THE MALGRANGE VANISHING THEOREM WITH SUPPORT CONDITIONS

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0. Introduction

Let X be a complex manifold of dimension n, and suppose an open subset Z of X is given such that $\overline{Z} \setminus Z$ is compact.

Denote by $\Phi = \Phi(Z)$ the family of all closed subsets C of X such that $C \cap \overline{Z}$ is compact. Then Φ is a family of supports in X (in the sense of Serre [S]). Note that Φ consists of all closed subsets of X if Z is relatively compact in X. If E is a holomorphic vector bundle over X, then we use the following notations:

- $C^0_{s,r}(\Phi; X, E)$ is the space of all E-valued continuous (s,r)-forms f on X with $\operatorname{supp} f \in \Phi$;
- $Z^0_{s,r}(\Phi; X, E)$ is the subspace of all $\overline{\partial}$-closed forms in $C^0_{s,r}(\Phi; X, E)$.
- $E^0_{s,r}(\Phi; X, E) := C^0_{s,r}(\Phi; X, E) \cap \overline{\partial} C^0_{s,r-1}(\Phi; X, E), \quad \text{if } r \geq 1;$
- $E^0_{s,r}(\Phi; X, E) := \{0\}, \quad \text{if } r = 0;$
- $H^{s,r}_\Phi(X, E) := Z^0_{s,r}(\Phi; X, E) \big/ E^0_{s,r}(\Phi; X, E).$

Note that $H^{s,r}_\Phi(X, E)$ is the usual Dolbeault cohomology group if Z is relatively compact.

0.1. Definition. We say X is $(n-1)$-*concave at the ends contained in* Z if either Z is relatively compact (and hence no end is contained in Z) or there exists a C^∞ function ρ on X such that:

(1) $\rho(z) > \inf_{\zeta \in X} \rho(\zeta)$ for all $z \in X$;

(2) the sets $\{\zeta \in \overline{Z} \mid \rho(\zeta) \geq \alpha\}$, $\alpha > \inf_{\zeta \in X} \rho(\zeta)$, are compact;

(3) there exists $\alpha_0 > \inf_{\zeta \in X} \rho(\zeta)$ such that $\{\zeta \in X \mid \rho(\zeta) \leq \alpha_0\} \subseteq Z$ and the Levi form of ρ has at least 2 positive eigenvalues on $\{\zeta \in X \mid \rho(\zeta) \leq \alpha_0\}$.

In this paper we prove the following

0.2. Theorem. *Suppose X is connected, $(n-1)$-concave at the ends contained in Z, and $X \setminus Z$ is not compact. Then, for each holomorphic vector bundle E over X,*

$$H^{0,n}_\Phi(X, E) = 0.$$

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Example [Mi]*:* Let $X = \tilde{X} \setminus S$ where $\tilde{X}$ is a compact complex space of dimension n whose singular points are isolated. Let S be the set of all singular points of $\tilde{X}$. Assume that S is divided into two non-empty subsets S_1 and S_2. Let $U \subset\subset \tilde{X} \setminus S_2$ be a neighborhood of S_1. Set $Z = U \setminus S_1$. Then X is $(n-1)$-concave at the ends defined by Z and $X \setminus Z$ is not compact.

If Z is relatively compact and hence Φ consists of all closed subsets, Theorem 0.2 is the classical vanishing theorem of Malgrange [M] (1955). Ohsawa [O] (1984) observed that this theorem can be obtained also by the following argument: Since, by a theorem of Green and Wu [G-W] (1975), any connected non-compact Riemannian manifold admits a C^∞ exhausting function with strictly positive Laplacian, every connected non-compact complex manifold of dimension n admits an n-convex exhausting function. Therefore, the theorem of Malgrange follows from Andreotti-Grauert theory [A-G] (1962). Ohsawa used this argument to give a new proof (the first proof was given by Siu [Siu-1,Siu-2] (1969)) for the Malgrange vanishing theorem on reduced complex analytic spaces of dimension n without compact n-dimensional irreducible branches, constructing an n-convex exhausting function on such spaces.

It seems to the authors that the proof of Theorem 0.2 given below is interesting also in this classical situation of Malgrange ($Z \subset\subset X$), because, in this case, we do not use any global complex-geometric properties of the manifold - it's true we also use the exhausting function of Green and Wu, but the difference to Ohsawa's argument is that we do not need the full information given by this function: We only use the consequence that *there exists a Morse function without local maxima.* By Green and Wu such a function exists on all connected non-compact C^∞-manifolds. This makes it possible to prove the Malgrange vanishing theorem also on CR-manifolds which are connected, non-compact, locally embeddable and 1-concave [La-L].

At the end of this paper (Sect. 4) we give an application of Theorem 0.2 to the Hartogs-Bochner phenomenon.

1. Notations

If X is a complex manifold and E is a holomorphic vector bundle over X, then we use the following notations:

If $D \subset\subset X$ is a relatively compact open subset of X, then:

$C^0_{s,r}(\overline{D}, E)$ is the Banach space of continuous E-valued (s,r)-forms on $\overline{D}$;

$C^\alpha_{s,r}(\overline{D}, E)$, $0 < \alpha < 1$, is the Banach space of E-valued (s,r)-forms on $\overline{D}$ which are Hölder continuous with exponent α;

$C^\alpha_{s,r}(\overline{D}; X, E)$, $0 \le \alpha < 1$, is the Banach space of forms $f \in C^0_{s,r}(X, E)$ with

$$\operatorname{supp} f \subseteq \overline{D} \qquad \text{and} \qquad f|_{\overline{D}} \in C^\alpha_{s,r}(\overline{D}, E),$$

endowed with the topology of $C^\alpha_{s,r}(\overline{D}, E)$;

$C^\alpha_{s,r}(X, E)$ is the Fréchet space of forms $f \in C^0_{s,r}(X, E)$ with $f|_{\overline{D}} \in C^\alpha_{s,r}(\overline{D}, E)$

for each open $D \subset\subset X$, endowed with the topology of uniform convergence in each $C^{\alpha}_{s,r}(\overline{D}, E)$.

If now Y is an arbitrary subset of X, then we denote by $C^{\alpha}_{s,r}(Y; X, E)$ the subspace of all $f \in C^{\alpha}_{s,r}(X, E)$ with $\operatorname{supp} f \subseteq Y$, endowed with the Fréchet topology of $C^{\alpha}_{s,r}(X, E)$. We set

$$Z^{\alpha}_{s,r}(Y; X, E) = Z^{\alpha}_{s,r}(X, E) \cap C^{\alpha}_{s,r}(Y; X, E).$$

$Z^{\alpha}_{s,r}(Y; X, E)$ will be considered also as Fréchet space endowed with the topology of $C^{\alpha}_{s,r}(X, E)$. Note that if Y is compact, then $C^{\alpha}_{s,r}(Y; X, E)$ and $Z^{\alpha}_{s,r}(Y; X, E)$ are Banach spaces, $0 \le \alpha < 1$.

If E is the trivial line bundle, then in the above notations we omit the letter E.

2. Approximation in degree $n-1$

2.1. Definition. Let V be a topological space and $U \subseteq V$. We shall say that *U has no holes with respect to V* if, for each compact subset K of U, there exists a compact set $K' \subseteq U$ such that $K \subseteq K'$ and the set $V \setminus K'$ has no connected component which is relatively compact in V.

2.2. Lemma. *Let X be a complex manifold of dimension n which is completely $(n-1)$-convex in the sense of Andreotti-Grauert (for the purpose of the present paper we may assume that X is a convex domain in $\mathbb{C}^n$), and let D be a domain in X which has no holes with respect to X. Then, for each holomorphic vector bundle E over X, the image of the restriction map*

$$Z^0_{0,n-1}(X, E) \longrightarrow Z^0_{0,n-1}(D, E)$$

is dense in $Z^0_{0,n-1}(D, E)$.

Proof. Assume the contrary, i.e. there exists a form $f \in Z^0_{0,n-1}(D, E)$ which does not belong to the closure of $Z^0_{0,n-1}(X, E)$ in $Z^0_{0,n-1}(D, E)$. By the Hahn-Banach theorem and regularity of $\overline{\partial}$, then there exists a $\overline{\partial}$-closed $C^{\infty}_{n,1}$-form u with values in the dual bundle E^* such that $\operatorname{supp} u \subset\subset D$ and

$$\int_D u \wedge f \neq 0, \tag{2.1}$$

but $\int_D u \wedge g = 0$ for all $g \in Z^0_{0,n-1}(X, E)$. Since X is completely $(n-1)$-convex, from Andreotti-Grauert theory we get an E^*-valued $C^{\infty}_{n,0}$-form φ with compact support in X such that $\overline{\partial}\varphi = u$. Since D has no holes with respect to X, there exists a compact set $K \subset\subset D$ such that $\operatorname{supp} u \subseteq K$ and $X \setminus K$ has no connected component which is relatively compact in X. Since φ is holomorphic outside K and $\operatorname{supp} \varphi$ is compact, this implies that $\operatorname{supp} \varphi \subset\subset D$. Hence, by Stokes' theorem, $\int_D u \wedge f = (-1)^{n-1} \int_D \varphi \wedge \overline{\partial} f = 0$ which contradicts (2.1). $\square$

Recall that, since the boundary integral in the Bochner-Martinelli-Koppelman formula vanishes for forms of maximal degree, there is the following lemma:

2.3. Lemma. *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain with piecewise C^1-boundary, and let B_D be the Bochner-Martinelli-Koppelman operator for D. Then $\overline{\partial} B_D f = f$ for all $f \in C^0_{0,n}(\overline{D})$.*

2.4. Lemma. *Let $D \subset\subset \mathbb{C}^n$ be a domain with piecewise C^1-boundary. Then, for each $f \in Z^0_{0,n-1}(\overline{D})$, there exists a sequence U_j of neighborhoods of $\overline{D}$ and a sequence of forms $f_j \in Z^0_{0,n-1}(U_j)$ which converges to f uniformly on $\overline{D}$.*

Proof. Since the boundary of D is piecewise smooth, locally, this approximation can be realized by small shifts. Patching together these local approximations by a partition of unity, we obtain a sequence $\tilde{U}_j$ of neighborhoods of $\overline{D}$ and a sequence of forms $\tilde{f}_j \in C^0_{0,n-1}(\tilde{U}_j)$ such that $\tilde{f}_j$ converges to f when j tends to ∞, uniformly on $\overline{D}$, and, moreover, the forms $\overline{\partial}\tilde{f}_j$ are continuous on $\tilde{U}_j$ and converge to zero for $j \to \infty$, uniformly on $\overline{D}$. Take bounded neighborhoods $U_j \subseteq \tilde{U}_j$ of $\overline{D}$ so small that also

$$\sup_{\zeta \in U_j} \| \overline{\partial}\tilde{f}_j(\zeta) \| \longrightarrow 0 \qquad \text{for} \quad j \to \infty. \tag{2.2}$$

(Here $\| \overline{\partial}\tilde{f}_j(\zeta) \|$ is the maximum of the moduli of the coefficients of $\overline{\partial}\tilde{f}_j$ at ζ.) Let B_{U_j} be the Bochner-Martinelli-Koppelman operator on U_j. Then, by Lemma 2.3, setting

$$f_j := \tilde{f}_j - B_{U_j}\overline{\partial}\tilde{f}_j$$

we obtain a sequence $f_j \in Z^0_{0,n-1}(U_j)$. Further, by the well known estimates for the Bochner-Martinelli operator, (2.2) implies that

$$\sup_{\zeta \in U_j} \| (B_{U_j}\overline{\partial}\tilde{f}_j)(\zeta) \| \longrightarrow 0 \qquad \text{for} \quad j \to \infty.$$

Hence, the sequence f_j converges to f uniformly on $\overline{D}$. □

2.5. Definition. Let X be a complex manifold. A triplet $[A, B, V]$ will be called an *extension element in* X if A, B and V are open subsets of X with compact C^1-boundaries such that: $A \subseteq B$, $\overline{B} \setminus A \subset\subset V$, V is convex with respect to some holomorphic coordinates in a neighborhood of $\overline{V}$, $A \cap V$ and $B \cap V$ have piecewise C^1-boundary, and either $\overline{A} \cap \overline{V} = \emptyset$ or $\overline{A}$ admits a basis of neighborhoods U such that $U \cap V$ has no holes with respect to V.

Finally, let us mention the following simple lemma, which is proved in [La-L]):

2.6. Lemma. *Let X be a C^∞ manifold and φ a real C^∞-function on X all critical points of which are non-degenerate such that the following conditions are fulfilled:*

(i) no critical point of φ lies on $\varphi^{-1}(0) \cup \varphi^{-1}(1)$;

(ii) $\varphi^{-1}([0,1])$ is compact;

(iii) φ has no points of local maxima in $\varphi^{-1}(]0,1[)$.

Then there exists a finite number of extension elements $[A_l, B_l, V_l]$, $l = 0, \ldots, N$, *such that* $A_0 = \varphi^{-1}(]-\infty, 0[)$, $B_l = A_{l+1}$ *for* $0 \le l \le N-1$ *and* $B_N = \varphi^{-1}(]-\infty, 1[)$. *Moreover these extension elements can be chosen so that the sets* V_l *are arbitrarily small, for example, so small that a given vector bundle on* X *is trivial over some neighborhood of each* $\overline{V}_l$.

3. Proof of Theorem 0.2

As already observed, for compact $\overline{Z}$ the theorem is well-known. Therefore we may restrict ourselves to the case when $\overline{Z}$ is not compact. (Note however that by an obvious modification of the arguments given below one obtains also a proof in the case of compact $\overline{Z}$, which is simpler than in the non-compact case.)

Suppose the hypotheses of Theorem 0.2 are fulfilled and let ρ and α_0 be as in Definition 0.1. Set

$$\rho_0 = \inf_{\zeta \in \overline{Z}} \rho(\zeta).$$

By a theorem of Green and Wu [G-W] (see also [O] for a proof in the case of a complex manifold), any connected non-compact C^∞-manifold admits an exhausting function whose Laplacian (with respect to an arbitrarily chosen Riemannian metric) is everywhere strictly positive. Therefore, we may assume that ρ has also the following properties (additional to those from Definition 0.1):

- the sets $\{\zeta \in X \setminus Z \,|\, \rho(\zeta) \le \alpha\}$, $\alpha < \infty$, are compact;

- there exists $\beta_0 > \alpha_0$ such that the Laplacian of ρ is everywhere strictly positive on $\rho^{-1}([\beta_0, \infty[)$ (which implies that ρ has no points of local maxima in $\rho^{-1}([\beta_0, \infty[))$.

Moreover, by the Morse pertubation argument (see, e.g., the theorem on page 43 in [G-P]), we may assume that all critical points of ρ are non-degenerate. Set

$$D_{\alpha\beta} = \{\zeta \in X \,|\, \alpha < \rho(\zeta) < \beta\}$$

for $\rho_0 \le \alpha < \beta \le \infty$. If $D \subseteq G$ are open subsets of X where D is relatively compact in X, then we denote by $C^\mu_{s,r}(\overline{D}; \overline{G}, E)$, $0 \le \mu < 1$, the Banach space of all continuous, resp. Hölder continuous with exponent μ, E-valued (s,r)-forms f on $\overline{G}$ with $\operatorname{supp} f \subseteq \overline{D}$ (endowed with the topology of $C^\mu_{s,r}(\overline{D}, E)$). Denote by $Z^\mu_{s,r}(\overline{D}; \overline{G}, E)$ the subspace of all $\overline{\partial}$-closed forms in $C^\mu_{s,r}(\overline{D}; \overline{G}, E)$.

3.1. Lemma. *Let* $\rho_0 < \alpha' < \alpha < \alpha_0$ *and let* D *be an open subset of* X *with smooth compact boundary such that* $\overline{D}_{\rho_0,\alpha_0} \subseteq D$. *Then:*

(i) There is a continuous linear operator

$$A_0 : C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E) \longrightarrow \bigcap_{\varepsilon > 0} C^{1-\varepsilon}_{0,n-1}(\overline{D} \cap \overline{D}_{\alpha',\infty}; \overline{D}, E)$$

such that $\overline{\partial} A_0 f = f$ *on* D_{ρ_0,α_0} *for all* $f \in C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$.

(ii) There exist continuous linear operators

$$A : C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-1}(\overline{D} \cap \overline{D}_{\alpha',\infty}; \overline{D}, E)$$

and

$$K : C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$$

such that

$$\overline{\partial} A f = f + K f \qquad \text{on} \quad D$$

for all $f \in C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$.

Proof. Part (i): Since the Levi form of ρ has at least 2 positive eigenvalues on $\overline{D}_{\rho_0,\alpha_0}$, Lemmas 12.3 and 12.4 (iii) in [H-L] immediately imply the following statement: *If* $f \in C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$, *then there exists* $u \in \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-1}(\overline{D} \cap \overline{D}_{\alpha',\infty}; \overline{D}, E)$ *with* $\overline{\partial} u = f$ *on* D_{ρ_0,α_0}. Moreover, the proof of Lemma 12.4 (iii) in [H-L] (page 107) shows that this solution u can be given by a continuous linear operator A_0 as required.

Part (ii): Choose α" with $\alpha < \alpha" < \alpha_0$, and take a finite number of C^1 domains $U_1, \dots, U_N \subseteq D_{\alpha",\infty}$ such that $D_{\rho_0,\alpha_0} \cup U_1 \cup \dots \cup U_N = D$ and some neighborhood of each $\overline{U}_j$ is biholomorphically equivalent to an open subset of $\mathbb{C}^n$. Then, by Lemma 2.3 and the well known estimates for the Bochner-Martinelli kernel, we have continuous linear operators

$$A_j : C^0_{0,n}(\overline{U}_j, E) \longrightarrow \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-1}(\overline{U}_j, E)$$

with $\overline{\partial} A_j f = f$ for all $f \in C^0_{0,n}(\overline{U}_j, E)$. Now we take real C^∞ functions $\chi_0, \dots, \chi_N$ on X such that supp $\chi_0 \subset\subset D_{\rho_0,\alpha_0}$, supp $\chi_j \subset\subset U_j$ $(j = 1, \dots, N)$ and $\chi_0 + \dots + \chi_N = 1$ on $\overline{D}$. Then the operators

$$A := \sum_{j=0}^{N} \chi_j A_j \quad \text{and} \quad K := \sum_{j=0}^{N} \overline{\partial} \chi_j \wedge A_j$$

have the required properties. □

By Ascoli's theorem and Fredholm theory, it follows from part (ii) of this lemma that the space $C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E) \cap \overline{\partial} \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-1}(\overline{D} \cap \overline{D}_{\alpha',\infty}; \overline{D}, E)$ is topologically closed and of finite codimension in $C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$, and from part (i) it follows that this finite codimension is independent of α and α'. Therefore, we have the following

3.2. Corollary. *If D is an open subset of X with smooth compact boundary such that $\overline{D}_{\rho_0,\alpha_0} \subseteq D$, then there exists a finite number N_D such that the following holds: If $\rho_0 < \alpha' < \alpha < \alpha_0$, then the space*

$$C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E) \cap \overline{\partial} \bigcap_{\varepsilon>0} C^{1-\varepsilon}_{0,n-1}(\overline{D} \cap \overline{D}_{\alpha',\infty}; \overline{D}, E)$$

is topologically closed and of codimension N_D in $C^0_{0,n}(\overline{D} \cap \overline{D}_{\alpha,\infty}; \overline{D}, E)$.

3.3. Lemma. *Let $\rho_0 < \alpha < \alpha_0$ and $\beta_0 < \beta < \infty$ such that $d\rho(\zeta) \neq 0$ if $\rho(\zeta) = \beta$. Then the space $Z^0_{0,n-1}(\overline{D}_{\alpha,\infty}; X, E)$ is dense in $Z^0_{0,n-1}(\overline{D}_{\alpha,\beta}; \overline{D}_{\rho_0,\beta}, E)$.*

Proof. By Lemma 2.6 there exists a sequence of extension elements $[A_l, B_l, V_l]$, $l = 0, 1, \ldots$, in X such that $A_0 = D_{\rho_0,\beta}$, $B_l = A_{l+1}$ for all l, and $X = \bigcup_{l\geq 0} A_l$. Moreover we may assume that, for all l, $\overline{V}_l \cap \overline{D}_{\rho_0,\alpha_0} = \emptyset$ and the vector bundle E is trivial over some neighborhood of $\overline{V}_l$. Now it is sufficient to prove that, for all l, the restriction map

$$Z^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap \overline{B}_l; \overline{B}_l, E) \longrightarrow Z^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap \overline{A}_l; \overline{A}_l, E)$$

has dense image. Let $l \in \mathbb{N}$ and $u \in Z^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap \overline{A}_l; \overline{A}_l, E)$ be given.

By Lemmas 2.2 and 2.4, then there exists a sequence $v_j \in Z^0_{0,n-1}(V_l, E)$ which converges to u uniformly on the compact subsets of $V_l \cap \overline{A}_l$. Take a C^∞-function χ on X with $\operatorname{supp}\chi \subset\subset V_l$ and $\chi \equiv 1$ in a neighborhood of $\overline{B}_l \setminus A_l$. Setting $\varphi_j = u + \chi(v_j - u)$, we define a sequence $\varphi_j \in C^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap \overline{B}_l; \overline{B}_l, E)$ which converges to u uniformly on $\overline{A}_l$. Moreover, then the sequence $\overline{\partial}\varphi_j = \overline{\partial}\chi \wedge (v_j - u)$ belongs to $C^0_{0,n}(\overline{D}_{\alpha_0,\infty} \cap \overline{B}_l; \overline{B}_l, E)$ and converges to zero in this space. Hence, by Corollary 3.2 and Banach's open mapping theorem, there is a sequence $\psi_j \in C^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap B_l; \overline{B}_l, E)$ which converges to zero in $C^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap B_l; B_l, E)$ such that $\overline{\partial}\psi_j = \overline{\partial}\varphi_j$ on $\overline{B}_l$. Setting $u_j = \varphi_j - \psi_j$, we obtain a sequence $u_j \in Z^0_{0,n-1}(\overline{D}_{\alpha,\infty} \cap B_l; \overline{B}_l, E)$ which converges to u uniformly on $\overline{A}_l$. $\square$

3.4. Lemma. *Let $\rho_0 < \alpha' < \alpha < \alpha_0$ and $\beta_0 < \beta_1 < \beta_2 < \infty$ such that $d\rho(\zeta) \neq 0$ if $\rho(\zeta) = \beta_1$ or $\rho(\zeta) = \beta_2$, and let $f \in C^0_{0,n}(\overline{D}_{\alpha,\beta_2}; \overline{D}_{\rho_0,\beta_2}, E)$ such that $f = \overline{\partial}u_1$ on D_{ρ_0,β_1} for certain $u_1 \in C^0_{0,n-1}(\overline{D}_{\alpha',\beta_1}; \overline{D}_{\rho_0,\beta_1}, E)$. Then there exists also $u_2 \in C^0_{0,n-1}(\overline{D}_{\alpha',\beta_2}; \overline{D}_{\rho_0,\beta_2}, E)$ with $f = \overline{\partial}u_2$ on D_{ρ_0,β_2}.*

Proof. By Lemma 2.6, there exists a finite number of extension elements $[A_l, B_l, V_l]$, $l = 0, \ldots, N$, in X such that $A_0 = D_{\rho_0,\beta_1}$, $B_l = A_{l+1}$ for all $l = 0, \ldots, N-1$, and $B_N = D_{\rho_0,\beta_2}$. Moreover we may assume that, for all l, $\overline{V}_l \cap \overline{D}_{\rho_0,\alpha_0} = \emptyset$ and the vector bundle E is trivial over a neighborhood of $\overline{V}_l$. Now we assume (proof by induction over l) that, for certain l with $0 \leq l \leq N$, we already have $u_l \in C^0_{0,n-1}(\overline{D}_{\alpha',\infty} \cap \overline{A}_l; \overline{A}_l, E)$ with $f = \overline{\partial}u_l$ on A_l. Then we have to find $u_{l+1} \in C^0_{0,n-1}(\overline{D}_{\alpha',\infty} \cap \overline{B}_l; \overline{B}_l, E)$ with $f = \overline{\partial}u_{l+1}$ on B_l. Since, by Corollary 3.2, the space

$$C^0_{0,n}(\overline{D}_{\alpha,\infty} \cap \overline{B}_l; \overline{B}_l, E) \cap \overline{\partial}C^0_{0,n-1}(\overline{D}_{\alpha',\infty} \cap \overline{B}_l; \overline{B}_l, E)$$

is closed with respect to the topology of the Banach space $C^0_{0,n}(\overline{D}_{\alpha,\infty} \cap \overline{B}_l; \overline{B}_l, E)$, for this it is sufficient to find a sequence $(\psi_\nu) \subseteq C^0_{0,n-1}(\overline{D}_{\alpha',\infty} \cap \overline{B}_l; \overline{B}_l, E)$ such that the sequence $(\overline{\partial}\psi_\nu)$ is contained in $C^0_{0,n}(\overline{D}_{\alpha,\infty} \cap \overline{B}_l; \overline{B}_l, E)$ and converges to f uniformly on $\overline{D}_{\alpha,\infty} \cap \overline{B}_l$.

By Lemma 2.3 there exists $v_l \in C^0_{0,n-1}(\overline{V_l \cap B_l}, E)$ with $\overline{\partial}v_l = f$ on $\overline{V_l \cap B_l}$. Take a C^∞-function χ on X such that $\operatorname{supp}\chi \subset\subset V_l$ and $\chi \equiv 1$ in a neighborhood of $\overline{B}_l \setminus A_l$. By Lemmas 2.2 and 2.4 we can find a sequence $(\varphi_\nu)_{\nu\in\mathbb{N}} \subseteq Z^0_{0,n-1}(V_l, E)$ which converges to $u_l - v_l$ uniformly on $\overline{A}_l \cap \operatorname{supp}\chi$. Setting $\psi_\nu = u_l + \chi(v_l - u_l - \varphi_\nu)$, we obtain the required sequence. □

3.5. Lemma. *Let $f \in C^0_{0,n}(\Phi; X, E)$. If there exist α, β with $\rho_0 < \alpha < \alpha_0$ and $\beta_0 < \beta < \infty$ such that $f = \overline{\partial} u_\beta$ on $D_{\rho_0,\beta}$ for some $u_\beta \in C^0_{0,n}(\overline{D}_{\alpha,\beta}; \overline{D}_{\rho_0,\beta}, E)$, then there exists $u \in C^0_{0,n-1}(\Phi; X, E)$ with $f = \overline{\partial} u$ on X.*

Proof. Take an increasing sequence $(\lambda_\nu)_{\nu\in\mathbb{N}}$, converging to infinity such that $\beta_0 < \lambda_0 < \beta$ and $d\rho(\zeta) \neq 0$ if $\rho(\zeta) \in \{\lambda_0, \lambda_1, \dots\}$, and take α', α with $\rho_0 < \alpha' < \alpha < \alpha_0$ such that $\operatorname{supp} f \subseteq \overline{D}_{\alpha,\infty}$. Then from Lemma 3.4 we get a sequence $u_\nu \in C^0_{0,n-1}(\overline{D}_{\alpha',\lambda_\nu}; \overline{D}_{\rho_0,\lambda_\nu}, E)$, $\nu = 1, 2, \dots$, with $f = \overline{\partial} u_\nu$ on D_{ρ_0,λ_ν}. In view of the approximation lemma 3.3, this sequence can be chosen converging uniformly on the compact subsets of X. This limit is the required solution u. □

End of proof of Theorem 0.2. We introduce on $C^0_{s,r}(\Phi; X, E)$ the inductive limit topology defined by the Fréchet spaces $C^0_{s,r}(C; X, E)$, $C \in \Phi$. Then $C^0_{s,r}(\Phi; X, E)$ is an LF-space. It follows from Corollary 3.2 and Lemma 3.5 that $E^0_{0,n}(\Phi; X, E)$ is of finite codimension in $C^0_{0,n}(\Phi; X, E)$. Since $E^0_{0,n}(\Phi; X, E)$ is the image of a closed operator between LF-spaces, this implies that $E^0_{0,n}(\Phi; X, E)$ is topologically closed in the LF-space $C^0_{0,n}(\Phi; X, E)$ and moreover, that $E^0_{0,n}(\Phi; X, E)$ itself is also an LF-space (with respect to the topology induced from $C^0_{0,n}(\Phi; X, E)$).

Now we consider a continuous linear functional L on $C^0_{0,n}(\Phi; X, E)$ such that

$$L(\varphi) = 0 \qquad \text{for all} \quad \varphi \in E^0_{0,n}(\Phi; X, E) \tag{3.1}$$

By the Hahn-Banach theorem, we have to show that L is the zero functional.

Let E^* be the dual of the bundle E. By (3.1), L defines a $\overline{\partial}$-closed E^*-valued current of bidegree $(n, 0)$ on X. By regularity of $\overline{\partial}$, this current is defined by an holomorphic E^*-valued $(n, 0)$-form h. L is continuous on $C^0_{0,n}(\Phi; X, E)$ and $X \setminus Z \in \Phi$ (recall that $\Phi = \Phi(Z)$). Hence, L is continuous on the Fréchet space $C^0_{0,n}(X \setminus Z; X, E)$. This implies that $(\operatorname{supp} h) \cap (X \setminus Z)$ is compact. Since $X \setminus \overline{Z}$ is not relatively compact in X and X is connected, it follows by uniqueness of holomorphic functions that $h \equiv 0$ on X, i.e. L is the zero functional. □

4. An application to the Hartogs-Bochner phenomenon

Using ideas of Lupacciolu [Lu] and Chirka-Stout (see Theorem 3.3.1 and its proof in [C-S]), Theorem 0.2 gives the following result on Hartogs-Bochner extension, which generalizes the theorem of Weinstock [W]:

4.1. Theorem. *Let X and Z be as in Theorem 0.2. Suppose D is an open subset of X such that: the boundary ∂D is of class C^1 (but not necessarily compact), $\overline{D} \setminus Z$ is compact and $X \setminus \overline{D}$ has not more than a finite number of components which are either compact or contained in Z. Then for any holomorphic vector bundle E over*

X and each continuous CR-section $f : \partial D \to E$ the following two conditions are equivalent:

(i) There exists a continuous section $F : \overline{D} \to E$ which is holomorphic over D such that $F|\partial D = f$.

(ii) $\int_{\partial D} f\varphi = 0$ for any continuous $\overline{\partial}$-closed $(n, n-1)$-form φ with values in E^* (the dual bundle of E) defined in a neighborhood of $\overline{D}$ such that $\operatorname{supp}\varphi \cap \partial D$ *is compact.*

In the proof of this theorem we use the following

4.2. Theorem. *Let X, Z and Φ be as in Theorem 0.2. Denote by Φ^* the family of closed subsets C of X such that $C \cap (X \setminus Z)$ is compact. Let E be a holomorphic vector bundle on X and let E^* be the dual bundle of E. Then the space $E^0_{0,1}(\Phi^*; X, E^*)$ consists of all $f \in Z^0_{0,1}(\Phi^*; X, E^*)$ such that*

$$\int_X f \wedge \psi = 0 \qquad \text{for all} \quad \psi \in Z^0_{n,n-1}(\Phi; X, E). \tag{4.1}$$

In particular, then $H^{0,1}_{\Phi^*}(X, E^*)$ is separated (with respect to the LF-topology of $C^0_{0,1}(\Phi^*; X, E^*)$).

Proof. That condition (4.1) is necessary, follows from Stokes' theorem. Conversely, let $f \in Z^0_{0,1}(\Phi^*; X, E)$ with (4.1) be given. By Theorem 0.2,

$$H^{n,n}_{\Phi}(X, E) = 0. \tag{4.2}$$

By (4.1) and (4.2), in the following way, a linear functional

$$u : C^0_{n,n}(\Phi; X, E) \longrightarrow \mathbb{C}$$

can be defined: For each $\varphi \in C^0_{n,n}(\Phi; X, E)$ we take $\psi \in C^0_{n,n-1}(\Phi; X, E)$ with $\overline{\partial}\psi = \varphi$ and set

$$u(\varphi) = \int_X f \wedge \psi.$$

Since both $C^0_{n,n-1}(\Phi; X, E)$ and $C^0_{n,n}(\Phi; X, E)$ are LF-spaces and the open mapping theorem holds for closed linear surjections between such spaces, it follows that u is continuous with respect to the topology of $C^0_{n,n}(\Phi; X, E)$. In particular, u is a $(0,0)$-current on X. From the definition of u it is clear that $\overline{\partial}u = f$. Since $X \setminus Z \in \Phi$, in particular, u is continuous on $C^0_{n,n}(X \setminus Z; X, E)$. This implies that, $(\operatorname{supp} u) \cap (X \setminus Z)$ is compact, i.e. $\operatorname{supp} u \in \Phi^*$. Moreover, by regularity of $\overline{\partial}$, u is a continuous section of E. $\square$

Remarks. I. Consider the example given after Theorem 0.2. Let Φ be the family of closed subsets C of X such that $C \cap \overline{Z}$ is compact. In [Mi] the following separation theorem is obtained: If $n \geq 3$ and E is a holomorphic vector bundle over X such that $K^{-1} \otimes E$ extends to S_1, then $H^{0,n-1}_{\Phi}(X, E)$ is separated. From Theorem 4.2

it follows that this is true also for $n = 2$, even without the extendability condition on E.

II. Consider the Rossi example [R]. This is a 2-dimensional complex manifold X, diffeomorphic to $\mathbb{C}^2 \setminus \{0\}$, such that $H^{0,1}(X)$ is not separated. If Φ is the family of closed subsets of $\mathbb{C}^2$ which do not meet the origin, then it follows from Theorem 4.2 that $H^{0,1}_{\Phi}(X)$ is separated.

Proof of Theorem 4.1. The conclusion (i)$\Rightarrow$(ii) follows from Stokes' theorem. Assume now that condition (ii) is satisfied.

Let $U_1, \dots, U_N$ be the connected components of $X \setminus \overline{D}$ which are either compact or contained in Z. Take points $z_j \in U_j$, $1 \leq j \leq N$, and set $X_0 = X \setminus \{z_1, \dots, z_N\}$. Let ρ and α_0 be as in Definition 0.1. Set $Z_0 = \{\zeta \in X \mid \rho(\zeta) < \alpha_0\}$. We may assume that $\overline{Z}_0 \cap \{z_1, \dots, z_N\} = \emptyset$. Denote by $\Phi_0 = \Phi_0(Z_0)$ the family of all closed subsets C of X_0 such that $C \cap Z_0$ is compact, and denote by Φ_0^* the family of closed subsets C^* of X such that $C^* \cap (X_0 \setminus Z_0)$ is compact. Then $\overline{D} \subseteq X_0$ and $X_0 \setminus \overline{D}$ has no connected component which belongs to the family Φ_0^*. By Theorem 0.2,

$$H^{n,n}_{\Phi_0}(X_0, E^*) = 0. \tag{4.1}$$

Consider the E-valued $(0,1)$-current $[i_* f]^{0,1}$ on X_0 defined by

$$[i_* f]^{0,1}(\varphi) = \int_{\partial D} f \wedge \varphi$$

for all E^*-valued $C^\infty_{n,n-1}$-forms φ with compact support in X_0. Since f is CR, this current is $\overline{\partial}$-closed. The support of $[i_* f]^{0,1}$ is $\overline{\partial} D$ and therefore contained in Φ_0^*. Hence, as f satisfies (ii), by Theorem 4.2, we can solve the equation

$$\overline{\partial} F = [i_* f]^{0,1}$$

with an E-valued $(0,0)$-current F on X_0 such that $\operatorname{supp} F \in \Phi_0^*$, i.e. $\operatorname{supp} F \setminus Z_0$ is compact. Since $\overline{D} \setminus Z_0$ is compact, it follows that also $\operatorname{supp} F \setminus (D \cup Z_0)$ is compact. Since no connected component of $X_0 \setminus \overline{D}$ is in Φ_0^* and F is holomorphic outside $\overline{\partial} D$, it follows that $F \equiv 0$ outside $\overline{D}$. Now it follows by standard arguments (see, e.g., the proof of Theorem 5.1 in [La]) that F is the required extension of f. □

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