Eisenman Intrinsic Measures and Algebraic Invariants

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ABSTRACT. We generalize the Sakai theorem that says that every complex algebraic manifold of general type is measure hyperbolic. We introduce the notion of k-measure hyperbolicity for every Eisenman k-measure and, following Sakai, we consider an analogue $\bar{\kappa}_k$ of the Kodaira logarithmic dimension which construction uses logarithmic k-forms. We show that a complex algebraic manifold is k-measure hyperbolic if $\bar{\kappa}_k(X) = \dim X$.

1. Introduction

In 1969 Eisenman introduced on each complex manifold X of dimension n intrinsic measures E_k^X $(k = 1, \ldots, n)$ which are biholomorphic invariants [E70], [Ko70]. The most important Eisenman measures are the first one E_1^X (the Kobayashi-Royden pseudometric) and the top one E_n^X (the Kobayashi-Eisenman pseudovolume) which have various applications in complex analysis [Ko70, Ko76, GW85]. The intermediate Eisenman measures turned out also to be useful in the analytic cancellation problem [Ka94, Z90].

In 1977 Iitaka [I77] introduced the Kodaira logarithmic dimension of algebraic varieties which is one of the main algebraic invariants now. In the same year, developing ideas from [CG72], Sakai found a remarkable connection between this algebraic invariant and the Kobayashi-Eisenman pseudovolume [Sa77]. His theorem says that if the Kodaira logarithmic dimension of a complex algebraic manifold X coincides

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with its standard dimension n then X is measure hyperbolic, that is, the Kobayashi-Eisenman pseudovolume E_n^X is volume everywhere except for, possibly, a subset of X of Hausdorff $2n$-measure zero.

Our aim is to generalize this theorem for all Eisenman measures. We use the notion of k-measure hyperbolicity (see section 2.3) which is compatible with the standard measure hyperbolicity. We consider the following algebraic invariants which were introduced by Sakai in another paper [Sa78] (see also [Ma95]). Let L be a holomorphic vector bundle on a compact algebraic manifold $\bar{X}$ of rank r. Let $S^m L$ be the symmetric m-th power of L. If no symmetric power of L has a non-trivial global section put $\lambda(L) = -\infty$. Otherwise put

$$\lambda(L) = \lim_{m \to +\infty} \sup \frac{\log \dim \Gamma(X, S^m L)}{\log m} - r + 1.$$

Let D be an SNC-divisor in $\bar{X}$ and $X = \bar{X} \setminus D$. Following Iitaka [I77] we consider the sheaf $\Omega^k(\bar{X}, D)$ of germs of logarithmic k-forms on $\bar{X}$ along D. We call $\lambda(\Omega^k(\bar{X}, D))$ the Kodaira-Iitaka-Sakai logarithmic k-dimension $\bar{\kappa}_k(X)$ of X (this definition does not depend on an SNC-completion of X). Sakai used these dimensions in a special case when X is compact and, judging by his remarks, he kept the logarithmic case in mind as well. The Kodaira-Iitaka-Sakai n-dimension is, of course, the standard Kodaira logarithmic dimension. The main result of this paper is

Theorem *Let X be an algebraic complex manifold such that $\bar{\kappa}_k(X)$ coincides with* $\dim X$. *Then X is k-measure hyperbolic.*

The proof of the Theorem follows the ideas from [CG72], [Sa74], [Sa77]. Each step of the proof is a generalization of a well-known fact and we try to reflect the names of these facts in the titles of sections.

2. Terminology and Notation

2.1 In this paper X is always a complex manifold of dimension n, TX is the holomorphic tangent bundle of X; T_xX is the holomorphic tangent space at $x \in X$; $\Lambda^k T_xX$ (resp. $\Lambda^k TX$) is the k-th exterior power of T_xX (resp. TX); $D_x^k X$ (resp. $D^k X$) is the set of decomposable elements in $\Lambda^k T_xX$ (resp. $\Lambda^k TX$). That is, the elements of $D_x^k X$ (decomposable k-vectors) are of form $v_1 \wedge \cdots \wedge v_k$ where $v_i \in T_xX$ for each i. Note that $\Lambda^n TX = D^n X$ is a holomorphic line bundle and each infinitesimal volume form generates a metric on this bundle. Suppose that B is the unit ball in $\mathbf{C}^k$ with center at the origin o. Let $\| \zeta \|$ be the metric on $\Lambda^k TB$ generated by the Euclidean volume.

Definition For every $x \in X$ and every $\nu \in D^k_x X$ $(1 \leq k \leq n)$ the intrinsic Eisenman k-measure of ν is

$$E^X_k(x, \nu) \equiv \inf\{\| \zeta \|^2\}$$

where infimum is taken over all $\zeta \in D^k_o B$ for which there exists a holomorphic mapping $f: B \to X$ with

$$f(o) = x \text{ and } f_*(\zeta) = \nu.$$

2.2 These measures can be also defined in the following way. Fix the standard Euclidean coordinate system in B and a k-dimensional subspace W in $T_x X$. One may suppose that there is a local coordinate system $w_1, ..., w_n$ in a neighborhood of x for which W is generated by vectors $\partial/\partial w_1, ..., \partial/\partial w_k$. Let $\Phi = (\frac{\sqrt{-1}}{2\pi})^k \prod_{i=1}^k dw_i \wedge d\bar{w}_i$. Consider $\nu \in D^k_x X$ such that ν generates W. Choose a mapping f as in Definition 2.1, then $w_1, ..., w_k$ can be treated as a local coordinate system on the germ of $N = f((B, o))$ at x. The mapping $f : B \to N$ is locally invertible at x, i.e. its Jacobian $Jf(o)$ is different from 0. Put

$$\Theta^X_k(x, W) = \inf_f \frac{1}{| Jf(o) |^2} \Phi,$$

where f runs over all holomorphic mappings $f: B \to X$ such that $f(o) = x$ and $f_*(T_o B) = W$. Then

$$E^X_k(x, \nu) = \Theta^X_k(x, W)(\nu \wedge \bar{\nu})$$

for every decomposable k-vector $\nu \in D^k_x X$ which generates W.

2.3 Definition We say that X is k-measure hyperbolic if $E^X_k(x, \nu) \neq 0$ for every $D^k X \setminus A$ where $A \subset D^k X$ has Hausdorff 2ℓ-measure zero with $\ell = \dim D^k X$.

Remark This definition is different form the one in [GW85] but it has an advantage: the n-measure hyperbolicity of X is equivalent to the standard measure hyperbolicity.

2.4 Let L be a holomorphic vector bundle over X. Then L_x is the fiber of this bundle over a point $x \in X$, and $\Gamma(X, L)$ is the space of holomorphic sections of L.

Denote by $\bar{L}$ the antiholomorphic vector bundle that corresponds to L. That is, the elements of the transition matrices of $\bar{L}$ are the complex conjugates of the corresponding elements of the transition matrices of L. The space of antiholomorphic section of $\bar{L}$ will be denoted by $\bar{\Gamma}(X, \bar{L})$. Put $|L|^2 = L \otimes \bar{L}$.

If K is a holomorphic line bundle then the transition functions of $|K|^2$ are positive and we can speak about smooth positive sections of $|K|^2$ which are also called metrics on K for the following reason.

Let $\{U_\alpha\}$ be a coordinate covering of X. Recall that a section $h = \{h_\alpha\}$ of $|K|^2$ is smooth positive if each trivialization function h_α on U_α is smooth positive. Let $\delta = \{\delta_\alpha\}$ be a section of K. Note that $||\delta(x)||^2 := |\delta_\alpha(x)|^2/h_\alpha(x)$, $x \in X$ is independent from α which gives a function $||\delta||^2$ on the whole X. Its value $||\delta(x)||^2$ can be treated as the square of the length of δ at x.

2.5 Let $\gamma = \{\gamma_\alpha\}$ be a nonnegative continuous section of $|K|^2$. That is, for each α the function γ_α on U_α is nonnegative and

$$\gamma_\alpha = |g_{\alpha\beta}|^2 \gamma_\beta \text{ in } U_\alpha \cap U_\beta$$

where $g_{\alpha\beta}$ is the transition function of K. Let A be the set of zeros of γ. Suppose that γ is smooth on $X \setminus A$. Recall that the exterior derivative can be written as $d = \partial + \bar{\partial}$, $d^c = \frac{\sqrt{-1}}{4\pi}(\partial - \bar{\partial})$, and, therefore, $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. One can see that $dd^c \log \gamma_\alpha = dd^c \log \gamma_\beta$. Hence we obtain a (1,1)-form on $X \setminus A$ which will be denoted by $dd^c \log \gamma$. (In the case when γ is a metric this form represents the first Chern class of K in the De Rham cohomology.)

Remark. We cannot consider γ as a metric on K when A is not empty. But in any case we can consider γ as a pseudometric on the dual bundle K^*. More precisely: if $\nu \in K^*_x$ and $\bar{\nu}$ is its conjugate in $\bar{K}^*_x$ then $\gamma(\nu \otimes \bar{\nu})$ is the square of the pseudolength of ν.

2.6 For every vector bundle L we shall denote by $S^m L$ the symmetric m-th power of this bundle. Let $\Omega^k(X)$ be the sheaf of germs of holomorphic k-forms on X and let $\overline{\Omega^k(X)}$ be the sheaf of germs of antiholomorphic k-forms on X. For every

$\varphi \in \Gamma(X, S^m\Omega^k(X))$ its conjugate in $\bar{\Gamma}(X, S^m\overline{\Omega^k(X)})$ will be denoted by $\bar{\varphi}$. Consider the fibration of antiholomorphic decomposable k-vectors $\overline{D^kX}$ over X. For a decomposable k-vector $\nu \in D^kX$ its conjugate in $\overline{D^kX}$ will be denoted by $\bar{\nu}$.

Suppose that $[H]$ is a holomorphic line bundle on X. Let γ be a pseudometric on $[-H]$, that is γ is a nonnegative continuous section of the smooth real bundle $|[H]|^2$. If $\varphi \in \Gamma(X, S^m\Omega^k(X)) \otimes [-H]$ then $\varphi \otimes \bar{\varphi}$ can be viewed as a continuous section of the smooth vector bundle $|S^m\Omega^k(X)|^2 \otimes |[-H]|^2$. Hence $\gamma \otimes \varphi \otimes \bar{\varphi}$ is a continuous section of $|S^m\Omega^k(X)|^2$. The last bundle can be embedded into $|(\Omega^k(X))^{\otimes m}|^2$ since there exists a natural embedding of $S^m\Omega^k(X)$ into into $(\Omega^k(X))^{\otimes m}$. The dual bundle of $\Omega^k(X)$ is Λ^kTX. Therefore for every $\nu \in D^kX$ the expression $(\gamma(x) \otimes \varphi \otimes \bar{\varphi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m})$ makes sense and defines a nonnegative number.

If $n = k$ then $\Omega^k(X)$ is a line bundle. Hence $S^m\Omega^k(X)$ is isomorphic to the line bundle $(\Omega^k(X))^{\otimes m}$ and $\gamma \otimes \varphi \otimes \bar{\varphi}$ is a nonnegative continuous section of the smooth real line bundle $|(\Omega^k(X))^{\otimes m}|^2$. This implies that $\psi = (\gamma \otimes \varphi \otimes \bar{\varphi})^{1/m}$ can be viewed as a pseudovolume form on X (recall that a pseudometric on Λ^kTX is called a pseudovolume).

2.7 Beginning from section 4 X will be the complement to an effective divisor $D = D_1 + \ldots + D_s$ of simple normal crossing (SNC) type in a projective algebraic manifold $\bar{X}$. In particular, for every point of X there exists a local coordinate system $(z, w) = (z_1, \ldots, z_l, w_1, \ldots, w_{n-l})$ such that

$$z_1 \cdots z_l = 0$$

defines the germ of D around this point ($0 \leq l \leq m$ and when $m = 0$ this point does not belong to D). In this case $\Omega^k(X)$ contains the subsheaf $\Omega^k(\bar{X}, D)$ of logarithmic k-forms along D (see [I77]). The germs of these forms can be written as

$$\sum_{r+q=k} c_{I,J}(z,w)\frac{dz_{i(1)}}{z_{i(1)}} \wedge \ldots \frac{dz_{i(r)}}{z_{i(r)}} \wedge \frac{dw_{j(1)}}{w_{j(1)}} \wedge \ldots \frac{dw_{j(q)}}{w_{j(q)}},$$

where $c_{I,J}(z, w)$ is the germ of a holomorphic function, and the indices of summation are $I = (i(1), \ldots, i(r))$ and $J = (j(1), \ldots, j(q))$.

2.8 Suppose that X is compact and the rank of a vector bundle L on X is r. If no symmetric power of L has a non-trivial global section put $\lambda(L) = -\infty$. Otherwise put

$$\lambda(L) = \lim_{m\to+\infty} \sup \frac{\log \dim \Gamma(X, S^m L)}{\log m} - r + 1.$$

Definition We shall call $\lambda(L)$ the Sakai dimension of L.

It is known [Sa78, Ma95] that

$$\lambda(L) \leq n.$$

This inequality will be crucial in section 5.

3. The Ahlfors-Griffiths-Kobayashi Lemma.

3.1 The result we are going to discuss in this section is well-known in the case of $k = n$ (e.g., see [GW85, Ko76, Sa74]). We shall follow to a great extend the argument in the top-dimensional case [Sa74], but it is a pleasure to repeat this proof.

Let $\{U_\alpha\}$ be a coordinate covering of X and let ω be a pseudovolume form on X. That is, locally ω is the product of the Euclidean volume form (in the coordinate system of U_α) and a nonnegative function ξ_α. The Ricci form of ω is the $(1,1)$-form $\operatorname{Ric}\omega = dd^c \log \xi_\alpha$ which is defined at those points where ξ_α is smooth and positive. Let $(z_1, \ldots, z_k)$ be a coordinate system in $\mathbf{C}^k$ and let $B[r]$ be the open ball of radius r in $\mathbf{C}^k$. Let Φ be the volume form on $\mathbf{C}^k$ defined by $\Phi = (\frac{\sqrt{-1}}{2\pi})^k \prod_{i=1}^k dz_i \wedge d\bar{z}_i$. Then the Poincaré volume V_r on $B[r]$ is given by $V_r = \mu_r(z) \cdot \Phi$ where

$$\mu_r(z) = (k+1)^k k! r^2 / (r^2 - \sum_{i=1}^k |z_i|^2)^{k+1}.$$

(Up to a constant factor, this volume is also called the Bergman volume or the Kobayashi-Eisenman volume.) It is known that $\operatorname{Ric} V_r = dd^c \log \mu_r > 0$ (i.e., this (1,1)-form is positive definite) and $(\operatorname{Ric} V_r)^k = V_r$ [Sa74].

3.2 Theorem *Let A be a proper closed analytic subset of X, let H be an effective divisor on X generating a line bundle $[H]$, and let γ be a nonnegative section of $|[H]|^2$ which vanish on A only and which is smooth on $X \setminus A$. Let $\varphi \in \Gamma(X, S^m \Omega^k(X) \otimes [-H])$. Suppose that the following conditions hold*

(1) $dd^c \log \gamma$ is a positive definite (1,1)-form on $X \setminus A$;

(2) for every $x \in X \setminus A$ and for every $\nu \in D^k_x X$ we have

$$[(-1)^{\frac{k(k-1)}{2}}(dd^c \log\gamma)^k(\nu\wedge\bar\nu)]^m \geq (\gamma(x)\otimes\varphi\otimes\bar\varphi)(\nu^{\otimes m}\otimes\bar\nu^{\otimes m}).$$

Let $f : B[r] \to X$ be a holomorphic mapping. Then $\psi := [f^*(\gamma\otimes\varphi\otimes\bar\varphi)]^{1/m}$ is a pseudovolume form on $B[r]$ such that $\psi \leq V_r$.

Proof. It was shown in section 2.6 that ψ is a pseudovolume. We can suppose that $\varphi|_{f(B[r])} \neq 0$ since otherwise $\psi \equiv 0$. Let Φ and μ_r be as in 3.1. Then $\psi(z) = \xi(z)\Phi$ where ξ is a nonnegative continuous function which is C^∞ on $B[r]\setminus f^{-1}(A)$. For every positive t such that $0 < t < r$ put

$$\eta_t(z) = \frac{\xi(z)}{\mu_t(z)}.$$

Then $\eta_t(z)$ is continuous on $B[t]\cup\partial B[t]$. Let $z_0 \in B[t]\cup\partial B[t]$ be a point where $\eta_t(z)$ attains its maximum. Note that $z_0 \notin f^{-1}(A)$ since $\xi|_{f^{-1}(A)}$ and, therefore, $\eta_t|_{f^{-1}(A)}$ are identically zero. Thus $\eta_t(z)$ is C^∞ near z_0.

In order to follow the proof in [Sa74] we need to show that

$$dd^c \log\xi(z)|_{z=z_0} = dd^c \log\gamma'(z)|_{z=z_0}$$

where $\gamma' = \gamma\circ f$ (that is, $dd^c\log\xi(z)|_{z=z_0}$ coincides with $f^*(dd^c\log\gamma)(z)|_{z=z_0}$). Note that z_0 cannot be a point where each $k\times k$-minor of the Jacobi matrix of f (in a suitable local coordinate system on X) is zero. In particular, by the implicit function theorem the corresponding irreducible branch N of the germ of $f(B[r])$ at $f(z_0)$ is smooth and k-dimensional. Consider $H_N = H\cap N$. The restriction φ_N of φ to $\Lambda^k N$ is a holomorphic section of the trivial line bundle $S^m\Omega^k(N)\otimes[-H_N]$ (i.e., it can be treated as a holomorphic function on N) and locally $\psi = [(\gamma_{\alpha_0}\cdot\varphi_N\cdot\bar\varphi_N)\circ f]^{1/m}$ where $z_0 \in U_{\alpha_0}$. Since z_0 is a point of maximum $\psi(z_0)\neq 0$ and thus $\varphi_N(f(z_0)) \neq 0$. Hence $\xi(z) = \gamma'_{\alpha_0}\cdot|h(z)|^2$ where $h(z)$ is a germ of a holomorphic function at z_0 such that $h(z_0)\neq 0$, and $\gamma'_{\alpha_0} = \gamma_{\alpha_0}\circ f$. This implies that $dd^c\log\xi = dd^c\log\gamma'$.

Now we can follow [Sa74] again. By the maximality, we have

$$dd^c\log\eta_t(z)|_{z=z_0} = [dd^c\log\xi(z) - dd^c\log\mu_t(z)]_{z=z_0} \leq 0.$$

Note that if A and B are nonnegative definite Hermitian matrices such that $A \leq B$ then $\det A \leq \det B$. Using the facts that

$$dd^c\log\xi(z)|_{z=z_0} = f^*(dd^c\log\gamma)(z)|_{z=z_0} \geq 0$$

and

$$dd^c \log \mu_t = \mathrm{Ric}\, V_t > 0,$$

we obtain

$$f^*((dd^c \log \gamma)^k) \leq (\mathrm{Ric}\, V_t)^k \text{ at } z_0.$$

Since $V_t = (\mathrm{Ric}\, V_t)^k$ it follows from condition (2) that $\psi \leq V_t$ at $z = z_0$. Hence $\xi(z_0) \leq \mu_t(z_0)$ and $\eta_t(z) \leq \eta_t(z_0) \leq 1$. Taking limit as $t \to r$ we have

$$\xi/\mu_r = \eta_r(z) \leq 1$$

which implies that $\psi = \xi(z)\Phi \leq \mu_r \Phi = V_r$.
QED.

3.3 Corollary *Under the assumption of Theorem 3.2 the manifold X is k-measure hyperbolic if φ is not identically zero.*

Proof. One can consider φ as an element of $\Gamma(X, S^m\Omega^k(X))$ which vanishes on H. Hence it makes sense to speak about the value of φ on $\nu^{\otimes m}$ where $\nu \in D^k X$. Consider the subset $\mathcal{A}$ of $D^k X$ such that $\varphi(\beta^{\otimes m}) = 0$ for every $\beta \in \mathcal{A}$. Since φ is a holomorphic nonzero section $\mathcal{A}$ is a proper closed analytic subset of $D^k X$. Consider a point $x \in X$ and a k-vector $\nu \in D^k_x X$ which is not in $\mathcal{A}$. Choose a local coordinate system $w_1, ..., w_n$ in a neighborhood of x such that the vectors $\partial/\partial w_1, ..., \partial/\partial w_k$ generate the same k-space W in $T_x X$ as ν. Put

$$\alpha = \partial/\partial w_1 \wedge ... \wedge \partial/\partial w_k.$$

Since α is proportional to ν we have $b := \varphi(\alpha^{\otimes m}) \neq 0$. Let $f : B \to X$ be a holomorphic mapping from the unit ball B in $\mathbf{C}^k$ into X such that $f(o) = x$ and $f_*(T_o B) = W$ where o is the origin. Consider the image N of the germ $(\mathbf{C}^k, o)$ of $\mathbf{C}^k$ under f. Since f is not degenerate at o one can see that N is smooth, k-dimensional, and $(w_1, \ldots, w_k)$ is a local coordinate system on N. Hence $\psi = [f^*(\gamma \otimes \varphi_N \otimes \bar{\varphi}_N)]^{1/m}$ where φ_N is the restriction of φ to N. Let ξ be the same as in the proof of Theorem 3.2. Then the above formula for ψ implies that $\xi(0) = |J_f|^2 |b|^{2/m}$ where J_f is the Jacobian of the mapping $f : (\mathbf{C}^k, o) \to N$. By Theorem 3.2 $\xi(0) \leq \mu_r(0) = (k+1)^k k!$. That is

$$\frac{1}{|J_f(o)|^2} \geq \frac{|b|^{2/m}}{(k+1)^k k!} > 0.$$

It follows from section 2.2 that the Eisenman k-measure $E_k^X(x,\nu) > 0$.
QED.

Remark This Corollary and Theorem 3.2 remain true if A from Theorem 3.2 is not necessarily an analytic subset of X, but only a closed subset of Hausdorff $2n$-measure zero.

4. The Kodaira-Iitaka-Sakai dimensions of algebraic varieties

4.1 From now on we suppose that $X, \bar{X}, D$, and $\Omega^k(\bar{X}, D)$ are the same as in section 2.7. Put $L = \Omega^k(\bar{X}, D)$. Consider the Sakai dimension $\lambda(L)$ of L (see section 2.8 for the definition).

Definition We shall call $\lambda(L)$ the Kodaira-Iitaka-Sakai logarithmic k-dimension $\bar{\kappa}_k(X)$ of X.

We need to show that this definition is correct which is a simple repetition of Iitaka's argument.

4.2 Proposition *Let $\bar{X}_1$ and $\bar{X}_2$ be complex complete algebraic manifolds, and D_1 and D_2 are divisor of SNC-type in $\bar{X}_1$ and $\bar{X}_2$ respectively. Suppose that $\bar{f} : \bar{X}_1 \to \bar{X}_2$ is a morphism and that $\bar{f}$ is the extension of a dominant morphism $f : X_1 \to X_2$ where $X_i = \bar{X}_i - D_i$. Then $\bar{f}$ generates a natural homomorphsim*

$$f^* : S^m\Omega^k(\bar{X}_2, D_2) \to S^m\Omega^k(\bar{X}_1, D_1).$$

Moreover, if $f : X_1 \to X_2$ is birational and proper then f^* is an isomorphism.

Proof. One way to show this fact is to repeat Proposition 1 from [I77] almost word for word. A shorter way is another reference to [I77]. Iitaka mentions that Proposition 1 holds for forms in $T_{m_1,\dots,m_n}(X)$ where

$$T_{m_1,\dots,m_n}(X) = \Gamma(\bar{X}, (\Omega^1(\bar{X}, D))^{\otimes m_1}) \otimes \dots \otimes \Gamma(\bar{X}, (\Omega^n(\bar{X}, D))^{\otimes m_n}).$$

If $m_k = m$ and $m_i = 0$ for every $i \neq k$ then

$$T_{m_1,\dots,m_n}(X) = \Gamma(\bar{X}, (\Omega^k(\bar{X}, D))^{\otimes m}).$$

Hence Proposition 1 from [I77] yields a natural homomorphsim

$$f^* : \Gamma(\bar{X}, (\Omega^k(\bar{X}_2, D_2))^{\otimes m}) \to \Gamma(\bar{X}, (\Omega^k(\bar{X}_1, D_1))^{\otimes m})$$

which is an isomorphism when f is birational and proper. Since f^* is natural (it is generated by the induced mapping of m-tuple k-forms) it commutes with the action of the symmetric group $S(m)$ on the corresponding vector bundles. Note that $\Gamma(\bar{X}, S^m\Omega^k(\bar{X}, D))$ can be treated as a subspace of $\Gamma(\bar{X}, (\Omega^k(\bar{X}, D))^{\otimes m})$ which is invariant relative to the action of $S(m)$. This implies the desired conclusion.
QED.

4.3 Corollary *The Kodaira-Iitaka-Sakai logarithmic k-dimesnion $\bar{\kappa}_k(X)$ does not depend on the choice of completion $\bar{X}$ if $D = \bar{X} \setminus X$ is of SNC-type.*

Proof. Let $\bar{X}'$ and $\bar{X}''$ be two smooth completions of X whose boundaries D' and D'' are of SNC-type. Then the identical automorphism of X generates a rational map $g : \bar{X}' \to \bar{X}''$. By Hironaka's Main Theorem II there exists a completion $\bar{X}$ of X with a boundary D of SNC-type and two birational morphism $\rho' : \bar{X} \to \bar{X}'$ and $\rho'' : \bar{X} \to \bar{X}''$ such that $\rho'' = g \circ \rho'$ and the restrictions of ρ' and ρ'' to X are automorphisms of X. Proposition 4.2 implies that

$$\lambda(\Omega^k(\bar{X}, D)) = \lambda(\Omega^k(\bar{X}', D')) = \lambda(\Omega^k(\bar{X}'', D'')).$$

QED.

Remark. One can define the Kodaira-Iitaka-Sakai dimension for a singular variety as this dimension of its smooth model.

5. The Kodaira Lemma

We remind that we use the assumption and notation from section 2.7. Let H be a divisor which generates a very ample line bundle $[H]$ on $\bar{X}$.

Proposition (cf. Kodaira's Lemma in [Sa74]) *Let L be a holomorphic vector bundle on $\bar{X}$ whose Sakai dimension $\lambda(L)$ is n. Then $\dim \Gamma(\bar{X}, S^m L \otimes [-H]) > 0$ for some large m.*

Proof. Since $[H]$ is very ample it has no base points, and by Bertini theorem we can choose H to be smooth. Hence we have the following exact sequence [GH78, ch. 1, section 1]

$$0 \to \Gamma(\bar{X}, S^m L \otimes [-H]) \to \Gamma(\bar{X}, S^m L) \to \Gamma(\bar{X}, S^m(L|_H)) \to$$

where $L|_H$ is the restricition of L to H. Since $\dim H = n-1$ it follows from section 2.8 that $\dim \Gamma(H, S^m(L|_H)) \leq O(m^{n-1})$. Using the assumption about the Sakai dimension of L we have $\dim \Gamma(\bar{X}, S^{mm_0} L) \geq am^n$, $a > 0$ for some m_0 and for any large m. Hence the exact sequence implies that $\dim \Gamma(\bar{X}, S^{mm_0} L \otimes [-H]) > 0$ for sufficiently large m.
QED.

Corollary *Let $\bar{\kappa}_k(X) = n$. Then there exists a nontrivial $\varphi \in \Gamma(\bar{X}, S^m L \otimes [-H])$ for some large m.*

6. The Carlson-Griffiths-Sakai Construction

6.1 Our aim is to apply Theorem 3.2 and Corollary 3.3. Hence besides a nontrivial φ which is now provided by Corollary 5 we need γ as in Theorem 3.2 which will be constructed in this section in the manner of [CG72] and [Sa77]. Let $\{U_\alpha\}$ be a coordinate covering of $\bar{X}$. Suppose that $[H]$ is a very ample bundle on $\bar{X}$ and that $h = \{h_\alpha\}$ is a metric on H such that $dd^c \log h$ is a positive (1,1)-form on $\bar{X}$. Let $\delta_i = \{\delta_{i,\alpha}\}$ be a holomorphic section of $[D_i]$ defining D_i where $i = 1, \ldots, s$. Let $||\delta_i||$ be the length of δ_i in some metric on $[D_i]$. Note that multiplying δ_i by small $\epsilon > 0$ we can make $||\delta_i||$ as small as we wish.

Recall that for every point x_0 of D there exists a coordinate system $(z, w) = (z_1, \ldots, z_l, w_1, \ldots, w_{n-l})$ in a neighborhood U of x_0 such that

$$z_1 \cdots z_l = 0$$

defines the germ of D around this point. Consider a k-vector $\nu \in D^k_x X$ where $x \in U$. It is of form

$$\nu = \sum_{I,J} a_{I,J} \Lambda_{t=1}^{q} \frac{\partial}{\partial z_{i(t)}} \wedge \Lambda_{t=1}^{k-q} \frac{\partial}{\partial w_{j(t)}}$$

where the indices of summation are $I = (i(1), \dots, i(q))$, $J = (j(1), \dots, j(k-q))$, and q runs from 0 to $\min(k, l)$ (we use the symbol Λ here to denote the exterior product).

We shall fix this notation $x_0, U, (z, w), \nu$, and Λ for the rest of the section.

6.2 Lemma *Let*

$$\gamma = \frac{ch}{\prod_{j=1}^{s}(\log ||\delta_j||^2)^{2m}}$$

where c is a positive constant, that is, γ is a nonnegative section of $|[H]|^2$ which vanishes on D only. If the lengths $||\delta_i||$ are sufficiently small then $dd^c \log \gamma$ is positive definite on $X = \bar{X} \setminus D$ and for every $x_0 \in D$ there exists a coordinate neighborhood U as before this lemma such that in this neighborhood

$$dd^c \log \gamma \geq bm\frac{\sqrt{-1}}{2\pi}\{\sum_{j=1}^{l} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2(\log |z_j|^2)^2} + \sum_{j=1}^{n-l} dw_j \wedge d\bar{w}_j\}\,,\; b > 0.$$

Proof. The argument is exactly the same as in Proposition 2.1 in [CG72]. The only difference is that Carlson and Griffiths considered the case where $[H]$ is the sum of the canonical bundle and another bundle. Thus one need to replace $\Omega / \prod_{j=1}^{l} ||\delta_j||^2$ in formula (2.3) from [CG72] by ch and $\prod_{j=1}^{l}(\log ||\delta_j||^2)^2$ by $\prod_{j=1}^{s}(\log ||\delta_j||^2)^{2m}$. After this the proof becomes the exact repetition and the desired inequality for $dd^c \log \gamma$ is formula (2.11) from [CG72] up to notation and the inessential factor m.
QED.

6.3 Consider the k-th exterior power of the right-hand side of the inequality in Lemma 6.2

$$\omega = \left(bm\frac{\sqrt{-1}}{2\pi}\right)^k \sum_{I,J} \left\{\Lambda_{t=1}^{q} \frac{dz_{i(t)} \wedge d\bar{z}_{i(t)}}{|z_{i(t)}|^2(\log |z_{i(t)}|^2)^2} \wedge \Lambda_{t=1}^{k-q}(dw_{j(t)} \wedge d\bar{w}_{j(t)})\right\}$$

where the indices of summation are $I = (i(1), \dots, i(q))$, $J = (j(1), \dots, j(k-q))$, and q is any natural number from 0 to $\min(k, l)$.

Corollary *Let $x \in U \setminus D$. Then for every k-vector $\nu \in D_x^k X$ we have*

$$(-1)^{\frac{k(k-1)}{2}}(dd^c \log \gamma)^k(\nu \wedge \bar{\nu}) \geq (-1)^{\frac{k(k-1)}{2}}\omega(\nu \wedge \bar{\nu}).$$

In particurlar,

$$[(-1)^{\frac{k(k-1)}{2}}(dd^c \log \gamma)^k(\nu \wedge \bar{\nu})]^m \geq b_1 \left(\sum_{I,J} \frac{|a_{I,J}|^2}{\prod_{t=1}^{q} |z_{i(t)}|^2(\log |z_{i(t)}|^2)^2}\right)^m$$

where b_1 is a positive constant.

Proof. Consider the subspace W of T_xX generated by ν. Then the left-hand side and the right-hand side of the inequality in Lemma 6.2 generate positive definite Hermitian matrices A and B on W. Since $A \geq B$ we have $\det A \geq \det B$ which implies the desired conclusion.
QED.

6.4 Let $\psi \in \Gamma(\bar{X}, S^m\Omega^k(\bar{X}, D))$. Then in U

$$\psi = \sum_{\mathcal{I},\mathcal{J}} c_{\mathcal{I},\mathcal{J}}(x) \prod_{r=1}^{m} (\Lambda_{t=1}^{q_r} \frac{dz_{i(r,t)}}{z_{i(r,t)}} \wedge \Lambda_{t=1}^{k-q_r} dw_{j(r,t)})$$

where the functions $c_{\mathcal{I},\mathcal{J}}$ are holomorphic in U, the indices of summation are $\mathcal{I} = \{i(r,t)\}_{r=1,t=1}^{m,q_r}$, $\mathcal{J} = \{j(r,t)\}_{r=1,t=1}^{m,k-q_r}$, and q_r is any natural number from 0 to $\min(k,l)$ which depends on r. Note that

$$\nu^{\otimes m} = \sum_{I_r,J_r,\mathcal{I},\mathcal{J}} \bigotimes_{r=1}^{m} (a_{I_r,J_r} \Lambda_{t=1}^{q_r} dz_{i(r,t)} \wedge \Lambda_{t=1}^{k-q_r} dw_{j(r,t)})$$

where the summation runs over indices $\mathcal{I}, \mathcal{J}, I_r, J_r$ such that

$$\mathcal{I}(r,\cdot) = I_r, \ \mathcal{J}(r,\cdot) = J_r$$

for every $r = 1, \dots, m$. Hence the triangle inequality implies

$$(\psi \otimes \bar{\psi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}) \leq \left[\sum_{I_r,J_r,\mathcal{I},\mathcal{J}} |c_{\mathcal{I},\mathcal{J}}(x)| \prod_{r=1}^{m} \frac{|a_{I_r,J_r}|}{\prod_{t=1}^{q_r} |z_{i(r,t)}|} \right]^2 .$$

Choosing a smaller neighborhood U, if necessary, we can suppose that $|c_{\mathcal{I},\mathcal{J}}(x)|^2$ does not exceed some positive constant b_2 for every $x \in U$ and every $\mathcal{I}, \mathcal{J}$. Hence

$$(\psi \otimes \bar{\psi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}) \leq b_2 \left[\sum_{I_r,J_r,\mathcal{I},\mathcal{J}} \prod_{r=1}^{m} \frac{|a_{I_r,J_r}|}{\prod_{t=1}^{q_r} |z_{i(r,t)}|} \right]^2 .$$

For every positive integer p there exists $d > 0$ such that for all numbers $d_1, \dots, d_p$ the following inequality between homogeneous forms holds

$$(d_1 + \ldots + d_p)^{2m} \leq d(d_1^{2m} + \ldots + d_p^{2m}).$$

Applying this inequality in the case when the set $\{d_i\}$ coincides with the set

$$\left\{\frac{|a_{I,J}|}{\prod_{t=1}^{q}|z_{i(t)}|}\right\}$$

where I, J are the same as in section 6.1 one obtains

Lemma *When U is sufficiently small, $x \in U \setminus D$, and $\nu \in D^k_x X$ we have*

$$(\psi \otimes \bar{\psi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}) \leq b_3 \left[\sum_{I,J} \frac{|a_{I,J}|^2}{\prod_{t=1}^{q}|z_{i(t)}|^2}\right]^m$$

for some positive b_3.

6.5 Proposition *Let $\varphi \in \Gamma(\bar{X}, S^m\Omega^k(\bar{X}, D)) \otimes [-H]$ and let γ be as in Lemma 6.2. If c in the definition of γ is sufficiently small then conditions (1) and (2) from Theorem 3.2 hold.*

Proof. Condition (1) follows from Lemma 6.2. Since $dd^c \log \gamma$ does not depend on c one can suppose that for each neighborhood of D we have

$$[(-1)^{\frac{k(k-1)}{2}}(dd^c \log \gamma)^k(\nu \wedge \bar{\nu})]^m \geq (\gamma(x) \otimes \varphi \otimes \bar{\varphi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m})$$

for every x which does not belong to this neighborhood and for every $\nu \in D^k_x X$. The compactness of D implies that it suffices to check the same inequality for every point x from $U \setminus D$.

Let h be as Lemma 6.2 and let θ be a nonvanishing section of $[-H]$ over U. Then $h \otimes \theta \otimes \bar{\theta}$ may be viewed as a section of a trivial real line bundle over U, i.e. as a real function. Hence, reducing U, if necessary, one can suppose that

$$\gamma \otimes \theta \otimes \bar{\theta} \leq \frac{cb_4}{\prod_{j=1}^{s}(\log ||\delta_j||^2)^{2m}}, \ b_4 > 0.$$

Furthermore, since on U the length $||\delta_j||$ of the section δ_j (recall that δ_j is a defining section for D_j) is the product of $|z_j|$ and a nonvanishing function for $i = 1, \dots, l$ we have

$$\gamma \otimes \theta \otimes \bar{\theta} \leq \frac{cb_5}{\prod_{j=1}^{s}(\log |z_j|^2)^{2m}}, \ b_5 > 0.$$

Since θ is nonvanishing we have $\varphi = \theta \otimes \psi$ where $\psi \in \Gamma(U, S^m\Omega^k(U, D \cap U))$. Then

$$(\gamma \otimes \varphi \otimes \bar{\varphi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}) = (\gamma \otimes \theta \otimes \bar{\theta}) \cdot (\psi \otimes \bar{\psi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}).$$

Using the upper bound on $\gamma \otimes \theta \otimes \bar{\theta}$ and Lemma 6.4, we have

$$(\gamma \otimes \varphi \otimes \bar{\varphi})(\nu^{\otimes m} \otimes \bar{\nu}^{\otimes m}) \leq \frac{cb_6}{\prod_{j=1}^{s}(\log |z_j|^2)^{2m}} \left[\sum_{I,J} \frac{|a_{I,J}|^2}{\prod_{t=1}^{q} |z_{i(t)}|^2} \right]^m, \; b_6 > 0.$$

In combination with Corollary 6.3 this implies the desired conclusion.
QED.

6.6 If $\bar{\kappa}_k(X) = n$ then by Corollary 5 there exists a nontrivial

$$\varphi \in \Gamma(\bar{X}, \Omega^k(\bar{X}, D)) \subset \Gamma(\Omega^k(X)).$$

By Proposition 6.5 there exists γ such that the assumption of Theorem 3.2 holds. Corollary 3.3 yields our main result.

Theorem *Let X be a complex algebraic manifold of dimension n. Suppose that the Kodaira-Iitaka-Sakai logarithmic k-dimension of X coincides with n. Then X is k-measure hyperbolic.*

Remark (1) It is worth mentioning that this theorem gives sufficient conditions only for k-measure hyperbolicity. It follows from [Sa78] that a surface Y can be 1-measure hyperbolic even in the case when $\bar{\kappa}_1(Y) = 0$.

(2) If the holomorphic mappings from the ball in the definition of Eisenman measures are replaced by meromorphic mappings which are reqular at the origin then we obtain other intrinsic measures which are called Yau measures [Y75]. One can introduce the notion of k-measure hyperbolicity with respect to the Yau measures (we call it meromorphic k-measure hyperbolicity) in the same way we did for Eisenman measures. It can be shown that under the assumption of Corollary 3.3 the manifold is meromorphically k-measure hyperbolic (the proof follows the argument in [Ko76]). This may be a stronger result since for a complex manifold the meromorphic k-measure hyperbolicity implies the k-measure hyperbolicity but not vice versa ([Ka96].

(3) Note that Theorem 6.6 has applications to the analytic cancellation problem. For instance, a manifold X whose Kodaira-Iitaka-Sakai logarithmic k-dimension coincides with its dimension n cannot be biholomorphic to $\mathbf{C}^m \times Y$ where the dimension

$n - m$ of Y is less than k (e.g., see [Ka94]). We discuss such applications further in the next paper.

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