Existence of Green Function and Bounded Harmonic Functions on Galois Covers of Riemann Surfaces*

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Abstract

This article consists of two parts. In the first one we prove that a rank ≥ 3 abelian covering of a Riemann surface has Green function, a result which in the case of a compact base is due to S. Mori. In the second one we show that any surface which has Green function admits a finite cover carrying bounded non-constant harmonic functions.

I. Existence of Green function on abelian covers.- A Galois covering with an abelian Galois group is called an abelian cover, the rank of the covering is defined to be the rank of this abelian group. In 1953, Mori [Mo] showed that a rank ≥ 3 abelian covering of a compact Riemann surface is of class P_G, that is, it carries Green function. We prove the following generalization:

THEOREM 1. — *A rank* ≥ 3 *abelian covering of a Riemann surface is of class* P_G.

In what follows, the notation $X \in O_G$ means that the surface X is not of class P_G.

Proof of Theorem 1. The proof is based on the following special case of a theorem of Kusunoki-Mori, which can be viewed as a generalization of the maximum principle. For a detailed exposition of this theorem, we also refer to [SaNa], III.1.2G.

THEOREM 2. — *(Kusunoki-Mori, [KuMo]) Let* X *be a surface of class* O_G*, and let* Y *be a subsurface of* X *with an analytic boundary* ∂Y*. Then the double* $\mathbf{dbl}Y$ *of* Y *along* ∂Y *is also of class* O_G.

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COROLLARY 1. — *Let Y be a subsurface of X with an analytic boundary. If the double* **dbl**Y *of Y is of class P_G, then X is also of class P_G.*

Let now R be a Riemann surface and $p' : Z \to R$ be a rank≥ 3 abelian covering. Then there exists an intermediate abelian covering $p : X \to R$ of pure rank 3 (i.e. a $\mathbb{Z}^3$-covering), such that Z is a covering of X. Since the property of being a P_G-surface passes to coverings, it suffices to prove the theorem for the case of pure rank 3.

Set $G := p^\star \pi_1(X) \subset \pi_1(R)$. Then $\pi_1(R)/G = \mathbb{Z}^3$. Let $\gamma_1, \gamma_2, \gamma_3$ be three loops in R such that $[\gamma_1], [\gamma_2], [\gamma_3] \in \pi_1(R)$ generate $\mathbb{Z}^3$ in the quotient. Let R_0 be a compact subsurface of R with analytic boundary ∂R which contains $\gamma_1, \gamma_2, \gamma_3$. Then the surface $Y := p^{-1}(R_0)$ above R_0 is connected. Thus, the double **dbl**Y of the subsurface Y of X is a $\mathbb{Z}^3$-covering of the double **dbl**R_0 of R_0. The surface **dbl**R_0 being compact, Mori's theorem implies that **dbl**Y is of class P_G, and the Corollary 1 above implies that X is of class P_G. This completes the proof.

Some Corollaries of Theorem 1.

1. On commutator subgroups of Fuchsian groups. It has been shown in [McKSu], [LyMcK] that the maximal abelian covering of $\mathbb{C}\backslash\{0,1\}$ is of class P_G. Together with Theorem 1, this supplies the list of surfaces which do not have an abelian cover of class P_G: Namely, the sphere $S^2 = \mathbb{P}^1_{\mathbb{C}}$, the complex plane $\mathbb{C}$, the punctured plane $\mathbb{C}\backslash\{0\}$, the tori $\mathbb{T}$, and the punctured tori $\mathbb{T}\backslash\{q\}$. The only non-trivial case is that of a punctured torus, so we describe its maximal abelian cover. If $p : \mathbb{C} \to \mathbb{T}$ is the universal covering of $\mathbb{T}$, then $\mathbb{C}\backslash p^{-1}(q)$ is the maximal abelian cover of $\mathbb{T}\backslash\{q\}$, which is easily seen to be not of class P_G. Also, note that $\mathbb{T}\backslash\{q\}$ is the only surface in the above list which is covered by the unit disc Δ.

According to a theorem of Myrberg [My], if a Riemann surface X is covered by the unit disc Δ, and G is the corresponding Fuchsian group acting on Δ, then $X \in P_G$ if and only if G is of convergence type; that is,

$$\sum_{g\in G}(1-|g(z)|) < \infty$$

for one, and hence for all $z \in \Delta$ (see also [Ts], X.13). The following is an immediate corollary of Theorem 1 and the remarks above:

Corollary 2. — *Let a Riemann surface X be covered by the unit disc Δ, and let $G \subset \mathbf{Aut}(\Delta)$ be its covering group. Then*

(i) If $X \neq \mathbb{T}\backslash\{q\}$, then the commutator subgroup $[G,G]$ is of convergence type.

(ii) If H is a subgroup of G such that $[G,G] \subset H \subset G$, and the rank of the abelian group G/H is ≥ 3, then H is of convergence type.

2. Carathéodory hyperbolicity of metabelian covers. A Riemann surface X is called Carathéodory hyperbolic if bounded holomorphic functions separate the points of X. It is interesting to know when Carathéodory hyperbolic surfaces appear as "small" covers of Riemann surfaces. In [LiZa] it is shown that if a Riemann surface R has an abelian cover Y of class P_G, then Y has a Carathéodory hyperbolic, abelian cover X, such that X is a metabelian (i.e. two-step solvable) Galois covering of R. Hence, Theorem 1 implies the following corollary.

Corollary 3. — *If R is not one of the surfaces S^2, $\mathbb{C}$, $\mathbb{C}\backslash\{0\}$, $\mathbb{T}$, $\mathbb{T}\backslash\{q\}$, then it admits a metabelian, Carathéodory hyperbolic Galois covering $X \to R$.*

The converse of this corollary is also true, for it is obvious that the surfaces S^2, $\mathbb{C}$, $\mathbb{C}\backslash\{0\}$, $\mathbb{T}$ do not possess *any* cover carrying bounded analytic functions. For the surface $\mathbb{T}\backslash\{q\}$, recall that its maximal abelian cover is of class O_G. One of the results in [LySu] asserts that an abelian cover of an O_G-surface is of class O_{HB}, that is it has no bounded non-constant harmonic functions. This shows that a metabelian cover of $\mathbb{T}\backslash\{q\}$ is of the class O_{HB}, so in particular it has no bounded analytic functions, and it cannot be Carathéodory hyperbolic.

On the other hand, applying Corollary 3 to the maximal abelian cover X of $\mathbb{T}\backslash\{q\}$, we see that X has a metabelian Carathéodory hyperbolic Galois cover Y, which is a 3-step solvable cover of $\mathbb{T}\backslash\{q\}$. This shows that the case of punctured tori is not very exceptional; any surface covered by the unit disc admits a solvable Carathéodory hyperbolic cover.

II. Existence of bounded harmonic functions on finite covers of P_G-surfaces.- Let R be a Riemann surface of class O_G, and let X be a rank ≥ 3 abelian cover of R. Then, as we have noticed above, $X \in O_{HB}$ by a result in [LySu]. On the other hand, Theorem 1 implies that $X \in P_G$. So there are many Riemann surfaces $X \in P_G \cap O_{HB}$. Let us denote by Z the maximal abelian (hence, $\mathbb{Z}^\infty$-)

cover of X. It has been observed in [LySu] that Z is of class P_{HB}, that is, it does carry a non-constant bounded harmonic function. The theorem below states that X has a *finite* Galois cover Y which is of class P_{HB}. However, by an argument due to V. Lin, Y does not carry any non-constant bounded analytic function (see Remark 2).

THEOREM 3. — *Any Riemann surface X of class P_G has a finite cover Y which is of class P_{HB} and, moreover, Y even carries a Diriclet finite bounded harmonic function.*

Proof of Theorem 3. A P_G-surface of genus $g = 0$ already carries a Dirichlet finite non-constant bounded harmonic function (see [SaNa], III.5G). Hence, setting $X = Y$ we are done. If $g \neq 0$ then there exists a closed analytic curve γ on X which does not divide the surface in two parts. We denote two sides of γ by γ^+, γ^-, and we cut X along γ. Let $\tilde{X}$ be a second copy of X, $\tilde{\gamma}$ be the copy of γ in $\tilde{X}$ with corresponding sides $\tilde{\gamma}^+$, $\tilde{\gamma}^-$. Gluing X to $\tilde{X}$ via identifications $\gamma^+ \leftrightarrow \tilde{\gamma}^-$, $\gamma^- \leftrightarrow \tilde{\gamma}^+$, we obtain a $\mathbb{Z}_2$-covering Y of X.

Claim. The surface Y is of class P_{HB}.

An immediate way to see this is to observe that the harmonic part of the Royden's boundary of Y consists of two points, which implies the existence of a Diriclet finite non-constant bounded harmonic function on Y (see [Ro], also [SaNa], III.3F). However, we shall give a more elementary proof based on the following theorem:

THEOREM 4. — *(Bader-Parreau [BaPa], Nevanlinna [Nev]) Let Y_1, Y_2 be two disjoint subsurfaces of Y with analytic boundaries ∂Y_1, ∂Y_2. Assume that there exists two non-constant bounded harmonic functions u_1 on Y_1 and u_2 on Y_2 such that $u_1 \equiv 0$ on ∂Y_1 and $u_2 \equiv 0$ on ∂Y_2. Then Y carries a non-constant bounded harmonic function. Moreover, if u_1, u_2 have finite Dirichlet integrals, then Y carries a non-constant bounded harmonic function with finite Dirichlet integral.*

Proof of the claim. The surface X being of class P_G, the harmonic measure ω of the ideal boundary of X with respect to γ is non-vanishing, that is, ω is a bounded non-constant harmonic function with finite Dirichlet integral on $X \backslash \gamma$ which vanishes on γ (See [Ts] X.1). Let $\tilde{\omega}$ be the same function on $\tilde{X} \backslash \tilde{\gamma}$. Setting $Y_1 := X \backslash \gamma$, $Y_2 := \tilde{X} \backslash \tilde{\gamma}$, $u_1 := \omega$, $u_2 := \tilde{\omega}$, the hypotheses of Theorem 2 are satisfied, hence Y carries a bounded non-constant harmonic function with finite Dirichlet integral.

Remarks.

1. A sufficient condition for a Galois cover Y of a P_G-surface X to be of class P_{HB} is the compactness of the boundary of a fundamental region of the corresponding group action on Y; this can be proved in the same way as in the proof above.

It should be noted that a finite (even infinite) cover of a P_G-surface can be of class O_{HB}. For example, if X is a rank-3 abelian cover of a compact surface K of genus $g = 2$, and Y is the maximal abelian (i.e. rank-4) cover of K, then as we have already remarked, X, Y are of class $P_G \cap O_{HB}$, but Y is a $\mathbb{Z}$-cover of X.

2. In contrast with the possible existence of bounded non-constant harmonic functions, a finite cover Y of an O_{HB} surface X cannot carry non-constant bounded analytic functions. The proof goes as follows: Let n be the degree of a finite covering $p : Y \to X$, and let f be a bounded analytic function on Y. Let $x \in X$ and $p^{-1}(x) = \{y_1, ..., y_n\}$. For $j = 1, ..., n$ define $a_j(x) := \sigma_j(f(y_1), ..., f(y_n))$, where σ_j is the elementary symmetric polynomial of degree j in n variables. Then each a_j is a bounded analytic function on X. The real parts of the a_j's, being bounded harmonic functions, are constant, hence $a_j = const$, which implies that $f = const$.

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