimensions of fixed points.

Let X be a variety with an action of a torus T and a fixed point x. In general, there is no inequality between $\dim_x(X) - \dim_x(X^T)$ and $\sum_{T'}(\dim_x(X^{T'}) - \dim_x(X^T))$ (sum over all subtori of codimension one), as shown by the following

Example. — Let X be the hypersurface in $\mathbf{A}^4$ with equation

$$xy + zt = 0.$$

Let $T = \mathbf{G}_m \times \mathbf{G}_m$ act on $\mathbf{A}^4$ by

$$(u, v) \cdot (x, y, z, t) = (ux, u^{-1}y, vz, v^{-1}t).$$

Then X is T-stable and the origin is the unique fixed point. The non trivial subsets $X^{T'}$ are: $xy = z = t = 0$ for $T' = \mathbf{G}_m \times \{1\}$, and $x = y = zt = 0$ for $T' = \{1\} \times \mathbf{G}_m$. Thus, X and the $X^{T'}$ are irreducible, and $\sum_{T'} \dim(X^{T'}) = 2$ whereas $\dim(X) = 3$.

On the other hand, consider the action of $T = \mathbf{G}_m \times \mathbf{G}_m$ on $\mathbf{A}^4$ by

$$(u, v) \cdot (x, y, z, t) = (u^3x, v^3y, u^2vz, uv^2t).$$

Then again X is T-stable and the origin is the unique fixed point; but now the $X^{T'}$ are the four coordinate lines, whence $\sum_{T'} \dim(X^{T'}) = 4$.

However, we will obtain an upper bound for $\dim_x(X) - \dim_x(X^T)$ in terms of the $X^{T'}$. Observe that T acts on $X^{T'}$ through its quotient T/T', which we can identify with $\mathbf{G}_m$. Denote by $X_+^{T'}(x)$ (resp. $X_-^{T'}(x)$) the set of all $y \in X^{T'}$ such that x is the limit of ty as $t \to 0$ (resp. $t^{-1} \to 0$) where $t \in \mathbf{G}_m$. Then both $X_+^{T'}(x)$ and $X_-^{T'}(x)$ are locally closed T-stable subsets of $X^{T'}$, and x is their unique common point.

Theorem. — *Notation being as above, there are only finitely many T' such that $X^{T'} \neq X^T$, and we have*

$$\dim_x(X) - \dim_x(X^T) \leq \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x))$$

(sum over all subtori of codimension one).

If moreover $X^{T'}$ is smooth at x, then

$$\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) = \dim_x(X^{T'}) - \dim_x(X^T).$$

In particular, if each $X^{T'}$ is smooth or attractive at x, then

$$\dim_x(X) - \dim_x(X^T) \leq \sum_{T'} (\dim_x(X^{T'}) - \dim_x(X^T)).$$

Proof. — We may assume that X is affine and that each irreducible component of X^T contains x. Then each $X_+^{T'}(x)$ is closed in X, and contains x as an attractive fixed point for the action of T/T'. Thus, there exists a T/T'-module $V_+^{T'}$ and an equivariant finite surjective morphism

$$\pi_+^{T'} : X_+^{T'}(x) \to V_+^{T'}$$

such that $\pi_+^{T'}(x) = 0$ (by a version of Noether normalization lemma, see e.g. Proposition A2). Because $X_+^{T'}(x)$ is T-stable and closed in X, we can extend $\pi_+^{T'}$ to an equivariant morphism

$$p_+^{T'} : X \to V_+^{T'}.$$

Similarly, we have $p_-^{T'} : X \to V_-^{T'}$.

Observe that there are only finitely many subtori $T' \subset T$ of codimension one, such that $V_\pm^{T'}$ is non zero: indeed, such a subtorus is contained in the kernel of a weight of T in the tangent space T_xX. Let V denote the product of all the $V_\pm^{T'}$, and let

$$p : X \to V$$

be the product morphism; then $p(X^T) = \{0\}$, because $p(x) = 0$ and $V^T = \{0\}$ by construction.

We claim that X^T is a connected component of the fiber $p^{-1}(0)$. Otherwise, there exists an irreducible T-stable curve $C \subset p^{-1}(0)$ such that x is an isolated fixed point of C

(see e.g. Proposition A3). Then T acts on C through some non trivial character χ. Thus, C is contained in $X^{T'}$ where $T' \subset T$ is the connected kernel of χ. Moreover, because T acts non trivially on C, this curve must be contained in $X_+^{T'}(x) \cup X_-^{T'}(x)$. But then $p(C)$ has dimension one, by construction of p.

From the claim, it follows that

$$\begin{aligned}\dim_x(X) - \dim_x(X^T) &\leq \dim(V) \\ &= \sum_{T'} (\dim(V_+^{T'}) + \dim(V_-^{T'})) \\ &= \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x)).\end{aligned}$$

If $X^{T'}$ is smooth at x, then there exists an equivariant morphism

$$f : X^{T'} \to T_x(X^{T'}),\ x \mapsto 0$$

which is étale at x. It follows that $X_+^{T'}(x)$, $X_-^{T'}(x)$ and X^T are smooth at x, and that

$$\begin{aligned}\dim_x(X^{T'}) = \dim T_x(X^{T'}) &= \dim T_x(X^{T'})_+ + \dim T_x(X^{T'})_- + \dim T_x(X^{T'})^T \\ &= \dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) + \dim_x(X^T).\end{aligned}$$

This implies our latter statement.

1.3. Rational smoothness at an attractive fixed point.

Consider as above a torus T acting on a variety X with a fixed point x. Call x *attractive* if it admits a Zariski open neighborhood U_x such that, for all $y \in U_x$, the orbit closure $\overline{Ty}$ contains x. Equivalently, all weights of T acting on the tangent space T_xX are contained in an open half-space. In particular, x is an isolated fixed point.

Theorem. — *Let X be a T-variety with an attractive fixed point x. Then we have*

$$\dim_x(X) \leq \sum_{T'} \dim_x(X^{T'})$$

(sum over all subtori of codimension one). Moreover, X is rationally smooth at x if and only if the following conditions hold:

(i) *A punctured neighborhood of x in X is rationally smooth.*

(ii) *$X^{T'}$ is rationally smooth at x for each subtorus $T' \subset T$ of codimension one.*

(iii) $\dim_x(X) = \sum_{T'} \dim_x(X^{T'})$.

Proof. — The first assertion follows from Theorem 1.2. If X is rationally smooth at x, then *(i)* certainly holds, and *(ii)*, *(iii)* follow from Theorem 1.1. Another proof of this result, and of the converse as well, is sketched in [Br2], using methods from the appendix in [K-L1]. We reproduce this proof with some changes, so that it adapts to arbitrary characteristic.

We may assume that X is affine; then any T-orbit closure contains x. Assume that *(i)* holds, and set $\dot{X} := X \setminus x$. Then $\dot{X}$ is rationally smooth, because x is an attractive fixed point. Moreover, we can find an injective one parameter subgroup $\lambda : \mathbf{G}_m \to T$ such that all weights of the $\mathbf{G}_m$-action in the tangent space T_xX are positive. Then the quotient

$$\dot{X}/\mathbf{G}_m := \mathbf{P}(X)$$

exists and is a projective variety. Indeed, because x is attractive, we can find a closed equivariant embedding of X into T_xX, and this defines a closed embedding of $\mathbf{P}(X)$ into $(T_xX \setminus \{0\})/\mathbf{G}_m$, a weighted projective space.

We claim that $\mathbf{P}(X)$ is rationally smooth. Indeed, $\mathbf{G}_m$ acts in $\dot{X}$ with finite isotropy groups. Using e.g. Proposition A4, it follows that $\dot{X}$ is covered by $\mathbf{G}$-invariant open subsets of the form

$$(\mathbf{G}_m \times Y)/\Gamma$$

where $\Gamma \subset \mathbf{G}_m$ is a finite subgroup, and $Y \subset X$ is a locally closed Γ-stable subvariety; here Γ acts diagonally on $\mathbf{G}_m \times Y$. Then $\mathbf{P}(X)$ is covered by the quotients Y/Γ. Because X is rationally smooth and the map $\mathbf{G}_m \times Y \to X : (t, y) \mapsto ty$ is étale, $\mathbf{G}_m \times Y$ is rationally smooth, too (see e.g. Proposition A1). Thus, Y is rationally smooth, and so is the quotient Y/Γ by Proposition A1 again.

The action of T on X induces an action on $\mathbf{P}(X)$, with fixed point set the disjoint union of the $\mathbf{P}(X^{T'})$ (where $T' \subset T$ is a subtorus of codimension one). Indeed, T-fixed points in $\mathbf{P}(X)$ correspond to T-orbits of dimension one in $\dot{X}$.

Observe that X is rationally smooth at x if and only if

$$H^m(\dot{X}) = \begin{cases} \mathbf{Q} & \text{if } m = 0 \text{ or } m = 2d-1 \\ 0 & \text{otherwise.} \end{cases}$$

where $d = \dim_x(X)$. Indeed, the action of $\mathbf{G}_m$ on X extends to a map $\mathbf{A}^1 \times X \to X$ sending $0 \times X$ to x, and restricting to the identity $1 \times X \to X$. Thus, $H^m(X) = 0$ for all $m > 0$. Now our observation follows from the long exact sequence

$$\cdots \to H^m(X) \to H^m(\dot{X}) \to H_x^{m+1}(X) \to H^{m+1}(X) \cdots$$

Observe finally that X is rationally smooth at x if and only if $\mathbf{P}(X)$ is a rational cohomology complex projective space of dimension $d-1$. Indeed, let

$$\pi : \dot{X} \to \mathbf{P}(X)$$

be the quotient map and let $\mathbf{Q}_{\dot{X}}$ be the constant sheaf on $\dot{X}$ associated with $\mathbf{Q}$. Let us compute the higher direct images $R^i\pi_*\mathbf{Q}_{\dot{X}}$. For this, consider the commutative square

$$\begin{array}{ccc} \mathbf{G}_m \times Y & \to & Y \\ \downarrow & & \downarrow \\ (\mathbf{G}_m \times Y)/\Gamma & \to & Y/\Gamma \end{array}$$

where Y and Γ are as above, and the downwards maps p, q are quotients by Γ. Then we have

$$R^i\pi_*(p_*^\Gamma \mathbf{Q}_{(\mathbf{G}_m\times Y)/\Gamma}) = q_*^\Gamma(R^i pr_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y})$$

where p_*^Γ, q_*^Γ denote invariant direct image. But $pr_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ and $R^1 pr_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ are isomorphic to $\mathbf{Q}_Y$, and $R^i pr_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ vanishes for $i \geq 2$. Moreover, $q_*^\Gamma\mathbf{Q}_Y$ is isomorphic to $\mathbf{Q}_{Y/\Gamma}$ via q^*, and a similar statement holds for p_*^Γ. It follows that $\pi_*\mathbf{Q}_{\dot{X}}$ and $R^1\pi_*\mathbf{Q}_{\dot{X}}$ are isomorphic to $\mathbf{Q}_{\mathbf{P}(X)}$, and that $R^i\pi_*\mathbf{Q}_{\dot{X}} = 0$ for $i \geq 2$. Thus, the Leray spectral sequence for π reduces to a Gysin long exact sequence

$$\cdots H^{m-1}(\dot{X}) \to H^m(\mathbf{P}(X)) \to H^{m-2}(\mathbf{P}(X)) \to H^m(\dot{X}) \to \cdots$$

which implies immediately our assertion.

Assume now that *(ii)* and *(iii)* hold. Then we claim that the rational cohomology of $\mathbf{P}(X)$ vanishes in odd degrees, and that the topological Euler characteristic $\chi(\mathbf{P}(X))$ is equal to d. To check this, we use equivariant cohomology again. Notation being as in the proof of Theorem 1.1, the map

$$\mathbf{P}(X) \times_T E_T \to E_T/T = B_T$$

is a fibration with fiber $\mathbf{P}(X)$. Because the latter is projective and rationally smooth, the associated spectral sequence degenerates (by the criterion of Deligne, see e.g. [J] Proposition 13). Thus, the $H^*(B_T)$-module $H_T^*(\mathbf{P}(X))$ is free, and $H^*(\mathbf{P}(X))$ is the quotient of this module by the submodule generated by all characters of T.

From the localization theorem in equivariant cohomology (see e.g. [H] Chapter III, or Proposition A5), it follows that the $H^*(B_T)$-module $H_T^*(\mathbf{P}(X))$ becomes isomorphic to

$$H_T^*(\mathbf{P}(X)^T) = H^*(B_T) \otimes H^*(\mathbf{P}(X)^T) = \bigoplus_{T'} H^*(B_T) \otimes H^*(\mathbf{P}(X^{T'}))$$

after inverting all non trivial characters of T. Moreover, by the preceding discussion and rational smoothness of the $X^{T'}$, each $H^*(\mathbf{P}(X^{T'}))$ is a rational cohomology projective space; in particular, its cohomology vanishes in odd degrees. Because $H^*(B_T)$ vanishes in odd degrees, too, it follows that the same holds for $H_T^*(\mathbf{P}(X))$, and for $H^*(\mathbf{P}(X))$ as well. Moreover, we have for the Euler characteristic of $\mathbf{P}(X)$:

$$\begin{aligned}\chi(\mathbf{P}(X)) &= \mathrm{rk}_{H^*(B_T)} H_T^*(\mathbf{P}(X)) = \mathrm{rk}_{H^*(B_T)} H_T^*(\mathbf{P}(X)^T) \\ &= \sum_{T'} \chi(\mathbf{P}(X^{T'})) = \sum_{T'} \dim(X^{T'}) = d\end{aligned}$$

which proves our claim.

Because $\mathbf{P}(X)$ is projective of dimension $d-1$, it has non trivial rational cohomology in degrees $0, 1, \ldots, d-1$. Thus, the claim implies that $\mathbf{P}(X)$ is a rational cohomology complex projective space of dimension $d-1$, so that X is rationally smooth at x.

Conversely, assume that X is rationally smooth at x. Then, reversing the previous arguments, we see that rational cohomology of each $\mathbf{P}(X^{T'})$ vanishes in odd degree, and that

$$d = \sum_{T'} \chi(\mathbf{P}(X^{T'})).$$

Because each $\mathbf{P}(X^{T'})$ is a projective algebraic variety of dimension $\dim_x(X^{T'}) - 1$, it follows that $\chi(\mathbf{P}(X^{T'})) \geq \dim_x(X^{T'})$. Thus, we have $d \geq \sum_{T'} \dim_x(X^{T'})$. But the reverse inequality holds by Theorem 1.2: so we must have

$$d = \sum_{T'} \dim_x(X^{T'}), \ \chi(\mathbf{P}(X^{T'})) = \dim_x(X^{T'}).$$

for all T'. It follows that each $\mathbf{P}(X^{T'})$ is a rational cohomology projective space, and that $X^{T'}$ is rationally smooth at x.

The methods of the proof above lead to the following

Corollary 1. — *Let T be a torus acting on an irreducible variety X of dimension two; let $x \in X$ be an attractive fixed point, contained in only finitely many irreducible T-stable curves. Then X is rationally smooth at x.*

Proof. — We may assume that X is affine and that T acts faithfully. Then $\dim(T) = 2$ (otherwise there are infinitely many irreducible T-invariant curves through x, namely, the T-orbit closures). Thus, X contains a dense T-orbit; in other words, the normalization of X is an affine toric surface. It follows that X contains exactly four T-orbits: the fixed point x, two orbits of dimension one, and the open orbit.

Let $\mathbf{P}(X)$ be as in the proof above, then $\mathbf{P}(X)$ is a projective irreducible curve with a dense T-orbit. Thus, $\mathbf{P}(X)$ is homeomorphic to projective line. Moreover, $\dot{X}$ is covered by two affine open subsets of the form $\mathbf{G}_m \times_\Gamma C$ where Γ is a finite group, and C is an irreducible affine curve admitting a non trivial action of $\mathbf{G}_m$ (this follows e.g. from Proposition A4). Thus, C is unibranched, and $\dot{X}$ is rationally smoth.

As another consequence of (the proof of) Theorem 1.3, let us derive the following refinement of a result due to Carrell and Peterson [C] Theorem D.

COROLLARY 2. — *Let T be a torus acting on a variety X with an isolated fixed point x, such that the number of irreducible T-stable curves through x is finite; denote this number by $n(X,x)$. Then*

$$\dim_x(X) \le n(X,x).$$

If moreover X is rationally smooth at x, then

$$\dim_x(X) = n(X,x)$$

and each irreducible T-stable curve through x is exactly the fixed point set of a subtorus of codimension one in T.

Conversely, if x is attractive and admits a rationally smooth punctured neighborhood, and if $\dim_x(X) = n(X,x)$, then X is rationally smooth at x.

Proof. — We may assume that X is affine and that $X^T = \{x\}$. Observe that each irreducible T-stable curve in X is fixed pointwise by a unique subtorus $T' \subset T$ of codimension one. Further, $X^{T'}$ contains only finitely irreducible T-stable curves through x, and all such curves must be contained in $X_+^{T'}(x) \cup X_-^{T'}(x)$. Thus, the dimension of both $X_+^{T'}(x)$ and $X_-^{T'}(x)$ is at most one, and $\dim X_+^{T'}(x) + \dim X_-^{T'}(x)$ is at most the number of irreducible T-stable curves through x in $X^{T'}$. Now the inequality $\dim_x(X) \le n(X,x)$ follows from Theorem 1.2.

If X is rationally smooth at x, then each $X^{T'}$ is rationally smooth at x as well by Theorem 1.1. Thus, $X^{T'}$ is irreducible. It follows that the connected component of x in $X^{T'}$ is either $\{x\}$ or an irreducible T-stable curve through x. Now the equality $\dim_x(X) = n(X,x)$ follows from Theorem 1.1.

For the converse, we argue as in the proof of Theorem 1.3: the T-fixed points in $\mathbf{P}(X)$ correspond to T-orbits of dimension one in X, that is, to irreducible T-invariant curves through x. Thus, the number of T-fixed points in $\mathbf{P}(X)$ is $\dim_x(X) = \dim \mathbf{P}(X) + 1$. It follows that $\mathbf{P}(X)$ is a rational cohomology complex projective space of dimension $\dim_x(X) - 1$.

Remark. — The assumption that x admits a rationally smooth punctured neighborhood cannot be omitted, as shown by the following example. Let X be the hypersurface in $\mathbf{A}^5$ with equation

$$x^2 + yz + xtw = 0.$$

Let $T = \mathbf{G}_m \times \mathbf{G}_m$ act on $\mathbf{A}^5$ by

$$(u,v)\cdot(x,y,z,t,w) = (u^2v^2x, u^3vy, uv^3z, u^2t, v^2w).$$

Then the origin of $\mathbf{A}^5$ is an attractive fixed point, X is T-stable of dimension four, and X contains four irreducible T-stable curves: the coordinate lines, except for the x-axis. But X is not rationally smooth at the origin. Indeed, consider the action of $\mathbf{G}_m$ on $\mathbf{A}^5$ by

$$u \cdot (x, y, z, t, w) = (x, uy, u^{-1}z, t, w).$$

Then X is $\mathbf{G}_m$-stable and $X^{\mathbf{G}_m}$ is defined by $y = z = x^2 + xtw = 0$. Thus, $X^{\mathbf{G}_m}$ is reducible at the origin, and we conclude by Theorem 1.1.

2. Rational smoothness of orbit closures in flag varieties

2.1. Attractive slices.

We will apply our criterion of rational smoothness to certain orbit closures. For this, we need the following

Definition. — Let X be a variety with an action of a linear algebraic group H and let $x \in X$. A *slice* to the orbit Hx at x is a locally closed affine subvariety $S \subset X$ which satisfies the following conditions.

(a) x is an isolated point of $S \cap Hx$.

(b) S is stable under a maximal torus T of the isotropy group H_x.

(c) The map

$$\begin{array}{cccc} \alpha: & H \times S & \to & X \\ & (h, s) & \mapsto & hs \end{array}$$

is smooth at $(1, x)$.

The slice S is *attractive* if

(d) x is an attractive fixed point for the T-action on S.

It is easy to see that there always exists a slice S. If moreover S is attractive, then $S \cap Hx = \{x\}$ and the map α is smooth everywhere.

Proposition. — *Let X be a variety with an action of a linear algebraic group H, let $x \in X$ and let S be a slice to Hx at x. If X is rationally smooth at all points of Hx, then the T-variety S satisfies conditions (i), (ii) and (iii) of Theorem 1.3. The converse holds if S is attractive.*

Proof. — The map α is H-equivariant; thus, it is smooth at all points (h, x) where $h \in H$, and the image of α is a neighborhood of Hx in X. Using Proposition A1, we see that X is rationally smooth along Hx if and only if S is rationally smooth at x.

As a first application, we give a direct proof of a criterion for rational smoothness of Schubert varieties, obtained by Carrell and Peterson using Kazhdan-Lusztig theory (see [C] Theorem E).

Let G be a connected semisimple group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus with Weyl group W. The T-fixed points in the flag variety G/B are indexed by W. For $w \in W$ we still denote by w the corresponding fixed point, and by $X(w) = \overline{BwB}/B$ the corresponding Schubert variety; then the dimension of $X(w)$ is the length of w, denoted by $\ell(w)$. Let $x \in W$, then $x \in X(w)$ if and only if $x \leq w$ for the Bruhat ordering on W.

We now recall the construction of slices to Schubert varieties, and the description of their T-stable curves. By the Bruhat decomposition, the map

$$\begin{array}{ccc} (U \cap xU^- x^{-1}) \times (U^- \cap xU^- x^{-1}) & \to & \mathcal{B}(G) \\ (g, h) & \mapsto & ghx \end{array}$$

is an open immersion, and its restriction

$$U \cap xU^- x^{-1} \to Bx : g \mapsto gx$$

is an isomorphism. Set

$$S := X(w) \cap (U^- \cap xU^- x^{-1})x,$$

then S is a T-stable attractive slice to Bx at x in $X(w)$.

Let $R \subset W$ be the set of reflections. For $r \in R$, let T^r be its fixed point set in T, and let G^{T^r} be the centralizer of T^r in G, a reductive group of semisimple rank one. Set

$$C(x, r) := G^{T^r} x.$$

Then, by [C] Theorem F, the $C(x, r)$ ($r \in R$) are the irreducible T-stable curves through x in G/B. Furthermore, $r \in R(x, w)$ if and only if $C(x, r)$ is contained in X. More precisely, we have $x < rx \leq w$ (resp. $rx < x$) if and only if $C(x, r) \subset S$ (resp. $C(x, r) \subset Bx$).

Now, combining the proposition above with Corollary 1.2 and 1.3.2, we obtain the following

Corollary. — *Let x, w in W such that $x \leq w$, and let $n(x, w)$ be the number of $r \in R$ such that $rx \leq w$. Then $l(w) \leq n(x, w)$. Moreover, $X(w)$ is rationally smooth at x if and only if $l(w) = n(y, w)$ for all $y \in W$ such that $x \leq y < w$.*

2.2. Orbits of spherical subgroups in flag varieties.

We still consider a connected semisimple group G and denote by $\mathcal{B}(G)$ its flag variety. Let $H \subset G$ be a spherical subgroup, that is, $\mathcal{B}(G)$ contains only finitely many H-orbits. Let H^0 be the connected component of 1 in H, then H^0 is spherical in G, too.

Easy but useful properties of H-orbits in $\mathcal{B}(G)$ are given by the following

Proposition.

(i) Each closed orbit is isomorphic to a finite union of copies of the flag variety $\mathcal{B}(H^0)$.

(ii) Let $X \subset \mathcal{B}(G)$ be an orbit closure and $X_0 \subset X_1 \subset \cdots \subset X_\ell = X$ a maximal chain of orbit closures. Then $\ell = \dim(X) - \dim \mathcal{B}(H^0)$.

(iii) Let $\tilde{H} \supset H$ be a subgroup of G which normalizes H and such that $\tilde{H}/H$ is connected. Then H and $\tilde{H}$ have the same orbits in $\mathcal{B}(G)$.

Proof.

(i) Let $x \in \mathcal{B}(G)$ be such that Hx is closed. Then the variety Hx is complete; thus, the same holds for its component H^0x. Moroever, the isotropy group $H_x = H \cap G_x$ is solvable. Thus, H_x^0 is a Borel subgroup of H.

(ii) Choose a Borel subgroup B of G, then the partially ordered sets of H-orbit closures in $\mathcal{B}(G)$ and of B-orbit closures in G/H are isomorphic. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_\ell = Y$ be a maximal chain of B-orbit closures in G/H. Then $Y_{\ell-1}$ is an irreducible component of the complement of the open B-orbit in Y_ℓ. Because that orbit is affine, we have $\dim(Y_{\ell-1}) = \dim(Y_\ell) - 1$. It follows that $\dim(Y_0) = \dim(Y) - \ell$. Back to H-orbits in $\mathcal{B}(G)$, we thus have $\dim(X_0) = \dim(X) - \ell$. Moreover, X_0 is a closed orbit, whence $\dim(X_0) = \dim \mathcal{B}(H^0)$.

(iii) Let $\mathcal{O} \subset \mathcal{B}(G)$ be an H-orbit and let c be its codimension in $\mathcal{B}(G)$. We show that $\mathcal{O}$ is $\tilde{H}$-stable, by induction on c.

If $c = 0$ then $\mathcal{O}$ is open in $\mathcal{B}(G)$. Choose $x \in \mathcal{O}$, then Hx is an open subset of $\tilde{H}x$, whence the product $H\tilde{H}_x$ is open in $\tilde{H}$. But $H\tilde{H}_x$ is a closed subgroup of $\tilde{H}$ containing H, because $\tilde{H}$ normalizes H. Thus, $H\tilde{H}_x$ is a union of components of $\tilde{H}$, and Hx is a union of components of $\tilde{H}x$. But $\tilde{H}/H$ is connected, whence $Hx = \tilde{H}x$.

For arbitrary c, observe that the closure $\overline{\mathcal{O}}$ is a union of components of the set of $x \in \mathcal{B}(G)$ such that the codimension of Hx in $\mathcal{B}(G)$ is at least c. The latter set is closed and $\tilde{H}$-stable, because $\tilde{H}$ normalizes H. As $\tilde{H}/H$ is connected, it follows that $\overline{\mathcal{O}}$ is $\tilde{H}$-stable. Now the argument above shows that $\mathcal{O}$ is $\tilde{H}$-stable.

Definition. — The *rank* $\ell(X)$ of an H-orbit closure $X \subset \mathcal{B}(G)$ is the codimension in X of any closed orbit, or equivalently, the common length of all maximal chains $X_0 \subset X_1 \subset \cdots \subset X_\ell = X$ of orbit closures.

In the case where $H = B$ as in 2.1, the closed orbits are fixed points, and the rank of $X = X(w)$ is the length of w.

For a *reductive* spherical subgroup $H \subset G$ and an H-orbit closure X in $\mathcal{B}(G)$, we will show that $\ell(X)$ satisfies an inequality similar to Corollary 2.1, with equality if X is rationally smooth. For this, we will analyze the fixed points in X of a maximal torus of H, and of its codimension one subtori.

2.3. Fixed points in flag varieties.

Let $H \subset G$ be a reductive spherical subgroup, and let $T_H \subset H$ be a maximal torus. For a subtorus $T' \subset T_H$, we denote by $G^{T'}$ (resp. $H^{T'}$) its centralizer in G (resp. H) and by $\mathcal{B}(G)^{T'}$ its fixed point in $\mathcal{B}(G)$. It is well known that $G^{T'}$ is connected and reductive, and that $\mathcal{B}(G)^{T'}$ contains only finitely many orbits of $G^{T'}$, each of them being isomorphic to the flag variety $\mathcal{B}(G^{T'})$. The torus T' is *regular in* G if $\mathcal{B}(G)^{T'}$ is finite, or equivalently, $G^{T'}$ is a maximal torus of G.

Lemma. — *Notation being as above, T_H is regular in G. Moreover, each $H^{T'}$ is a reductive spherical subgroup of $G^{T'}$.*

Proof. — Because H^0 acts on $\mathcal{B}(G)$ with only finitely many orbits, $(H^0)^{T'}$ acts on $\mathcal{B}(G)^{T'}$ with only finitely many orbits, too; see [R]. It follows that $(H^0)^{T'}$ is spherical in $G^{T'}$. In particular, $(H^0)^{T_H} = T_H$ is spherical and central in G^{T_H}. Thus, G^{T_H} is a torus, and T_H is regular in G.

Now assume that the codimension of T' in T_H is one, and that T' is singular in G. Then $T' \subset H^{T'} \subset G^{T'}$ and the quotient $H^{T'}/T'$ has rank one. Let G' be the quotient of $G^{T'}$ by its center, and let H' be the image of $H^{T'}$ in G'. Then $G^{T'}$ and G' have the same flag variety, which we denote by $\mathcal{B}'$. Moreover, H' is a reductive spherical subgroup, of rank at most one, of the non trivial connected adjoint semisimple group G'. Thus, H'^0 is either the multiplicative group or $(\mathrm{P})\mathrm{SL}_2$. Because H' has finitely many orbits in $\mathcal{B}'$, we have $\dim(\mathcal{B}') \leq 1$ in the former case, and $\dim(\mathcal{B}') \leq 3$ in the latter case. Thus, G' is isomorphic to $(\mathrm{PSL}_2)^n$ with $n \leq 3$, or to PSL_3. A closer look leads to the following classification.

(1) $H' = G' = \mathrm{PSL}_2$. Then $\mathcal{B}'$ is projective line $\mathbf{P}^1$ with transitive action of H'.

(2) H'^0 is a one dimensional torus of $G' = \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1$, and the H'^0-orbits in $\mathcal{B}'$ are two fixed points and their complement. If H' is not connected, then it is the normalizer of H'^0, and it exchanges both fixed points in $\mathcal{B}'$.

(3) $H' = \mathrm{PSL}_2$, the diagonal in $G' = \mathrm{PSL}_2 \times \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1 \times \mathbf{P}^1$ with diagonal action of H'. The H'-orbits in $\mathcal{B}'$ are the diagonal and its complement.

(4) $H' = \mathrm{PSL}_2 = \mathrm{SO}_3$ embedded into $\mathrm{PSL}_3 = G'$. We can consider $\mathcal{B}'$ as the variety of flags in projective plane $\mathbf{P}^2$, and H' as the stabilizer in PSL_3 of a smooth conic C_0. Then the H'-orbits in $\mathcal{B}'$ are given by the position of a flag (p, d) (where p is a point of $\mathbf{P}^2$ and d a line containing p) with respect to C_0. So there is a unique closed orbit: the set of flags (p, d) such that d is tangent to C_0 at p_0. This orbit is isomorphic to $\mathbf{P}^1$. And there are two orbit closures of dimension two, defined by: p is in C_0, resp. d is tangent to C_0. It is easy to see that the maps $(p, d) \mapsto p$, resp. $(p, d) \mapsto d$ identify these orbit closures to the rational ruled surface of index two, denoted by $\mathbf{F}_2$.

(5) $H'^0 = \mathrm{SL}_2$ and $G' = \mathrm{PSL}_3$, where H'^0 is embedded as the image of matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{pmatrix}$ with $ad - bc = 1$. Denote by $\tilde{H}'$ the normalizer of H'^0 in G', then $\tilde{H}'$ is the image of matrices of the form $\begin{pmatrix} t & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{pmatrix}$ with $t(ad - bc) = 1$. Observe that $\tilde{H}'$ normalizes H', and that the quotient $\tilde{H}'/H'$ is the multiplicative group. Thus, H' and $\tilde{H}'$ have the same orbits in $\mathcal{B}'$, by Proposition 2.2. Observe that $\tilde{H}'$ is the stabilizer in G' of a point p_0 in $\mathbf{P}^2$, represented by the first basis vector of $\mathbf{C}^3$, and of a line l_0 in $\mathbf{P}^2$, represented by the first dual basis vector. Thus, $\tilde{H}'$ has three closed orbits in $\mathcal{B}'$: the set of flags (p, d) such that $p = p_0$ (resp. $d = d_0$; $p \in d_0$ and $d \in p_0$). These orbits are isomorphic to $\mathbf{P}^1$. Moreover, there are two $\tilde{H}'$-orbit closures of dimension two, consisting of flags (p, d) such that $p_0 \in d$ (resp. $p \in d_0$). The maps $(p, d) \mapsto p$ (resp. $(p, d) \mapsto d$) identify theses orbit closures to the blow-up of $\mathbf{P}^2$ at the point p_0 (resp. the blow-up of the dual projective plane at the point d_0). Thus, both orbit closures are isomorphic to the rational ruled surface $\mathbf{F}_1$ of index 1.

(6) $H' = \mathrm{PSL}_2$, the small diagonal in $G' = \mathrm{PSL}_2 \times \mathrm{PSL}_2 \times \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with diagonal action of H'. The H'-orbit closures in $\mathcal{B}'$ are the small diagonal $\mathbf{P}^1$, three partial diagonals isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathcal{B}'$.

Remarks.

(i) For a symmetric subgroup H of G, we will see in 2.5 that only types (1) to (4) can occur. It can be checked that the same holds if G is simple and $H^0 \subset G$ is a maximal connected reductive spherical subgroup; for this, one uses Krämer's classification of reductive spherical subgroups of simple groups [Kr]. But types (5) and (6) do occur in general, e.g. type (5) for

$$H = \mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n+1} = G,$$

and type (6) for

$$H = \mathrm{SO}_{2n+1} \subset \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+2} = G$$

where H is embedded in G by $h \mapsto (h,(h,1))$, or for $H = G_2 \subset \mathrm{SO}_8 = G$ embedded by its defining representation.

(ii) By [M-S] 6.4, orbits of symmetric subgroups in flag varieties admit attractive slices at all points. But this fails for arbitrary reductive subgroups: consider for example $G = \mathrm{PSL}_3$ and $H = \mathrm{SL}_2$ as in type (5). Then we can take for T_H the image of diagonal matrices with eigenvalues $(1, t, t^{-1})$ where $t \in \mathbf{G}_m$. Let $x \in \mathcal{B}(G)$ be the standard flag in $\mathbf{C}^3$, then the weights of the T_H-action on the normal space $T_x\mathcal{B}(G)/T_xHx$ are 1 and -1. Thus, Hx admits no attractive slice at x. Moreover, both H-orbits of dimension two have unipotent isotropy groups, so that they admit no attractive slice either.

2.4. A criterion for rational smoothness.

Notation being as in 2.3, we will describe fixed point subsets in an H-orbit closure $X \subset \mathcal{B}(G)$, and deduce a necessary condition for rational smoothness of X.

Let T be the centralizer of T_H in G; it is a maximal torus of G. Let W be the Weyl group of (G, T), and $R \subset W$ the subset of reflections. For $r \in R$, denote by T^r its fixed point subgroup in T_G, and set

$$T_H^r := (T_H \cap T^r)^0.$$

Then the T_H^r are exactly the subtori of codimension one of T_H which are singular in G. For $1 \le t \le 6$, call r of type (t) if T_H^r has type (t) in the classification above.

Finally, for $T' = T_H^r$ as above, let $\ell_{T'}(X, x)$ be the sum of the ranks of the irreducible components of the $H^{T'}$-varieties $X^{T'}$ which contain x. Observe that the maximal value of $\ell_{T'}(X, x)$ is 0 in type (1), 1 in types (2) and (3), 2 in types (4) and (5), and 3 in type (6).

Proposition. — *Let $X \subset \mathcal{B}(G)$ be the closure of an H-orbit.*

(i) For any $r \in R$, each irreducible component of $X^{T_H^r}$ is smooth, and is either a point (this may occur in type (1)), or $\mathbf{P}^1$ (this may occur in all types), or $\mathbf{P}^1 \times \mathbf{P}^1$ (in types (3) and (6)), or $\mathbf{F}_1$ (in case (5)), or $\mathbf{F}_2$ (in type (4)), or $\mathcal{B}(\mathrm{PSL}_3)$ (in types (4) and (5)), or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ (in type (6)).

(ii) We have

$$\ell(X) \le \sum_{T'} \ell_{T'}(X, x)$$

with equality if X is rationally smooth at x.

Proof.

(i) is checked by inspection.

For *(ii)*, by Theorems 1.1 and 1.2, we have

$$\dim \mathcal{B}(H^0) = \sum_{T'} \dim \mathcal{B}(H^{T',0}),\ \dim(X) \le \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x)).$$

Moreover, we claim that

$$\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) \leq \dim \mathcal{B}(H^{T',0}) + \sum_Y \ell(Y)$$

(sum over all irreducible components Y of $X^{T'}$ which contain x). Indeed, if $X^{T'}$ is irreducible at x, then it is smooth there by *(i)*. Thus, we have

$$\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) = \dim_x(X^{T'}) = \dim \mathcal{B}(H^{T',0}) + \ell(X^{T'})$$

where the first equality follows from Theorem 1.2, and the second one is the definition of the rank. If $X^{T'}$ is reducible at x, then we are in case (4), (5) or (6), and moreover $H'x$ is closed in $\mathcal{B}'$. In cases (4) and (6), x is attractive in $\mathcal{B}'$ and the claim is clear; in case (5), it is checked by inspection. Our inequality follows.

If moreover X is rationally smooth at x, then each $X^{T'}$ is irreducible at x, and we conclude by Theorem 1.1.

2.5. The symmetric case.

Consider now a connected semisimple group G with an involutive automorphism θ. Then the fixed point set $H = G^\theta$ is called a symmetric subgroup; it is a reductive spherical subgroup of G. We refer to [Sp1] for this and for other results on symmetric spaces, to be used below.

We will obtain a precise version of Proposition 2.3 *(ii)*, in terms of the combinatorics of H-orbits in $\mathcal{B}(G)$. We begin by relating the approach of 2.3 to the structure of symmetric spaces.

Let $T_H \subset H$ be a maximal torus, then its centralizer T is a θ-stable maximal torus of G. Thus, θ acts on the Weyl group W and on the subset R of reflections. For $r \in R$, let G_r be the derived subgroup of G^{T^r}. Then G_r is a semisimple group of rank one, containing a representative of r.

LEMMA.

(i) There exists a θ-stable Borel subgroup B of G containing T; then $B^{\theta,0}$ is a Borel subgoup of H. Any two such Borel subgroups of G are conjugated by W^θ.

(ii) Let $r \in R$. Then
r has type (1) if and only if: $\theta(r) = r$, and G_r is contained in H.
r has type (2) if and only if: $\theta(r) = r$, and G_r is not contained in H.
r has type (3) if and only if: $\theta(r)$ commutes with r, and $\theta(r) \neq r$.
r has type (4) if and only if: $\theta(r)$ does not commute with r.
There are no reflections of type (5) or (6).

Proof.

(i) There exists a pair (B_0, T_0) where B_0 is a θ-stable Borel subgroup of G, and T_0 is a θ-stable maximal torus of B_0. Let U_0 be the unipotent radical of B_0, and let B_0^- be the opposite Borel subgroup, with unipotent radical B_0^-. Then the product map $U_0^- \times T_0 \times U_0 \to G$ is an open immersion. Thus, the same holds for the product map $(U_0^-)^{\theta,0} \times T_0^{\theta,0} \times U_0^{\theta,0} \to H$. It follows that $B_0^{\theta,0}$ and $(B_0^-)^{\theta,0}$ are opposite Borel subgroups of H. In particular, $T_0^{\theta,0}$ is a maximal torus of H. Thus, we can write $T_H = hT_0^{\theta,0}h^{-1}$ for some $h \in H$. Taking centralizers in G, we obtain $T = hT_0h^{-1}$; then we can take $B = hB_0h^{-1}$. If B' is another Borel subgroup containing T, there exists a unique $w \in W$ such that $B' = wBw^{-1}$; now B' is θ-stable if and only if $\theta(w) = w$.

(ii) Let $T' = T_H^r$, then θ acts on the group $G^{T'}$ and on its quotient G' by its center. Let H' be the image of H in G'; then H' is a subgroup of finite index in G'. It follows that (G', H') is not of type (5) or (6), because SL_2 is not a subgroup of finite index of a symmetric subgroup of PSL_3 or of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$. The description of types (1) to (4) follows from [Sp1] § 2.

Reflections of type (1) (resp. (2)) are called *compact imaginary* (resp. *non-compact imaginary*), whereas reflections of type (3) or (4) are called *complex*; for B as in the lemma above, the pair (T, B) is called *standard*. We then identify $\mathcal{B}(G)$ with G/B; the point $x \in (G/B)^T$ is identified with an element of W, still denoted by x.

We now recall the parametrization of H-orbits in G/B; our notation differs from that in [Sp1] by an inverse, because B-orbits in G/H are considered there. Let N be the normalizer of T in G, then N is θ-stable. Set

$$\mathcal{V} := \{g \in G \mid g^{-1}\theta(g) \in N\}.$$

Then $\mathcal{V}$ is stable by the $H \times T$-action: $(h, t)g = hgt^{-1}$, and each $(H \times B)$-orbit in G meets $\mathcal{V}$ in a unique $(H \times T)$-orbit. As a consequence, H-orbits in G/B are parametrized by the set of double classes

$$V := H\backslash\mathcal{V}/T.$$

There is a base point $v_0 \in V$, the image of $1 \in N$; the corresponding H-orbit is closed, by the lemma above. Observe that $\mathcal{V}$ is stable under right multiplication by N; this defines an action of W on V, denoted by $(w, v) \mapsto w \cdot v$.

For $v \in V$, we denote by $X(v) \subset G/B$ the corresponding H-orbit closure, and by $\ell(v)$ its rank. We write $v' \leq v$ if $X(v') \subset X(v)$. This defines a partial order on V, which is studied in [R-S].

Finally, we will need the following result, see [Sp2] 2.5: For any $r \in R$ of type (2), there exists $g(r) \in G_r$ such that $g(r)^{-1}\theta(g(r))$ is a representative of r in N. In particular, $g(r) \in \mathcal{V}$. Let $v(r)$ be its image in V.

THEOREM.. — *Let $v \in V$ and let $x \in W$ such that $x \cdot v_0 \leq v$. Let $n_2(v,x)$ be the number of reflections r of type (2) such that $x \cdot v(r) \leq v$. For $t = 3,\ 4$, let $n_t(v,x)$ be the number of reflections r of type (t) such that $rx \cdot v_0 \leq v$. Then we have:*

$$\ell(v) \leq n_2(v,x) + \frac{1}{2} n_3(v,x) + n_4(v,x),$$

with equality if $X(v)$ is rationally smooth at x.

Proof. — We wish to apply Proposition 2.4 *(ii)* combined with the lemma above. For this, given a subtorus $T' \subset T_H$ of codimension one, we analyze the contribution of T' to the formula in that Proposition. We denote by $\ell_{T'}(v,x)$ the sum of the ranks of the irreducible components of $X(v)^{T'}$ which contain x, and by $X(v)^{T'}_x$ the union of these components, i.e., the connected component of x in $X(v)^{T'}$.

If $T' = T^r_H$ for r of type (2), the component of x in $(G/B)^{T'}$ is the curve $C(x,r)$ considered in 2.1. By [Sp2] 3.1, this curve is contained in $X(v)$ if and only if $x \cdot v(r) \leq v$. In other words, we have $\ell_{T'}(v,x) = 1$ if $x \cdot v(r) \leq v$, and $\ell_{T'}(v,x) = 0$ otherwise.

If $T' = T^r_H$ for r of type (3) or (4), observe that $X(v) \cap G^{T'} x$ is connected by the explicit description in 2.3. Thus, we have $X(v) \cap G^{T'} x = X(v)^{T'}_x$. Now $rx \cdot v_0 \leq v$ iff $rx \in X(v)$ iff $rx \in X(v)^{T'}_x$ (because $rx \in G^{T'} x$ anyway). For T' of type (3), one checks that $\ell_{T'}(v,x)$ is the half of the number of $r \in R$ such that $T^r_H = T'$ and that $rx \in X(v)^{T'}_x$.

If r has type (4), then one checks that $\ell_{T'}(v,x)$ is at most the number of r as above, with equality if $X(v)^{T'}_x$ is irreducible.

In the case where T_H is a maximal torus of G (that is, θ is inner), only types (1) and (2) occur, and we recover the following result of Springer [Sp2]: the rank of $X(v)$ is at most the number of non compact imaginary reflections r such that $rx \cdot v_0 \leq v$, with equality if $X(v)$ is rationally smooth at x.

Remark. — More generally, consider a point $x \in X$ non necessarily fixed by a maximal torus of H. Then the orbit Hx admits an attractive slice at x, by [M-S] 6.4. Thus, a criterion for rational smoothness of X along Hx can be derived from Proposition 2.1. This leads to the following question: For a subtorus T' of codimension one in a maximal torus of H_x, is $X^{T'}$ rationally smooth at x ?

3. Closures of double classes in regular group completions

3.1. Construction of slices.

Let G be a connected reductive group. The action of $G \times G$ by $(g_1, g_2)\gamma = g_1 \gamma g_2^{-1}$ identifies G with the homogeneous space $(G \times G)/\operatorname{diag} G$ where $\operatorname{diag} G$ denotes the diagonal in $G \times G$. Let $T \subset G$ be a maximal torus, W its Weyl group, and B, B^- two opposite

Borel subgroups containing T. Then $B\times B^-$ acts on G as above, the orbits being the double classes BwB^- where $w\in W$. In particular, the open orbit is BB^-.

Let X be a $(G\times G)$-equivariant completion of G which is regular in the sense of [B-D-P]. Then $B\times B^-$ acts on X with finitely many orbits, whose study was initiated in [Br1]. We will construct attractive slices to these orbits. For this, we need more notation and results, adapted from [Br1] 2.1.

Each $(G\times G)$-orbit $\mathcal{O}\subset X$ contains a unique point y such that: $(B\times B^-)y$ is open in $\mathcal{O}$, and y is the limit of a one parameter subgroup of T. We refer to y as the base point of $\mathcal{O}$.

Moreover, $\mathcal{O}$ determines two opposite parabolic subgroups $P\supset B$ and $Q\supset B^-$, with unipotent radicals $R_u(P)$, $R_u(Q)$ and common Levi subgroup $L=P\cap Q$, by requiring that the isotropy group $(G\times G)_y$ is the semidirect product of $R_u(Q)\times R_u(P)$ with $(\mathrm{diag}\ L)(T\times 1)_y$. In particular, $(T\times T)_y=(\mathrm{diag}\ T)(T\times 1)_y$ is a maximal torus in $(G\times G)_y$. In fact, $(T\times 1)_y=(Z\times 1)_y$ where Z denotes the connected center of L.

Let Φ be the root system of (G,T), then we have the subsets Φ^+ (resp. Φ_L) of roots of (B,T) (resp. (L,T)). Let W^L be the set of all $w\in W$ such that $w(\Phi_L^+)$ is contained in Φ^+. Then each $(B\times B^-)$-orbit in $\mathcal{O}=(G\times G)y$ can be written uniquely as

$$(B\times B^-)(w,\tau)y$$

for $w\in W$ and $\tau\in W^L$.

Choose representatives $\tilde{w}$, $\tilde{\tau}$ in the normalizer of T, and set $x:=(\tilde{w},\tilde{\tau})y$. Then

$$(T\times T)_x=(w,\tau)(T\times T)_y(w^{-1},\tau^{-1})$$

is a maximal torus in $(G\times G)_x$ and thus in $(B\times B^-)_x$. The codimension of $(B\times B^-)x$ in $(G\times G)x$ is $\ell(w)+\ell(\tau)$.

For simplicity, set

$$Z_y:=(Z\times 1)_y,$$

then Z_y is the isotropy group of y for the left action of T on $\overline{T}$. Let $\Sigma(y)$ be the set of all $z\in\overline{T}$ such that y belongs to $\overline{Z_y z}$. Then $\Sigma(y)$ is a Z_y-stable slice to Ty at y in $\overline{T}$, and also a slice to $\mathcal{O}$ in X. Moreover, $\Sigma(y)$ is isomorphic to affine space $\mathbf{A}^d$ where $d=\mathrm{codim}_{\overline{T}}(Ty)=\mathrm{codim}_X(\mathcal{O})$, and Z_y acts linearly on $\mathbf{A}^d$ by d independent characters. Thus, $\Sigma(y)$ contains exactly d irreducible Z_y-stable curves through y: the coordinate lines $C_1(y),\dots,C_d(y)$.

For any $\alpha\in\Phi$, let $U_\alpha\subset G$ be the corresponding unipotent subgroup. If $w^{-1}(\alpha)\in\Phi^+\cup\Phi_L$, then $U_\alpha\times 1$ does not fix x. Thus,

$$C(x,\alpha):=(U_\alpha\times 1)x$$

is an irreducible locally closed curve through x, stable by $(T\times T)_x$. We define similarly

$$C(x,\alpha)^- := (1\times U_\alpha)x$$

for $\alpha\in\Phi$ such that $\tau^{-1}(\alpha)\in\Phi^-\cup\Phi_L$. Finally, we set

$$C_i(x) := (\tilde{w},\tilde{\tau})C_i(y)$$

for $1\le i\le d$.

THEOREM. — *Notation being as above, the map*

$$\begin{array}{ccc} (U^-\cap wUw^{-1})\times(U\cap\tau U^-\tau^{-1})\times\Sigma(y) & \to & X \\ (g,h,z) & \mapsto & (g\tilde{w},h\tilde{\tau})z \end{array}$$

is an embedding, and its image S is an attractive $(T\times T)_x$-stable slice to $(B\times B^-)x$ at x. Moreover, the irreducible $(T\times T)_x$-stable curves through x in S are the $C(x,\alpha)$ ($\alpha\in\Phi^-\cap w(\Phi^+)$), the $C(x,\alpha)^-$ ($\alpha\in\Phi^+\cap\tau(\Phi^-)$), and the $C_i(x)$ ($1\le i\le d$).

Proof. — After multiplication by $(\tilde{w},\tilde{\tau})^{-1}$, we reduce to the somewhat simpler study of X along the orbit $(w^{-1}Bw,\tau^{-1}B^-\tau)y$. For this, set

$$\tilde{S} := (U\cap w^{-1}U^-w)\times(U^-\cap\tau^{-1}U\tau)\times\Sigma(y),\ \tilde{y} := (1,1,y).$$

Consider the map

$$\begin{array}{cccc} \pi: & \tilde{S} & \to & X \\ & (g,h,z) & \mapsto & (g,h)z \end{array}$$

and denote by $S(y)$ its image.

The group $(T\times T)_y$ acts on $\tilde{S}$ by

$$(u,v)\cdot(g,h,z) = (ugu^{-1},vhv^{-1},uv^{-1}z)$$

with fixed point $\tilde{y}$, and π is equivariant. Identifying $\tilde{S}$ with affine space of dimension $\ell(w)+\ell(\tau)+d$, the action of $(T\times T)_y$ is linear, with weights: $(\alpha,0)$ ($\alpha\in\Phi^+\cap w^{-1}(\Phi^-)$), $(0,\alpha)$ ($\alpha\in\Phi^-\cap\tau^{-1}(\Phi^+)$), and the weights of $C_1(y),\ldots,C_d(y)$. Moreover, the multiplicity of each weight is one, and $(T\times T)_y = (\operatorname{diag} T)Z_y$ where Z_y acts on $C_1(y),\ldots,C_d(y)$ through d linearly independent weights. It follows that $\tilde{y}$ is attractive, and that the $(T\times T)_y$-invariant curves in $\tilde{S}$ are the $(U_\alpha\times 1)y$ ($\alpha\in\Phi^+\cap w^{-1}(\Phi^-)$), the $(1\times U_\alpha)y$ ($\alpha\in\Phi^-\cap\tau^{-1}(\Phi^+)$), and $C_1(y),\ldots,C_d(y)$.

Moreover, from the description of $(G\times G)_y$ and the fact that $\Sigma(y)$ is transversal to $(G\times G)y$ at y, it follows that π is étale at $\tilde{y}$, and that $\pi^{-1}(\pi(\tilde{y})) = \{\tilde{y}\}$. Because $\tilde{y}$ is attractive, π is an isomorphism onto $S(y)$, a locally closed subvariety of X.

Finally, we check that the action map

$$w^{-1}Bw\times\tau^{-1}B^-\tau\times S(y)\to X$$

is smooth at $(1,1,y)$: this follows from the decompositions of tangent spaces

$$T_yX = T_y(G\times G)y \oplus T_y\Sigma(y) = T_y(B\times B^-)y \oplus T_y\Sigma(y)$$
$$= T_y(w^{-1}Bw \times \tau^{-1}B^-\tau)y \oplus T_y((U\cap w^{-1}U^-w)\times(U^-\cap\tau^{-1}U\tau)y) \oplus T_y\Sigma(y)$$
$$= T_y(w^{-1}Bw\times\tau^{-1}B^-\tau)y \oplus T_yS(y)$$

which follow in turn from the structure of $(G\times G)_y$ described above.

Applying Corollary 1.3.1, we obtain immediately the following

Corollary. — *Any $(B\times B^-)$-orbit closure in a regular completion of G is rationally smooth in codimension two.*

In contrast, $(B\times B^-)$-orbit closures in regular completions are singular in codimension two, apart from very few exceptions [Br1] Corollary 2.2.

3.2. More on slices and closures of double classes.

We just saw that closures of double classes in regular group completions admit attractive slices at all points; further, these slices contain only finitely many invariant curves. Therefore, we can obtain a criterion for rational smoothness of these closures, similar to that for Schubert varieties (Corollary 2.1). To make this explicit, we need to know more about invariant curves, and to describe the inclusion relation between closures of double classes as well.

Notation being as in 3.1, we begin by analyzing the irreducible $(T\times T)_x$-stable curves through x in the slice S. Because X is regular, the $(G\times G)$-orbit $\mathcal{O}$ of codimension d is contained in the closure of d orbits $\mathcal{O}_1,\ldots,\mathcal{O}_d$ of codimension $d-1$. Furthermore, we can index these orbits so that the base point y_i of each $\mathcal{O}_i$ belongs to the curve $C_i(y)$. Thus, we have $C_i(y)=\overline{Z_yy_i}=Z_yy_i\cup\{y\}$, and $C_i(x)\setminus\{x\}$ is contained in $(B\times B^-)(w,\tau)y_i$.

The behaviour of the other curves is given by the following

Proposition. — *Notation being as above, $C(x,\alpha)\setminus\{x\}$ is contained in $(B\times B^-)(s_\alpha w,\tau)y$ for any $\alpha\in\Phi^-\cap w(\Phi^+)$. Similarly, $C(x,\alpha)^-\setminus\{x\}$ is contained in $(B\times B^-)(w,s_\beta\tau)y$ for any $\alpha\in\Phi^+\cap\tau(\Phi^-)$.*

Proof. — Set $\dot U_\alpha := U_\alpha\setminus\{1\}$, then $C(x,\alpha)\setminus\{x\}=\dot U_\alpha x$. Let $s_\alpha\in R$ be the reflection associated with α, then

$$\dot U_\alpha \subset U_{-\alpha}s_\alpha TU_{-\alpha} = U_{-\alpha}Ts_\alpha wU_{-w^{-1}(\alpha)}w^{-1}\subset Bs_\alpha wU_{-w^{-1}(\alpha)}w^{-1}.$$

Set $\beta := w^{-1}(\alpha)$, then $\beta\in\Phi^+$. If $\beta\notin\Phi_L^+$ then $U_{-\beta}\times 1$ fixes y and the assertion follows. Otherwise, $(U_{-\beta}\times 1)y=(1\times U_{-\beta})y$ because $\beta\in\Phi_L^+$. Thus, we have

$$(\dot U_\alpha\times 1)x\subset(Bs_\alpha wU_{-\beta},\tau)y=(Bs_\alpha w,\tau U_{-\beta})y\subset(B\times B^-)(s_\alpha w,\tau)y$$

because $\tau U_{-\beta} = U_{-\tau(\beta)}\tau$ is contained in $B^-\tau$. The proof of the second assertion is similar.

We now describe the inclusion order between closures of $(B \times B^-)$-orbits in X. This is given by the lemma below, where $w_{0,L}$ denotes the longest element in W_L. A closely related statement is obtained in [P-P-R] for reductive algebraic monoids; the latter can be considered as affine embeddings of connected reductive groups.

LEMMA. — *Notation being as above, the closure of $(B \times B^-)(w, \tau)y$ in $\mathcal{O} = (G \times G)y$ is the union of the $(B \times B^-)(w', \tau')y$ where $w', \tau' \in W$ satisfy $w' \geq w$ and $\tau' w_{0,L} \geq \tau w_{0,L}$.*

If moreover $\mathcal{O}' \subset \overline{\mathcal{O}}$ is a $(G \times G)$-orbit with base point y' and associated Levi subgroup L', then

$$\overline{(B \times B^-)(w, \tau)y} \cap \mathcal{O}' = \bigcup \overline{(B \times B^-)(wv, \tau v)y'}$$

(decomposition into irreducible components), where the union is over all $v \in W_L$ such that $\tau v \in W^{L'}$ and $\ell(w) = \ell(wv) + \ell(v)$.

Proof. — Consider the $(B^- \times B)$-orbits in $\mathcal{O}$. We claim that the orbit $(B^- \times B)(1, w_{0,L})y$ is closed. Indeed, we have $B^- = B_L^- R_u(Q)$ and $B = B_L R_u(P)$, whence

$$(B^- \times B)(1, w_{0,L})y = (B_L^- \times B_L)(1, w_{0,L})y = (1, w_{0,L})(B_L^- \times B_L^-)y = (1, w_{0,L})(1 \times B_L^-)y$$

and $(1 \times B_L^-)y$ identifies with the image of B_L^- in L/Z_y, which is closed there.

Now we have $B^-\tau = B^-\tau B_L^-$ (because $\tau B_L^- \tau^{-1} \subset B^-$), whence

$$(B \times B^-)(w, \tau)y = (B \times B^-)(w, \tau w_{0,L})(1, w_{0,L})B_L^- y.$$

Equivalently,

$$(B \times B^-)(w, \tau)y = (B \times B^-)(w, \tau w_{0,L})(B^- \times B)y.$$

So the canonical map

$$\overline{(B \times B^-)(w, \tau w_{0,L})(B^- \times B)} \times_{B^- \times B} (B^- \times B)(1, w_{0,L})y \to \overline{(B \times B^-)(w, \tau)y}$$

is dominant and proper, hence surjective. By the Bruhat decomposition, the closure in G of $(B \times B^-)(w, \tau w_{0,L})(B^- \times B)$ is the union of the double classes $(B \times B^-)(w', \tau' w_{0,L})(B^- \times B)$ with $w' \geq w$ and $\tau' w_{0,L} \geq \tau w_{0,L}$. This implies the first assertion, whereas the second assertion follows from [Br1] Theorem 2.1.

3.3. Singularities of closures of double classes.

Using the combinatorics of 3.2, we show that the closure of a double class BwB^- at a fixed point of $B\times B^-$ contains in general all irreducible $(T\times T)$-stable curves through that point (this improves on [Br1] Theorem 2.2, with a more natural proof). Thus, this closure is not rationally smooth, as a rule.

An exception to that rule is the case where $G = \mathrm{PGL}(2)$. Indeed, that group has a unique regular completion X, the projectivization of the space of 2×2 matrices. Moreover, the closure in X of the standard Borel subgroup B is isomorphic to $\mathbf{P}^2$ and hence smooth; it contains only two irreducible $(T\times T)$-stable curves through the $(B\times B)$-fixed point.

Similarly, the group SL_2 has a unique regular completion X, a quadric in the projective completion of the space of 2×2 matrices. Moreover, the closure in X of the standard Borel subgroup B is a non-degenerate quadratic cone of dimension two. Thus, $\overline{B}$ is singular, but rationally smooth; again, it contains only two irreducible $(T\times T)$-stable curves through the $(B\times B)$-fixed point.

We will see that all exceptions arise from both examples above. To state our result in a precise way, we need the following

Definition. — A simple root α is called *isolated* if α is not connected to any simple root in the Dynkin diagram of G. In particular, G has no isolated simple root if and only if the quotient of G by its center contains no direct factor isomorphic to $\mathrm{PGL}(2)$.

Theorem. — *Let X be a regular completion of G, let $w\in W$ and let $x\in X$ be a fixed point of $B\times B^-$. If G has no isolated simple root, then $\overline{BwB^-}$ contains all irreducible $(T\times T)$-invariant curves through x. In particular, the tangent space to $\overline{BwB^-}$ at x is the whole tangent space to X at x, and $\overline{BwB^-}$ is not rationally smooth there unless $w=1$, that is, $\overline{BwB^-}=X$.*

Proof. — Because $\overline{BwB^-}$ contains Bw_0B^-, we may assume that $w=w_0$. Then the slice S at x is a $(T\times T)$-stable open neighborhood of x, and the irreducible $(T\times T)$-stable curves through x in S are: the $C(x,\alpha) = (U_\alpha\times 1)x$ $(\alpha\in\Phi^-)$, the $C(x,\alpha)^- = (1\times U_\alpha)x$ $(\alpha\in\Phi^+)$, and $C_1(x),\dots,C_l(x)$ where l is the rank of G. Moreover, $C(x,\alpha)\setminus\{x\}$ is contained in $(B\times B^-)(s_\alpha,1)x$ by Proposition 3.2, and similarly for $C(x,\alpha)^-$.

Let z be the base point of the closed orbit $Z := (G\times G)x$, then $x = (w_0,w_0)z$ where $w_0\in W$ is the longest element. We have

$$(B\times B^-)(s_\alpha,1)x = (B\times B^-)(s_\alpha w_0, w_0)z \subset \overline{Bw_0B^-}$$

where the inclusion follows from Lemma 3.2. Thus, $C(x,\alpha)$ is contained in $\overline{Bw_0B^-}$. The argument for $C(x,\alpha)^-$ is similar.

Consider now a curve $C_i(x)$ where $1 \le i \le l$. By Proposition 3.2, there exists a $(G \times G)$-orbit $\mathcal{O}_i$ with base point z_i such that $\dim(\mathcal{O}_i) = \dim(Z) + 1$ and that $C_i(x) \setminus \{x\}$ is contained in $(T \times T)(w_0, w_0)z_i$. Let P, Q, L, Z be associated to $\mathcal{O}_i$ as in 3.1. Then $\dim(Z) \ge \dim(Z_{y_i}) = \dim(T) - 1$. Thus, either $P = B$, or P is a minimal parabolic subgroup containing B.

In the former case, $(G \times G)_{y_i}$ is the kernel of a character of $B^- \times B$. Arguing as above, we obtain that $C_i(x)$ is contained in $\overline{BwB^-}$.

In the latter case, let α be the simple root corresponding to P, and set

$$W^\alpha := \{w \in W \mid w(\alpha) \in R^+\}.$$

Then we have by Lemma 3.2:

$$\overline{Bw_0B^-} \cap \mathcal{O}_i = \bigcup_{v \in W^\alpha} \overline{(B \times B^-)(w_0v, v)z_i}.$$

Choose a simple root β which is connected to α in the Dynkin diagram. Then $s_\alpha s_\beta$ and $w_0 s_\alpha s_\beta s_\alpha$ are in W^α. Thus,

$$\overline{Bw_0B^-} \supset \overline{(B \times B^-)(s_\alpha s_\beta s_\alpha, w_0 s_\alpha s_\beta s_\alpha)z_i} \supset (B \times B^-)(w_0, w_0)z_i$$

where the first inclusion follows from Lemma 3.2, and the second one from that Lemma applied to $w = s_\alpha s_\beta s_\alpha$, $\tau = w_0 s_\alpha s_\beta s_\alpha$, $w' = \tau' = w_0$. Indeed, $w' \ge w$ is clear, and $\tau' s_\alpha = w_0 s_\alpha \ge w_0 s_\alpha s_\beta = \tau s_\alpha$ because $s_\alpha \le s_\alpha s_\beta$.

So we conclude that $C_i(x)$ is contained in $\overline{Bw_0B^-}$. The remaining assertions follow now from Corollary 1.3.2.

Appendix

Proposition A1. — *Let X be an algebraic variety of dimension d and let $x \in X$.*

(i) The dimension of the space $H^{2d}_x(X)$ is the number of d-dimensional irreducible components of X through x.

(ii) If X is rationally smooth at x, then it is irreducible at x.

(iii) Let $\pi : X \to Y$ be the quotient by the action of a finite group G. If X is rationally smooth at x, then Y is rationally smooth at $\pi(x)$.

(iv) Let $\pi : X \to Y$ be a smooth morphism. Then X is rationally smooth at x if and only if Y is rationally smooth at $\pi(x)$.

Proof.

(i) Let $\mathcal{T}_{X,\mathbf{Q}}$ be the dualizing complex of X for sheaves of $\mathbf{Q}$-vector spaces [V]. For each integer m, the homology sheaf $\mathcal{H}_m(\mathcal{T}_{X,\mathbf{Q}})$ is associated with the presheaf $U \mapsto H^m_c(U)^*$ (the dual of cohomology with compact supports). This presheaf vanishes for $m > 2d$, and is a sheaf for $m = 2d$. Moreover, by [V] Corollaire 2.6.5, the stalk of $\mathcal{T}_{X,\mathbf{Q}}$ at x is the dual of $R\Gamma_x(\mathbf{Q}_X)$ where $\mathbf{Q}_X$ denotes the constant sheaf on X associated with $\mathbf{Q}$. It follows that $U \mapsto H^{2d}_c(U)$ is a sheaf, and that its stalk at x is $H^{2d}_x(X)$. This implies our assertion.

(ii) It follows from *(i)* that X has a unique irrreducible component Y of dimension d which contains x. If X has another irreducible component Z of dimension $e < d$ which contains x, then we can choose a smooth point $z \in Z \setminus Y$ arbitrarily close to x. Now $H^{2e}_z(X) = H^{2e}_z(Z)$ is non zero, a contradiction.

(iii) Denote by $\mathbf{Q}_X$ the constant sheaf on X associated with $\mathbf{Q}$. Then G acts on the direct image $\pi_*\mathbf{Q}_X$ and the subsheaf of invariants $\pi^G_*\mathbf{Q}_X$ is isomorphic to $\mathbf{Q}_Y$ via the map $\mathbf{Q}_Y \to \pi_*\mathbf{Q}_X$ (indeed, this map induces an isomorphism on stalks). Moreover, $R^i\pi_*\mathbf{Q}_X = 0$ for $i \geq 1$. It follows that $\pi_* : H^*(X) \to H^*(Y)$ restricts to an isomorphism

$$H^*(X)^G \cong H^*(Y).$$

Considering the isomorphisms above for X and $X \setminus \pi^{-1}\pi(x) = X \setminus Gx$, we obtain an isomorphism

$$H^*_{Gx}(X)^G \cong H^*_{\pi(x)}(Y).$$

Further, the left hand side is isomorphic to

$$\Big(\bigoplus_{g\in G/G_x} H^*_{gx}(X)\Big)^G \cong H^*_x(X)^{G_x}.$$

Because X is rationally smooth at x, the vector space $H^*_x(X)$ is one-dimensional, concentrated in degree $2\dim_x(X)$, and G_x acts trivially there. Thus, Y is rationally smooth at $\pi(x)$.

(iv) Shrinking X and Y, we can factor π as an étale morphism $f : X \to Y \times \mathbf{A}^n$ followed by projection $g : Y \times \mathbf{A}^n \to Y$. By excision, we have $H^*_x(X) \cong H^*_{f(x)}(Y \times \mathbf{A}^n)$. Moreover, by the Thom isomorphism, we have $H^*_{(y,z)}(Y \times \mathbf{A}^n) \cong H^{*-2n}_y(Y)$. It follows that $H^*_x(X) \cong H^{*-2n}_{\pi(x)}(Y)$.

Proposition A2.. — *Let X be an affine variety with a $\mathbf{G}_m$-action and an attractive fixed point x. Then there exists a $\mathbf{G}_m$-module V and a finite equivariant surjective morphism $\pi : X \to V$ such that $\pi^{-1}(0) = \{x\}$ (as a set).*

Proof. — Let A be the algebra of regular functions over X, then $A = \bigoplus_{n=0}^{\infty} A_n$ is positively graded by the $\mathbf{G}_m$-action. For any positive integer r, set $A^{(r)} = \bigoplus_{n=0}^{\infty} A_{nr}$. Then A is a finite module over $A^{(r)}$, and there exists r such that $A^{(r)}$ is generated by its elements of minimal degree. So we can assume that A is generated by its elements of degree 1.

For any irreducible component Y of X, the set of $f \in A_1$ such that $f(Y) = 0$ is a proper linear subspace of A_1. So there exists $f \in A_1$ such that $f(Y) \neq 0$ for all such Y. Let $X' \subset X$ be the zero set of f, then $x \in X'$ and $\dim(X') = d - 1$ where $d = \dim(X)$. So we construct inductively $f = f_1, f_2, \ldots, f_d \in A_1$ such that x is their unique common zero. Consider the morphism

$$\pi = (f_1, f_2, \ldots, f_d) : X \to \mathbf{A}^d.$$

Then π is equivariant for the $\mathbf{G}_m$-action on $\mathbf{A}^d$ by multiplication, and $\pi^{-1}(0) = \{x\}$: the quotient of A by its ideal generated by $f_1, \ldots, f_d$ is finite dimensional. By the graded Nakayama lemma, it follows that π is finite. Because $\dim(X) = d$, the map π is dominant, and hence surjective.

Proposition A3. — *Let X be a variety with a non trivial action of a torus T and a fixed point x belonging to all irreducible components of X. Then there exists an irreducible T-stable curve $C \subset X$ which contains x as an isolated fixed point.*

Proof. — By induction on the dimension of X, the case of dimension one being trivial. We may assume that X is affine and irreducible. Let $\pi : X \to X//T$ be the quotient in the sense of geometric invariant theory. Then π is surjective, and its fibers are infinite (because T acts non trivially on X). In particular, the set

$$\pi^{-1}\pi(x) = \{y \in X \mid x \in \overline{Ty}\}$$

is infinite. Thus, we may assume that $X = \overline{Ty}$. If $\dim(X) = 1$, we can take $C = X$; otherwise, we can choose $z \in X \setminus Ty$ such that $z \neq x$. Then $x \in \overline{Tz}$ with $\dim(Tz) < \dim(Ty)$. We conclude by induction.

Proposition A4. — *Let T be a torus acting on a variety X and let $\mathcal{O} \subset X$ be an orbit. Then $\mathcal{O}$ admits an open affine T-stable neighborhood U in X, with an equivariant retraction $\pi : U \to \mathcal{O}$.*

Proof. — We may assume that X is affine. Let f be a regular function on X which vanishes identically on $\overline{\mathcal{O}} \setminus \mathcal{O}$, and is an eigenvector of T. Then f has no zero in the orbit $\mathcal{O}$, and therefore $\mathcal{O}$ is closed in the open affine T-invariant subset $X \cap (f \neq 0)$. Thus, we may assume that $\mathcal{O}$ is closed in X.

The orbit $\mathcal{O}$ is isomorphic to a torus. Choose an isomorphism $f : \mathcal{O} \to \mathbf{G}_m^n$, then the coordinate functions $f_1, \ldots, f_n$ are eigenvectors of T. Because $\mathcal{O}$ is closed in X, we can extend $f_1, \ldots, f_n$ to regular functions on X, eigenvectors of T. They define an equivariant morphism $F : X \to \mathbf{A}^n$ which maps $\mathcal{O}$ isomorphically to $\mathbf{G}_m^n$. Then we can take $U = F^{-1}(\mathbf{G}_m^n)$.

Proposition A5. — *Let X be a T-variety, $T' \subset T$ a subtorus, and $i_{T'} : X^{T'} \to X$ the inclusion of the fixed point set. Then the map*

$$i_{T'}^* : H_T^*(X) \to H_T^*(X^{T'})$$

becomes an isomorphism after inverting finitely many characters of T which restrict non trivially to T'.

Proof. — Observe that the kernel and cokernel of $i_{T'}^*$ are modules over $H_T^*(X \setminus X^{T'})$. Thus, it is enough to prove that $H_T^*(X \setminus X^{T'})$ is killed by a product of characters which restrict non trivially to T'. In other words, we may assume that T' fixes no point of X.

Let $U \subset X$ and $\mathcal{O}$ be as in Proposition 4 above. Then $H_T^*(U)$ is a module over $H_T^*(\mathcal{O})$ and the latter is killed by all characters which restrict trivially to the isotropy group Γ of $\mathcal{O}$. Because T' fixes no point of $\mathcal{O}$, we can find a character χ which restricts trivially to Γ but not to T'. Now the kernel and cokernel of the map $H_T^*(X) \to H_T^*(U)$ are modules over $H_T^*(X \setminus U)$, and we conclude by Noetherian induction.

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