Affine modifications and affine hypersurfaces with a very transitive automorphism group

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Abstract

We study a kind of modification of an affine domain which produces another affine domain. First appeared in passing in the basic paper of O. Zariski [Zar], it was further considered by E.D. Davis [Da]. In [Ka 1] its geometric counterpart was applied to construct contractible smooth affine varieties non-isomorphic to Euclidean spaces. Here we provide certain conditions (more general then those in [Ka 1]) which guarantee preservation of the topology under a modification.

As an application, we show that the group of biregular automorphisms of the affine hypersurface $X \subset \mathbf{C}^{k+2}$ given by the equation $uv = p(x_1, \ldots, x_k)$ where $p \in \mathbf{C}[x_1, \ldots, x_k]$, acts $m-$ transitively on the smooth part reg X of X for any $m \in \mathbf{N}$. We present examples of such hypersurfaces diffeomorphic to Euclidean spaces.

Contents

*Partially supported by the NSA grant MDA904-96-01-0012.
Classification math. : 32H15, 32H99, 14E99.
Mots-clés : blow up, modification, Rees algebra, affine domain, automorphism group, $\mathbf{C}_+$ -action, affine hypersurface, contractible variety, exotic $\mathbf{C}^n$, Zariski cancellation problem.

Introduction

It is well known (and elementary) that for $n > 1$ the automorphism group Aut $\mathbf{C}^n$ of the affine space $\mathbf{C}^n$ (or, which is the same, of the polynomial ring in n variables $\mathbf{C}^{[n]} := \mathbf{C}[x_1, \ldots, x_n]$) acts $m-$ transitively on $\mathbf{C}^n$ for any $m \in \mathbf{N}$. That is, the diagonal action of the group Aut $\mathbf{C}^n$ on the $m-$ th symmetric power $S^m\mathbf{C}^n$, and even on the $m-$ th Cartesian power $(\mathbf{C}^n)^m$, is transitive outside of the diagonals for any[1] $m \in \mathbf{N}$. Clearly, every Zariski open subset of the form $\mathbf{C}^n \setminus K$ where $K \subset \mathbf{C}^n$ is a finite set of points, possesses this property.

Let V be a compact complex space. By a generalized Bochner-Montgomery Theorem [Akh, (2.3)], the automorphism group Aut V is a complex Lie group. Therefore, it cannot be $m-$ transitive on V for $m >$ dim Aut V. Thus, the question arises:

For which complex manifolds or, at least, for which quasi-projective varieties X the group Aut X is $m-$ transitive on X for any $m \in \mathbf{N}$?

Or, more restrictively (to exclude the above examples of type $\mathbf{C}^n \setminus K$)

Which affine algebraic varieties possess this property?

In sections 5 and 6 we describe a class of affine hypersurfaces non-homeomorphic, in general, to the affine spaces, with very transitive automorphism groups (see The Transitivity Theorem in sect. 6).

One more remark is in order. One might specify the above problem by asking whether, for m given, there exists an $m-$ transitive algebraic group action on X. For m sufficiently large with respect to the dimension of X, the class of such varieties X seems to be rather poor. Actually, already the affine plane $\mathbf{C}^2$ does not admit an $m-$ transitive algebraic group action for $m \geq 3$. Indeed, by the Jung-van der Kulk Theorem [Ju, vdK], the group Aut $\mathbf{C}^2$ can be represented as the amalgamated product of the affine group Aff $\mathbf{C}^2$ and the Jonquière subgroup $J(\mathbf{C}^2) \subset$ Aut $\mathbf{C}^2$ of the triangular transformations $(x, y) \longmapsto (x, y + p(x))$ where $p \in \mathbf{C}[x]$. By the theorem of Serre [Se], any subgroup of finite length $G \subset$ Aut $\mathbf{C}^2$ is conjugate either with a subgroup of Aff $\mathbf{C}^2$ or with a subgroup of $J(\mathbf{C}^2)$. Since the polynomial x is an invariant of the Jonquière subgroup $J(\mathbf{C}^2)$, in the latter case G possesses a non-constant invariant function, and therefore, it is not even transitive on $\mathbf{C}^2$. In the former case, G is at most $2-$ transitive; indeed, the collinearity of a triple of points is preserved by the action of the affine group Aff $\mathbf{C}^2$. It remains to observe following D. Wright [Wr 1] that any algebraic subgroup $G \subset$ Aut $\mathbf{C}^2$ is of finite length.

The proof of Theorem 6.1 on $\infty-$ transitivity of the automorphism group of a smooth hypersurface $X \subset \mathbf{C}^{k+2}$, $k \geq 2$, given by the equation $uv - p(\overline{x}) = 0$ goes as follows. This hypersurface X can be naturally presented in two different ways as an affine modification of $\mathbf{C}^{k+1}$ (see below). Due to Corollary 2.2, one can lift to X those automorphisms of the affine space $\mathbf{C}^{k+1}$ which preserve the locus of at least one of these modifications. In

[1] See [RoRu] for an $\infty-$ transitivity property of the group of analytic automorphisms of $\mathbf{C}^n$.

such a way we obtain two subgroups $\widehat{G}_1$ and $\widehat{G}_2$ of the automorphism group $\mathrm{Aut}\, X$. It turns out that the subgroup $\widehat{G}$ of $\mathrm{Aut}\, X$ generated by $\widehat{G}_1$ and $\widehat{G}_2$ acts $m-$ transitively on X for any $m \in \mathbf{N}$.

While a general hypersurface $X \subset \mathbf{C}^{k+2}$ as above has quite a rich topology, in sections 5 and 7 we give a series of examples of hypersurfaces of this type which are diffeomorphic to the affine space $\mathbf{C}^{k+1}$, $k \geq 3$. Presumably, they are not, in general, isomorphic to $\mathbf{C}^{k+1}$, and so, they should present new exotic algebraic structures on $\mathbf{C}^{k+1}$ (see e.g. [Za 2]). As well, this would show that the Miyanishi's characterization of the affine 3-space $\mathbf{A}^3$ [Miy] cannot be applied to $\mathbf{A}^n$ for $n \geq 4$ (see Remark 5.3). But if one of them were isomorphic to $\mathbf{C}^{k+1}$ this would answer in negative alternatively, either to the Zariski Cancellation Problem or to the Abhyankar-Sathaye Embedding Problem (see below).

Sections 1-3 are devoted to a kind of modification which acts on the affine rings producing new affine rings; we call it *affine modification*. It appeared in passing in the classical Zariski paper [Zar] (see also [Hiro, III.2]), but apparently, the first proper study of this transform was done in [Da]. A geometric counterpart of the affine modification occured to be useful in constructing exotic algebraic structures on the affine spaces, as it was done in [Ka 1] (see also [Za 2]).

In Corollary 2.2 we lift automorphisms to an affine modification. In section 3 we provide certain conditions (more general then those in [Ka 1]) which guarantee preservation of the topology under a modification.

Recall the Abhyankar-Sathaye Embedding Problem:

Is every closed embedding of $\mathbf{C}^k$ *into* $\mathbf{C}^n$ *rectifiable, i.e. equivalent to a linear one up to the actions of the automorphisms groups Aut* $\mathbf{C}^k$ *resp. Aut* $\mathbf{C}^n$*?*

In section 4 we give a generalization of the Sathaye-Wright Theorem [Sat, Wr 2] which guarantees rectifiability of the special embeddings $\mathbf{C}^2 \hookrightarrow \mathbf{C}^3$ given by the equations of the form $f(x, y)z^n + g(x, y) = 0$. Notice that for $n = 1$ such a surface is an affine modification of the affine plane $\mathbf{C}^2$ along the divisor $D_f = (f)$ with center at the ideal $I = (f, g) \subset \mathbf{C}[x, y]$. More generally, in Theorem 4.2 we show that any smooth acyclic surface $X \subset \mathbf{C}^3$ given by the equation $f(x, y)z^n + g(x, y) = 0$ is isomorphic to $\mathbf{C}^2$ (and can be rectified). This is no longer true for $\mathbf{Q}-$ acyclic surfaces; see Example 4.1.

1. Affine modifications

Consider a triple (A, I, f) consisting of a commutative ring A with unity, an ideal I of A, and an element f of I which is not a zero divisor. We call it a *Noetherian triple* if A is Noetherian, and an *affine triple* (over $\mathbf{C}$) if A is an affine domain $/\, \mathbf{C}$, that is, a finitely generated commutative $\mathbf{C}-$ algebra which is a domain. With t being a new symbol, denote by

$$A[It] = A \oplus \bigoplus_{n=1}^{\infty} (It)^n \simeq A \oplus I \oplus I^2 \oplus \ldots = Bl_I(A) \simeq \mathrm{Sym}_A I$$

the blowup algebra, or the Rees algebra [Ei, §5.2], [Va].

Definition 1.1. By the *affine modification* $\Sigma_{I,f}(A)$ *of the ring* A *along* (f) *with center* I (or, shortly, with the locus (I, f)) we mean the quotient of the blowup algebra $A[It]$ by the principal ideal generated by the element $1 - ft \in A[It]$:

$$\Sigma_{I,f}(A) = A[It]/(1-ft)\,.$$

When f and I are fixed, without abuse of notation we denote $A' = \Sigma_{I,f}(A)$. Clearly, if (A, I, f) is a Noetherian triple, then A' is again a commutative Noetherian ring with unity.

Our purposes in this paper are mainly of geometric nature[2]. Thereby, we adopt the following

Conventions. Hereafter, (A, I, f) is assumed to be an affine triple / $\mathbf{C}$. Besides, we fix two systems of generators $a_1, \ldots, a_r$ of the algebra A resp. $b_0 = f, b_1, \ldots, b_s$ of the ideal I. Denoting $\mathbf{C}^{[r]}$ the polynomial ring in r variables, consider the surjective homomorphisms

$$\varphi : \mathbf{C}^{[r]} := \mathbf{C}[x_1, \ldots, x_r] \to A, \ \ x_i \longmapsto a_i, \ \ i = 1, \ldots, r,$$

resp.

$$\varphi_I : \mathbf{C}^{[r+s+1]} = \mathbf{C}[x_1, \ldots, x_r, y_0, \ldots, y_s] \to A[It]\,,$$
$$x_i \longmapsto a_i, \ \ y_j \longmapsto b_j t, \ \ i = 1, \ldots, r, \ \ j = 0, \ldots, s.$$

Let X resp. Y be the image of the associated closed embedding $\widehat{\varphi}$: spec $A \hookrightarrow \mathbf{C}^r$ resp. $\widehat{\varphi}_I$: spec $A[It] \hookrightarrow \mathbf{C}^{r+s+1}$. Then X and Y are reduced irreducible affine varieties [3] (indeed, $A[It] \subset A[t]$ is an affine domain / $\mathbf{C}$). The principal ideal $(1-ft) \subset A[It]$ being prime[4], the $\mathbf{C}-$ algebra A' is an affine domain / $\mathbf{C}$, too. Thus, spec A' is a reduced irreducible affine variety isomorphic to the hypersurface $V(1-ft) \subset$ spec $A[It] \simeq Y$.

Denote by $\rho : A[It] \to A' = A[It]/(1-ft)$ the canonical surjection. Since $\varphi_I(1-ft) = 1 - y_0$, the image X' of the closed embedding $\widehat{\varphi_I \circ \rho}$: spec $A' \hookrightarrow \mathbf{C}^{r+s+1}$ coincides with the hyperplane section $X' = H' \cdot Y \simeq V(1-ft)$ of Y defined by the affine hyperplane $H' := \{y_0 = 1\} \simeq \mathbf{C}^{r+s}$.

Definition 1.2. We call $X' \subset \mathbf{C}^{r+s}$ as above the *affine modification of the affine variety* $X \subset \mathbf{C}^r$ *with the locus* (I, f) (or in other words, *along the divisor* D_f *with center* I); we denote $X' = \Sigma_{I,f}(X)$.

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[2]By this reason, we do not consider here possible generalizations, for instance, such as replacing the filtration of A by the powers $\{I^n\}_{n\in\mathbf{N}}$ of the ideal I (resp. the multiplicative system $\{f^n\}_{n\in\mathbf{N}} \subset I$) by a more general one.

[3]We do not suppose the divisor D_f resp. the affine subscheme spec A/I in X being reduced or irreducible; that is, the ideals (f) and I are not assumed being radical or primary.

[4]Indeed, the principal ideal generated by the regular function $1-ft$ in the algebra $A[t] = \mathbf{C}[X \times \mathbf{C}]$ is prime. That is, $p(t)q(t) = (1-ft)r(t)$ where $p, q, r \in A[t]$, implies that $(1-ft)$ divides $p(t)$ or $(1-ft)$ divides $q(t)$ in $A[t]$. But if, say, $(1-ft)$ divides $p(t)$ in $A[t]$, and $p \in A[It]$, then, as it is easily seen, $(1-ft)$ divides $p(t)$ in $A[It]$.

Thus, this definition takes into account the distinguished systems of generators $a_1, \ldots, a_r$ of the algebra A resp. $b_0 = f,\, b_1, \ldots, b_s$ of the ideal I, that is, the closed embeddings $\mathrm{spec}\, A \simeq X \hookrightarrow \mathbf{C}^r$ and $\mathrm{spec}\, A' \simeq X' \hookrightarrow \mathbf{C}^{r+s}$.

Remarks

1.1. For a Noetherian triple $(A,\, I,\, f)$ and a fixed system of generators $b_0 = f,\, b_1, \ldots, b_s$ of the ideal I one may consider the closed embedding $\widehat{\beta} : \mathrm{spec} A' \hookrightarrow \mathrm{spec} A \times \mathbf{C}^s$ associated with the surjective homomorphism of the polynomial algebra $A^{[s]} := A[y_1, \ldots, y_s]$ over A :

$$\beta : A^{[s]} \to A', \quad \beta(y_i) = \rho(b_i t), \quad i = 1, \ldots, s\,.$$

For the affine triple $(A,\, I,\, f)$ the embedding $\widehat{\varphi}_I\,$: $\mathrm{spec} A' \hookrightarrow \mathbf{C}^{r+s}$ as above is the composition

$$\mathrm{spec} A' \overset{\widehat{\beta}}{\hookrightarrow} : \mathrm{spec} A \times \mathbf{C}^s \overset{\widehat{\varphi}\times \mathrm{id}}{\hookrightarrow} \mathbf{C}^r \times \mathbf{C}^s\,.$$

1.2. Set $\overline{x} = (x_1, \ldots, x_r)$ and $\overline{y} = (y_0,\, y_1, \ldots, y_s) = (y_0,\, \overline{y}')$. The affine cone $Y \subset \mathbf{C}^{r+s+1}$ being homogeneous in $\overline{y}$ its ideal $I(Y)$ admits a system of generators

$$h_1(\overline{x},\, \overline{y}), \ldots, h_N(\overline{x},\, \overline{y}) \in \mathbf{C}^{[r+s+1]}$$

which are polynomials homogeneous in $\overline{y}$. Then the ideal $I(X') \subset \mathbf{C}^{[r+s]}$ is generated by the polynomials $\widetilde{h}_1(\overline{x},\, \overline{y}'), \ldots, \widetilde{h}_N(\overline{x},\, \overline{y}')$, where

$$\widetilde{h}_i(x_1, \ldots, x_r,\, y_1, \ldots, y_s) := h_i(x_1, \ldots, x_r, 1,\, y_1, \ldots, y_s).$$

1.3. Denote $Y^* = Y \setminus V_Y$ where $Y \subset X \times \mathbf{C}^{s+1} \subset \mathbf{C}^{r+s+1}$ is the affine cone introduced above with the vertex set $V_Y := X \times \{\overline{0}\}$. Then the blow up

$$Z = Bl_I X = \mathrm{Proj}_A Bl_I A = \mathrm{Proj}_A A[It] \subset X \times \mathbf{P}^s \subset \mathbf{C}^r \times \mathbf{P}^s$$

of X with center I is the canonical projection of Y^*, that is, $Z = Y^*/\mathbf{C}^*$ [Ei, §5.2]. This projection restricted to the hyperplane section $X' = H' \cap Y$ yields an isomorphism $X' \overset{\simeq}{\longrightarrow} Z \setminus H_0$ where $H_0 \subset \mathbf{P}^s$ is the coordinate hyperplane $y_0 = 0$.

1.4. In the sequel we distinguish between two notions of proper transform of the divisor D_f; for that we use two terms, proper transform and strict transform, respectively. Namely, by the *proper transform* D_f^{pr} of D_f in Z we mean the set of homogeneous prime ideals $p \in Z = \mathrm{Proj}\, A[It]$ such that $ft \in p$. Thus, $\mathrm{supp} D_f^{pr} = \{y_0 = ft = 0\} = H_0 \cap Z$, and so,[5] $X' \simeq Z \setminus H_0 = Z \setminus D_f^{pr}$. Whereas the *strict transform* D_f' of D_f in Z we understand as usually, i.e. as the closure in Z of the preimage $\sigma_I^{-1}(D \setminus V(I))$. In general, these two transforms are different; see Examples 1.3 and 3.4 below.

We denote by E the exceptional divisor[6] $\sigma_I^{-1}(C) \subset Z$ where $C := V(I) \subset X$ and $\sigma_I : Z \to X$ is the blowup morphism, and by E' its affine part $E' = E \setminus H_0 = E \setminus D_f^{pr} \subset X'$.

[5] By abuse of notation, we write $Z \setminus D_f^{pr}$ instead of $Z \setminus \mathrm{supp} D_f^{pr}$.

[6] By abuse of language, usually by the exceptional divisor we mean its support.

1.5. In the case when $D := D_f$ is a reduced effective divisor and I is the (radical) ideal of a closed reduced subvariety $C = V(I) \subset \mathrm{reg} D$ in X, the affine modification $X' = \Sigma_{I,\,f}(X) =: \Sigma_{C,\,D}(X)$ coincides with the one considered in [Ka 1].

Notation. Denote Frac$A$ the field of fractions of a domain A. For an affine triple $(A,\, I,\, f)$, $A[I/f]$ denotes the subalgebra of the field Frac$A$ generated over A by the elements g/f where g runs over the ideal I. Under the above conventions we have

$$A[I/f] = A[b_1/f, \dots, b_s/f] = \{a/f^k \in \mathrm{Frac}A \,|\, a \in I^k,\ k = 0,\, 1, \dots\}\,.$$

For an affine variety X we denote by $\mathbf{C}[X]$ the algebra of regular functions on X, and by $\mathbf{C}(X)$ the rational function field Frac $\mathbf{C}[X]$ of X.

Proposition 1.1: The first properties of the affine modification.

(a) In the notation as above, the affine modification $X' = \Sigma_{I,\,f}(X)$ is isomorphic to the complement $Z \setminus D_f^{pr}$.

(b) There is a canonical isomorphism $\alpha\, :\, A' \xrightarrow{\simeq} A[I/f] = A[b_1/f, \dots, b_s/f]$ which sends $\rho(A) \simeq A$ isomorphically onto A and $\rho(b_i t)$ into b_i/f, $i = 1, \dots, s$. Thus, $X' = \mathrm{spec}\ A' \simeq \mathrm{spec}\ A[I/f]$.

(c) (E.D. Davis [Da]) *Consider the surjective homomorphism*

$$\beta\, :\, A^{[s]} = A[y_1, \dots, y_s] \to A[I/f] = A[b_1/f, \dots, b_s/f] \simeq A',\ \ \beta(y_i) = b_i/f,\ \ i = 1, \dots, s\,.$$

Denote by I' the ideal of the polynomial algebra $A^{[s]}$ generated by the elements $L_1, \dots, L_s \in \ker\beta$ where $L_i = fy_i - b_i$. Then $\ker\beta = I'$ (that is, the subvariety $X' \subset X \times \mathbf{C}^s$ is defined by the equations $fy_i - b_i = 0$, $i = 1, \dots, s$) iff I' is a prime ideal, i.e. $\mathrm{spec}A^{[s]}/I'$ is a reduced irreducible subvariety in $X \times \mathbf{C}^s$. The latter is true, for instance, if the system of generators $b_0 = f,\, b_1, \dots, b_s$ of the ideal I is regular[7].

(d) Furthermore, if $\ker\beta = I'$ then we have an isomorphism $E' \simeq C \times \mathbf{C}^s$ where $E' \subset X'$ is the exceptional divisor and $C = V(I)$.

Proof. For the proof of (a) see Remark 1.4 above, and for that of (c) see [Da, Prop. 2], [Ful, (A.6.1)] or also [Mik]. The statement (d) immediately follows from (c). Indeed, by (c), we have

$$E' = \sigma_I^{-1}(C) = \{(x,\, \bar{y}) \in X \times \mathbf{C}^s \,|\, b_i(x) = 0,\ i = 0, \dots, s\} = C \times \mathbf{C}^s \subset X \times \mathbf{C}^s\,.$$

(b) On $X' = H' \cap Y$ we have $t = 1/f$, and $y_i = b_i t$, $i = 1, \dots, s$. Thus, $A' = \mathbf{C}[X'] = A[1,\, y_1, \dots, y_s] \,|\, X' = A[b_1/f, \dots, b_s/f]$, as stated. □

[7] I.e. for each $i = 1, \dots, s$ the image of b_i is not a zero divisor in $A/(b_0, \dots, b_{i-1})$.

Remarks

1.6. Recall that the blowup morphism $\sigma_I : Z \to X$ coincides with the restriction to Z of the first projection $\mathrm{pr}_1 : X \times \mathbf{P}^s \to X$. The composition

$$X' \hookrightarrow Z \subset X \times \mathbf{P}^s \stackrel{\mathrm{pr}_1}{\longrightarrow} X$$

associated with the composition of homomorphisms $A \stackrel{i}{\hookrightarrow} A[It] \stackrel{\rho}{\longrightarrow} A'$ yields a birational morphism $\sigma_I : X' \to X$. It coincides with the restriction to X' of the first projection $\mathrm{pr}_1 : \mathbf{C}^r \times \mathbf{C}^s \to \mathbf{C}^r$. The induced isomorphism of the rational function fields $\mathbf{C}(X) = \mathrm{Frac}\ A \stackrel{\simeq}{\longrightarrow} \mathrm{Frac}\ A' = \mathbf{C}(X')$ canonically identifies the algebra A' with the subalgebra $A[I/f] \subset \mathbf{C}(X)$ (cf. Proposition 1.1(b)).

The isomorphism

$$\sigma_I^{-1} : X \setminus D_f \stackrel{\simeq}{\longrightarrow} X' \setminus E' \hookrightarrow X' \hookrightarrow \mathbf{C}^r \times \mathbf{C}^s$$

is given by the formulas $x_i = a_i,\ y_j = b_j/f,\ i = 1, \ldots, r,\ j = 1, \ldots, s$.

1.7. Actually, the prime ideal $\ker \beta$ as in Proposition 1.1(c) which defines the affine subvariety $X' \subset X \times \mathbf{C}^s$ coincides with the radical $\mathrm{rad} I'$, and $I' \supset f^r \ker \beta$ for a sufficiently large r [Da].

Example 1.1. If $I = A$, that is, the height $\mathrm{ht}\ I = 0$, then we have $A' \simeq A[1/f]$, $Z \simeq X$ and $X' \simeq X \setminus D_f \simeq Z \setminus D_f^{pr}$.

Example 1.2. If $I = (f)$ is a principal ideal (and so, $\mathrm{ht}\ I = 1$), we obtain $A[I/f] = A$, whence, $X' = X \simeq Z = X \times \mathbf{P}^0$ whereas $\mathrm{supp} D_f^{pr} = \emptyset$. Thus, the equality $X' = Z \setminus D_f^{pr}$ still holds.

Example 1.3. Furthermore, even if $\mathrm{ht}\ I \geq 2$ it may happen that $\mathrm{supp}\ D_f^{pr} = \sigma_I^{-1}$ ($\mathrm{supp}\ D_f$), and so, $X' \simeq X \setminus D_f$, i.e. that the hyperplane section $H_0 \cap Z$ contains the exceptional divisor E of the blow up σ_I. For instance, take $A = \mathbf{C}^{[2]} = \mathbf{C}[x,\, y]$, that is, $X = \mathbf{C}^2$, $I = (x,\, y)$ and $f = x^2 \in I$. Then we have $Z = \{((x,\, y),\, (u : v)) \in \mathbf{C}^2 \times \mathbf{P}^1 \mid xv = yu\}$, and the curve $\mathrm{supp} D_f^{pr}$ given in Z by the equation $ux = 0$ consists of two lines, the first one $l' := \{u = 0\}$ being the strict transform in Z of the affine line $l := \mathrm{supp} D_f = \{x = 0\}$ and the second one $\{x = y = 0\}$ being the exceptional divisor $E \subset Z$. Thus, $X' = Z \setminus (l' \cup E) \simeq X \setminus l \simeq \mathbf{C}^* \times \mathbf{C} \subset \mathbf{C}^2 \simeq X$. While for the affine triple $(A,\, I,\, x)$ we obtain $X' = \Sigma_{I,x}(X) \simeq \mathbf{C}^2$.

It is well known (see e.g. [Ha, 7.17]) that every birational projective morphism of quasiprojective varieties $Y \to X$ is a blow up of X with center at a subsheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Similarly, we have the following theorem.

Theorem 1.1: Birational morphisms as affine modifications.

Any birational morphism $q : Y \to X$ of affine varieties is an affine modification. More precisely, there are an ideal $I \subset A = \mathbf{C}[X]$, an element $f \in I$ and an isomorphism

$\alpha : Y \xrightarrow{\simeq} X' := \Sigma_{I,f}(X)$ *such that* $q = \sigma_I \circ \alpha$ *where* $\sigma_I : X' \to X$ *is the blowup morphism with center* I.

Proof. Denote $A_1 = (q^*)^{-1}(\mathbf{C}[Y]) \subset \mathbf{C}(X) = \mathrm{Frac}\, A$, where $(q^*)^{-1} : \mathbf{C}(Y) \to \mathbf{C}(X)$ is the induced isomorphism of the rational function fields. Each function $a \in A = \mathbf{C}[X]$ being lifted by q^* to a function in $\mathbf{C}[Y]$ comes back under the inverse homomorphism $(q^*)^{-1}$. Therefore, $A \subset A_1 \subset \mathrm{Frac}\, A$. It is enough to show that $A_1 = A[I/f]$ for some ideal I of A and some $f \in I$.

The affine domain A_1 being finitely generated we have $A_1 = A[a_1/f_1, \ldots, a_k/f_k]$ with some $a_i, f_i \in A$. Put $f = f_1 \cdot \ldots \cdot f_k \in A$, and let $a_i/f_i = b_i/f$. Set $I = (f, b_1, \ldots, b_k) \subset A$. Then we have $A_1 = A[a_1/f_1, \ldots, a_k/f_k] = A[b_1/f, \ldots, b_k/f] = A[I/f]$, as desired. $\square$

Next we give several examples of affine modifications.

Example 1.4: Modification of an affine space along a hyperplane with center at a point. Let $A = \mathbf{C}[x_1, \ldots, x_n, y]$, $f = y$, and let $I = (x_1, \ldots, x_n, y)$ be the maximal ideal corresponding to the origin of $X = \mathbf{C}^{n+1}$. Then $A' \simeq \mathbf{C}[x_1 y^{-1}, \ldots, x_n y^{-1}, y] \subset \mathbf{C}(x_1, \ldots, x_n, y)$. By Proposition 1.1(c), $X' \simeq \mathbf{C}^{n+1}$ is given in $\mathbf{C}^{2n+1}$ by the equations $x_i = y y_i$, $i = 1, \ldots, n$, and the exceptional divisor $E' = E \setminus D_f^{pr} \simeq \mathbf{C}^n$ is given in X' as $y = 0$. In the coordinates $(y_1, \ldots, y_n, y)$ of $X' \simeq \mathbf{C}^{n+1}$ the blowup morphism $\sigma_I : X' \to X$ is given as $\sigma_I(y_1, \ldots, y_n, y) = (y y_1, \ldots, y y_n, y)$.

A modification of $\mathbf{C}^{n+1}$ along a coordinate hyperplane with center at a coordinate subspace can be described in the same way.

Example 1.5: A singular modification of the affine plane. Set $A = \mathbf{C}[x, y]$ (that is, $X = \mathbf{C}^2$), $f = x$, $I = (x, y^2)$. The system of generators $b_0 := x$, $b_1 = y^2$ of the ideal I is regular. Hence, by Proposition 1.1(c), X' is the affine surface $xz = y^2$ in $\mathbf{C}^3$. It is isomorphic to the quotient of the affine plane $\mathbf{C}^2$ by the involution $(x, z) \longmapsto (-x, -z)$.

Example 1.6: Modification of an affine space along a divisor with center at a codimension two complete intersection. Set $A = \mathbf{C}^{[r]}$, that is, $X = \mathbf{C}^r$, and $I = (f, g)$, where $f, g \in A$ are non-constant polynomials without common factor. Then f, g form a regular system of generators of the ideal I. Thus, by Proposition 1.1(c), the affine modification $X' = \Sigma_{I,f}(X)$ is the hypersurface in $\mathbf{C}^{r+1}$ with the equation $f(\overline{x})y - g(\overline{x}) = 0$ where $\overline{x} = (x_1, \ldots, x_r)$. The blowup morphism $\sigma_I : X' \to X$ is the restriction to X of the projection $\mathbf{C}^{r+1} \to \mathbf{C}^r$, $(\overline{x}, y) \to \overline{x}$. The exceptional divisor $E' \subset X'$ is given in $\mathbf{C}^{r+1}$ by the equations $f(\overline{x}) = g(\overline{x}) = 0$; so, $E' \simeq V(I) \times \mathbf{C}$.

Example 1.7: The Russell cubic threefold. In particular, set $A = \mathbf{C}[x, z, t]$ (that is, $X = \mathbf{C}^3$), $f = -x^2$ and $I = (f, g)$ where $g = x + z^2 + t^3$. Then the affine modification $X' = \Sigma_{I,f}(X)$ is the smooth 3-fold $x + x^2 y + z^2 + t^3 = 0$ in $\mathbf{C}^4$. We call it the *Russell cubic* (see [Ru 1]). It birationally dominates the affine space $\mathbf{C}^3$ via the blowup morphism $\sigma_I : X' \to X \simeq \mathbf{C}^3$, $\sigma_I : (x, y, z, t) \longmapsto (x, z, t)$.

In turn, the Russell cubic threefold $X' \subset \mathbf{C}^4$ is birationally dominated by[8] $\mathbf{C}^3$. Indeed, for any $x \neq 0$, y is expressed in terms of z and t; whence, the part $\{x \neq 0\}$ of the threefold X' is isomorphic to $\mathbf{C}^2 \times \mathbf{C}^*$. The 'book-surface' $B := \{x = 0\} \subset X'$ is the product $\mathbf{C} \times \Gamma_{2,3}$ where $\Gamma_{2,3} := \{z^2 + t^3 = 0\} \subset \mathbf{C}^2$. Fix a smooth point $\rho \in \Gamma_{2,3}$, and perform the affine modification $\sigma' : X'' \longrightarrow X'$ of X' along B with the center $C := \mathbf{C} \times \{\rho\}$. In this way we replace B by a smooth surface $E' \simeq \mathbf{C}^2$ and replace the function x by a function $h : X'' \longrightarrow \mathbf{C}$ such that all the fibers of h are smooth reduced surfaces isomorphic to $\mathbf{C}^2$. Using an explicit presentation of X'' it can be checked that $X'' \simeq \mathbf{C}^3$ (see Example 7.3 below), and so, $\sigma' : \mathbf{C}^3 \simeq X'' \longrightarrow X'$ is a birational (whence, dominant) morphism.

It is known that the Russell cubic is diffeomorphic to $\mathbf{C}^3$ (see e.g. [Ru 1, Ka 1, Za 2] and Example 3.2 below). However, by a theorem of Makar-Limanov [ML] (see also [De, KaML 1, Za 2]), it is not isomorphic to $\mathbf{C}^3$. A smooth affine variety which is diffeomorphic but non-isomorphic to $\mathbf{C}^n$ is called an *exotic* $\mathbf{C}^n$. Thus, the Russell cubic X' is an exotic $\mathbf{C}^3$ *of sandwich type*, that is, there are birational morphisms $\mathbf{C}^3 \to X' \to \mathbf{C}^3$.

An affine modification may possess the following decomposition.

Proposition 1.2: A decomposition of an affine modification. *Let* $f = f_1 f_2 \in I$*. Denote* $I_1 = (I, f_1)$ *and* $A_1 = \Sigma_{I_1, f_1}(A)$*. Consider the ideal* I_2 *of the algebra* $A[I_1/f_1] \simeq A_1$ *generated by the subspace* I/f_1*. Then we have* $A' := \Sigma_{I,f}(A) \simeq A_1' := \Sigma_{I_2, f_2}(A_1)$.

Respectively, denoting $X = \mathrm{spec} A$, $X' = \mathrm{spec} A'$ *and* $X_1 = \mathrm{spec} A_1$ *we obtain the decomposition* $\sigma_I = \sigma_{I_2} \circ \sigma_{I_1}$ *where* $\sigma_I : X' \to X$, $\sigma_{I_1} : X_1 \to X$ *and* $\sigma_{I_2} : X' \to X_1$ *are the corresponding blowup morphisms.*

Proof. Let $I = (f, b_1, \ldots, b_s)$. By Proposition 1.1(b), we have:

$$A' \simeq A[I/f] = A[b_1/f, \ldots, b_s/f] = A[b_1/f_1 f_2, \ldots, b_s/f_1 f_2] \subset \mathrm{Frac}\, A\,,$$

$$A_1 \simeq A[I_1/f_1] = A[b_1/f_1, \ldots, b_s/f_1] \subset A[I/f] \subset \mathrm{Frac}\, A\;,$$

and $I_1 = (f_2, b_1/f_1, \ldots, b_s/f_1) \subset A_1$. Hence, $A_1' \simeq A_1[I_2/f_2] = A_1[b_1/f_1 f_2, \ldots, b_s/f_1 f_2] \subset \mathrm{Frac}\, A$. Clearly, the latter subalgebra coincides with $A[I/f] \simeq A'$, as claimed. □

Example 1.8. Let $A = \mathbf{C}^{[2]} = \mathbf{C}[x, y]$, $I = (f, g)$ where $f, g \in \mathbf{C}[x, y]$ and $f = f_1 f_2$. Then $A_1 = \mathbf{C}[x, y, z]/(f_1 z - g)$ (see Example 1.6), and σ_I is the restriction of $\sigma_{I_2} \circ \sigma_{I_1}$ to the surface $fz - g = 0$, where $\sigma_{I_1} : \mathbf{C}^3 \to \mathbf{C}^2$, $\sigma_{I_1} : (x, y, z) \longmapsto (x, y)$, and $\sigma_{I_2} : \mathbf{C}^3 \to \mathbf{C}^3$, $\sigma_{I_2} : (x, y, z) \longmapsto (x, y, f_2(x, y)z)$ is the affine modification of $\mathbf{C}^3$ along the divisor $D_{f_2} \times \mathbf{C}$ with center $D_{f_2} \simeq \mathbf{C}^{[3]}/(f_2, z)$.

[8] This observation is due to P. Russell.

2. The universal property of affine modifications

Here we show that the universal property of the blow ups (see e.g. [Hiro]) is still valid for the affine modifications.

Proposition 2.1: Lifting a homomorpism to affine modifications. *Consider two affine triples* (A, I, f) *and* (A_1, I_1, f_1) *and their affine modifications* $A' = \Sigma_{I,f}(A) \simeq A[I/f]$ *resp.* $A_1' = \Sigma_{I_1,f_1}(A_1) \simeq A_1[I_1/f_1]$*. Let* $\mu : A \to A_1$ *be a homomorphism such that* $\mu(I) \subset I_1$ *and* $\mu(f) = \alpha f_1$ *where* $\alpha \in A_1$ *is an invertible element. Then there exists a unique homomorphism* $\mu' : A' \to A_1'$ *which extends* μ*; it can be defined as follows:*

$$\mu' : A[I/f] \ni a/f^k \longmapsto \alpha^{-k}\mu(a)/f_1^k \in A_1[I_1/f_1].$$

If, in addition, $\mu(A) = A_1$ *and* $\mu(I) = I_1$*, then* $\mu'(A') = A_1'$*.*

Proof. By Proposition 1.1(b) (see also Remark 1.6 above), there are canonical isomorphisms $A' \simeq A[I/f]$ resp. $A_1' \simeq A_1[I_1/f_1]$. Thus, it is enough to show that μ admits a unique extension $\mu' : A[I/f] \to A_1[I_1/f_1]$, which is surjective if so are μ and $\mu | I : I \to I_1$. Furthermore, since $A_1[I_1/f_1] = A_1[I_1/\alpha f_1]$, replacing αf_1 by f_1 we may suppose in the sequel that $\alpha = 1$, i.e. that $f_1 = \mu(f)$.

Consider the natural extension

$$\tilde{\mu} : A[t] \to A_1[\tau], \quad \tilde{\mu} | A = \mu, \ \ \tilde{\mu}(t) = \tau .$$

Since $\tilde{\mu}$ sends the principal ideal $(1-ft) \subset A$ into the principal ideal $(1-f_1\tau) \subset A_1$, it induces a homomorphism $\widehat{\mu} : A[t]/(1-ft) \to A_1[\tau]/(1-f_1\tau)$. In this way, via the canonical isomorphisms $A[t]/(1-ft) \simeq A[1/f] \subset \text{Frac}\ A$ resp. $A_1[\tau]/(1-f_1\tau) \simeq A_1[1/f_1] \subset \text{Frac}\ A_1$, we obtain an extension $\widehat{\mu} : A[1/f] \to A_1[1/f_1]$ of μ. Any such extension sends a generic element $a/f^k \in A[1/f]$ into the element $\mu(a)/\mu(f^k) = \mu(a)/f_1^k \in A_1[1/f_1]$. Therefore, $\widehat{\mu}$ is uniquely defined on $A[1/f]$ by the formula $\widehat{\mu}(a/f^k) = \mu(a)/f_1^k$.

Observe that an element $a/f^k \in A[1/f]$ belongs to $A[I/f]$ iff $a \in I^k$. Since by assumption, $\mu(I^k) \subset I_1^k$, $k = 0, 1, \ldots$, we have $\widehat{\mu}(A[I/f]) \subset A_1[I_1/f_1]$. So, $\mu' := \widehat{\mu} | A[I/f]$ is a desired extension, and, clearly, it is unique.

Now, if $\mu(A) = A_1$ and $\mu(I) = I_1$, then $\mu(I^k) = I_1^k$, $k = 0, 1, \ldots$. Hence, any element $a_1/f_1^k \in A_1[I_1/f_1]$ can be written as $\mu'(a/f^k)$ with $a \in I^k$. Therefore, in this case μ' is a surjection. □

Definition 2.1. Let $J \subset A$ be a prime ideal such that $f \notin J$. We define the *strict transform* J' of J in $A' = A[I/f]$ as follows:

$$J' = \{a' \in A' \mid f^k a' \in J \ \text{for some}\ k \in \mathbf{N}\} .$$

It is easily seen that $J' \subset A'$ is a prime ideal containing J.

The following statements complete Proposition 2.1.

Proposition 2.2: How does a modification affect a subvariety.

(a) Let (A, I, f) be an affine triple, and let J be a prime ideal in A such that $f \notin J$. Denote by $\mu : A \to A_1 := A/J$ the canonical surjection. Set $I_1 = \mu(I)$, $f_1 = \mu(f)$, and $A'_1 = \Sigma_{I_1,f_1}(A_1)$. Then $A'/J' \simeq A'_1$.

(b) Furthermore, set $X = \mathrm{spec}\,A$, and let X_1 be an irreducible closed subvariety of X which is not contained in the support of D_f. Denote by $J = I(X_1)$ the defining ideal of $X_1 = \mathrm{spec}\,A_1$, and set $X' = \Sigma_{I,f}(X)$, $X'_1 = \Sigma_{I_1,f_1}(X_1)$ where I_1, f_1 are as in (a). Then the strict transform J' of the ideal J in $A' = \mathbf{C}[X']$ coincides with the defining ideal of the strict transform X_1^{st} of X_1 in X', and the variety X_1^{st} is isomorphic to X'_1.

Proof. (a) Let $\mu' : A' \to A'_1$ be a surjective extension of μ as in Proposition 2.1. We have to show that $\ker \mu' = J'$. Note that $J = \ker \mu \subset \ker \mu'$, and $f \notin \ker \mu'$, since $\mu'(f) = \mu(f) \neq 0$ in $A_1 \subset A'_1$. For an element $a' \in \ker \mu'$ chose $k \in \mathbf{N}$ so that $f^k a' \in A$. Then $0 = \mu'(f^k a') = \mu(f^k a')$, that is, $f^k a' \in J$, and hence, $a' \in J'$. Thus, $\ker \mu' \subset J'$.

Vice versa, let $a' \in J'$, and let $k \in \mathbf{N}$ be such that $f^k a' \in J = \ker \mu$. Thus, $\mu'(f^k a') = \mu(f^k a') = 0$. But $\mu(f^k) \neq 0$. Since A_1 and A'_1 are domains, the equality $\mu'(f^k)\mu'(a') = 0$ implies $\mu'(a') = 0$, i.e. $a' \in \ker \mu'$. Therefore, $J' \subset \ker \mu'$, and hence, $J' = \ker \mu'$. This proves (a).

(b) Observe that $g' \in A' = \mathbf{C}[X']$ vanishes on the strict transform X_1^{st} of X_1 iff so does $f^k g'$. If k is large enough, then we have $f^k g' \in A = \mathbf{C}[X] \hookrightarrow \mathbf{C}[X']$, and $f^k g' \mid X_1^{\mathrm{st}} = 0$ implies that $f^k g' \mid X_1 = 0$. Thus, $g' \in I(X_1^{\mathrm{st}})$ iff $f^k g' \in I(X_1) = J$ for large enough k, i.e. iff $g' \in J'$. Hence, $I(X_1^{\mathrm{st}}) \subset J'$.

Conversely, let $g' \in J'$, i.e. $f^k g' \in J = I(X_1)$ for large enough k. Thus, $f^k g' \mid X_1 = 0$, which implies that $g' \mid X_1^{\mathrm{st}} = 0$. Therefore, $J' \subset I(X_1^{\mathrm{st}})$, or $J' = I(X_1^{\mathrm{st}})$. Now (a) provides an isomorphism $X_1^{\mathrm{st}} = \mathrm{spec}\,A'/J' \simeq \mathrm{spec}\,A'_1 = X'_1$. The proof is completed. □

Corollary 2.1: Restricting affine modification to a subvariety. *Let $X = \mathrm{spec}\,A$ be an affine variety, and $X_1 = \mathrm{spec}\,A_1$ be an irreducible closed subvariety of X. Fix a proper ideal $I \subset A$, and let the ideal $I_1 \subset A_1$ consists of the restrictions to X_1 of the elements of I. Fix also an element $f \in I$ such that $f_1 := f \mid X_1 \neq 0$. Then there is a unique closed embedding $i' : X'_1 \hookrightarrow X'$ making the following diagram commutative:*

$$\begin{array}{ccccc} X'_1 & \overset{i'}{\hookrightarrow} & X' & \hookrightarrow & \mathbf{C}^r \times \mathbf{C}^s \\ \downarrow \sigma_1 & & \downarrow \sigma & & \downarrow \mathrm{pr}_1 \\ X_1 & \overset{i}{\hookrightarrow} & X & \hookrightarrow & \mathbf{C}^r \end{array},$$

where $\sigma : X' \to X$ resp. $\sigma_1 : X'_1 \to X_1$ is the blowup morphism of the affine modification $X' = \Sigma_{I,f}(X)$ resp. $X'_1 = \Sigma_{I_1,f_1}(X_1)$, $i : X_1 \hookrightarrow X$ is the identical *embedding, and* pr_1 *is the first projection.*

Remark 2.1. In particular, affine modifications commute with direct products.

Example 2.1: Modification of an affine hypersurface along a hyperplane section with center at a point. Let $g \in \mathbf{C}[x_1, \dots, x_n, y]$ be an irreducible polynomial such that $g(\overline{0}) = 0$. Set $X = \mathbf{C}^{n+1}$ and $X_1 = g^{-1}(0) \subset X$. That is, $X = \operatorname{spec} A$ resp. $X_1 = \operatorname{spec} A_1$ where $A = \mathbf{C}^{[n+1]}$ resp. $A_1 := \mathbf{C}^{[n+1]}/J,\ J := (g)$. Set $I = (x_1, \dots, x_n, y) \subset A$, $I_1 = (x_1 + (g), \dots, x_n + (g), y + (g)) \subset A_1$, $f = y \in I$ and $f_1 = y + (g) \in I_1$. As in Example 1.4 above consider the new coordinates $(y_1, \dots, y_n, y)$ in $X' \simeq \mathbf{C}^{n+1}$ where the blowup morphism $\sigma_I : X' \to X$ is given by $x_i = yy_i,\ i = 1, \dots, n$. Represent

$$g(yy_1, \dots, yy_n, y) = y^{\mu} g_1(y_1, \dots, y_n, y)\,,$$

where $g_1 \in \mathbf{C}[y_1, \dots, y_n, y]$ is not divisible by y. Then the equation of the affine modification $X_1' = X_1^{\mathrm{st}}$ of X_1 along the divisor D_{f_1} with center I_1 in $X' \simeq \mathbf{C}^{n+1} = \Sigma_{I,f}(X)$ is $g_1 = 0$.

Corollary 2.2: Lifting automorphisms to an affine modification.

(a) Let (A, I, f) *be an affine triple, and let* $\varphi \in \operatorname{Aut} A$ *be an automorphism such that* $\varphi(I) = I$ *and* $\varphi(f) = \alpha f$ *where* $\alpha \in A$ *is an invertible element. Then there exists a unique extension* $\varphi' \in \operatorname{Aut} A'$ *of* φ *to an automorphism of the affine modification* $A' = \Sigma_{I,f}(A)$.

(b) Furthermore, the induced automorphism $\widehat{\varphi}$ *of* $X = \operatorname{spec} A$ *preserves the divisor* $D = D_f$ *and the center* $V(I)$ *of the blow up. It can be lifted in a unique way to an automorphism* $\widehat{\varphi}'$ *of the affine modification* $X' = \Sigma_{I,f}(X)$ *which preserves the exceptional divisor* $E' \subset X'$ *of the blow up; in the complement* $X' \setminus E'$ *we have* $\widehat{\varphi}' = \sigma_I^{-1} \circ \varphi' \circ \sigma_I$ *where* $\sigma_I : X' \to X$ *is the blowup morphism.*

(c) In particular, let (X, D, C) *be a triple as in Remark 1.5 above, and let* $\widehat{\varphi} \in \operatorname{Aut} X$ *be an automorphism of* X *such that* $\widehat{\varphi}(D) = D,\ \widehat{\varphi}(C) = C$. *Then there exists a unique automorphism* $\widehat{\varphi}' \in \operatorname{Aut} X'$ *of the affine modification* $X' = \Sigma_{C,D}(X)$ *such that* $\widehat{\varphi}' \mid (X' \setminus E') = \widehat{\varphi} \mid (X \setminus D)$ *under the natural identification* $\sigma_C \mid (X' \setminus E') : X' \setminus E' \xrightarrow{\simeq} X \setminus D$.

Example 2.2. Let $A = \mathbf{C}^{[3]} = \mathbf{C}[x, y, z]$, that is, $X = \mathbf{C}^3$, $I = (x, z)$, and $f = x$. Then $X' \simeq \mathbf{C}^3$, and $\sigma_I^* : A \to A' = \mathbf{C}[x', y', z']$ is given as

$$\sigma_I^* : \quad \begin{aligned} x &\longmapsto x' \\ y &\longmapsto y' \\ z &\longmapsto x'z' \end{aligned}$$

(cf. Examples 1.4, 1.6). Let $\mu \in \operatorname{Aut} A$ be given by

$$\mu : \quad \begin{aligned} x &\longmapsto x \\ y &\longmapsto y + x g_1(x, y, z) \\ z &\longmapsto z + x g_2(x, y, z)\,, \end{aligned}$$

where $g_1, g_2 \in \mathbf{C}^{[3]}$. Then we have $\mu(I) = I$, $\mu(f) = f$, and the extension $\mu' \in \mathrm{Aut}\, A'$ can be given as

$$\mu' : \quad \begin{array}{l} x' \longmapsto x' \\ y' \longmapsto y' + x' g_1(x', y', x'z') \\ z' \longmapsto z' + g_2(x', y', x'z') \,. \end{array}$$

Recall the following notion.

Definition 2.2. Let A be a commutative algebra over $\mathbf{C}$, and let ∂ be a derivation of A. It is called *locally nilpotent* if for any $a \in A$ we have $\partial^n(a) = 0$ for some $n = n(\partial, a)$. For an algebra A denote by LND (A) the set of all locally nilpotent derivations (LND-s for short) of A. Giving an LND ∂ on a Noetherian $\mathbf{C}-$ algebra A is the same as giving a regular $\mathbf{C}_+-$ action[9] φ_∂ on spec A. For $p \in A$ and $t \in \mathbf{C}_+$ we put

$$\varphi_\partial(t, p) = \exp(t\partial)(p) = \sum_{i=0}^{n(\partial, a)-1} \frac{t^i \partial^i p}{i!}$$

[Re]. The kernel $\ker \partial$ coincides with the subalgebra $A^{\varphi_\partial} \subset A$ of $\varphi_\partial-$ invariants.

Corollary 2.3: Lifting LND-s and $\mathbf{C}_+-$ actions to an affine modification.

(a) Let (A, I, f) be an affine triple, and set $X = \mathrm{spec} A$. Let ∂ be an LND of the algebra $A = \mathbf{C}[X]$ such that $\partial(f) = 0$ and $\partial(I) \subset I$. Then ∂ can be lifted in a unique way to an LND ∂' of the affine modification $A' = \mathbf{C}[X'] = \Sigma_{I,f}(A)$.

(b) Let the notation be as in Corollary 2.2(c) above. Then any $\mathbf{C}_+-$ action φ_∂ on X which leaves invariant the subvarieties D and C can be lifted in a unique way to a $\mathbf{C}_+-$ action φ'_∂ on the affine modification $X' = \Sigma_{C,D}(X)$ which leaves invariant the exceptional divisor E'.

Example 2.3. Let $A = \mathbf{C}[x, y]$, i.e. $X = \mathbf{C}^2$, and $A' = \Sigma_{I,x}(A)$ where $I = (x, q(y))$, $q \in \mathbf{C}[y]$. Then $X' = \Sigma_{I,x}(\mathbf{C}^2)$ is the surface given in $\mathbf{C}^3$ by the equation $xz - q(y) = 0$, and the blowup morphism $\sigma_I : X \to \mathbf{C}^2$ is the restriction to X of the standard projection $\mathbf{C}^3 \to \mathbf{C}^2$, $(x, y, z) \longmapsto (x, y)$ (see Example 1.6).

Consider the triangular $\mathbf{C}_+-$ action $\varphi(t, (x, y)) = (x, y + txp(x))$ on $\mathbf{C}^2$ where $p \in \mathbf{C}[x]$, and the corresponding LND $\partial : A \to A$,

$$\partial : \quad \begin{array}{l} x \longmapsto 0 \\ y \longmapsto xp(x) \,. \end{array}$$

Clearly, $\partial(I) \subset I$. By Corollary 2.3, there exist unique extensions ∂' resp. φ' of ∂ resp. φ to A'; they are given, respectively, as follows:

$$\partial' : \quad \begin{array}{l} x \longmapsto 0 \\ y \longmapsto xp(x) \\ z = q(y)/x \longmapsto q'(y)p(x) \end{array}$$

[9] Here $\mathbf{C}_+$ stands for the additive group of the complex number field.

and

$$\varphi' : \begin{array}{l} x \longmapsto x \\ y \longmapsto y + txp(x) \\ z \longmapsto q(y+txp(x))/x = z + \sum\limits_{i=0}^{\deg q} t^i \frac{q^{(i)}(y)}{i!} x^{i-1} p^i(x)\,. \end{array}$$

3. Topology of affine modifications

Hereafter $e(Y)$ denotes the Euler characteristic of a topological space Y; for a divisor D by $e(D)$ we mean $e(\mathrm{supp} D)$. We start with the next simple observation.

Lemma 3.1: The Euler characteristic of an affine modification.

Let $\sigma_I : X' \to X$ be the affine modification of an affine variety X along a divisor $D = D_f$ with center I. As before, we denote by $E' \subset X'$ the exceptional divisor of the blow up σ_I. Then we have $e(X') - e(X) = e(E') - e(D)$.

Proof. The isomorphism $\sigma_I : X' \setminus E' \simeq X \setminus D_f$ (see Proposition 1.1(a)) provides the equality $e(X' \setminus E') = e(X \setminus D_f)$. By the additivity of the Euler characteristic with respect to a disjoint constructive decomposition of a quasiprojective variety [Du], we obtain $e(X') = e(X' \setminus E') + e(E')$ and $e(X) = e(X \setminus D_f) + e(D_f)$, and the statement follows. □

Example 3.1. Let $X = \mathbf{C}^n$ and $X' = \Sigma_{I,f}(X)$ be as in Example 1.6. That is, $I = (f, g)$ where $f, g \in \mathbf{C}^{[n]}$, and $X' = \{f(\overline{x})z - g(\overline{x}) = 0\} \subset \mathbf{C}^{n+1}$. Put $D = \{f = 0\} \subset \mathbf{C}^n$ and $C = V(I) = \{f = g = 0\} \subset \mathbf{C}^n$. Then we have $E' \simeq C \times \mathbf{C}$, so that $e(E') = e(C)$, and hence

$$e(X') = 1 + e(C) - e(D).$$

In particular, $e(X') = 1$ iff $e(D) - e(C) = e(D \setminus C) = 0$.

Till the end of this section we appropriate the complex analytic point of view. Observe that, with the language of schemes, one can naturally extend the notion of the affine modification to quasiprojective varieties and to more general ring spaces, and obtain results analogous to those of the previous sections. Instead, to simplify things, according to Proposition 1.1(a) and Theorem 1.1 we adopt the following definitions.

Definition 3.1. Consider a triple (M, C, D) resp. a pair (M', E') where M resp. M' is a reduced connected complex space, $D \subset M$ resp. $E' \subset M'$ is a closed hypersurface, and $C \subset D$ is a closed analytic subvariety of codimension at least two in M. Let $\sigma : M' \to M$ be a surjective morphism such that the restriction $\sigma \,|\, (M' \backslash E') : M' \backslash E' \to M \setminus D$ is a biholomorphism, and $\sigma(E') \subset C$. Then we say that the pair (M', E') is a *pseudoaffine modification* (via σ) of the triple (M, C, D) with a locus subordinated to (C, D) and with the exceptional divisor E'.

In particular, we consider the *pseudoaffine modification* $M' = \Sigma_{C,D}(M)$ *of* M *along* D *with center* C, where $M' = \widehat{M} \setminus D'$, $\widehat{M}$ is the blow up of M at C and D' is the strict transform of D in $\widehat{M}$.

Anyhow, modifying M we replace the divisor D by the new one E'. In the latter case $E' = E \setminus D'$ where $E = \sigma_C^{-1}(C) \subset \widehat{M}$ is the exceptional divisor of the blow up $\sigma_C : \widehat{M} \to M$ with center C. While in the former case we blow up M with center being an ideal sheaf supported by C.

Recall that by a *vanishing loop* of a divisor D in a complex manifold M one means any loop in $\pi_1(M \setminus D)$ whose image in the homology group $H_1(M \setminus D)$ coincides with a fibre of the normal circle bundle of the smooth part $\operatorname{reg} D$.

The next proposition, in particular, gives a generalization of Lemma 3.4 in [Ka 1].

Proposition 3.1: Preserving the fundamental group under a modification. *Let a pair (M', E') be a pseudoaffine modification of a triple (M, C, D) via a morphism $\sigma : M' \to M$ where M, M' are complex manifolds, and let the divisors D, E' admit finite decompositions into irreducible components $D = \bigcup_{i=1}^{n} D_i$ resp. $E' = \bigcup_{j=1}^{n'} E'_j$. Let $D_i^* = \sum_{j=1}^{n'} m_{ij} E'_j$. Assume that*

(i) $\sigma(E'_j) \cap \operatorname{reg} D_i \neq \emptyset$ as soon as $m_{ij} > 0$.

Then the following statements (a) and (b) hold.

(a) If (α) the lattice vectors $b_j = (m_{1j}, \dots, m_{nj}) \in \mathbf{Z}^n$, $j = 1, \dots, n'$, generate the lattice $\mathbf{Z}^n$, then (β) $\sigma_* : H_1(M'; \mathbf{Z}) \to H_1(M; \mathbf{Z})$ is an isomorphism.

The converse is true if

(ii) D_i is a principal divisor defined by a holomorphic function f_i on M, i.e. $D_i = f_i^*(0)$, $i = 1, \dots, n$.

(b) Assume further that

(iii) there is a disjoint partition $\{1, \dots, n'\} = J_1 \cup \ldots \cup J_n$ such that $D_i^* = \sum_{j \in J_i} m_{ij} E'_j \neq 0$, $i = 1, \dots, n$.

Set $d_i =$ g.c.d. $(m_{ij} \mid j \in J_i)$, $i = 1, \dots, n$.

Then (γ) $\sigma_* : \pi_1(M') \to \pi_1(M)$ is an isomorphism if (δ) $d_1 = \ldots = d_n = 1$.

Under the condition (ii) the converse is also true.

Proof. Let α_i be a vanishing loop of D_i in M, $i = 1, \dots, n$, and β_j be a vanishing loop of E'_j in M', $j = 1, \dots, n'$. Then the kernel of the natural homomorphism $\pi_1(M \setminus D) \to \pi_1(M)$ coincides with the minimal normal subgroup $H := << \alpha_1, \dots, \alpha_n >>$ of the group $G := \pi_1(M \setminus D)$ generated by $\alpha_1, \dots, \alpha_n$, i.e. with the subgroup generated by the conjugacy classes of $\alpha_1, \dots, \alpha_n$ (see e.g. [Za 2, (2.3.a)]). Similarly, the kernel of the natural homomorphism $\pi_1(M' \setminus E') \to \pi_1(M')$ is the normal subgroup $H' = << \beta_1, \dots, \beta_{n'} >>$ of the group $\pi_1(M' \setminus E') \overset{\sigma_*}{\cong} \pi_1(M \setminus D) = G$. Thus, $\pi_1(M) \simeq G/H$ and $\pi_1(M') \simeq G/H'$. Moreover, σ_* identifies H' with a subgroup of H, the surjection

$\sigma_* : \pi_1(M') \to \pi_1(M)$ coincides with the canonical surjection $G/H \to G/H'$, and hence, $\ker \sigma_* \simeq H/H'$. It follows that $\sigma_* : \pi_1(M') \to \pi_1(M)$ is an isomorphism iff $H = H'$.

Proof of (a). Denote $G' = [G, G]$, and let $\rho : G \to G/G' \simeq H_1(M \setminus D; \mathbf{Z})$ be the canonical surjection. Set $\widetilde{H} = \rho(H)$, $\widetilde{H'} = \rho(H')$ and $\tilde{\alpha}_i = \rho(\alpha_i)$, $\tilde{\beta}_j = \rho(\beta_j)$. Clearly, $\ker(\sigma_* : H_1(M'; \mathbf{Z}) \to H_1(M; \mathbf{Z})) \simeq \widetilde{H}/\widetilde{H'}$. In view of the condition (i) we have $\tilde{\beta}_j = \sum_{i=1}^n m_{ij}\tilde{\alpha}_i$, $j = 1, \ldots, n'$. Under the condition (α), the elements $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ can be expressed in terms of $\tilde{\beta}_1, \ldots, \tilde{\beta}_{n'}$, and so, $\widetilde{H'} = \widetilde{H}$. This proves the implication (α) $\Longrightarrow$ (β).

Assuming the condition (ii) we have an isomorphism

$$f_* := (f_1, \ldots, f_n)_* : \widetilde{H} \xrightarrow{\simeq} \mathbf{Z}^n$$

which sends $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ to the standard generators of the lattice $\mathbf{Z}^n$, and identifies the subgroup $\widetilde{H'}$ with the sublattice $\Lambda \subset \mathbf{Z}^n$ spanned by the vectors $b_j = (f \circ \sigma)_*(\tilde{\beta}_j)$, $j = 1, \ldots, n'$. Therefore, $\widetilde{H} = \widetilde{H'}$ iff $\Lambda = \mathbf{Z}^n$, that is, iff the condition (α) holds. This yields the converse implication (β) $\Longrightarrow$ (α).

Proof of (b). Under the assumptions (i) and (iii) the image $\sigma_*(\beta_J) \in G$ where $j \in J_i$, is conjugate with the element $\alpha_i^{m_{ij}}$. Thus,

$$H' = << \beta_1, \ldots, \beta_{n'} >> = << \alpha_i^{m_{ij}} \,|\, j \in J_i,\ i = 1, \ldots, n >> = << \alpha_i^{d_i} \,|\, i = 1, \ldots, n >> .$$

Therefore, the condition (δ) implies the coincidence $H = H'$, and so, it implies (γ). If (i)-(iii) hold, then, clearly, we have the implications (γ) $\Longrightarrow$ (β) $\Longrightarrow$ (α) $\Longleftrightarrow$ (δ), and hence, (γ) $\Longrightarrow$ (δ). □

The next theorem and its corollary generalize Theorem 3.5 in [Ka 1] (see also [Za 2, Thm. 5.1]).

Theorem 3.1: Preserving the homology under a modification.

Let a pair (M', E') be a pseudoaffine modification of a triple (M, C, D) via a morphism $\sigma : M' \to M$ (see Definition 3.1). Suppose that

(i) M, M' are complex manifolds and D, E' are topological manifolds admitting finite decompositions into irreducible components $D = \sum\limits_{i=1}^{n} D_i$ resp. $E' = \sum\limits_{j=1}^{m} E'_j$ where $m = n$ and $E'_i = \sigma^*(D_i)$, $i = 1, \ldots, n$;

(ii) $\sigma(E_i) \cap \text{reg}\, D_i \neq \emptyset$, $i = 1, \ldots, n$.

Put $\tau = i \circ (\sigma \,|\, E') : E' \to D$ where $i : C \hookrightarrow D$ is the identical embedding. Then $\sigma_* : H_*(M'; \mathbf{Z}) \longrightarrow H_*(M; \mathbf{Z})$ is an isomorphism iff

(iii) $\tau_* : H_*(E'_i; \mathbf{Z}) \longrightarrow H_*(D_i; \mathbf{Z})$ is an isomorphism for all $i = 1, \ldots, n$.

Proof. Set $\breve{M} = M \setminus D$, $\breve{M}' = M' \setminus E'$ and $\breve{\sigma} = \sigma \,|\, \breve{M}' : \breve{M}' \xrightarrow{\simeq} \breve{M}$. Consider the following commutative diagram where the horizontal lines are exact homology sequences of pairs with $\mathbf{Z}-$ coefficients:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & H_{j+1}(M',\, \breve{M}') & \longrightarrow & H_j(\breve{M}') & \longrightarrow & H_j(M') & \longrightarrow & H_j(M',\, \breve{M}') & \longrightarrow & H_{j-1}(\breve{M}') & \longrightarrow \ldots \\
& \Big\downarrow (\sigma,\, \breve{\sigma})_* & & \Big\downarrow \breve{\sigma}_* & & \Big\downarrow \sigma_* & & \Big\downarrow (\sigma,\, \breve{\sigma})_* & & \Big\downarrow \breve{\sigma}_* & \qquad (*) \\
\ldots \longrightarrow & H_{j+1}(M,\, \breve{M}) & \longrightarrow & H_j(\breve{M}) & \longrightarrow & H_j(M) & \longrightarrow & H_j(M,\, \breve{M}) & \longrightarrow & H_{j-1}(\breve{M}) & \longrightarrow \ldots
\end{array}$$

Due to the Thom isomorphism, it can be replaced by the following one:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & H_{j-1}(E') & \longrightarrow & H_j(\breve{M}') & \longrightarrow & H_j(M') & \longrightarrow & H_{j-2}(E') & \longrightarrow & H_{j-1}(\breve{M}') & \longrightarrow \ldots \\
& \Big\downarrow \tau_* & & \simeq \Big\downarrow \breve{\sigma}_* & & \Big\downarrow \sigma_* & & \Big\downarrow \tau_* & & \simeq \Big\downarrow \breve{\sigma}_* & \qquad (**) \\
\ldots \longrightarrow & H_{j-1}(D) & \longrightarrow & H_j(\breve{M}) & \longrightarrow & H_j(M) & \longrightarrow & H_{j-2}(D) & \longrightarrow & H_{j-1}(\breve{M}) & \longrightarrow \ldots
\end{array}$$

which is still commutative. Indeed, let Δ_i' be a small complex disc in M' which meets E_i' transversally at its origin, and let this origin be a generic point of E_i'. Then the image $\Delta_i = \sigma(\Delta_i')$ is a complex disc in M transversal to $\operatorname{reg} D_i$ at a generic point of C_i. Hence, for the Thom classes $u_i' \in H^2(M',\, M' \setminus E_i')$ of E_i' resp. $u_i \in H^2(M,\, M \setminus D_i)$ of D_i, uniquely defined by the conditions $u_i'(\Delta_i') = 1$ resp. $u_i(\Delta_i) = 1$, we have $\sigma^*(u_i) = u_i'$, $i = 1, \ldots, n$. Recall that the Thom isomorphism $H_{j+2}(M',\, \breve{M}') \simeq H_j(E')$ resp. $H_{j+2}(M,\, \breve{M}) \simeq H_j(D')$ is defined as the cap-product with the Thom class $u' = \sum_{i=1}^n u_i'$ resp. $u = \sum_{i=1}^n u_i$: $\eta' \mapsto u' \cap \eta'$ resp. $\eta \mapsto u \cap \eta$ (see e.g. [Do, VIII.11.21], [MilSta]). Now it is easily seen that the diagram $(**)$ is commutative.

It remains to note that by the Five Lemma, τ_* in $(**)$ yields an isomorphism for all j iff $\sigma_* : H_j(M') \to H_j(M)$ is an isomorphism for all j. The proof is completed. $\square$

Remark 3.1. If under the assumptions of Theorem 3.1 the morphism σ is the restriction of the blow up morphism σ_C with a smooth center C contained in $\operatorname{reg} D$, then $\tau : E' \to C$ is a smooth fibration over C with a fibre $\mathbf{C}^k$, where $k = \operatorname{codim}_D C$, and so, the contraction $\sigma \mid E' : E' \to C$ is a homotopy equivalence. Thus, in this case $\tau_* : H_*(E'; \mathbf{Z}) \longrightarrow H_*(D; \mathbf{Z})$ is an isomorphism iff $i_* : H_*(C; \mathbf{Z}) \longrightarrow H_*(D; \mathbf{Z})$ is [10].

Corollary 3.1: Preserving contractibility under a modification.

Suppose that the conditions (i), (ii) and (iii) of Theorem 3.1 are fulfilled. Then the pseudoaffine modification M' of M is acyclic resp. contractible iff M is.

Proof. The equivalence of acyclicity of M' and of M follows immediately from Theorem 3.1.

By the Hurewicz and Whitehead Theorems [FoFu, (2.11.5), (2.14.2)], contractibility of M resp. M' is equivalent to acyclicity and simply connectedness of M resp. M'.

[10] Cf. also Proposition 1.1(d).

Notice that the conditions (i), (iii) and (δ) of Proposition 3.1 are fulfilled. By (b) of this proposition, $\sigma : M' \to M$ induces an isomorphism of the fundamental groups. Thus, M' is simply connected iff M is, and the statement follows. □

Remark 3.2. Suppose that $\pi_1(M) = \mathbf{1}$ or $\pi_1(M') = \mathbf{1}$, the homomorphism $\sigma_* : \pi_2(M') \to \pi_2(M)$ is surjective and the conditions (i) - (iii) of Theorem 3.1 are fulfilled. Then by the Whitehead Theorem [FoFu, (2.14.5)], $\sigma : M' \to M$ is a homotopy equivalence.

We give below two examples of applications of Corollary 3.1.

Example 3.2: Russell's cubic is contractible. Recall (see Example 1.7) that the Russell cubic threefold $X = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbf{C}^4$ is the affine modification of $\mathbf{C}^3$ along the divisor $2D$, where $D := \{x = 0\} \subset \mathbf{C}^3$, with center at the ideal $I = (-x^2, x + z^2 + t^3) \subset \mathbf{C}^{[3]}$ supported by the plane curve $C = \Gamma_{2,3} = \{x = z^2 + t^3 = 0\} \subset D \simeq \mathbf{C}^2$. The exceptional divisor E' coincides with the book-surface $B = \{x = 0\} \subset X'$, $B \simeq \mathbf{C} \times \Gamma_{2,3}$. Therefore, the condition (iii) of Theorem 3.1 is fulfilled. As well, the conditions (i) and (ii) hold, and so, by Corollary 3.1, X is contractible. Moreover, by the Dimca-Ramanujam Theorem [Di 1, Ram] (see also [Za 2, Thm 4.2]), X is diffeomorphic to $\mathbf{R}^6$.

Example 3.3: Let $M = f^*(0)$, where $f \in \mathcal{O}(\mathbf{C}^n)$, be a smooth reduced analytic hypersurface in $\mathbf{C}^n$, and let $f_1, \ldots, f_k \in \mathcal{O}(\mathbf{C}^n)$ be holomorphic functions without common zeros on M. Consider the smooth analytic subvarieties

$$D := M \times \mathbf{C}^k \subset \mathbf{C}^{n+k}_{(\bar{x},\,\bar{u})} = \mathbf{C}^n_{\bar{x}} \times \mathbf{C}^k_{\bar{u}} \qquad \text{and} \qquad C := \{f(\bar{x}) = 0 = g(\bar{x},\,\bar{u})\} \subset \mathbf{C}^{n+k}_{(\bar{x},\,\bar{u})}$$

where $g(\bar{x},\,\bar{u}) := \sum_{i=1}^k u_i f_i \in \mathcal{O}(\mathbf{C}^{n+k})$. The natural embeddings $M \times \bar{0} \hookrightarrow C \hookrightarrow D$ being homotopy equivalences, by Corollary 3.1 and Remark 3.1, the pseudoaffine modification $X' = \Sigma_{C,D}(\mathbf{C}^{n+k})$ of $\mathbf{C}^{n+k}$ is a smooth contractible analytic hypersurface given in $\mathbf{C}^{n+k+1}_{(\bar{x},\,\bar{u},\,v)}$ by the equation $f(\bar{x})v - g(\bar{x},\,\bar{u}) = 0$ (cf. Example 1.6).

Since M is supposed being smooth one may take, for instance, $k = n$ and $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n$; then $C \simeq TM$ and $D \simeq T\mathbf{C}^n \,|\, M$.

The next example shows that in general, Corollary 3.1 does not hold if some of the conditions (i) – (iii) of Theorem 3.1 are violated.

Example 3.4: (see [Ka 1, Remark on p. 418]). Let $\sigma : X \to \mathbf{C}^2$ be a pseudoaffine modification of $\mathbf{C}^2$ along the cuspidal cubic $\Gamma := \Gamma_{2,3} = \{x^2 - y^3 = 0\}$ with center at the cusp $\bar{0} \in \Gamma$, that is, σ is the restriction to $X := \widehat{X} \setminus \Gamma'$ of the blowup morphism $\hat{\sigma} : \widehat{X} \to \mathbf{C}^2$ with center at the origin whereas $\Gamma' \subset \widehat{X}$ is the strict transform of Γ in $\widehat{X}$. Then X is neither simply connected nor acyclic. Indeed, set $X^* = X \setminus E' = \widehat{X} \setminus (E \cup \Gamma')$ where $E \subset \widehat{X}$ is the exceptional divisor of $\hat{\sigma}$, and let $\beta \in H_1(X^*; \mathbf{Z})$ be a vanishing loop of Γ' in X^*. It is easily seen that $\sigma_*(\beta) = 2\alpha$ where $\alpha \in H_1(\mathbf{C}^2 \setminus \Gamma; \mathbf{Z}) \simeq \mathbf{Z}$ is a vanishing loop of Γ in $\mathbf{C}^2$. In virtue of the isomorphism $\sigma\,|\,X^* : X^* \xrightarrow{\simeq} \mathbf{C}^2 \setminus \Gamma$ and of the exact sequence

$$\mathbf{0} \longrightarrow <\beta> \longrightarrow H_1(X^*;\,\mathbf{Z}) \longrightarrow H_1(X;\,\mathbf{Z}) \longrightarrow \mathbf{0}$$

(cf. Lemma 4.3 below) we have

$$H_1(X;\,\mathbf{Z}) \simeq H_1(X^*;\,\mathbf{Z})/ <\beta> \simeq H_1(\mathbf{C}^2 \setminus \Gamma;\,\mathbf{Z})/ <2\alpha> \simeq \mathbf{Z}/2\mathbf{Z}.$$

Remark 3.3. Notice that the proper transform $\Gamma^{\rm pr}$ of Γ in $\widehat{X}$ (see Remark 1.4) coincides with the union $E \cup \Gamma'$, and so, it differs from the strict transform Γ'. In turn, X differs from the affine modification $\Sigma_{I,f}(\mathbf{C}^2) \simeq \mathbf{C}^2 \setminus \Gamma$ where $I := (x^2 - y^3,\, x,\, y)$. Indeed, the blow up $Z = \widehat{X}$ can be given in $\mathbf{C}^2_{(x,\,y)} \times \mathbf{P}^2_{(u:v:w)}$ by the equations $xv - yu = 0,\ w = xu - y^2 v$, and hence, $\Gamma^{\rm pr} = \{w = 0\}$ contains the exceptional divisor[11] $E = \{x = y = 0\}$.

4. On the theorems of Sathaye and Wright

We propose the following

Questions

A *When an affine modification* $X = \Sigma_{I,f}(\mathbf{C}^n)$ *of* $\mathbf{C}^n$ *is isomorphic to* $\mathbf{C}^n$?

B *Let* $X = \Sigma_{I,f}(\mathbf{C}^n)$ *be an affine modification of* $\mathbf{C}^n$ *canonically embedded into* $\mathbf{C}^{n+k}$ *(see Definition 1.2). If* $X \simeq \mathbf{C}^n$, *is it necessarily rectifiable in* $\mathbf{C}^{n+k}$ *?*

C *Which embeddings* $\mathbf{C}^n \hookrightarrow \mathbf{C}^{n+k}$ *appear in this way?*

Question **B** is a specialization of the Abhyankar-Sathaye Embedding Problem mentioned in the Introduction. This problem is known to be answered in affirmative for $k \ge n+2$ (the Jelonek-Kaliman-Nori-Srinivas Theorem [Je, Ka 2, Sr]) and for n=k=1 (the Abhyankar-Moh and Suzuki Embedding Theorem [AM, Suz]). In a special case when $n = 2$ and $k = 1$ the positive answer to Question **B** is provided by a theorem of A. Sathaye[12] [Sat], generalized by D. Wright[13] [Wr 2] as follows.

Theorem 4.1 (Sathaye–Wright).

Let $X = X_{n,\,f,\,g}$ *be a surface in* $\mathbf{C}^3$ *given by the equation* $f(x,\,y)z^n + g(x,\,y) = 0$ *where* $f,\,g \in \mathbf{C}[x,\,y]$, $n \in \mathbf{N}$. *Suppose that* $X \simeq \mathbf{C}^2$. *Then* X *is rectifiable, i.e. there exists an automorphism* $\alpha \in Aut\ \mathbf{C}^2$ *which transforms* X *into a coordinate plane.*

We give below a new proof of Theorem 4.1, as well as some generalization. The proof is easy if one of the polynomials $f,\,g$ is constant; in the sequel we do not consider this

[11] More generally, one can prove the following: *If the center* C *of a blow up of a manifold* X *is smooth, reduced and it is contained in the singular locus of a reduced divisor* D_f, *then the proper transform* $D_f^{\rm pr}$ *contains the exceptional divisor* E *of the blow up.*

[12] which corresponds to the case $n = 1$ in Theorem 4.1 below.

[13] See also [Ru 2, RuSat] for some generalizations.

possibility. Observe that for $n = 1$ the surface $X = X_{1,f,g}$ as in Theorem 4.1 is the affine modification $\Sigma_{I,f}(\mathbf{C}^2)$ of $\mathbf{C}^2$ along the divisor $D_f = f^*(0)$ with center $I = (f, g)$ (see Example 1.6) whereas the surface $X_{n,f,g}$ can be presented as a cyclic covering of $X_{1,f,g}$ ramified to order n on $D_g = g^*(0)$. So, in the proof we use affine modifications.

Theorem 4.2. *Let $X = X_{n,f,g}$ be an irreducible smooth surface in $\mathbf{C}^3$ given by the equation $f(x, y)z^n + g(x, y) = 0$ where $f, g \in \mathbf{C}[x, y]$, $n \in \mathbf{N}$. The following conditions (i) - (iv) are equivalent:*

(i) $e(X) = 1$ and $H_1(X; \mathbf{Z}) = 0$.

(ii) X is acyclic, i.e. $\tilde{H}_*(X; \mathbf{Z}) = 0$.

(iii) $X \simeq \mathbf{C}^2$.

(iv) X is rectifiable.

If $n > 1$, then the above conditions are equivalent to the following one:

(v) The pair (f, g) is rectifiable in the following sense: there exists an automorphisms $\alpha \in Aut\, \mathbf{C}^{[2]}$ such that $(\alpha(f), \alpha(g)) = (p(x), y)$.

The implications $(v) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$ are easy; in the sequel we only prove $(i) \Longrightarrow (iv)$ in the case $n = 1$ and $(i) \Longrightarrow (v)$ in the case $n > 1$.

Remarks

4.1. There are examples of acyclic or even contractible smooth algebraic surfaces in $\mathbf{C}^3$ non-isomorphic to $\mathbf{C}^2$, see e.g. [tDP, KaML 2] and Example 7.1 below. Moreover, any smooth contractible affine algebraic surface of logarithmic Kodaira dimension 1 admits such an embedding into $\mathbf{C}^3$ [KaML 2]. However, Theorem 4.2 shows that the image of this embedding cannot be given by a 'binomial' equation $f(x, y)z^n + g(x, y) = 0$.

4.2. As a corollary we obtain that if the zero fibre of the polynomial $f(x, y)z^n + g(x, y)$ is acyclic (resp. isomorphic to $\mathbf{C}^2$) then so is every fibre. Notice that in general, a polynomial $p \in \mathbf{C}^{[3]}$ with a smooth acyclic, or even contractible, zero fibre may have non-acyclic generic fibres; see Example 7.3 below.

4.3. The next simple observation will be useful in what follows. Let $X_1 = X_{1,f,g}$ be a surface as in Theorem 4.2 with $n = 1$. Since X_1 is supposed being irreducible, the divisors D_f and D_g have no irreducible component in common. Thus, the center $D_f \cdot D_g$ of the blow up $\sigma_I : X_1 \to \mathbf{C}^2$, $\sigma_I(x, y, z) = (x, y)$, is supported by the finite set $\mathrm{supp}\, D_f \cap \mathrm{supp}\, D_g$ (see Example 1.6). The exceptional curve $E' \subset X_1$, $E' = \{f(x, y) = g(x, y) = 0\}$, is isomorphic to the product $\mathbf{C} \times (\mathrm{supp}\, D_f \cap \mathrm{supp}\, D_g)$, and hence, it consists of κ vertical lines in X_1 where $\kappa := \mathrm{card}\, (\mathrm{supp}\, D_f \cap \mathrm{supp}\, D_g)$.

The proof of Theorem 4.2 is based on the following lemmas 4.1 - 4.6.

Lemma 4.1. *Let X be a smooth irreducible affine surface. Then*

(a) The following conditions (i) and (ii) (resp. (i′) and (ii′)) are equivalent:

(i) $e(X)=1$ and $H_1(X;\,\mathbf{Z})=0$ resp. (i′) $e(X)=1$ and $b_1(X)=0$;

(ii) X is acyclic, i.e. $\tilde H_*(X;\,\mathbf{Z})=0$, resp. (ii′) X is $\mathbf{Q}-$ acyclic, i.e. $\tilde H_*(X;\,\mathbf{Q})=0$.

(b) If (i) holds, then the algebra $A=\mathbf{C}[X]$ is UFD.

Proof. (a) The implications (*ii*) $\Longrightarrow$ (*i*) resp. (*ii′*) $\Longrightarrow$ (*i′*) are straightforward. To prove the converse ones, notice that by the Lefchetz Hyperplane Section Theorem [Mil 1, Thm. 7.2], X has homotopy type of a finite cell complex of real dimension at most two. Hence, $H_3(X;\,\mathbf{Z})=H_4(X;\,\mathbf{Z})=0$ and $H_2(X;\,\mathbf{Z})$ is a free abelian group. Therefore, $e(X)=1-b_1(X)+b_2(X)$, and so, if $e(X)=1$ and $b_1(X)=0$, then $b_2(X)=0$, and moreover, $H_2(X;\,\mathbf{Z})=0$. Thus, (*i*) implies that $\tilde H_*(X;\,\mathbf{Z})=0$, i.e. X is acyclic; in turn, (*i′*) implies that $\tilde H_*(X;\,\mathbf{Q})=0$, i.e. X is $\mathbf{Q}-$ acyclic. This proves (a).

In view of (a), (b) follows from [Fuj, (1.17)–(1.20)] (see also [Ka 1, Prop. 3.2]). □

Lemma 4.2. *Let $f,\,g\in\mathbf{C}^{[2]}\setminus\mathbf{C}$ be two non-constant polynomials without common factor*[14]. *Then the surface $X_n=X_{n,\,f,\,g}=\{fz^n+g=0\}\subset\mathbf{C}^3$ is smooth iff the following two conditions are fulfilled:*

(i) For any point $P_0\in supp\, D_f\cap supp\, D_g$ the divisor D_g is non-singular and reduced at P_0. If so is the divisor D_f at P_0 too, then D_f and D_g are transversal at P_0.

(ii) If $n>1$, then D_g is a smooth reduced divisor.

Proof. The statement easily follows from the equality

$$\mathrm{grad}\,(fz^n-g)=(z^n\mathrm{grad}\,f-\mathrm{grad}\,g,\,nz^{n-1}f)\,.$$

□

Lemma 4.3. *Let X be a connected complex manifold, and let $D_1,\dots,D_s$ be reduced irreducible principal divisors in X. Set $D=\bigcup_{i=1}^s D_i$ and $X^*=X\setminus D$. Then there is an exact sequence*

$$\mathbf{0}\longrightarrow\mathbf{Z}^s\stackrel{\mu}{\longrightarrow}H_1(X^*;\,\mathbf{Z})\stackrel{i'_*}{\longrightarrow}H_1(X;\,\mathbf{Z})\longrightarrow\mathbf{0}\ ,$$

where $i:X^*\hookrightarrow X$ is the identical embedding, and μ sends the standard basis vectors $(e_1,\dots,e_s)$ of the lattice $\mathbf{Z}^s$ into the the vanishing loop classes $\alpha_1,\dots,\alpha_s$ of $D_1,\dots,D_s$, respectively. Moreover, this sequence splits.

Proof. We have the following exact sequence of the fundamental groups:

$$\mathbf{1}\longrightarrow<<\alpha_1,\dots,\alpha_s>>\longrightarrow\pi_1(X^*)\stackrel{i'_*}{\longrightarrow}\pi_1(X)\longrightarrow\mathbf{1}\ ,$$

[14]For instance, this is so if the surface $X_n=X_{n,\,f,\,g}$ is irreducible.

where $<<\alpha_1,\ldots,\alpha_s>>$ denotes the normal subgroup of the group $\pi_1(X^*)$ generated by these vanishing loops (see e.g. [Za 2, (2.3.a)]). Passing to the abelianizations shows that the above homology sequence is exact besides, possibly, at the second term. The divisor D'_i being principal we have $D'_i = g_i^*(0)$ where g_i is a holomorphic function on X, $i = 1, \ldots, s$. The morphism $\varphi = (g_1, \ldots, g_s) : X_1^* \to (\mathbf{C}^*)^s$ yields a surjection

$$\varphi_* : H_1(X^*; \mathbf{Z}) \twoheadrightarrow \mathbf{Z}^s, \ \varphi_*(\alpha(D'_i)) = e_i, \ i = 1, \ldots, s.$$

This provides the exactness at the second term, the splitting $H_1(X^*; \mathbf{Z}) \simeq H \oplus \ker\varphi_*$ where $H =<\alpha_1,\ldots,\alpha_s>$, and an isomorphism $H_1(X; \mathbf{Z}) \simeq \ker\varphi_*$. □

Lemma 4.4. *(a) If the surface $X_n = X_{n,f,g}$ is smooth resp. irreducible resp. $\mathbf{Q}$-acyclic, then so is $X_1 = X_{1,f,g}$.*

(b) Suppose that the surface X_n where $n>1$, is smooth and irreducible, and $H_1(X_n; \mathbf{Z}) = 0$. Then also $H_1(X_1; \mathbf{Z}) = 0$, and the divisor D_g is irreducible. Hence, if X_n is acyclic, then so is X_1.

Proof. (a) The first two statements of (a) easily follow from Lemma 4.2. As for the third one, consider the cyclic ramified covering $\mathbf{C}^3 \ni (x, y, z) \longmapsto (x, y, z^n) \in \mathbf{C}^3$, which restricts to a cyclic covering $\sigma_n : X_n \to X_1$ branched to order n over the curve $D'_g = \{z = 0\}$ in X_1. For any prime p which does not divide n the transfer provides an isomorphism of the homology group $H_*(X_1; \mathbf{Z}_p)$ and the subgroup of the homology group $H_*(X_n; \mathbf{Z}_p)$ fixed by the monodromy action [Bre, III(2.4)]. Thus, $\mathbf{Z}_p-$ acyclicity of X_n implies $\mathbf{Z}_p-$ acyclicity of X_1. If X_n is $\mathbf{Q}-$ acyclic, then it is $\mathbf{Z}_p-$ acyclic for all but finite number of the primes p, and the same holds for X_1. Therefore, X_1 is $\mathbf{Q}-$ acyclic, too. This proves (a).

(b) Let D'_g resp. D''_g be the divisor in X_1 resp. in X_n given by the equation $z = 0$. Set $X_1^* = X_1 \setminus D'_g$ and $X_n^* = X_n \setminus D''_g$. Then $\sigma_n \,|\, X_n^* : X_n^* \to X_1^*$ is an $n-$ sheeted unramified covering, and hence, $(\sigma_n)_*(H_1(X_n^*; \mathbf{Z}))$ is a subgroup of index at most n of the group $H_1(X_1^*; \mathbf{Z})$.

Let $\alpha(D'_i) \in H_1(X_1^*; \mathbf{Z})$ resp. $\alpha(D''_i) \in H_1(X_n^*; \mathbf{Z})$ be the vanishing loop class of the irreducible component D'_i of D'_g resp. D''_i of D''_g, $i = 1, \ldots, s$. By Lemma 4.3, we have the following commutative diagram where the horizontal lines are exact sequences:

$$\begin{array}{ccccccccc}
\mathbf{0} \longrightarrow & \mathbf{Z}^s & \stackrel{\simeq}{\longrightarrow} & H_1(X_n^*; \mathbf{Z}) & \stackrel{i''_*}{\longrightarrow} & H_1(X_n; \mathbf{Z}) = \mathbf{0} \longrightarrow \mathbf{0} \\
 & \downarrow \tau & & \downarrow (\sigma_n)_* & & \downarrow (\sigma_n)_* \\
\mathbf{0} \longrightarrow & \mathbf{Z}^s & \stackrel{\mu}{\longrightarrow} & H_1(X_1^*; \mathbf{Z}) & \stackrel{i'_*}{\longrightarrow} & H_1(X_1; \mathbf{Z}) \longrightarrow \mathbf{0}\ .
\end{array}$$

Clearly, $(\sigma_n)_*(\alpha(D''_i)) = n\alpha(D'_i)$, and so, $\tau(e_i) = ne_i$, $i = 1, \ldots, s$. Hence, the image $(\sigma_n)_*(H_1(X_n^*; \mathbf{Z}))$ is a subgroup of index at least n^s of the group $\ker i'_* \subset H_1(X_1^*; \mathbf{Z})$. On the other hand, it should be a subgroup of index at most n. Therefore, $s = 1$, i.e. D_g is irreducible, and, furthermore, $\ker i'_* = H_1(X_1^*; \mathbf{Z})$. Hence, $H_1(X_1; \mathbf{Z}) = 0$.

The last statement of (b) follows from (a) and Lemma 4.1(a). The lemma is proven. □

Lemma 4.5. *If the surface* $X_n = X_{n,f,g}$ *is smooth, irreducible and* $\mathbf{Q}-$ *acyclic, then the following assertions hold.*

(a) $e(D_f \setminus D_g) = 0$, *or, which is equivalent,* $e(D_f) = \kappa := e(D_f \cap D_g)$.

Furthermore, if $n > 1$, *then* $e(D_g) = e(D_f \cup D_g) = 1$. *If, in addition,* $H_1(X_n;\, \mathbf{Z}) = 0$, *then the divisor* D_g *is smooth, reduced, irreducible and isomorphic to* $\mathbf{C}$.

(b) The irreducible components $D_f^{(i)}$, $i = 1, \ldots, k$, *of the divisor* D_f *are disjoint simply connected curves smooth outside of* D_g; *each of them meets* suppD_g *at one point, which is smooth and reduced on* D_g. *In particular,* $k = e(D_f) = e(D_f \cap D_g) = \kappa \geq 1$.

If, in addition, $H_1(X_1;\, \mathbf{Z}) = 0$, *then each of the curves* $D_f^{(i)}$, $i = 1, \ldots, k$, *is smooth and meets* suppD_g *transversally.*

Proof. (a) Denote by E'' resp. by D_g'' the curve in X_n given by the equations $f = g = 0$ resp. $z = 0$. Consider the disjoint constructive decompositions

$$X := X_n = X' \cup X'' \cup X''' \qquad \text{and} \qquad Y := \mathbf{C}^2 = Y' \cup Y'' \cup Y''' ,$$

where

$$X' = X \setminus (E'' \cup D_g''), \quad X'' = D_g'' \simeq \mathrm{supp} D_g, \quad X''' = E'' \setminus D_g''$$

(so, X''' is a disjoint union of curves isomorphic to $\mathbf{C}^*$), and

$$Y' = \mathbf{C}^2 \setminus (D_f \cup D_g), \quad Y'' = D_g, \quad Y''' = D_f \setminus D_g \,.$$

Clearly, $\sigma_I \,|\, X' : X' \to Y'$ is a non-ramified $n-$ sheeted covering, and hence, $e(X') = ne(Y')$. By the additivity of the Euler characteristic [Du], we have

$$1 = e(X) = e(X') + e(X'') + e(X''') = ne(Y') + e(D_g) \,, \tag{4.1}$$

and

$$1 = e(Y) = e(Y') + e(Y'') + e(Y''') = e(Y') + e(D_g) + e(D_f \setminus D_g) \,. \tag{4.2}$$

Subtracting (2) from (1) we obtain

$$(n-1)e(Y') = e(D_f \setminus D_g) = e(D_f) - \kappa \,. \tag{4.3}$$

Putting here $n = 1$ we obtain the equalities in (a). Since by Lemma 4.4(a), the surface $X_1 := X_{1,f,g}$ is still $\mathbf{Q}-$ acyclic, the case $n > 1$ can be reduced to the case $n = 1$.

Now, if $n > 1$ we obtain from (3) the equality $e(Y') = 0$. By the definition of Y', this yields the equality $e(D_f \cup D_g) = 1$, and also, by (1), we have $e(D_g) = 1$.

If, in addition, $H_1(X_n;\, \mathbf{Z}) = 0$, then by Lemmas 4.2 and 4.4(b), the divisor D_g is smooth, reduced and irreducible. Since $e(D_g) = 1$ it is isomorphic to $\mathbf{C}$. This proves (a).

(b) is proven in Claims 1-6 below.

Claim 1. *Each irreducible component of* D_f *meets* suppD_g.

Proof. Assuming that there exists an irreducible component of D_f which does not meet suppD_g we would get a decomposition $f = f_1 f_2$ where $f_1 \neq$ const and $(f_1, g) = \mathbf{C}^{[2]}$. Let $\xi, \eta \in \mathbf{C}^{[2]}$ be polynomials such that $\xi f_1 + \eta g = 1$. Replacing $\eta g = 1 - \xi f_1$ we obtain the relation $f_1(\xi + \eta f_2 z^n) \,|\, X = 1$, i.e. $f_1 \,|\, X \in \mathbf{C}[X]$ is a non-constant invertible regular function, which is impossible. Indeed, since $b_1(X) = 0$ the regular function $f_1 \,|\, X$ can be expressed as $\exp(\psi)$ where ψ is a non-constant holomorphic function on X. But then $f_1 \,|\, X$ cannot be regular, a contradiction. □

Claim 2. *Each connected component of the curve* suppD_f *is simply connected and meets* D_g *at one point only.*

Proof. Let $D_1, \ldots, D_s$ be the connected components of the curve supp D_f. For a connected affine curve Γ we always have $e(\Gamma) \leq 1$, and $e(\Gamma) = 1$ iff $b_1(\Gamma) = 0$, i.e. iff Γ is simply connected (see e.g. [Za 1]). >From this observation and Claim 1 it follows that $e(D_i \setminus D_g) \leq 0$, $i = 1, \ldots, s$. Since we have by (a),

$$0 = e(D_f \setminus D_g) = \sum_{i=1}^{s} e(D_i \setminus D_g)\,,$$

we obtain that all the summands in the latter sum vanish. Thus, $0 = e(D_i \setminus D_g) = e(D_i) - e(D_i \cap D_g)$. Together with Claim 1 and the above observation this yields the inequalities $1 \leq e(D_i \cap D_g) = e(D_i) \leq 1$. Therefore, $e(D_i \cap D_g) = e(D_i) = 1$, and thus $b_1(D_i) = 0$, i.e. D_i is simply connected and meets D_g at one point only, $i = 1, \ldots, s$. □

Claim 3. $k \leq \kappa$ *where* k *is the number of irreducible components of* D_f.

Proof. Denote $D_f^{(i)}$, $i = 1, \ldots, k$, resp. $D_g^{(j)}$, $j = 1, \ldots, l$, the irreducible components of the divisor D_f resp. D_g. In the notation as in the proof of (a) above, consider the non-ramified $n-$ sheeted covering $\sigma := \sigma_I \,|\, X' \,:\, X' \to Y'$. It is easily seen that $H_1(Y'; \mathbf{Z}) \simeq \mathbf{Z}^{k+l}$. As in the proof of Lemma 4.3 we have the exact sequence

$$\mathbf{Z}^{\kappa+l} \longrightarrow H_1(X', \mathbf{Z}) \longrightarrow H_1(X, \mathbf{Z}) \longrightarrow \mathbf{0}\;.$$

Since $H_1(X, \mathbf{Z})$ is a torsion group it follows that $H_1(X', \mathbf{Z})$ contains a subgroup of a finite index which is a homomorphic image of $\mathbf{Z}^{\kappa+l}$. The image $\sigma_*(H_1(X'; \mathbf{Z}))$ is a finite index subgroup of the group $H_1(Y'; \mathbf{Z})$. Henceforth, $\kappa + l \geq k + l$, or $\kappa \geq k$. □

Claim 4. $k = \kappa$, *and the connected components of the divisor* D_f *coincide with its irreducible components.*

Proof. Indeed, by (a) and Claims 2, 3 we have $\kappa = e(D_f) = s \leq k \leq \kappa$. Hence, $s = k = \kappa$, and the claim follows. □

Claim 5. $D_f \setminus D_g$ *is a smooth divisor. Furthermore, if* $H_1(X_1; \mathbf{Z}) = 0$, *then the divisor* D_f *is smooth.*

Proof. Let $D = D_f^{(i)}$ be an irreducible component of the divisor D_f. It is simply connected and meets D_g at a unique point, say, P. Let $\nu : \mathbf{C} \to D$ be a normalization map such that $\nu(0) = P$. Then the polynomial $r := g \circ \nu \in \mathbf{C}^{[1]}$ vanishes only at zero. This implies that the curve $D \setminus \{P\}$ is smooth (indeed, as $r(t) = ct^m$ for some $c \in \mathbf{C}^*$ and $m \in \mathbf{N}$, the derivative r' vanishes only at the origin).

It remains to show that D is smooth at P providing that $H_1(X_1; \mathbf{Z}) = 0$. Assume on the contrary that P is a singular point of D. Denote $A_1 = \mathbf{C}[X_1]$. Set $\varphi = f_1 \circ \sigma_I \in A_1$ where $f_1 \in \mathbf{C}^{[2]}$ is an irreducible polynomial which defines D, and $\sigma_I : X_1 \to \mathbf{C}^2$, $I = (f, g) \subset \mathbf{C}^{[2]}$, is the blowup morphism. Then $\varphi^{-1}(0) = \sigma_I^{-1}(P) =: E'_P$ is the irreducible component over P of the exceptional divisor $E' \subset X_1$. We claim that E'_P is a multiple fibre of φ whereas its generic fibres are irreducible. Indeed, since the generic fibres of $f_1 \,|\, (\mathbf{C}^2 \setminus D_f)$ are irreducible, in view of the isomorphism $\sigma_I \,|\, (X_1 \setminus E') : X_1 \setminus E' \xrightarrow{\simeq} \mathbf{C}^2 \setminus D_f$ the same is true for $\varphi \,|\, (X_1 \setminus E')$. Further, we have the equalities

$$\text{grad}\,(f_1 \circ \pi) \,|\, E'_P = (\text{grad}_P\, f_1,\, 0) = \overline{0}\,,$$

where $\pi : \mathbf{C}^3 \to \mathbf{C}^2$, $\pi(x, y, z) = (x, y)$. It follows that $(\text{grad}\ \varphi) \,|\, E'_P = 0$ in local coordinates in X_1.

By Lemma 4.4(a), the surface X_1 is $\mathbf{Q}-$ acyclic. Since $H_1(X_1; \mathbf{Z}) = 0$, by Lemma 4.1, actually it is acyclic, and the algebra A_1 is UFD. Thus, in view of $\varphi^*(0) = mE'_P$, where $m > 1$, we have that $\varphi = \varphi_1^m$ for a certain $\varphi_1 \in A_1$. Therefore, the generic fibres of φ cannot be irreducible, which is a contradiction. □

Claim 6. *If $H_1(X_1; \mathbf{Z}) = 0$, then the curves supp D_f and supp D_g meet transversally.*

Proof. We keep all the notation from the proof of Claim 5. Assume that an irreducible component $D := \text{supp}\ D_f^{(i)}$ is tangent to D_g at their unique intersection point P. By Lemma 4.2, the divisor D_g is smooth and reduced at P, and $D_f = mD + \ldots$ for some $m > 1$. Thus, we have $f = f_1^m f_2$ where $f_1 \in \mathbf{C}^{[2]}$ is an irreducible polynomial which defines D. As above, the generic fibres of the regular function $\varphi = f_1 \circ \pi \,|\, X_1 \in A_1 = \mathbf{C}[X_1]$ are irreducible. We claim that $E'_P := \varphi^{-1}(0)$ is a multiple fibre of φ, which contradicts to the fact that A_1 is UFD (see the proof of Claim 5).

Indeed, by our assumption, we have $\text{grad}_P\, f = \gamma\ \text{grad}_P\, g$ for some $\gamma \in \mathbf{C}$. This implies that $(\text{grad}\ \varphi) \,|\, E'_P = 0$ in local coordinates in X, or, which is the same, that $\text{grad}\,(f_1 \circ \pi) \,|\, E'_P$ is proportional to $(\text{grad}\ F) \,|\, E'_P$ where $F := f_1^m f_2 z - g \in \mathbf{C}^{[3]}$ is the defining polynomial of the surface X_1 in $\mathbf{C}^3$. The latter follows from the equalities

$$(\text{grad}\, F) \,|\, E'_P = (mzf_1^{m-1} f_2\, \text{grad}\, f_1 + zf_1^m\, \text{grad}\, f_2 - \text{grad}\, g,\ f_1^m f_2) \,|\, E'_P$$

$$= (-\text{grad}_P\, g,\, 0) = (-\gamma\, \text{grad}_P\, f_1,\, 0) = -\gamma\, \text{grad}\,(f_1 \circ \pi) \,|\, E'_P\,.$$

Henceforth, $(\text{grad}\ \varphi) \,|\, E'_P = 0$, as claimed. □

Now the proof of (b) is completed. □

Lemma 4.6. *(a) Suppose that the surface X_n where $n > 1$, is smooth and irreducible. Then the implication (i) $\Longrightarrow$ (v) of Theorem 4.2 holds.*

(b) Suppose further that the surface X_1 is smooth, irreducible and $\mathbf{Q}-$ acyclic. Assume also that the curves supp D_f and supp D_g meet transversally[15]*. Then, by an appropriate automorphism $\alpha' = (\alpha, \mathrm{id}) \in \mathrm{Aut}\,(\mathbf{C}^2 \times \mathbf{C})$ of $\mathbf{C}^3$, the equation of X can be reduced to the following form:*

$$p(x)z = y + xh(x,\, y) + \mathrm{const.}$$

Proof. (a) If the condition $(\ i\ )$ of Theorem 4.2 is fulfilled, then by Lemmas 4.1(a) and 4.4(b), both surfaces X_1 and X_n are acyclic. Thus, by Lemma 4.5, $D_g \simeq \mathbf{C}$ and $D_f^{(i)} \simeq \mathbf{C}$ for each $i = 1, \ldots, \kappa$.

Let $f_1 \in \mathbf{C}^{[2]}$ be an irreducible polynomial which defines the irreducible component $D_f^{(1)}$ of D_f. Since by Lemma 4.5(b), the components $D_f^{(i)}$, $i = 1, \ldots, \kappa$, of D_f are disjoint and simply connected, for each $i = 1, \ldots, \kappa$ we have $f_1 \,|\, D_f^{(i)} \equiv \mathrm{const} = c_i$, and therefore, $f = p(f_1)$ for some polynomial $p \in \mathbf{C}^{[1]}$. In view of Lemma 4.5(b), the curves $D_f^{(1)} \simeq \mathbf{C}$ and $D_g \simeq \mathbf{C}$ meet transversally at a unique point. By the Abhyankar-Moh and Suzuki Embedding Theorem, it follows that $\alpha := (f_1,\, g) \,:\, \mathbf{C}^2 \to \mathbf{C}^2$ is an automorphism which transforms the pair $(f_1,\, g)$ into a pair of coordinate functions $(x,\, y)$, and the pair $(f,\, g)$ into the pair $(p(x),\, y)$, as desired. This proves the implication $(\ i\ ) \implies (\ v\ )$ of Theorem 4.2.

(b) By the Abhyankar-Moh and Suzuki Embedding Theorem, the smooth simply connected curve $D_f^{(1)} \simeq \mathbf{C}$ can be transformed into a coordinate line $x = 0$ by an automorphism of $\mathbf{C}^2$. Thus, we may assume that $f_1(x,\, y) = x$, and so, as above, $f(x,\, y) = p(x)$ and $X = \{p(x)z = g(x,\, y)\} \subset \mathbf{C}^3$.

If x_i is a root of p, that is, $D_f^{(i)} = \{x = x_i\}$, then by Lemma 4.5(b), the polynomial $g(x_i,\, y) \in \mathbf{C}[y]$ has only one root, say, c_i, and this root is simple, that is, $g(x_i,\, y) = \gamma_i(y - c_i)$. Hence, $g(x,\, y) = \gamma_i(y - c_i) + (x - x_i)h_i(x,\, y)$ for a certain $h_i \in \mathbf{C}^{[2]}$. Plugging here $x_1 = 0$ and replacing z by z/γ_1, we obtain the desired presentation. □

This lemma and the next proposition yield the implication $(i) \implies (iv)$ of Theorem 4.2 for $n = 1$. Denote by G the subgroup of the group Aut $\mathbf{C}^3$ which consists of the automorphisms of the type $(x,\, y,\, z) \longmapsto (x,\, \gamma_1 y + xg_1,\, \gamma_2 z + xg_2)$ where $g_i \in \mathbf{C}^{[3]}$ and $\gamma_i \in \mathbf{C}^*$, $i = 1,\, 2$.

Proposition 4.1. *Let X be an irreducible smooth $\mathbf{Q}-$ acyclic surface given in $\mathbf{C}^3$ by the equation $p(x)z = g(x,\, y)$ where $p \in \mathbf{C}^{[1]} \setminus \mathbf{C}$ and $g \in \mathbf{C}^{[2]} \setminus \mathbf{C}$. Suppose that the curves supp D_p and supp D_g meet transversally*[16]*. Then X can be transformed into a plane $L_c := \{y = c\}$, $c \in \mathbf{C}$, by an automorphism $\alpha \in G$.*

The proof proceeds by induction on deg p.

Claim 1. *The statement is true if* $\deg p = 1$.

[15] In particular, as follows from Lemma 4.5(b), supp D_f is smooth.
[16] By Lemma 4.5(b), this is true if X satisfies the condition $(\ i\ )$ of Theorem 4.2.

Proof. We may assume that $p(x) = x$, and by Lemma 4.5(b), that $g(x, y) = y + xh(x, y) - c$, $c \in \mathbf{C}$. Then the automorphism $\alpha \in \operatorname{Aut} \mathbf{C}^3$, $\alpha(x, y, z) = (x, y, z + h)$, transforms X into the surface $X' := \{xz = y - c\}$. Furthermore, the automorphism $\beta \in G$, $\beta(x, y, z) = (x, y + xz, z)$, transforms X' into the plane L_c. The resulting automorphism $\alpha' := \beta \circ \alpha$, $\alpha'(x, y, z) = (x, y + x(z + h_1), z + h_1)$ where $h_1 = h(x, y + xz)$, does not belong, in general, to the group G. But composing it further with the automorphism $\beta' \in \operatorname{Aut} \mathbf{C}^3$, $\beta'(x, y, z) = (x, y, z - h_1(0, y))$, we obtain a new one $\alpha'' = \beta' \circ \alpha'$ which does belong to G. It remains to note that β' preserves the plane L_c, hence, $\alpha''(X) = \alpha'(X) = L_c$, and we are done. $\square$

Claim 2. *Let $p(x) = xq(x)$ where $\deg q \geq 1$. Consider the surface $Y = \{q(x)z = g(x, y)\}$ in $\mathbf{C}^3$. Then $X = \Sigma_{I_1, \xi}(Y)$ is the affine modification of Y along the divisor D_ξ with center $I_1 = (\xi, \eta) \subset \mathbf{C}[Y]$ where $\xi = x \,|\, Y$ and $\eta = z \,|\, Y$. Furthermore, Y is a smooth irreducible $\mathbf{Q}-$ acyclic surface; it is acyclic if X is.*

Proof. The modification $\Sigma_{I, x}(\mathbf{C}^3)$ along the plane $D_x = \{x = 0\}$ with center $I := (x, z) \subset \mathbf{C}^{[3]}$ is isomorphic to $\mathbf{C}^3$, and the blowup morphism $\sigma_I : \mathbf{C}^3 \to \mathbf{C}^3$ is given as $(x, y, z) \longmapsto (x, y, xz)$ (cf. Examples 1.4, 2.1). By Proposition 2.2(b), the strict transform $X = Y'$ of Y under σ_I coincides with the modification $\Sigma_{I_1, \xi}(Y)$. This proves the first assertion.

It is easily seen that under our assumptions, X being smooth and irreducible implies the same for Y (cf. Lemma 4.2).

Due to our assumptions and to Lemma 4.5(b), the irreducible component $D_p^1 := \{x = 0\}$ of the divisor $D_p = p^*(0)$ in $\mathbf{C}^2$ meets the divisor D_g transversally at one point, say, $P_0 = (0, y_0)$, and D_g is reduced at P_0. Thus, we have $g(0, y) = \gamma(y - y_0)$. Hence, the polynomials $x, g(0, y), z \in \mathbf{C}^{[3]}$ generate the maximal ideal of the point $P_0' := (0, y_0, 0) \in Y \subset \mathbf{C}^3$. It follows that $I_1 = (\xi, \eta) = I(P_0')$ is a maximal ideal of the algebra $\mathbf{C}[Y]$. Thereby, the blowup morphism $\sigma : X \to Y$, $\sigma := \sigma_{I_1} = \sigma_I \,|\, X$, consists of the usual blow up at P_0' and deleting the strict transform of the curve $l_0 := \{\xi = 0\} \subset Y$, $l_0 \simeq \mathbf{C}$, passing through P_0' (see Proposition 1.1). By Theorem 3.1 (see also [Ka 1, Thm. 3.5], [Za 2, Thm. 5.1]), this modification preserves the homology, i.e. $\sigma_* : H_*(X; \mathbf{Z}) \longrightarrow H_*(Y; \mathbf{Z})$ is an isomorphism. Thus, Y is $\mathbf{Q}-$ acyclic resp. acyclic iff X is. $\square$

Claim 3: The induction step. *Let Y be as in Claim 2. Suppose that Y can be transformed into a plane $L_c := \{y = c\}$, $c \in \mathbf{C}$, by an automorphism $\alpha \in G$. Then the same is true for X.*

Proof. Notice that the action of the group G on $\mathbf{C}^{[3]}$ preserves the ideal $I = (x, z)$ and fixes the element $x \in I$. Therefore, by Corollary 2.2, α can be lifted in a unique way to an automorphism α' of $\mathbf{C}^3 = \Sigma_{I, x}(\mathbf{C}^3)$ such that $\alpha \circ \sigma_I = \sigma_I \circ \alpha'$. The automorphism α' is of the form

$$\alpha' : (x, y, z) \longmapsto (x, \gamma_1 y + x g_1(x, y, xz), \gamma_2 z + g_2(x, y, xz))$$

(see Example 2.2). It sends the surface $X = Y'$ onto the strict transform $L_c' = L_c$ of the plane L_c, i.e. again onto the plane L_c (indeed, $\sigma_I^*(y) = y$). Now, composing α'

with the automorphism $\beta \in \text{Aut}\, \mathbf{C}^3$, $\beta(x, y, z) = (x, y, z - g_2(0, \gamma_1^{-1}y, 0))$, we get an automorphism $\alpha'' := \beta \circ \alpha'$ which, as it can be easily seen, belongs to the group G. Since β preserves the plane L_c, α'' still transforms X into this plane. This proves Claim 3 and completes the proof of the proposition. $\square$.

Now the proof of Theorem 4.2 is completed.

Example 4.1. Consider the surface $X \subset \mathbf{C}^3$ given by the equation $x^2 z = x + y^2$. It is easily seen that X is $\mathbf{Q}-$ acyclic but not acyclic. Indeed, in the notation as in Proposition 4.1 we have $E' \simeq \mathbf{C}$, $X \setminus E' \simeq \mathbf{C}^* \times \mathbf{C}$ and $\sigma^*(D_p) = 2E'$, which provides that $H_i(X;\, \mathbf{Z}) = 0$ for $i \ge 2$ and $H_1(X;\, \mathbf{Z}) \simeq \mathbf{Z}/2\mathbf{Z}$ (cf. the proof of Theorem 3.1). Thus, the assumption of Proposition 4.1 that the curves $\text{supp}\, D_p$ and $\text{supp}\, D_g$ meet transversally is essential, as well as the condition $E' = \sigma^*(D)$ in (i) of Theorem 3.1. This example also shows that Theorem 4.2 cannot be extended to $\mathbf{Q}-$ acyclic surfaces.

5. Topology of the hypersurfaces $uv = p(x_1, \dots, x_k)$

Notation. Let $X = X(p) \subset \mathbf{C}^{k+2}$ be the irreducible hypersurface given by the equation $uv = p(\bar{x})$ where $\bar{x} = (x_1, \dots, x_k) \in \mathbf{C}^k$, $k \ge 1$, and $p \in \mathbf{C}^{[k]}$, $\deg p > 0$. For $c \in \mathbf{C}$ denote $U_c = X \cap \{u = c\}$, $V_c = X \cap \{v = c\}$, and $X_0 = p^{-1}(0) \subset \mathbf{C}^k$. We also regard X_0 as the subvariety of X given in $\mathbf{C}^{k+2}$ by the equations $u = v = p(\bar{x}) = 0$.

Remark 5.1. The variety $X(p)$ is the affine modification of $\mathbf{C}^{k+1}_{\bar{x},u}$ along the hyperplane $D_u = \{u = 0\}$ with center $I = (p, u) \subset \mathbf{C}^{[k+1]}$ and with the exceptional divisor $U_0 \subset X(p)$ (see Example 1.6). If the divisor $p^*(0)$ is reduced, then $X(p) = \Sigma_{X_0, D_u}(\mathbf{C}^{k+1})$.

The following lemma is easy, and we omit the proof.

Lemma 5.1. *(a) There are natural isomorphisms* $X \setminus U_0 \simeq X \setminus V_0 \simeq \mathbf{C}^k \times \mathbf{C}^*$, $U_0 \simeq V_0 \simeq X_0 \times \mathbf{C}$, *and* $U_0 \cap V_0 = X_0$.

(b) X *is smooth iff* $p^*(0)$ *is a smooth reduced (not necessarily irreducible) divisor in* $\mathbf{C}^k$ *(supported by* X_0 *). Furthermore, if the divisor* $p^*(0)$ *is reduced, then we have*

$$\text{sing}\, X = \text{sing}\, U_0 \cap \text{sing}\, V_0 = \text{sing}\, X_0, \quad \textit{and} \quad \text{sing}\, U_0 \simeq (\text{sing}\, X_0) \times \mathbf{C} \simeq \text{sing}\, V_0\,.$$

Corollary 5.1. $e(X) = e(X_0)$.

Proposition 5.1. *(a) Let* $p = p_1^{d_1} \dots p_l^{d_l}$ *be the canonical factorization of the polynomial* p. *Then* $\pi_1(X) \simeq \mathbf{Z}/(d)$ *where* $d = g.c.d.\ (d_1, \dots, d_l)$.

(b) Assume that X *is smooth. Then there is an isomorphism of the reduced homology groups* $\tilde{H}_*(X;\, \mathbf{Z}) \simeq \tilde{H}_{*-2}(X_0;\, \mathbf{Z})$.

Proof. (a) By Lemma 5.1(a), we have that $\pi_1(X \setminus U_0) \simeq \pi_1(\mathbf{C}^k \times \mathbf{C}^*) \simeq \mathbf{Z}$. The kernel of the natural surjection $i_* : \pi_1(X \setminus U_0) \to \pi_1(X)$ is generated, as a normal subgroup, by vanishing loops, say, α_i of the irreducible components $U_0^{(i)} \simeq X_0^{(i)} \times \mathbf{C}$ of U_0 where $X_0^{(i)} := p_i^{-1}(0) \subset \mathbf{C}^k$, $i = 1, \ldots, l$ (see e.g. [Za 2, (2.3.a)]).

Set $\rho = (u \,|\, (X \setminus U_0))_* : \pi_1(X \setminus U_0) \to \pi_1(\mathbf{C}^*) \simeq \mathbf{Z}$. Fix a generator t of the group $\pi_1(X \setminus U_0) \simeq \mathbf{Z}$ such that $\rho(t) = 1$. Clearly, $\rho(\alpha_i) = d_i \in \mathbf{Z} = \pi_1(\mathbf{C}^*)$; that is, $\alpha_i = d_i \cdot t$, $i = 1, \ldots, l$, and hence, $\mathrm{Ker}\, i_* = \langle \alpha_1, \ldots, \alpha_l \rangle = d\mathbf{Z}$. Thus, $\pi_1(X) \simeq \mathbf{Z}/(d)$, as claimed. □

(b) Since X is assumed being smooth, by Lemma 5.1(b), X_0 is smooth and reduced, and so, $d = 1$. Then by (a), $\pi_1(X) = \mathbf{1}$, and hence, also $H_1(X; \mathbf{Z}) = 0$. Due to Lemma 5.1(a), from the exact homology sequence of the pair $(X, X \setminus U_0)$ we obtain the isomorphisms of the $\mathbf{Z}-$ homology groups: $H_i(X) \simeq H_i(X, X \setminus U_0)$ for all $i \geq 3$. In view of the Thom isomorphism, which we denote below by τ, this yields

$$H_i(X) \simeq H_i(X, X \setminus U_0) \overset{\tau}{\simeq} H_{i-2}(U_0) \simeq H_{i-2}(X_0) \;\; \forall \; i \geq 3\,.$$

For $i = 2$ we get:

$$0 = H_2(X \setminus U_0) \to H_2(X) \to H_2(X, X \setminus U_0) \overset{\tau}{\simeq} H_0(U_0)$$

$$\simeq H_0(X_0) \xrightarrow{\partial_*} \mathbf{Z} \simeq H_1(X \setminus U_0) \to 0 = H_1(X)\,.$$

Since X_0 is a smooth reduced divisor, the number $l = \deg p$ of its irreducible components coincides with the number $b_0(X_0)$ of its connected components. Recall (see e.g. [MilSta]) that the Thom isomorphism $\tau : H_0(U_0) \xrightarrow{\simeq} H_2(X, X \setminus U_0)$ to each point $P \in U_0$ associates the relative homology class $[\Delta_P] \in H_2(X, X \setminus U_0)$ of a disc $\Delta_P \subset X$ centered at P and transversal to U_0; furthermore, $\partial_*[\Delta_P] = [\partial \Delta_P] \in H_1(X \setminus U_0)$. Thus, for $\beta = (\beta_1, \ldots, \beta_l) \in H_2(X, X \setminus U_0) \simeq H_0(X_0) \simeq \mathbf{Z}^l$ we have $\partial_*(\beta) = \sum_{i=1}^l \beta_i \in H_1(X \setminus U_0) \simeq \mathbf{Z}$. It follows that $H_2(X) \simeq \ker \partial_* \simeq \mathbf{Z}^{b_0(X_0)-1} \simeq \tilde{H}_0(X_0)$; in particular, $H_2(X) = 0$ iff X_0 is irreducible. Since X is irreducible, we conclude that $\tilde{H}_*(X) \simeq \tilde{H}_{*-2}(X_0)$, and the statement of (b) follows. □

Corollary 5.2. *(a) X is simply connected iff $d = g.c.d.\,(d_1, \ldots, d_l) = 1$.*

(b) Suppose that X is smooth. Then X is contractible iff X_0 is acyclic; actually, in this case X is diffeomorphic to $\mathbf{R}^{2k+2}$.

For $k > 1$ the last statement of (b) follows from the Dimca-Ramanujam Theorem [Di 1, Ram] (see also [Za 2, Thm 4.2]), and for $k = 1$ it follows from the next proposition.

Proposition 5.2. *(a) Let $k = 1$. The following conditions are equivalent:*

(i) $X \simeq \mathbf{C}^2$;

(ii) $X \subset \mathbf{C}^3$ is a smooth acyclic surface;

(iii) $\deg p = 1$.

(b) Let $k = 2$. The following conditions are equivalent:

(i) $X \simeq \mathbf{C}^3$;

(ii) $X \subset \mathbf{C}^4$ *is a smooth acyclic 3-fold;*

(iii) the divisor $p^*(0)$ *is reduced and* $X_0 \simeq \mathbf{C}$;

(iv) the polynomial $p \in \mathbf{C}^{[2]}$ *is equivalent to a linear one, i.e. for some* $\alpha \in Aut\,\mathbf{C}^2$ *we have* $p \circ \alpha = q$ *where* $q(x_1, x_2) = x_1$.

Proof. (a) Let $k=1$. The implications (*iii*) $\Longrightarrow$ (*i*) $\Longrightarrow$ (*ii*) are clear. Conversely, by Corollary 5.1 and Lemma 5.1(b), under the assumption (*ii*) we have $1 = e(X) = e(X_0) = \deg\, p$. This proves (a).

(b) The theorem of Abhyankar-Moh and Suzuki [AM, Suz] asserts the equivalence (*iii*) $\Longleftrightarrow$ (*iv*). The implications (*iv*) $\Longrightarrow$ (*i*) $\Longrightarrow$ (*ii*) are easy; (*ii*) $\Longrightarrow$ (*iii*) follows from Lemma 5.1(b) and Proposition 5.1(b), and so, we are done. □

Remark 5.2. However, starting with $k=3$ the equivalences of Proposition 5.2 fail; see Examples 7.1 and 7.2 below.

A generalization. More generally, consider the variety $X = X(p_1, \dots, p_m) \subset \mathbf{C}^{k+m+1}$ given by a system of equations $uv_i = p_i(\bar{x})$, $i = 1, \dots, m$ where $\bar{x} = (x_1, \dots, x_k) \in \mathbf{C}^k$, $k \geq m \geq 1$, and $p_i \in \mathbf{C}^{[k]} \setminus \mathbf{C}$, $i = 1, \dots, m$. This variety X is the affine modification of $\mathbf{C}^{k+1}_{(\bar{x},\,u)}$ along the hyperplane $D_u = \{u = 0\}$ with center at the ideal $I = (p_1, \dots, p_m,\, u) \subset \mathbf{C}^{[k+1]}$ supported by the affine variety $X_0 := \{p_1(\bar{x}) = \ldots = p_m(\bar{x}) = 0\} \subset D_u \simeq \mathbf{C}^k$. Clearly, X is irreducible iff X_0 is a set-theoretic complete intersection, i.e. all its irreducible components are codimension m subvarieties of $\mathbf{C}^k$, which will be always assumed in the sequel. Under this assumption most of the results proved above for $m=1$ remain true in this more general setting. Namely, we have the following statements.

Proposition 5.3. *(a) Denote by* U_0 *the hypersurface in* $X = X(p_1, \dots, p_m)$ *given by the equation* $u = 0$. *Then we have* $U_0 \simeq X_0 \times \mathbf{C}^m$ *and* $X \setminus U_0 \simeq \mathbf{C}^* \times \mathbf{C}^m$. *In particular,* $e(X) = e(X_0)$.

(b) Assume further that

(i) X_0 *is the ideal-theoretic complete intersection of the divisors* $p_i^*(0)$, $i = 1, \dots, m$, *that is,* $I(X_0) = (p_1, \dots, p_m)$.

Then we have

$$\operatorname{sing} X_0 \subset \operatorname{sing} X \subset \operatorname{sing} U_0 \simeq (\operatorname{sing} X_0) \times \mathbf{C}^m \,.$$

In particular, X *is smooth iff* X_0 *is.*

(c) Suppose that the assumption (i) is fulfilled and X *is smooth. Then we have* $\pi_1(X) = \mathbf{1}$ *and* $\tilde{H}_*(X; \mathbf{Z}) \simeq \tilde{H}_{*-2}(X_0; \mathbf{Z})$. *In particular,* X *is contractible (and, moreover, diffeomorphic to* $\mathbf{R}^{2k+2}$ *for* $k \geq 2$ *) iff* X_0 *is acyclic.*

The proof goes exactly in the same way as before, and so, we leave it to the reader.

Remark 5.3. Presumably, even being diffeomorphic to the affine space, the variety $X = X(p_1, \ldots, p_m)$ is not, in general, isomorphic to $\mathbf{C}^{k+1}$ for $k \geq 3$, and so, it should provide an exotic algebraic structure on $\mathbf{C}^{k+1}$ (see e.g. [Za 2]). But at present we have no invariant available to distinguish X from the affine space (see Remark 6.1 below).

Suppose that, indeed, for a certain smooth acyclic complete intersection $X_0 \subset \mathbf{C}^k$ the variety $X = X(p_1, \ldots, p_m)$ is not isomorphic to $\mathbf{C}^{k+1}$. Then we would have an example showing that Miyanishi's characterization of the affine 3-space $\mathbf{A}^3_k$ [Miy] does not hold any more in higher dimensions. Indeed, by Proposition 5.3, the varieties X and $U_0 \simeq X_0 \times \mathbf{C}^m$ being smooth and acyclic we have $e(X) = e(U_0) = 1$, $X \setminus U_0 \simeq \mathbf{C}^* \times \mathbf{C}^k$, the algebras $\mathbf{C}[X]$ and $\mathbf{C}[U_0]$ are UFD and have only constants as the units (see e.g. [Ka 1, Prop. 3.2]). Thus, all the assumptions of the Miyanishi Theorem are fulfilled, whereas $X \not\simeq \mathbf{C}^{k+1}$.

Further, if for a certain codimension m smooth acyclic complete intersection $X_0 \subset \mathbf{C}^k$ non-isomorphic to $\mathbf{C}^{k-m}$ the variety X were isomorphic to $\mathbf{C}^{k+1}$, this would answer in negative, alternatively, either to the Zariski Cancellation Problem[17] or to the Abhyankar-Sathaye Embedding Problem[18]. Indeed, the former happens if the hypersurface $U_0 = X_0 \times \mathbf{C}^m \subset X$ is isomorphic to $\mathbf{C}^k$ (which is only possible if X_0 was contractible). Otherwise, the latter takes place since U_0 is the zero fibre of the polynomial $u \,|\, X \in \mathbf{C}[X] \simeq \mathbf{C}^{[k+1]}$ with all other fibres U_c, $c \neq 0$, isomorphic to $\mathbf{C}^k$. Observe that due to the Miyanishi-Sugie and Fujita Cancellation Theorem, for $k - m \leq 2$ only the second possibility might happen.

Formally, there is also a possibility that $X \not\simeq \mathbf{C}^{k+1}$ whereas $U_0 \simeq \mathbf{C}^k$. In that case we would have an example of an exotic $\mathbf{C}^n$, $n = k + 1 \geq 4$, fibered by the affine spaces $U_c \simeq \mathbf{C}^{n-1}$.

We may enlarge our collection of contractible affine varieties passing to ramified cyclic coverings over $X = X(p_1, \ldots, p_m)$, as follows.

Proposition 5.4. *(a) Suppose that the variety $X = X(p_1, \ldots, p_m)$ as in Proposition 5.3(c) above is smooth and contractible. Then for any $n \in \mathbf{N}$ the variety $X_n \subset \mathbf{C}^{k+m+1}$ given by the system of equations $u^n v_i = p_i(\bar{x})$, $i = 1, \ldots, m$, is smooth and contractible, too.*

(b) For a sequence of integers $s_0, \ldots, s_m \in \mathbf{N}$ such that gcd $(s_i, s_j) = 1$ for all $i \neq j$, consider the variety

$$Y = Y_{s_0,\ldots,s_m}(p_1, \ldots, p_m) := \{u^{s_0} v_i^{s_i} = p_i(\bar{x}),\ i = 1, \ldots, m\} \subset \mathbf{C}^{k+m+1}$$

[17] In the particular case when $k = 3$ and $X_0 \simeq \mathbf{C}$ is a complete intersection given by $p_1(\bar{x}) = p_2(\bar{x}) = 0$ in $\mathbf{C}^3$, the smooth contractible 4-folds $X = X(p_1, p_2) \subset \mathbf{C}^5$ were studied (in algebraic fashion) in [As] as potential counterexamples to the Zariski Cancellation Problem. Indeed, in [As] an isomorphism $X \times \mathbf{C} \simeq \mathbf{C}^5$ was established.

[18] Cf. also Remark 7.3 below for another conjectural counterexample to the Abhyankar-Sathaye Embedding Problem.

where $p_i \in \mathbf{C}^{[k]}$, $i = 1, \ldots, m$. *Suppose that the following conditions are fulfilled:*

(i) $p := p_1 \cdot \ldots \cdot p_m$ *is a prime decomposition, and* $D := p^*(\bar{0})$ *is a reduced simple normal crossing divisor in* $\mathbf{C}^k$;

(ii) the divisor $D_i := D_{p_i}$ *is* $\mathbf{Z}_q-$ *acyclic for any prime divisor* q *of* s_i,, $i = 1, \ldots m$, *and* $X_0 = \cap_{i=1}^m D_i \subset \mathbf{C}^k$ *is a smooth acyclic complete intersection;*

(iii) the group $\pi_1(\mathbf{C}^k \setminus D)$ *is abelian (and hence, isomorphic to* $\mathbf{Z}^m$ *).*

Then Y *is a smooth contractible variety (diffeomorphic to* $\mathbf{R}^{2k+2}$ *if* $k \geq 2$ *).*

Proof. (a) The variety X_n is a cyclic covering of X ramified to order n on U_0 with the covering morphism $\rho : X_n \to X$, $\rho : (\bar{x}, u, \bar{v}) \longmapsto (\bar{x}, u^n, \bar{v})$. By Proposition 5.3(c), the ramification divisor $U_0 = X_0 \times \mathbf{C}^m$ is acyclic, the fundamental group of its complement $\pi_1(X \setminus U_0) \simeq \mathbf{Z}$ is abelian, and the regular function $u \,|\, (X \setminus U_0)$ where $X \setminus U_0 \simeq \mathbf{C}^* \times \mathbf{C}^k$, is a quasi-invariant of weight 1 of the natural $\mathbf{C}^*-$ action. Now the assertion of (a) follows from Theorem A in [Ka 3] (see also [Za 2, Thm. 7.1]).

(b) Denote $V_0^{(i)} = X \cap \{v_i = 0\}$, $i = 1, \ldots, m$ where $X = X(p_1, \ldots, p_m)$. The morphism $\mathbf{C}^{k+m+1} \to \mathbf{C}^{k+m+1}$, $(\bar{x}, u, v_1, \ldots, v_m) \longmapsto (\bar{x}, u^{so}, v_1^{s_1}, \ldots, v_m^{s_m})$, restricted to Y makes Y a multicyclic covering of X branched to order s_0 over U_0 resp. to order s_i over $V_0^{(i)}$, $i = 1, \ldots, m$. Thus, we may use Theorem 8.1 in [Za 2] which provides conditions to guarantee contractability of a multicyclic covering over a contractible manifold. To see that these conditions are satisfied, first of all, we observe that the function u resp. v_i, $i = 1, \ldots, m$, is a quasi-invariant of weight 1 resp. -1 of the $\mathbf{C}^*-$ action $(\lambda, (\bar{x}, u, \bar{v})) \longmapsto (\bar{x}, \lambda u, \lambda^{-1}\bar{v})$ on $\mathbf{C}^{k+m+1}$ which leaves the variety X invariant.

Further, the hypersurface $U_0 \simeq X_0 \times \mathbf{C}^m$ is smooth and acyclic since X_0 is. For each $i = 1, \ldots, m$ the hypersurface $V_0^{(i)}$ is the affine modification of the smooth variety $D_i \times \mathbf{C}$ along the divisor D_i with center $X_0 \subset D_i$ and with the exceptional divisor $E_i := U_0 \cap V_0^{(i)} \simeq X_0 \times \mathbf{C}^{m-1}$ via the morphism $\sigma_i : V_0^{(i)} \to D_i \times \mathbf{C}$ which is the restriction to $V_0^{(i)}$ of the projection $\pi : \mathbf{C}^{k+m+1} \to \mathbf{C}^{k+1}$, $\pi(\bar{x}, u, \bar{v}) = (\bar{x}, u)$ (see Corollary 2.1).

Notice that the proof of Theorem 3.1 on preservation of the homology under a modification goes equally for the $\mathbf{Z}_p-$ homology groups. In virtue of the condition (ii) above, by this Theorem, the smooth hypersurface $V_0^{(i)} \subset X$ is $\mathbf{Z}_q-$ acyclic for any prime divisor q of s_i, $i = 1, \ldots, m$.

We have an isomorphism $X^* := X \setminus (U_0 \cup V_0^{(i)} \cup \ldots \cup V_0^{(i)}) \simeq (\mathbf{C}^k \setminus D) \times \mathbf{C}^*$. Hence by the condition (iii), the fundamental group $\pi_1(X^*)$ is abelian. Now all the assumptions of Theorem 8.1 in [Za 2] are varified. By this theorem, Y is contractible. □

Corollary 5.3. *Let* $X_0 = p^*(0)$, $p \in \mathbf{C}[k]$, *be a smooth reduced acyclic hypersurface in* $\mathbf{C}^k$. *Then for any* $n \in \mathbf{N}$ *the hypersurface* $X_n := \{u^n v = p(\bar{x})\}$ *in* $\mathbf{C}^{k+2}$ *is smooth and contractible.*

If, furthermore, $\pi_1(\mathbf{C}^k \setminus X) \simeq \mathbf{Z}$, *then for any relatively prime integers* $s_0, s_1 \in \mathbf{N}$ *the hypersurface* $Y_{s_0, s_1} := \{u^{s_0}v^{s_1} = p(\bar{x})\}$ *in* $\mathbf{C}^{k+2}$ *is smooth and contractible.*

Remark 5.4. For instance, one may take as X_0 the tom Dieck-Petrie surface $X_{k,l} \subset \mathbf{C}^3$ (see Example 7.1 below). Indeed, it is easily seen that it satisfies all the conditions of Corollary 5.3.

6. $\mathbf{C}_+-$ actions on the hypersurfaces $uv = p(x_1, \dots, x_k)$

This section is devoted to the proof of the following theorem.

The Transitivity Theorem. *Let* $X = \{uv - p(\bar{x}) = 0\} \subset \mathbf{C}^{k+2}$ *where* $k \geq 2$ *and* $p \in \mathbf{C}^{[k]} \setminus \mathbf{C}$. *Then the automorphism group Aut* X *acts* $m-$ *transitively on* $X \setminus \operatorname{sing} X$ *for any* $m \in \mathbf{N}$.

We keep all the notation from Section 5. Set $\sigma_i = \pi_i \,|\, X$, $i = 1, 2$ where $\pi_i : \mathbf{C}^{k+2} \to \mathbf{C}^{k+1}$, $\pi_1 : (\bar{x}, u, v) \longmapsto (\bar{x}, v)$, and $\pi_2 : (\bar{x}, u, v) \longmapsto (\bar{x}, u)$, are the canonical projections. Then $\sigma_1 : X \to \mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$ resp. $\sigma_2 : X \to \mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_u$ is the affine modification of $\mathbf{C}^{k+1}$ along the hyperplane $D_v = \{v = 0\}$ resp. $D_u = \{u = 0\}$ with center[19] $I_1 = (p, v) \subset \mathbf{C}^{[k+1]}$ resp. $I_2 = (p, u) \subset \mathbf{C}^{[k+1]}$ and with the exceptional divisor $V_0 \subset X$ resp. $U_0 \subset X$ (see Example 1.6).

Concretizing Corollaries 2.2 and 2.3 in our setting we obtain the following statement.

Lemma 6.1. *Let* $\varphi : \mathbf{C}_+ \times \mathbf{C}^{k+1} \to \mathbf{C}^{k+1}$ *resp.* $\varphi : \mathbf{C}^{k+1} \to \mathbf{C}^{k+1}$ *be a regular* $\mathbf{C}_+-$ *action on* $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$ *resp. an automorphism of* $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$. *Suppose that* φ *leaves the subvarieties* $D_v = \{v = 0\}$ *and* X_0 *invariant. Then there is a unique regular* $\mathbf{C}_+-$ *action* $\widehat{\varphi}$ *on* X *resp. an automorphism* $\widehat{\varphi}$ *of* X *which leaves the hypersurface* V_0 *invariant and such that the restriction* $\widehat{\varphi} \,|\, (X \setminus V_0)$ *coincides with* $\sigma_1^{-1}\varphi\sigma_1$.

Notation. Let G_1 resp. G_2 be the subgroup of the group Aut $\mathbf{C}^{k+1}$ where $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$ resp. $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_u$, generated by all the $\mathbf{C}_+-$ subgroups T of Aut $\mathbf{C}^{k+1}$ such that the function v resp. u is a $T-$ invariant, and the restriction of T to the invariant hyperplane D_v resp. D_u leaves the subvariety X_0 invariant. Denote $\widehat{G}_i$, $i = 1, 2$, the subgroup of the group Aut X which corresponds to G_i in view of Lemma 6.1, and let $\widehat{G} \subset$ Aut X be the subgroup generated by $\widehat{G}_1$ and $\widehat{G}_2$. Notice that $\widehat{G}_2 = \varepsilon\widehat{G}_1\varepsilon^{-1}$ where $\varepsilon \in$ Aut$X$, $\varepsilon : (\bar{x}, u, v) \longmapsto (\bar{x}, v, u)$.

The Transitivity Theorem can be precised as follows.

Theorem 6.1. *The group* $\widehat{G}$ *acts* $m-$ *transitively on* $X \setminus \operatorname{sing} X$ *for any* $m \in \mathbf{N}$.

[19] with center $C_1 = X_0 \subset D_v$ resp. $C_2 = X_0 \subset D_u$ if the divisor $p^*(0)$ is reduced.

Remark 6.1. Recall [KaML 1, Za 2, (9.2)] that the *Makar-Limanov invariant* of an algebra A over $\mathbf{C}$ is the subalgebra $\mathrm{ML}\,(A) \subset A$ which consists of all the elements invariant under every $\mathbf{C}_+-$ subgroup of the automorphism group $\mathrm{Aut}\,A$; or, which is the same, $\mathrm{ML}\,(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \ker \partial$ (see Definition 2.2). From Theorem 6.1 it follows that for the algebra $A = \mathbf{C}[X]$ where X is as above, this invariant is trivial: $\mathrm{ML}\,(A) = \mathbf{C}$. The problem arises to find a substitution of the Makar-Limanov invariant which would permit to distinguish the varieties $X = X(p)$ up to isomorphism, espacially those diffeomorphic to the affine spaces.

The proof of Theorem 6.1 is based on Lemmas 6.2 - 6.6 below. In the next lemma for a class of $\mathbf{C}_+-$ actions $\varphi = \varphi_\partial$ on $\mathbf{C}^{k+1}$ which preserve the decomposition $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$ we specify the lifts $\hat{\varphi} \in \mathrm{Aut}\,X$ of φ, cf. Lemma 6.1.

Lemma 6.2. *Let δ be an LND of the polynomial algebra $\mathbf{C}^{[k]} = \mathbf{C}[x_1, \dots, x_k]$, and let $q \in \mathbf{C}[z]$ be a degree d polynomial with the roots $z_1 = 0, z_2, \dots, z_m$ where $z_i \neq z_j$ for $i \neq j$. Then*

(a) the formulas

$$\partial(x_i) = q(v)\delta(x_i),\ i = 1, \dots, k, \quad \partial(v) = 0 \tag{6.4}$$

define an LND $\partial = \partial_{\delta,\,q}$ on $\mathbf{C}^{[k+1]} = \mathbf{C}[x_1, \dots, x_k,\, v]$. The associated $\mathbf{C}_+-$ action $\varphi_\partial \subset G_1$ on $\mathbf{C}^{k+1}$ fixes each point of the hyperplanes $D_j = \{v = z_j\}$, $j = 1, \dots, m$.

(b) The corresponding lifted LND $\hat{\partial}$ on $A = \mathbf{C}[X]$ is the restriction to X of the LND on $\mathbf{C}^{[k+2]} = \mathbf{C}[\bar{x},\, u,\, v]$ (denote it also by $\hat{\partial}$) given by the formulas

$$\hat{\partial}(x_j) = \partial(x_j),\ \hat{\partial}(v) = \partial(v) = 0,\ \hat{\partial}(u) = \frac{1}{v}\partial(p(\bar{x})) = \frac{q(v)}{v}\delta p(\bar{x})\,. \tag{6.5}$$

Furthermore, the fixed point set Fix $\hat{\varphi}_\partial$ of the associated lifted $\mathbf{C}_+-$ action $\hat{\varphi}_\partial \subset \hat{G}_1$ on X contains the union of the hypersurfaces $V_{z_i} = \{v = z_i\}$ for all the nonzero roots z_i, $i = 2, \dots, m$, of the polynomial q; if $q'(0) = 0$, then it contains V_0, too.

Proof. It is easy to check that ∂ resp. $\hat{\partial}$ defined by the formulas (4) resp. (5) is, indeed, an LND of the algebra $\mathbf{C}^{[k+1]}$ resp. $\mathbf{C}^{[k+2]} = \mathbf{C}[\bar{x},\, u,\, v]$. >From (5) it follows that

$$\hat{\partial}(uv - p(\bar{x})) = v \cdot \frac{q(v)}{v}\delta(p(\bar{x})) - q(v)\delta(p(\bar{x})) = 0\,,$$

and so, $\hat{\partial}(J) \subset J$ where $J \subset \mathbf{C}^{[k+2]} = \mathbf{C}[\bar{x},\, u,\, v]$ is the principal ideal generated by the polynomial $uv - p(\bar{x})$. Therefore, $\hat{\partial}$ induces an LND of the quotient $\mathbf{C}[X] = \mathbf{C}[\bar{x},\, u,\, v]/J$. The other statements of the lemma can be verified without difficulty. □

Example 6.1. In particular, putting in Lemma 6.2 $\delta = \delta_i := \frac{\partial}{\partial x_i} \in \mathrm{LND}(\mathbf{C}^{[k]})$ we obtain $\partial = \partial_i = \partial_{i,\,q} := q(v)\frac{\partial}{\partial x_i} \in \mathrm{LND}(\mathbf{C}^{[k+1]})$ where $\mathrm{LND}(A)$ denotes the set of all LND's of an algebra A.

For the 'lifted' LND $\widehat{\partial} \in \mathrm{LND}(\mathbf{C}^{[k+2]})$ we have

$$\widehat{\partial}_i(x_i) = q(v),\ \ \widehat{\partial}_i(x_j) = 0\ \ if\ \ j \neq i,\ \ \widehat{\partial}_i(v) = 0,\ \ \widehat{\partial}_i(u) = v^{-1}q(v)\frac{\partial}{\partial x_i}p(\bar{x})\,.$$

It follows that $\widehat{\partial}_i^2(x_i) = 0$ and $\widehat{\partial}_i^n(u) = v^{-1}q^n(v)\frac{\partial^n}{\partial x_i^n}p(\bar{x})$. The associated $\mathbf{C}_+-$ action $\widehat{\varphi}_i := \widehat{\varphi}_{\partial_i}$ on $\mathbf{C}^{[k+2]}$ is given as

$$\widehat{\varphi}_i^t(x_i) = x_i + tq(v),\ \ \widehat{\varphi}_i^t(x_j) = x_j\ \ if\ \ j \neq i,\ \ \widehat{\varphi}_i^t(v) = v,\ \text{ and}$$

$$\widehat{\varphi}_i^t(u) = u + \frac{1}{v}\sum_{n=1}^{\deg p}\frac{t^n}{n!}q^n(v)\frac{\partial^n}{\partial x_i^n}p(\bar{x}), \qquad \text{i.e.} \tag{6.6}$$

$$\widehat{\varphi}_i^t(\bar{x}_i,\, u,\, v) = (\bar{x} + tq(v)\bar{e}_i,\, u + \frac{1}{v}[p(\bar{x} + tq(v)\bar{e}_i) - p(\bar{x})],\, v)\,,$$

where $(\bar{e}_1, \ldots, \bar{e}_k)$ is the standard basis of $\mathbf{C}^k$.

In the following lemma we keep all the notation from Lemma 6.2 and Example 6.1. We also denote by $\mathcal{O}_{\widehat{\varphi}}(P)$ the orbit of a point $P \in X$ under a $\mathbf{C}_+-$ action $\widehat{\varphi}$ on X.

Lemma 6.3. *Put* $q(v) = v$. *Then:*

(a) For any point $P \in U_0 \setminus \operatorname{sing} U_0$ *there exists* $i \in \{1, \ldots, k\}$ *such that* $\mathcal{O}_{\widehat{\varphi}_i}(P) \not\subset U_0$.

(b) For any point $P \in \operatorname{sing} U_0 \setminus \operatorname{sing} X$ *there exists* $\delta \in \mathrm{LND}(\mathbf{C}^{[k]})$ *such that for the associated lifted* $\mathbf{C}_+-$ *action* $\widehat{\varphi}_\partial$ *on* X *one has* $\mathcal{O}_{\widehat{\varphi}_\partial}(P) \not\subset \operatorname{sing} U_0$.

Proof. (a) As follows from Lemma 5.1(b), $P = (\bar{x}^0,\, 0,\, v^0) \notin \operatorname{sing} U_0$ iff $\operatorname{grad}_{\bar{x}^0} p \neq 0$, i.e. $\frac{\partial p}{\partial x_i}(\bar{x}^0) \neq 0$ for some $i \in \{1, \ldots, k\}$. By (6), the $u-$ coordinate $u_i(t)$ of the point $\widehat{\varphi}_i^t(P_0) \in \mathcal{O}_{\widehat{\varphi}_i}(P)$ is a polynomial in t with the non-zero linear term $(\frac{\partial p}{\partial x_i}(\bar{x}^0))t$. Hence, $\mathcal{O}_{\widehat{\varphi}_i}(P) \not\subset U_0$, as claimed.

(b) Let $P = (\bar{x}^0,\, 0,\, v^0) \in \operatorname{sing} U_0 \setminus \operatorname{sing} X$, that is, $\operatorname{grad}_{\bar{x}^0} p = 0$, and $v^0 \neq 0$. Fix a line l through $\bar{x}^0$ in $\mathbf{C}^k$ such that the restriction $p\,|\,l$ is non-constant. Chose a new coordinate system in $\mathbf{C}^k$ with the first coordinate axis being parallel to l. Take $\delta = \delta_1 = \frac{\partial}{\partial x_1} \in \mathrm{LND}\ (\mathbf{C}^{[k]})$ with respect to the new coordinates, so that $\widehat{\varphi}_\partial = \widehat{\varphi}_1$. By (6), the projection $\bar{x}^t = \bar{x}^0 + tv^0\bar{e}_1$ of the point $\widehat{\varphi}_1^t(P) \in \mathcal{O}_{\widehat{\varphi}_1}(P)$ to $\mathbf{C}^k_{\bar{x}}$ runs over l. Since $\operatorname{grad} p$ does not vanish identically on l, we obtain that $\mathcal{O}_{\widehat{\varphi}_1}(P) \not\subset \operatorname{sing} U_0$. The proof is completed. $\square$

Lemma 6.4. *For any set of* m *distinct points* $P_1, \ldots, P_m \in X \setminus \operatorname{sing} X$ *there exists an automorphism* $\widehat{\varphi} \in \widehat{G}_1$ *of* X *close enough to the identity such that* $\widehat{\varphi}(P_i) \notin U_0$, $i = 1, \ldots, m$.

Proof. Starting with $m = 0$ assume, by induction, that the statement is true for any set of m distinct points in $X \setminus \operatorname{sing} X$. Take an arbitrary set of $m + 1$ distinct points $P_1, \ldots, P_{m+1} \in X \setminus \operatorname{sing} X$. By the inductive hypothesis, we may suppose that $P_1, \ldots, P_m \notin U_0$ and $P_{m+1} \in X \setminus \operatorname{sing} X$. After applying to $P = P_{m+1}$, if necessary, an automorphism $\widehat{\varphi}_\delta^t \in \widehat{G}_1$ of Lemma 6.3(b) with $|t|$ small enough, we can achieve that $P_{m+1} \notin \operatorname{sing} U_0$, while still keeping $P_1, \ldots, P_m \notin U_0$. If it occurs that $P_{m+1} \in U_0 \backslash \operatorname{sing} U_0$, then applying Lemma 6.3(a) to $P = P_{m+1}$ in the same way as above, we are done. $\square$

Denote by G_0 the subgroup of the group Aut $\mathbf{C}^k$ generated by all the $\mathbf{C}_+$ -subgroups of Aut $\mathbf{C}^k$. The proof of the following lemma can be found in [Je, Ka 2]; for the sake of completeness we reproduce it here.

Lemma 6.5. *For any $k \geq 2$ and any $m \in \mathbf{N}$ the group G_0 acts $m-$ transitively on* $\mathbf{C}^k$.

Proof. The proof goes by induction on k, $k \geq 2$. Fix two arbitrary sets of m distinct points $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ in $\mathbf{C}^k$ where $P_i = (\bar{x}^{(i)}, x_n^{(i)})$ and $Q_i = (\bar{y}^{(i)}, y_n^{(i)})$ for certain $\bar{x}^{(i)}, \bar{y}^{(i)} \in \mathbf{C}^{k-1}$ and $x_n^{(i)}, y_n^{(i)} \in \mathbf{C}$, $i = 1, \ldots, m$. Choosing a generic coordinate system in $\mathbf{C}^k$ we may suppose that $\bar{x}^{(i)} \neq \bar{x}^{(j)}$, $x_n^{(i)} \neq x_n^{(j)}$, $\bar{y}^{(i)} \neq \bar{y}^{(j)}$ and $y_n^{(i)} \neq y_n^{(j)}$ for all $i \neq j$.

If $n > 2$ then by the inductive hypothesis, we can find an automorphism $\alpha' \in G_0(\mathbf{C}^{k-1})$ such that $\alpha'(\bar{x}^{(i)}) = \bar{y}^{(i)}$, $i = 1, \ldots, m$. After applying the automorphism $\alpha = (\alpha', \mathrm{id}_{\mathbf{C}}) \in G_0$ we may suppose that $\bar{y}^{(i)} = \bar{x}^{(i)}$, $i = 1, \ldots, m$. Let $p \in \mathbf{C}^{[k-1]}$ be a polynomial such that $p(\bar{x}^{(i)}) = y_n^{(i)} - x_n^{(i)}$, $i = 1, \ldots, m$. Consider the triangular $\mathbf{C}_+-$ action β^t on $\mathbf{C}^k$ given as $\beta^t : (\bar{x}, x_n) \longmapsto (\bar{x}, x_n + tp(\bar{x}))$. Then the automorphism $\beta := \beta^1 \in G_0$ sends P_i to Q_i, $i = 1, \ldots, m$. This provides the induction step.

For $k = 2$ we start with the triangular automorphism $\alpha \in G_0$, $\alpha : (x_1, x_2) \longmapsto (x_1 + q(x_2), x_2)$ where $q \in \mathbf{C}[z]$ is a polynomial such that $q(x_2^{(i)}) = y_1^{(i)} - x_1^{(i)}$, $i = 1, \ldots, m$, and then we apply $\beta \in G_0$, as above. This completes the proof. $\square$

For any set of $n+1$ distinct non-zero complex numbers $c_0, c_1, \ldots, c_n \in \mathbf{C} \backslash \{0\}$ denote by $\operatorname{Stab}_{c_1,\ldots,c_n}(V_{c_0})$ the subgroup of the group $\widehat{G}_1$ which consists of the automorphisms of X leaving the hypersurface V_{c_0} invariant and fixing each point of the hypersurfaces V_{c_i}, $i = 1, \ldots, n$.

Lemma 6.6. *(a) The group $\operatorname{Stab}_{c_1,\ldots,c_n}(V_{c_0})$ acts $m-$ transitively on V_{c_0} for any $m \in$* $\mathbf{N}$.

(b) For any set of m distinct points $P_1, \ldots, P_m \in X \setminus \operatorname{sing} X$ there exists an automorphism $\widehat{\varphi} \in \widehat{G}$ of X such that $\widehat{\varphi}(P_i) \in U_1$, $i = 1, \ldots, m$.

Proof. (a) Fix two arbitrary sets of m distinct points $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ in V_{c_0} where $P_i = (\bar{x}^{(i)}, u^{(i)}, c_0)$, $u^{(i)} = p(\bar{x}^{(i)})/c_0$, and $Q_i = (\bar{y}^{(i)}, \tilde{u}^{(i)}, c_0)$, $\tilde{u}^{(i)} = p(\bar{y}^{(i)})/c_0$, $i = 1, \ldots, m$. By Lemma 6.5, there exists an automorphism $\alpha \in G_0$ such that $\alpha(\bar{x}^{(i)}) = \bar{y}^{(i)}$, $i = 1, \ldots, m$.

Decompose $\alpha = \psi_1^{t_1} \circ \ldots \circ \psi_l^{t_l}$ into a product of elements of $\mathbf{C}_+-$ subgroups of the group Aut $\mathbf{C}^k$, and let $\tilde{\delta}_j \in \operatorname{LND}(\mathbf{C}^{[k]})$ be the infinitesimal generator of the subgroup $\{\psi_j^t\}_{t \in \mathbf{C}_+}$, $j = 1, \ldots, l$.

Let also $q \in \mathbf{C}[z]$ be the degree n polynomial with the roots $c_1, \ldots, c_n$ such that $q(c_0) = 1$. Denote by $\tilde{\partial}_j = \partial_{\tilde{\delta}_j, q}$ the LND of the algebra $\mathbf{C}^{[k+1]} = \mathbf{C}[\bar{x}, v]$ defined as in (5) of Lemma 6.2(a), and by $\tilde{\varphi}_j$ the corresponding $\mathbf{C}_+-$ action on $\mathbf{C}^{k+1} = \mathbf{C}^k_{\bar{x}} \times \mathbf{C}_v$, $j = 1, \ldots, l$. Then we have: $\tilde{\varphi}_j : (\bar{x}, c_0) \longmapsto (\psi_j(\bar{x}), c_0)$, and $\tilde{\varphi}_j$ fixes each point of the hyperplanes $D_{c_i} = \{v = c_i\}$, $i = 1, \ldots, n$. The composition $\tilde{\alpha} = \tilde{\varphi}_1^{t_1} \circ \ldots \circ \tilde{\varphi}_l^{t_l}$ also fixes each point of the union $\bigcup_{i=1}^n D_{c_i}$, stabilizes the hyperplane D_{c_0}, and $\tilde{\alpha} \mid D_{c_0} = \alpha$, i.e. $\tilde{\alpha}(\bar{x}, c_0) = (\alpha(\bar{x}), c_0)$. Therefore, $\tilde{\alpha}(\bar{x}^{(i)}, c_0) = (\bar{y}^{(i)}, c_0)$, $i = 1, \ldots, m$.

By Lemma 6.2(b), the lift $\widehat{\alpha} = \widehat{\varphi}_1^{t_1} \circ \ldots \circ \widehat{\varphi}_l^{t_l} \in \widehat{G}_1$ fixes each point of the union $\bigcup_{i=1}^n V_{c_i}$ and stabilizes the hypersurface V_{c_0}, that is, $\widehat{\alpha} \in \mathrm{Stab}_{c_1, \ldots, c_n}(V_{c_0})$. Moreover, since $\widehat{\alpha} \mid V_{c_0} = \sigma_1^{-1} \tilde{\alpha} \sigma_1 \mid V_{c_0}$ (see Lemma 6.1) we have $\widehat{\alpha}(P_i) = Q_i$, $i = 1, \ldots, m$. This proves (a).

(b) By Lemma 6.4, we may assume that $P_i = (\bar{x}^{(i)}, u^{(i)}, v^{(i)}) \in X \setminus V_0$, $i = 1, \ldots, m$. Reordering, if necessary, the points P_i we may also suppose that $v^{(1)} = \ldots = v^{(m')} =: c_0 \neq v^{(j)}$, $j = m' + 1, \ldots, m$. By (a), there exists an automorphism $\widehat{\varphi} \in \widehat{G}_1$ such that $\widehat{\varphi}(P_j) \in V_{c_0} \cap U_1$, $j = 1, \ldots, m'$, and $\widehat{\varphi}(P_j) = P_j$, $j = m' + 1, \ldots, m$. Hence, proceeding by induction, we are done. $\square$

Proof of Theorem 6.1. Fix an arbitrary set of m distinct points $P_1, \ldots, P_m \in X \setminus \mathrm{sing}\, X$ and another such set $Q_1, \ldots, Q_m \in U_1$. By Lemma 6.6(b), after applying, if necessary, to the points $P_1, \ldots, P_m$ an automorphism $\widehat{\varphi}_1 \in \widehat{G}$ we may suppose that also $P_1, \ldots, P_m \in U_1$. Then, exchanging in Lemma 6.6(a) the roles of U_1 and V_1, by this lemma, we can find an automorphism $\widehat{\varphi}_2 \in \widehat{G}_2$ such that $\widehat{\varphi}_2(P_i) = Q_i$, $i = 1, \ldots, m$. The proof is completed. $\square$

7. Examples of acyclic surfaces in $\mathbf{C}^3$ and of smooth contractible 4-folds $uv = p(x, y, z)$ in $\mathbf{C}^5$

In the examples of smooth acyclic surfaces in $\mathbf{C}^3$ with big fundamental groups (see Example 7.1 below) we use the following simple lemma (cf. [Za 2, Lemma 7.2]).

Lemma 7.1. *Let Y be a connected simply connected complex manifold, F be a smooth irreducible hypersurface in Y, and $p : X \to Y$ be a branched cyclic covering over Y ramified to order s on F. Denote $G_X = \pi_1(X^*)$ resp. $G_Y = \pi_1(Y^*)$, where $X^* := X \setminus p^{-1}(F)$ and $Y^* := Y \setminus F$. We identify G_X with the index s subgroup $p_*(G_X) \subset G_Y$, and we denote by β_F a vanishing loop of F in Y. Then*

$$\pi_1(X) \simeq G_X / << (\beta_F)^s >>$$

where $<< (\beta_F)^s >>$ denotes the normal closure in G_Y of the cyclic subgroup generated by $(\beta_F)^s$. Furthermore, $\pi_1(X) \simeq G_X / << (\beta_F)^s >>$ is an index s subgroup of the group $G_Y / << (\beta_F)^s >>$ with a cyclic quotient.

Proof. We have the following commutative diagram of group homomorphisms

$$\begin{array}{ccccccccc}
 & & \mathbf{1} & & \mathbf{1} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\mathbf{1} & \longrightarrow & N_X & \longrightarrow & G_X = \pi_1(X^*) & \stackrel{(i_X)_*}{\longrightarrow} & \pi_1(X) & \longrightarrow & \mathbf{1} \\
 & & \downarrow p_* & & \downarrow p_* & & \downarrow p_* & & \\
\mathbf{1} & \longrightarrow & N_Y & \stackrel{\simeq}{\longrightarrow} & G_Y = \pi_1(Y^*) & \stackrel{(i_Y)_*}{\longrightarrow} & \pi_1(Y) = \mathbf{1} & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathbf{Z}/s\mathbf{Z} & & & &
\end{array}$$

where $i_X : X^* \hookrightarrow X$ resp. $i_Y : Y^* \hookrightarrow Y$ denotes the identical embedding and $N_X := \ker\,(i_X)_*$ resp. $N_Y := \ker\,(i_Y)_*$. Thus, $\pi_1(X) \simeq G_X/N_X$, $\mathbf{1} = \pi_1(Y) \simeq G_Y/N_Y$ where

$$N_X =<< \alpha_F >> \qquad \text{and} \qquad N_Y =<< \beta_F >>$$

with $\alpha_F \in N_X$ resp. $\beta_F \in N_Y$ being a vanishing loop of $p^{-1}(F) \simeq F$ in X resp. of F in Y; see e.g. [Za 2, (2.3.a)]. Moreover, identifying G_X with the subgroup $p_*(G_X) \subset G_Y$ we may assume that $\alpha_F = (\beta_F)^s$.

Since the quotient $G_Y/G_X \simeq \mathbf{Z}/s\mathbf{Z}$ is abelian we have $K_Y := [G_Y, G_Y] \subset G_X$. Since $G_Y = N_Y =<< \beta_F >>$ the abelianization $(G_Y)_{\text{ab}} := G_Y/K_Y$ is a cyclic group generated by the class $K_Y\beta_F$. Hence, any element $g \in G_Y$ can be written as $g = g'(\beta_F)^t$ where $g' \in K_Y \subset G_X$ and $t \in \mathbf{Z}$. Thus, we have $g(\beta_F)^s g^{-1} = g'(\beta_F)^s g'^{-1}$ for any $g \in G$. Therefore, the normal closure N_X of the cyclic subgroup $< (\beta_F)^s = \alpha_F >$ in the group G_X coincides with its normal closure in the bigger group G_Y, i.e. $N_X \subset G_Y$ is a normal subgroup and it coincides with the subgroup $<< (\beta_F)^s >>$. Hence, we have $\pi_1(X) \simeq G_X/N_X \simeq G_X/ << (\beta_F)^s >>$, as required.

The chain of normal subgroups $N_X \subset G_X \subset G_Y$ yields the short exact sequence

$$\mathbf{1} \longrightarrow \pi_1(X) \simeq G_X/N_X \longrightarrow G_Y/N_X \longrightarrow G_Y/G_X \simeq \mathbf{Z}/n\mathbf{Z} \longrightarrow \mathbf{1}\,,$$

and the last assertion of the lemma follows. □

Example 7.1. Let $X_{k,l,s,m} = p^{-1}_{k,l,s,m}(0) \subset \mathbf{C}^3$ be the surface defined by the polynomial

$$p_{k,l,s,m} = \frac{(xz^m+1)^k - (yz^m+1)^l - z^s}{z^m} \in \mathbf{C}[x,\, y,\, z]$$

where $0 \le m \le s$. It is smooth if $m > 0$, and it has at most one singular point $P_0 = (1,\, 1,\, 0)$ if $m = 0$. For gcd $(k,\, l) =$ gcd $(k,\, s) =$ gcd $(l,\, s) = 1$ and $m = s$ the surface $Y_{k,l,s} := X_{k,l,s,s}$ is acyclic. Indeed, it can be presented as a cyclic $\mathbf{C}^*-$ covering over a contractible *tom Dieck-Petrie surface* $X_{k,l} := X_{k,l,1,1} \subset \mathbf{C}^3$ (see [tDP]) branched to order s along the line $L_{k,l} := X_{k,l} \cap \{z = 0\} \simeq \mathbf{C}$ in $X_{k,l}$, and the acyclicity follows as in the proof of Theorem A in [Ka 1] (see also [Za 2, §5]).

However, in general the acyclic surface $Y_{k,l,s}$ is not contractible and possesses quite a big fundamental group. Indeed, let $\sigma : \mathbf{C}^3 \to \mathbf{C}^3$, $\sigma(x, y, z) = (xz, yz, z)$, be the affine modification of $\mathbf{C}^3$ along the plane $z = 0$ with center at the origin (see Example 1.4). Then the restriction $\sigma \,|\, X_{k,l,s,m} : X_{k,l,s,m} \to X_{k,l,s,m-1}$ is the affine modification of $X_{k,l,s,m-1}$ along the line $D := X_{k,l,s,m-1} \cap \{z = 0\}$ with center at the origin (see Example 2.1). Furthermore, the surface $X_{k,l,s,1}$ coincides with the pseudoaffine modification of the smooth quasiprojective surface $X'_{k,l,s,0} := X_{k,l,s,0} \setminus \{P_0\}$ along the smooth curve $D^* := D \setminus \{P_0\}$ with center $\overline{0} \in D^*$ (see Definition 3.1).

By Corollary 3.1, the surface $Y_{k,l,s} = X_{k,l,s,s}$ being acyclic also the surfaces $X_{k,l,s,m}$ are acyclic for all $m = 1, \ldots, s$. The repeated application of Lemma 3.4 in [Ka 1] (or of Proposition 3.1 above) yields the isomorphisms

$$\pi_1(Y_{k,l,s}) = \pi_1(X_{k,l,s,s}) \simeq \pi_1(X_{k,l,s,s-1}) \simeq \ldots \simeq \pi_1(X_{k,l,s,1}) \simeq \pi_1(X'_{k,l,s,0})\,.$$

The surface $X_{k,l,s,0} \simeq X_{k,l,s} := \{x^k - y^l - z^s = 0\} \subset \mathbf{C}^3$ is homotopically equivalent to the cone over the Pham–Brieskorn 3-manifold $M_{k,l,s} := X_{k,l,s} \cap S^5$, that is, over the link of the surface singularity of $X_{k,l,s}$ in the sphere S^5. In turn, $X'_{k,l,s,0} \simeq X_{k,l,s} \setminus \{\overline{0}\}$ is homotopically equivalent to the link $M_{k,l,s}$, and thus $\pi_1(Y_{k,l,s}) \simeq \pi_1(M_{k,l,s})$; denote the latter group by $G'_{k,l,s}$.

The structure of the group $G'_{k,l,s}$ is well known [Mil 2]. It is finite iff $1/k + 1/l + 1/s > 1$, infinite nilpotent iff $1/k + 1/l + 1/s = 1$. If $1/k + 1/l + 1/s \neq 1$, then $G'_{k,l,s} = [G_{k,l,s}, G_{k,l,s}]$ where

$$G_{k,l,s} := < \gamma_1, \gamma_2, \gamma_3 \,|\, \gamma_1^k = \gamma_2^l = \gamma_3^s = \gamma_1\gamma_2\gamma_3 >$$

is a central extension of the Schwarz triangular group

$$T_{k,l,s} := < b_1, b_2, b_3 \,|\, b_1^2 = b_2^2 = b_3^2 = 1, \ (b_1b_2)^k = (b_2b_3)^l = (b_3b_1)^s = 1 > \ .$$

Note that for $1/k + 1/l + 1/s < 1$ the Schwarz triangular group $T_{k,l,s}$ contains a free subgroup with two generators. Therefore, the group $G'_{k,l,s}$ also contains such a subgroup; in particular, it is not solvable. It is known [Bri] that the Pham–Brieskorn manifold $M_{k,l,s}$ is a homology 3-sphere iff gcd $(k, l) =$ gcd $(k, s) =$ gcd $(l, s) = 1$. Under this condition the group $G'_{k,l,s}$ is perfect, i.e. coincides with its commutator subgroup; indeed, its abelianization $H_1(Y_{k,l,s}; \mathbf{Z})$ is trivial. Notice that the condition of relative primeness never holds in the Euclidean case $1/k+1/l+1/s = 1$; in the spherical one $1/k+1/l+1/s > 1$ it holds only for the Kleinian icosahedral triple $(k, l, s) = (2, 3, 5)$.

The isomorphism $X^*_{k,l} := X_{k,l} \setminus L_{k,l} \simeq \mathbf{C}^2 \setminus \Gamma_{k,l}$, where $\Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbf{C}^2$ [tDP], provides the presentation

$$B_{k,l} := \pi_1(X^*_{k,l}) =< a, b \,|\, a^k = b^l >$$

(see e.g. [Di 2]). Since $Y^*_{k,l,s} := Y_{k,l,s} \setminus \{z = 0\} \to X^*_{k,l}$, $(x, y, z) \longmapsto (x, y, z^s)$, is a non-ramified cyclic covering, the group $\pi_1(Y^*_{k,l,s})$ is isomorphic to an index s subgroup, say, $\tilde{C}_{k,l,s}$ of the group $B_{k,l}$ with the cyclic quotient $B_{k,l}/\tilde{C}_{k,l,s} \simeq \mathbf{Z}/s\mathbf{Z}$. We have

$$\ker\left(i_* : \pi_1(Y^*_{k,l,s}) \to \pi_1(Y_{k,l,s})\right) = << \alpha^s >>,$$

where $\alpha \in B_{k,l}$ is a vanishing loop of the line $L_{k,l} \subset X_{k,l}$ [Za 2, (2.3.a)]. It can be shown that $\alpha = a^q b^p \in B_{k,l}$ where $p, q \in \mathbf{Z}$ are such that $kp + lq = 1$.

Therefore, for gcd $(k, l) =$ gcd $(k, s) =$ gcd $(l, s) = 1$ the group $G'_{k,l,s} \simeq \pi_1(Y_{k,l,s})$ is isomorphic to an index s subgroup $C_{k,l,s}$ of the quotient

$$B_{k,l,s} := B_{k,l}/ << \alpha^s >>=< a, b \,|\, a^k = b^l, (a^q b^p)^s = 1 >$$

(see Lemma 7.1 above). In particular, for $k = 2, l = 3$ in view of the isomorphism $X^*_{2,3} \simeq \mathbf{C}^2 \setminus \Gamma_{2,3}$ we have that $B_{2,3} = B_3$ is the 3-braid group with the generators $\sigma_1, \sigma_2 \in B_3$ being vanishing loops of $L_{2,3}$ in $X_{2,3}$, $a = \sigma_1\sigma_2\sigma_1$, $b = \sigma_1\sigma_2$. Therefore, $G'_{2,3,s}$ is isomorphic to an index s subgroup $C_{2,3,s}$ of the group

$$B_{2,3,s} = B_3/ << \sigma_1^s >>=< \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_1^s = \sigma_2^s = 1 >$$

which consists of the elements of algebraic length[20] divisible by s.

Remark 7.1. Observe that $X_{k,l,s,m}$ being an acyclic surface of logarithmic Kodaira dimension 1, it can be obtained starting with a Hirzebruch surface under the broken chains construction (see [FlZa] for terminology). Actually, the construction of $X_{k,l,s,m}$ needs only three broken chains. More generally, every acyclic surface of logarithmic Kodaira dimension 1 with three broken chains can be presented as a hypersurface in $\mathbf{C}^3$ (cf. [KaML 2]).

Example 7.2. Consider, further, the 4-fold $X = X^{k,l,s,m} = \{uv - p_{k,l,s,m}(x, y, z) = 0\}$ in $\mathbf{C}^5$. By Corollary 5.1, the surface $X_{k,l,s,m}$ being acyclic implies that the hypersurface $X \subset \mathbf{C}^5$ is diffeomorphic to the Euclidean space $\mathbf{R}^8$. But in general, as we have seen above, the surface $X_0 = X_{k,l,s,m} \subset \mathbf{C}^3$ is not contractible and possesses quite a big fundamental group. This shows that Proposition 5.2 cannot be extended to $k = 3$.

Next we give an example of a polynomial $p = p^0_{2,3} \in \mathbf{C}^{[3]}$ with a smooth acyclic (even contractible) zero fibre $F_0 = p^{-1}(0)$ and non-acyclic generic fibres $F_c = p^{-1}(c)$. In fact, the surface F_0 in this example is isomorphic to the tom Dieck-Petrie surface $X_{2,3}$ (see Example 7.1 above), but the embedding $X_{2,3} \xrightarrow{\simeq} F_0 \subset \mathbf{C}^3$ is not equivalent to the standard one up to the action on $\mathbf{C}^3$ of the automorphism group Aut$\,\mathbf{C}^3$. This provides also examples of non-equivalent embeddings of an exotic $\mathbf{C}^n$ into $\mathbf{C}^{n+1}$.

Example 7.3. Consider the affine modification $\sigma = \sigma_{\bar{0},H} : \mathbf{C}^4 \to \mathbf{C}^4$, $\sigma : (x, y, z, t) \mapsto (x, xy, xz, xt)$, of $\mathbf{C}^4$ along the hyperplane $H_0 = \{x = 0\}$ with center at the origin (cf. Example 1.4). Consider also the Russell cubic $X \subset \mathbf{C}^4$ with the equation

$$-x + x^2y + (z + 1)^2 - (t + 1)^3 = 0$$

[20] Recall that the algebraic length of an element $\prod_{n=1}^m \sigma_{i_n}^{a_n}$ is the integer $\sum_{n=1}^m a_n$.

(cf. Example 1.7). By Corollary 2.1, the restriction of σ to the strict transform X' of X yields the affine modification $\sigma_{\bar{0},B} : X' = \Sigma_{\bar{0},B}(X) \to X$ of X along the book-surface $B = H_0 \cap X \simeq \Gamma_{2,3} \times \mathbf{C}$ with center $\bar{0} \in B \setminus \operatorname{sing} B$. The hypersurface $X' \subset \mathbf{C}^4$ is given by the equation

$$-1 + x^2 y + \frac{(xz+1)^2 - (xt+1)^3}{x} = 0$$

(cf. Example 2.1). The isomorphism $\mathbf{C}^3 \simeq X'$ as in Example 1.7 provides an embedding of $\mathbf{C}^3$ into $\mathbf{C}^4$. A direct computation shows that this embedding is rectifiable.[21]

The hyperplane section $X^0_{2,3} := X' \cap D_0$ where $D_0 = \{y = 0\}$, is isomorphic to the tom Dieck-Petrie surface $X_{2,3}$ (see Example 7.1). But the embedding $X_{2,3} \xrightarrow{\simeq} X^0_{2,3} \hookrightarrow X' \simeq \mathbf{C}^3$ of the tom Dieck-Petrie surface into $\mathbf{C}^3$ is not equivalent to the standard one $X_{2,3} \hookrightarrow D_0 = \mathbf{C}^3$. Indeed, the latter one is defined by the polynomial $p_{2,3} = \frac{(xz+1)^2-(xt+1)^3}{x} - 1 \in \mathbf{C}^{[3]}$ with all the fibres being contractible surfaces; see e.g. [Za 2, Example 6.1]. On the other hand, it is easily seen that for a generic $c \in \mathbf{C}$ the fibre $F_c = p^{-1}(c)$ of the regular function $p = p^0_{2,3} := y \mid X' \in \mathbf{C}[X'] \simeq \mathbf{C}^{[3]}$ which defines the surface $X^0_{2,3}$ in X' has the Euler characteristic $e(F_c) = 5$ (*hint*: use the fibration $F_c \to \mathbf{C}$ defined by the restriction $x \mid F_c$). In particular, the surfaces F_c for a generic $c \in \mathbf{C}$ are not acyclic.

Consider further the exotic product-structure $X_{2,3,n} := X_{2,3} \times \mathbf{C}^{n-2}$ on $\mathbf{C}^n$, $n \geq 3$ (see [Za 2, §4]). By the similar arguments as above, two realizations $X_{2,3,n} := X_{2,3} \times \mathbf{C}^{n-2} \hookrightarrow \mathbf{C}^{n+1} = \mathbf{C}^3 \times \mathbf{C}^{n-2}$ and $X^0_{2,3,n} := X^0_{2,3} \times \mathbf{C}^{n-2} \hookrightarrow \mathbf{C}^{n+1}$ of this exotic $\mathbf{C}^n$ as a hypersurface in $\mathbf{C}^{n+1}$ are not equivalent modulo the action on $\mathbf{C}^{n+1}$ of the automorphism group $\operatorname{Aut} \mathbf{C}^{n+1}$.

Remarks

7.2. By Corollary 5.2(b), the zero fibres of the polynomials $uv - p_{2,3}(x, y, z) \in \mathbf{C}^{[5]}$ and $uv - p^0_{2,3}(x, y, z) \in \mathbf{C}^{[5]}$, as well as the generic fibres of the first one, are smooth contractible hypersurfaces in $\mathbf{C}^5$, whereas the generic fibres of the second one have Euler characteristic 5 (see Corollary 5.1).

7.3. More generally, we have presented above a collection of examples of smooth acyclic hypersurfaces in $\mathbf{C}^n$ (see also [Za 2]). Most of them are not rectifiable; actually, their defining polynomials have a fibre non-isomorphic to $\mathbf{C}^{n-1}$. By Corollary 3.1 (see also Remark 3.1), performing the affine modification of $\mathbf{C}^n$ along such a hypersurface $D = p^*(0)$, $p \in \mathbf{C}^{[n]}$, with a smooth reduced acyclic center $C \subset D$ leads to a smooth contractible affine $n-$ fold X. The question arises when X itself is isomorphic to the affine space $\mathbf{C}^n$ (this is a specialization of the more general Question A in sect. 4).

Suppose that this is the case, and that, moreover, $C \simeq \mathbf{C}^k$. Then the exceptional divisor $E = \sigma_C^{-1}(C) = q^*(0) \subset X$ where $\sigma_C : X \to \mathbf{C}^n$ is the blowup morphism and

[21] It is rectifiable e.g. via the composition $\gamma \circ \beta \circ \alpha$ of the triangular automorphisms
$\alpha : (x, y, z, t) \longmapsto (x, y, u, t)$ where $u = z + f(x, t) = z - xt^2(xt+3)/2$,
$\beta : (x, y, u, t) \longmapsto (x, v, u, t)$ where $v = y + g(x, u, t) = y + ut^2(xt+3) + xt^4(xt+3)^2/4$, and
$\gamma : (x, v, u, t) \longmapsto (x, v, u, w)$ where $w = -3t + h(x, u, v) = -3t + x^2 v + xu^2 + 2u - 1$.

$q := p \circ \sigma_C \in \mathbf{C}[X] \simeq \mathbf{C}^{[n]}$, would be isomorphic to $\mathbf{C}^{n-1}$. But q having a fibre non-isomorphic to $\mathbf{C}^{n-1}$, the hypersurface $E \simeq \mathbf{C}^{n-1}$ in $X \simeq \mathbf{C}^n$ could not be rectifiable. Thus, this would answer in negative to the Abhyankar-Sathaye Embedding Problem (cf. Remark 5.3)[22].

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[22]Recall that the varieties discussed in Remark 5.3 arise as affine modifications of the affine space $\mathbf{C}^n$ along a hyperplane with non-linear acyclic centers. Whereas here we consider, in particular, affine modifications of $\mathbf{C}^n$ along acyclic hypersurfaces with center at a linear subspace.

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