ARCS AND WEDGES ON SANDWICHED SURFACE SINGULARITIES

by Monique LEJEUNE-JALABERT & Ana REGUERA

ABSTRACT. — A wedge on a surface singularity (S, P) is a formal parametrization of S by power series in two variables, locally at P. A genericity condition, which is expected to be enough to guarantee that a wedge lifts to the minimal desingularization of the surface, is proved to be so if the singularity is sandwiched.

To the memory of O. Zariski and to P. Samuel

Introduction

The following question is motivated by the challenging problem of understanding Nash's families of arcs on a surface singularity in relation with its minimal desingularization (see [N]).

Does a wedge centered at a "general" arc on a normal surface singularity (S, P) lift to its minimal desingularization $\mathfrak{X}$?

Let us explain our terminology. An *arc* on (S, P) is an algebroid curve going through P on S, given by formal power series in one variable $\underline{x}(t)$. We say that an arc is *general* on (S, P) if its strict transform on $\mathfrak{X}$ is smooth and intersects transversally the exceptional curve E at a point lying on a Zariski dense open set of regular points of E. A *wedge* on (S, P) is a parametrization of S locally at P by formal power series in two variables $\underline{x}(u_1, u_2)$. We say that a wedge is *centered* at an arc if its parametrization $\underline{x}(t)$ comes from the wedge by substituting series $u_1(t), u_2(t)$.

The aim of the present paper is to give an affirmative answer to the above question when (S, P) is a sandwiched singularity. By Zariski's complete ideal theory, such a singularity is the birational join of finitely many primitive ones. (Definitions are given below.)

Key words: sandwiched surface singularities, arcs, wedges, desingularization.

Math. classification: 14B05, 14J17, 14E15, 32S45.

Most of our discussion consists in extending the combinatorial argument working for toric surface singularities developed in [L-J] to primitive ones.

A system of approximate roots of a general element in the simple complete ideal I whose blowing-up p_0 produces the primitive singularity (S,P), or equivalently a generating sequence for the divisorial valuation ν associated to the irreducible exceptional curve of p_0, helps us drawing a *toric environment*. A similar construction has been given independently by Golding and Teissier in [G.T]. Roughly speaking, we interpret p_0 and various modifications of S including its minimal desingularization as being strict transforms by suitable equivariant modifications between toric varieties. More precisely this goes as follows.

Pick a generating sequence $\{x_0,\dots,x_{g+1}\}$ for ν and set $\delta_{g+1}=(\overline{\beta}_0,\dots,\overline{\beta}_{g+1})$ with $\overline{\beta}_i=\nu(x_i)$. First, sending the variable X_i to x_i defines an embedding of the non singular algebroid surface $\widehat{\mathfrak{X}}_0$ supporting I in $\widehat{\mathcal{Z}}_0 := \operatorname{Spec} k[[X_0,\dots,X_{g+1}]]$. Let Σ_0 be the fan, elementary subdivision of the cone $\mathbb{R}^{g+2}_{\geq 0}$ by $\mathbb{R}_{\geq 0}\delta_{g+1}$, let $\pi_0 : \mathcal{Z}_{\Sigma_0} \longrightarrow \mathbb{A}^{g+2}_k$ be the resulting equivariant modification and let $\widehat{\mathfrak{X}}_{\Sigma_0}$ be the strict transform of $\widehat{\mathfrak{X}}_0$ by π_0. We may assume without loss of generality that $\widehat{\mathfrak{X}}_{\Sigma_0}$ has a unique singular point $\widehat{P}$. The main result of [Sp1] may be reformulated by saying that (S,P) and $(\widehat{\mathfrak{X}}_{\Sigma_0},\widehat{P})$ have isomorphic formal neighborhoods.

Next we derive from the explicit formulas giving x_{i+1} in terms of $(x_0,\dots,x_i)$ that $\widehat{\mathfrak{X}}_0$ is defined in $\widehat{\mathcal{Z}}_0$ by a system of g functions which is not degenerate for its Newton polyhedra. Following Khovanskii [Kh], we consider the least fine subdivision $\Sigma_{\mathcal{T}}$ of its "Newton fans" and of Σ_0, and the resulting equivariant modification $\mathcal{Z}_{\Sigma_{\mathcal{T}}} \to \mathcal{Z}_{\Sigma_0}$. We show that the cones $\sigma\in\Sigma_{\mathcal{T}}$ whose associated orbit $\mathbb{O}_\sigma$ intersects the strict transform $\widehat{\mathfrak{X}}_{\mathcal{T}}$ of $\widehat{\mathfrak{X}}_{\Sigma_0}$ form a fan Θ consisting of $\sigma_{i,1}=\langle\delta_{i-1},\delta_i\rangle$ and $\sigma_{i,2}=\langle\varepsilon_i,\delta_i\rangle$ for $1\leq i\leq g+1$ and their faces, where $\langle\ ,\ \rangle$ denotes the cone generated by the vectors written inside, ε_i is the unit vector on the X_i-axis and

$$\delta_i=(\overline{\beta}_0,\dots,\overline{\beta}_i,n_i\overline{\beta}_i,\dots,n_i\cdots n_g\overline{\beta}_i),\ \ 0\leq i\leq g+1$$

with $n_0=0$, $n_i=e_{i-1}/e_i$, $1\leq i\leq g$ and $e_i=g.c.d.(\overline{\beta}_0,\dots,\overline{\beta}_i)$, $0\leq i\leq g$. By removing $\sigma_{g+1,2}$ and $\mathbb{R}_{\geq 0}\varepsilon_{g+1}$ from Θ, we get the $\sigma\in\Sigma_{\mathcal{T}}$ such that $\widehat{P}$ lies in the Zariski closure of the image of $\mathbb{O}_\sigma\cap\widehat{\mathfrak{X}}_{\mathcal{T}}$ in $\widehat{\mathfrak{X}}_{\Sigma_0}$. These cones form another fan Ξ which we call the *skeleton* of the primitive singularity. As in [G.L1], the non degeneracy property implies that $\widehat{\mathfrak{X}}_{\mathcal{T}}$ is a partial desingularization of $\widehat{\mathfrak{X}}_{\Sigma_0}$ with only toric singularities, namely its intersection points with the orbits $\mathbb{O}_{\sigma_{i,j}}$, for $\sigma_{i,j}\in\Xi$.

Finally for any regular subdivision $\Sigma_{\mathcal{R}}$ of $\Sigma_{\mathcal{T}}$ whose trace on the fan Θ above is its minimal regular subdivision, the map $\mathcal{Z}_{\Sigma_{\mathcal{R}}}\to\mathcal{Z}_{\Sigma_0}$ is an embedded desingularization of $\widehat{\mathfrak{X}}_{\Sigma_0}$ and its strict transform $\widehat{\mathfrak{X}}_{\Sigma_{\mathcal{R}}}$ is its minimal desingularization. This is detailed in section 1.

We begin the next section by relating arcs and wedges on (S,P) with its skeleton. Indeed, as a trivial consequence of the triangular inequality, the *characteristic vector* $\alpha = \operatorname{ord}_t x_i(t)_{0\leq i\leq g+1}$ of an *arc* on (S,P) meeting the torus, lies on Ξ. Similarly, p being any irreducible element in $k[[u_1,u_2]]$ and ν_p denoting the p-adic valuation, for any wedge on (S,P), $\alpha_p = \nu_p x_i(u_1,u_2)_{0\leq i\leq g+1}$ lies on Ξ. The finitely many $\alpha_p \neq 0$ are defined to be the *characteristic vectors* of the *wedge*.

The "general" arcs on (S,P) are exactly those whose characteristic vector belongs to the minimal generating system of some $\sigma \in \Xi$ (that is of the semigroup $\sigma \cap \mathbb{Z}^{g+2} \setminus 0$). By the way, this is a combinatorial translation of Cor. 8.4 [Sp1]. One may also decide whether a wedge lifts to the minimal desingularization of (S,P) by looking at its characteristic vectors. This happens if and only if there exists a cone τ in the minimal regular subdivision of Ξ which contains them all. So one may read off the skeleton of a primitive singularity, similar informations about its arcs and wedges to those encoded in the cone giving rise to a toric surface singularity.

The last part of section 2 is a proposition expressing additional restrictions on the location of the characteristic vectors of a wedge. This observation is crucial in the proof of the main theorem, which is given in section 3. The results of this paper were announced in [L.R].

Acknowledgements. — We would like to thank C. Bouvier, E. Casas, G. Gonzalez-Sprinberg, B. Teissier, M. Spivakovsky for their suggestions and encouragement.

This work has grown up during reciprocal visits at Grenoble and Valladolid partly supported by the French-Spanish cooperation program "Singularities of algebraic varieties" n° 92127 and at the Fields Institute Toronto, on the occasion of the Singularity Semester in 97. We are grateful to these institutions for their hospitality and financial support. The typesetting has been realized by A. Guttin-Lombard with her usual skill and patience. It is a pleasure for us to acknowledge her participation.

0. Terminology, notation

In this preliminary section, we fix our terminology and notation, and we make a review of some useful results.

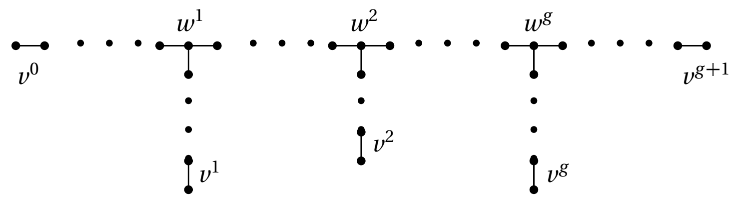
0.1. — By a *sandwiched* (resp. primitive sandwiched, for short *primitive* in the sequel) *surface singularity*, we mean the formal neighborhood $\widehat{S}$ of a singular point P on a surface S obtained by blowing-up a complete (= integrally closed) (resp. simple complete, for short simple in the sequel) ideal I in the local ring of a closed point O on a non singular algebraic surface $\mathfrak{X}_0$ defined over an algebraically closed field k.

We may assume I to be primary for the maximal ideal of $R := \mathcal{O}_{\mathfrak{X}_0,O}$. In addition, $\mathfrak{X}_0$ and I can be chosen in such a way that P is the only singular point on S and that any irreducible curve on the minimal desingularization $\mathfrak{X}$ of S which is mapped onto an exceptional curve for $S \to \mathfrak{X}_0$ is a (-1)-curve ([Sp2], Cor. II.1.14).

0.2. — From now on until the end of section 2, we focus our attention on a primitive singularity $\widehat{S}$.

0.3. — Recall that the exceptional curve of the blowing-up $p_0 : S \to \mathfrak{X}_0$ of a simple ideal I is irreducible. Any *minimal generating sequence* $\{x_0, \ldots, x_{g+1}\}$ for the associated divisorial valuation ν enjoys the following geometrical and arithmetical properties:

0.3.1. — Let $q : \mathfrak{X} \to S$ be the minimal desingularization of S. The dual graph Γ of the configuration of irreducible exceptional curves for $p := q \circ p_0$ has the form



where we have labelled the ends $v^0, \ldots, v^{g+1}$ and the stars $w^1, \ldots, w^g$. By deleting v^{g+1} from Γ, we get the dual graph associated to the exceptional curves for q.

For $0 \leq i \leq g+1$, the strict transform on $\mathfrak{X}$ of the curve C_i defined by x_i in $(\mathfrak{X}_0, O)$ is smooth and intersects transversally the curve represented by v^i and no other exceptional curve for p. The curve $C = C_{g+1}$ is analytically irreducible at O, and has g Puiseux exponents; C_0 is smooth and transversal to C at O, while for $1 \leq i \leq g$, C_i has $i-1$ Puiseux exponents and maximal contact with C.

0.3.2. — The family $\overline{\beta}_i := \nu(x_i)$, $0 \leq i \leq g$, is the minimal generating system for the semigroup $\nu(R \setminus 0)$ of the valuation ν, and $\overline{\beta}_{g+1} := \nu(x_{g+1}) = \nu(I)$. We set $e_i := g.c.d.(\overline{\beta}_0, \ldots, \overline{\beta}_i)$, $0 \leq i \leq g+1$, $n_0 = 0$ and $n_i := e_{i-1}/e_i$, $1 \leq i \leq g+1$. Recall that $e_0 > e_1 > \cdots > e_g = e_{g+1} = 1$ and the following conditions hold:

i) $n_i\overline{\beta}_i$ belongs to the semigroup generated by $\overline{\beta}_0, \ldots, \overline{\beta}_{i-1}$, $1 \leq i \leq g+1$, *ii)* $\overline{\beta}_i > n_{i-1}\overline{\beta}_{i-1}$, $1 \leq i \leq g+1$,

For $1 \leq i \leq g$, these conditions characterize the semigroup of a plane curve singularity; the strict inequality for $i = g+1$ expresses that the blowing-up S of I has a unique singular point.

In addition, for $1 \leq i \leq g+1$, there exists a unique system of non negative integers b_{ij}, $0 \leq j < i$ such that $b_{ij} < n_j$ for $1 \leq j < i$ and $n_i\overline{\beta}_i = \sum_{0\leq j<i} b_{ij}\overline{\beta}_j$. In fact, one may choose $x_i \in R$, $0 \leq i < g+1$, in such a way that they satisfy polynomial identities of the form

$$x_{i+1} = x_i^{n_i} - c_i x_0^{b_{i0}} \cdots x_{i-1}^{b_{ii-1}} - \sum_{\gamma=(\gamma_0,\ldots,\gamma_i)} c_{i,\gamma} x_0^{\gamma_0} \cdots x_i^{\gamma_i},$$

with $o \leq \gamma_j < n_j$, for $1 \leq j \leq i$, and $\sum_j \gamma_j\overline{\beta}_j > n_i\overline{\beta}_i$, and with $c_{i\gamma}$, $c_i \in k$ and $c_i \neq 0$.

For details concerning the material of 0.3, see [Ab] and [Sp1]. In particular a proof of the above assertion follows from [Ab] 9.4 in characteristic 0 and from [Sp1] Remark 8.16 and Lemma 8.10. See also [Z].

0.4. — As far as *toric geometry* is concerned, we only need the basic elements of the dictionary given in [T.E] or [Oda], Chap. I and the characterization of the minimal regular subdivision $\sigma_{\mathcal{R}}$ of a 2-dimensional cone σ through the minimal generating system G_σ of the semigroup $\sigma \cap \mathbb{Z}^2 \setminus 0$. To be precise, G_σ consists of the integral points on the compact edges of the boundary polygon of the convex hull of $\sigma \cap \mathbb{Z}^2 \setminus 0$ and the one dimensional cones in $\sigma_{\mathcal{R}}$ are the half lines through the points in G_σ [G-S].

We denote by $\mathcal{Z}_\Sigma$ the toric variety which corresponds to the fan Σ; we denote by $\mathbb{O}_\sigma$ the T-orbit on $\mathcal{Z}_\Sigma$ which corresponds to $\sigma \in \Sigma$. The cones $\sigma^\vee$, $\sigma^\perp$ are respectively the dual and the orthogonal of σ, and $\mathring{\sigma}$ denotes its relative interior.

We set $\Delta := \mathbb{R}_{\geq 0}^{g+2}$. Two sets of vectors with integral coordinates in Δ will play a special role: the unit vector ε_i on the X_i-axis and $\delta_i = (\overline{\beta}_0, \ldots, \overline{\beta}_i, n_i\overline{\beta}_i, \ldots, n_i \cdots n_g\overline{\beta}_i)$, $0 \leq i \leq g+1$. We denote by $\langle \ldots, a_i, \ldots \rangle$ the cone generated by the $a_i \in \Delta$, while (a_i, a_j) is used for the usual scalar product of a_i, a_j.

We consider polynomials $F \in k[X_0, \ldots, X_{g+1}]$. If $F = \sum_\alpha c_\alpha \underline{X}^\alpha$, the support of F, (for short Supp F), is $\{\alpha \in \mathbb{Z}_{\geq 0}^{g+2}; c_\alpha \neq 0\}$; its Newton polyhedron $\mathcal{N}$ is the convex hull of $\bigcup_{\alpha \in \mathrm{Supp}\, F} \alpha + \Delta$. Its support function $h_{\mathcal{N}} : \Delta \to \mathbb{R}_{\geq 0}$ is defined by

$$h_{\mathcal{N}}(n) = \inf_{m \in \mathcal{N}} (m, n).$$

1. A toric environment

Let $\mathcal{Z}_0 := \mathcal{Z}_\Delta = \mathbb{A}_k^{g+2}$ endowed with its natural $T = k^{*g+2}$ action. The formal neighborhood $\widehat{\mathfrak{X}}_0$ of O in $\mathfrak{X}_0$ is a non singular complete intersection surface in the formal neighborhood $\widehat{\mathcal{Z}}_0 = \operatorname{Spec} k[[X_0, \ldots, X_{g+1}]]$ of O in $\mathcal{Z}_0$ defined by the polynomials

$$F_i = X_{i+1} - X_i^{n_i} + c_i X_0^{b_{i0}} \cdots X_{i-1}^{b_{ii-1}} + \sum_{\gamma} c_{i\gamma} X_0^{\gamma_0} \cdots X_i^{\gamma_i}, \ 1 \leq i \leq g.$$

Indeed by 0.3.1, (x_0, x_1) is a regular system of parameters of R.

We will successively introduce four subdivisions of Δ. Each arrow represented on the diagram below goes from a fan to a subdivision of this fan.

$$\begin{array}{ccccc} & \nearrow & \Sigma_0 & \searrow & & \\ \Delta & & & & \Sigma_{\mathcal{T}} & \longrightarrow \Sigma_{\mathcal{R}} \\ & \searrow & \Sigma_{\mathcal{N}} & \nearrow & & \end{array}$$

For each one of these subdivisions, we then decide which singularities appear on the strict transform of $\widehat{\mathfrak{X}}_0$ by the resulting equivariant modification.

1.0. — We begin by an elementary observation which is valid for any subdivision Σ of Δ. Since for $0 \leq i \leq g+1$, $\mathbb{R}_{\geq 0}\varepsilon_i$ is both a 1-dimensional face of Δ and a cone in Σ, the map $\pi_\Sigma : \mathcal{Z}_\Sigma \to \mathcal{Z}_0$ is an isomorphism over the corresponding orbit in $\mathcal{Z}_0$, namely $X_i = 0$, $X_j \neq 0$, $j \neq i$. Now the image of the generic point of C_i by the canonical map $\widehat{\mathfrak{X}}_0 \hookrightarrow \widehat{\mathcal{Z}}_0 \to \mathcal{Z}_0$ lies in this orbit; so π_Σ is an isomorphism over $\widehat{\mathfrak{X}}_0 \setminus O$, the exceptional locus of its strict transform $\widehat{p}_\Sigma : \widehat{\mathfrak{X}}_\Sigma \to \widehat{\mathfrak{X}}_0$ is its fiber over O and the only $\sigma \in \Sigma$ in $\Delta \setminus \mathring{\Delta}$ such that $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_\Sigma \neq \emptyset$ are the $\mathbb{R}_{\geq 0}\varepsilon_i$, $0 \leq i \leq g+1$.

We first deal with the elementary subdivision Σ_0 of Δ by $\mathbb{R}_{\geq 0}\delta_{g+1}$.

Proposition 1.1. — *The unique singular point $\widehat{P}$ of the strict transform $\widehat{\mathfrak{X}}_{\Sigma_0}$ of $\widehat{\mathfrak{X}}_0$ by the equivariant map $\pi_0 : \mathcal{Z}_{\Sigma_0} \to \mathcal{Z}_0$ is the closed orbit $\mathbb{O}_{\langle \varepsilon_0, \ldots, \varepsilon_g, \delta_{g+1} \rangle}$. Its formal neighborhood in $\widehat{\mathfrak{X}}_{\Sigma_0}$ is isomorphic to the formal neighborhood $\widehat{S}$ of S at P.*

Proof. — Let us check that the map $\widehat{\mathfrak{X}}_{\Sigma_0} \to \widehat{\mathfrak{X}}_0$ is the blowing-up $\widehat{p}_0$ with center $\widehat{I} := I\mathcal{O}_{\widehat{\mathfrak{X}}_0}$. First, by [T.E] Chap. I, th. 10, π_0 is the normalized blowing-up of the monomial ideal J generated by $\underline{X}^m = X_0^{m_0} \cdots X_{g+1}^{m_{g+1}}$ with $(m, \delta_{g+1}) \geq \prod_{0 \leq i \leq g+1} \overline{\beta}_i$. On the other hand, $\widehat{\nu}$ being the divisorial valuation of the function field $K(\widehat{\mathfrak{X}}_0)$ associated to the unique irreducible exceptional curve for $\widehat{p}_0$, for any $b \in \mathbb{N}$, one has

$$\widehat{I}^{\,b} = \{f \in \widehat{R};\ \widehat{\nu}(f) \geq b\overline{\beta}_{g+1}\}.$$

Indeed, $\widehat{I}$ is simple, complete and the product of any two complete ideals in $\widehat{R}$ is again complete. Therefore, in view of [Sp1], Th. 8.6, $\widehat{p}_0$ is the strict transform of $\widehat{\mathfrak{X}}_0$ by the blowing-up of J; but also by π_0 because of the normality of its top space. The claim follows except for the identification of $\widehat{P}$ with a closed orbit for the action of T on $\mathcal{Z}_{\Sigma_0}$. This will be proved later in Prop. 1.6, ii). ■

1.2. — We now come to the definition of the *Newton fan* $\Sigma_{\mathcal{N}}$ of $(F_1, \ldots, F_g)$. Let $\mathcal{N}_i$ be the Newton polyhedron of F_i and let $h^i : \Delta \to \mathbb{R}_{\geq 0}$ be its support function.

For any non empty face φ of $\mathcal{N}_i$,

$$\sigma_\varphi := \{n \in \Delta; (m, n) = h^i(n) \text{ for any } m \in \varphi\}$$

is a strongly convex cone in Δ and the resulting family $\{\sigma_\varphi\}$ is a fan $\Sigma_{\mathcal{N}_i}$ which subdivides Δ. Note that

$$H^i_n := \{m \in \mathbb{R}^{g+2}; (m, n) = h^i(n)\}$$

being the supporting hyperplane of $\mathcal{N}_i$ in the direction n, one has

$$(1.2.1) \qquad \mathcal{N}_i \cap H^i_n = \varphi \text{ if and only if } n \in \mathring{\sigma}_\varphi$$

The least fine subdivision of the $\Sigma_{\mathcal{N}_i}$, $1 \leq i \leq g$, is, by definition, the Newton fan $\Sigma_{\mathcal{N}}$ of $(F_1, \ldots, F_g)$. It consists of the intersections of the cones of the fans $\Sigma_{\mathcal{N}_i}$. We denote by $\pi_{\mathcal{N}} : \mathcal{Z}_{\Sigma_{\mathcal{N}}} \to \mathcal{Z}_0$ the resulting equivariant modification and by $\widehat{p}_{\mathcal{N}} : \widehat{\mathfrak{X}}_{\mathcal{N}} \to \widehat{\mathfrak{X}}_0$ the strict transform of $\widehat{\mathfrak{X}}_0$ (instead of $\widehat{p}_{\Sigma_{\mathcal{N}}}, \widehat{\mathfrak{X}}_{\Sigma_{\mathcal{N}}}$). Note that $\pi^{-1}_{\mathcal{N}}(O)$ is the union of the orbits $\mathbb{O}_\sigma$, $\sigma \in \Sigma_{\mathcal{N}}$, such that $\sigma \cap \mathring{\Delta} \neq \emptyset$. We now characterize which one of these orbits intersect $\widehat{\mathfrak{X}}_{\mathcal{N}}$.

Proposition 1.3. — *Let $\sigma \in \Sigma_{\mathcal{N}}$ such that $\sigma \cap \mathring{\Delta} \neq \emptyset$ and for any i, $1 \leq i \leq g$, let σ_{φ_i} be the smallest cone of $\Sigma_{\mathcal{N}_i}$ containing σ. The following conditions are equivalent:*

i) The intersection of the orbit $\mathbb{O}_\sigma$ with $\widehat{\mathfrak{X}}_{\mathcal{N}}$ is not empty.

ii) For any i, $1 \leq i \leq g$, φ_i is not a vertex of $\mathcal{N}_i$.

iii) σ is one of the following $3g+1$ cones $\langle \delta_g, \varepsilon_{g+1}\rangle$; $\langle \delta_{i-1}, \delta_i\rangle$, $\langle \varepsilon_i, \delta_i\rangle$, $\mathbb{R}_{\geq 0}\delta_i$, $1 \leq i \leq g$.

iv) Either $\dim \sigma = 1$ and $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}}$ is isomorphic to k^* as a scheme, or $\dim \sigma = 2$ and $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}}$ is a simple point.

Proof. — First we exhibit g functions in the ideal defining $\mathbb{O}_\sigma \cap \widehat{p}^{-1}_{\mathcal{N}}(O)$ in $\mathbb{O}_\sigma$. Since $\sigma \subset \sigma_{\varphi_i}$, we have that $\sigma^\vee_{\varphi_i} \subset \sigma^\vee$ and $\sigma^\perp_{\varphi_i} \subset \sigma^\perp$. Moreover, by the definition of σ_{φ_i}, there exists a half-line $n \subset \sigma \cap \mathring{\sigma}_{\varphi_i}$. As a consequence $\sigma^\perp_{\varphi_i} \subset \sigma^\vee_{\varphi_i} \cap \sigma^\perp \subset \sigma^\vee_{\varphi_i} \cap n^\perp = \sigma^\perp_{\varphi_i}$ ([Oda] Lemma A4), hence $\sigma^\perp_{\varphi_i} = \sigma^\vee_{\varphi_i} \cap \sigma^\perp$.

Recall that the ring of regular functions on the affine open set $\mathcal{Z}_\sigma$ of $\mathcal{Z}_{\Sigma_\mathcal{N}}$ (resp. on $\mathbb{O}_\sigma$) is $k[\sigma^\vee \cap \mathbb{Z}^{g+2}]$ (resp. $k[\sigma^\perp \cap \mathbb{Z}^{g+2}]$) and the monomials $\underline{X}^m$ with $m \in \sigma^\vee \setminus \sigma^\perp$ generate the ideal defining the closed orbit $\mathbb{O}_\sigma$ in $\mathcal{Z}_\sigma$.

Since $\sigma \cap \overset{\circ}{\Delta} \neq \emptyset$, φ_i is a compact face of $\mathcal{N}_i$ and there exists a unique $F_{i,\varphi_i} \in k[X_0, \ldots, X_{g+1}]$ such that $F_i - F_{i,\varphi_i} \in (\underline{X}^m)_{m \in \mathcal{N}_i \setminus \varphi_i}$. Now pick any $m_i \in \varphi_i \cap \mathbb{Z}^{g+2}$ and let F_i' (resp. F'_{i,φ_i}) $:= \underline{X}^{-m_i} F_i$ (resp. F_{i,φ_i}). Then $F_i' \in k[\sigma^\vee_{\varphi_i} \cap \mathbb{Z}^{g+2}]$ and $F'_{i,\varphi_i} \in k[\sigma^\perp_{\varphi_i} \cap \mathbb{Z}^{g+2}]$ by the definition of σ_{φ_i} and $F_i' - F'_{i,\varphi_i} \in (\underline{X}^m)_{m \in \sigma^\vee_{\varphi_i} \setminus \sigma^\perp_{\varphi_i}}$ because of (1.2.1).

It follows from the discussion just above that $F_i' \in k[\sigma^\vee \cap \mathbb{Z}^{g+2}]$, $F'_{i,\varphi_i} \in k[\sigma^\perp \cap \mathbb{Z}^{g+2}]$ and $F_i' - F'_{i,\varphi_i} \in (\underline{X}^m)_{m \in \sigma^\vee \setminus \sigma^\perp}$, which implies that $(F'_{i,\varphi_i})_{1 \leq i \leq g}$ belong to the ideal defining $\mathbb{O}_\sigma \cap \widehat{p_\mathcal{N}}^{-1}(O)$ in $\mathbb{O}_\sigma$.

$i) \Longrightarrow ii)$ If φ_i is a vertex, then F'_{i,φ_i} is a monomial, hence invertible in $k[\mathbb{O}_\sigma]$.

$ii) \Longrightarrow iii)$ The first step in the proof is an elementary observation which does not depend on the explicit expression of the F_i : a subset σ of Δ satisfies the hypothesis and condition *ii)* if and only if, for each i, $1 \leq i \leq g$, there exists a compact face φ_i of dimension at least one of $\mathcal{N}_i$ such that $\bigcap_{1 \leq i \leq g} \overset{\circ}{\sigma}_{\varphi_i} \neq \emptyset$ and $\sigma = \bigcap_{1 \leq i \leq g} \sigma_{\varphi_i}$. This equivalence follows from the definition of $\Sigma_\mathcal{N}$ in terms of the $\Sigma_{\mathcal{N}_i}$ and the following general fact: if a fan Σ' is a subdivision of a fan Σ and if σ is the smallest cone in Σ containing $\sigma' \in \Sigma'$, then $\overset{\circ}{\sigma} \supset \overset{\circ}{\sigma}{}'$. Note that we also have $\overset{\circ}{\sigma} = \cap \overset{\circ}{\sigma}_{\varphi_i}$.

Before going further, we need to name some compact faces of dimension one and two of each $\mathcal{N}_i$, which show up immediately. The hyperplane H^i in $\mathbb{R}^{g+2}$ given by

$$\overline{\beta}_0 X_0 + \cdots + \overline{\beta}_i X_i + n_i \overline{\beta}_i X_{i+1} = n_i \overline{\beta}_i$$

is the supporting hyperplane of $\mathcal{N}_i$ in the direction $\delta_i^{i+1} := (\overline{\beta}_0, \ldots, \overline{\beta}_i, n_i\overline{\beta}_i, 0, \ldots, 0)$. The triangle ψ_i whose vertices M_i^*, $* \in \{0, 1, 2\}$, are such that $\underline{X}^{M_i^0} = X_0^{b_{i0}} \cdots X_{i-1}^{b_{ii-1}}$, $\underline{X}^{M_i^1} = X_i^{n_i}$, $\underline{X}^{M_i^2} = X_{i+1}$ is the only 2-dimensional compact face of $\mathcal{N}_i$ in H^i. We set $\psi_i^0 = [M_i^0, M_i^1]$, $\psi_i^1 = [M_i^1, M_i^2]$, $\psi_i^2 = [M_i^2, M_i^0]$. For $i \neq 1$, $\mathcal{N}_i$ may have other compact faces of dimension at least one than the above, but we will see that they do not contribute to $\sigma \in \Sigma_\mathcal{N}$ meeting $\overset{\circ}{\Delta}$ which fulfill *ii)*.

The structure of the computation is the following:

a) We prove that $\bigcap_{1 \leq i \leq j} \overset{\circ}{\sigma}_{\psi_i^0}$ is the relative interior of $\langle \delta_j^{j+1}, \varepsilon_{j+1} \rangle \times \mathbb{R}^{g-j}_{\geq 0}$, $1 \leq j \leq g$;

b) The only compact faces ψ of dimension ≥ 1 of $\mathcal{N}_{j+1}$ such that $\overset{\circ}{\sigma}_\psi$ has a non empty intersection with $\bigcap_{1 \leq i \leq j} \overset{\circ}{\sigma}_{\psi_i^0}$ are ψ^*_{j+1} with $* = 0, 1, 2$ or empty for $0 \leq j < g$;

c) The only compact face ψ of dimension ≥ 1 of $\mathcal{N}_{j+k+1}$ such that $\overset{\circ}{\sigma}_\psi$ has a non empty intersection with $\bigcap_{1 \leq i \leq j} \overset{\circ}{\sigma}_{\psi_i^0} \cap \overset{\circ}{\sigma}_{\psi^*_{j+1}} \cap \overset{\circ}{\sigma}_{\psi^1_{j+2}} \cap \cdots \cap \overset{\circ}{\sigma}_{\psi^1_{j+k}}$ with $* = 1, 2$ or empty is $\psi_{j+k+1}^{\;1}$ for $j+1 \leq j+k < g$.

Proof of a). — By induction on j. This is clear for $j=1$. Assume the claim for $j-1$. For any i, $1 \leq i \leq g$, the cone $\sigma_{\psi_i^0}$ is contained in the hyperplace B_i^0 given by

$$b_{i0}X_0 + \cdots b_{ii-1}X_{i-1} = n_i X_i.$$

So

$$\bigcap_{1\leq i\leq j} \sigma_{\psi_i^0} \subset \bigcap_{1\leq i\leq j} B_i^0 \cap \Delta = \mathbb{R}_{\geq 0}(\overline{\beta}_0, \ldots, \overline{\beta}_j) \times \mathbb{R}_{\geq 0}^{g-j+1}.$$

Since $\overline{\beta}_j > n_{j-1}\overline{\beta}_{j-1}$, the inductive hypothesis implies that:

$$\mathbb{R}_{>0}(\overline{\beta}_0, \ldots, \overline{\beta}_j) \times \mathbb{R}_{>0}^{g-j+1} \subset \bigcap_{1\leq i<j} \mathring{\sigma}_{\psi_i^0}.$$

Hence

$$\bigcap_{1\leq i\leq j} \mathring{\sigma}_{\psi_i^0} = \mathbb{R}_{>0}(\overline{\beta}_0, \ldots, \overline{\beta}_j) \times \mathbb{R}_{>0}^{g-j+1} \cap \mathring{\sigma}_{\psi_j^0}.$$

Now $n := (\overline{\beta}_0, \ldots, \overline{\beta}_j, \alpha_{j+1}, \ldots, \alpha_{g+1}) \in \mathring{\sigma}_{\psi_j^0}$ if and only if the face of $\mathcal{N}_j$ in its supporting hyperplane H_n^j in the direction n is ψ_j^0. By definition of its support function h^j, we have

$$h^j(n) = \inf_{m\in\mathcal{N}_j}(m,n) = \min(\alpha_{j+1}, n_j\overline{\beta}_j),$$

and $H_n^j \cap \mathcal{N}_j = \psi_j^0$ if and only if $\alpha_{j+1} > n_j\overline{\beta}_j$, which completes the proof of *a)*.

Proof of b). — For any n as above in $\mathring{\sigma}_{\psi_j^0}$, we have

$$h^{j+1}(n) = \inf_{m\in\mathcal{N}_{j+1}}(m,n) = \min\left(\alpha_{j+2}, n_{j+1}\alpha_{j+1}, n_{j+1}\overline{\beta}_{j+1}, \inf_{m\in\mathcal{N}_{j+1}\smallsetminus\psi_{j+1}}(m,n)\right).$$

Now either, $\alpha_{j+1} \geq \overline{\beta}_{j+1}$, so

$$\inf_{m\in\mathcal{N}_{j+1}\smallsetminus\psi_{j+1}}(m,n) \geq \inf_{m\in\mathcal{N}_{j+1}\smallsetminus\psi_{j+1}}(m,\delta_{j+1}^{j+2}) > n_{j+1}\overline{\beta}_{j+1}$$

and

$$h^{j+1}(n) = \min(\alpha_{j+2}, n_{j+1}\overline{\beta}_{j+1}).$$

This implies that $\mathcal{N}_{j+1} \cap H_n^{j+1}$ may only be $\psi_{j+1}^{\;2}$, ψ_{j+1} or $\psi_{j+1}^{\;0}$; and the following three equivalences:

$$\begin{array}{llll} n \in \mathring{\sigma}_{\psi_{j+1}^{\;2}} & \Longleftrightarrow & \alpha_{j+1} > \overline{\beta}_{j+1} \quad \text{and} & \alpha_{j+2} = n_{j+1}\overline{\beta}_{j+1} \\ n \in \mathring{\sigma}_{\psi_{j+1}} & \Longleftrightarrow & \alpha_{j+1} = \overline{\beta}_{j+1} \quad \text{and} & \alpha_{j+2} = n_{j+1}\overline{\beta}_{j+1} \\ n \in \mathring{\sigma}_{\psi_{j+1}^{\;0}} & \Longleftrightarrow & \alpha_{j+1} = \overline{\beta}_{j+1} \quad \text{and} & \alpha_{j+2} > n_{j+1}\overline{\beta}_{j+1}. \end{array}$$

Or, $n_j\overline{\beta}_j < \alpha_{j+1} < \overline{\beta}_{j+1}$. The hyperplane given by $\overline{\beta}_0X_0 + \cdots + \overline{\beta}_jX_j + \alpha_{j+1}X_{j+1} = n_{j+1}\alpha_{j+1}$ intersects the X_i-axis at $X_i = n_{j+1}\alpha_{j+1}/\overline{\beta}_i$ for $0 \leq i \leq j$ and the X_{j+1}-axis at $X_{j+1} = n_{j+1}$.

On the other hand $H^{j+1} \cap (X_{j+2} = \cdots = X_{g+1} = 0)$ is a supporting hyperplane of $\mathcal{N}_{j+1} \cap (X_{j+2} = \cdots = X_{g+1} = 0)$ and intersects the X_i-axis at $n_{j+1}\overline{\beta}_{j+1}/\overline{\beta}_i$ for $0 \le i \le j+1$. So

$$\inf_{m \in \mathcal{N}_{j+1} \setminus \psi_{j+1}} (m, n) > n_{j+1}\alpha_{j+1}.$$

This implies that $\mathcal{N}_{j+1} \cap H_n^{j+1}$ may only be ψ_{j+1}^1 and the equivalence

$$n \in \overset{\circ}{\sigma}_{\psi_{j+1}^1} \iff n_j\overline{\beta}_j < \alpha_{j+1} < \overline{\beta}_{j+1} \text{ and } \alpha_{j+2} = n_{j+1}\alpha_{j+1}.$$

Proof of c). — By induction on k. In addition to our inductive hypothesis, we may assume that

$$\Big(\bigcap_{1 \le i \le j} \overset{\circ}{\sigma}_{\psi_i^0}\Big) \cap \overset{\circ}{\sigma}_{\psi_{j+1}^*} \cap \overset{\circ}{\sigma}_{\psi_{j+2}^1} \cap \cdots \cap \overset{\circ}{\sigma}_{\psi_{j+k}^1}$$

is the relative interior of

$$\begin{cases} \nu_{j+1}^k := \mathbb{R}_{>0}\delta_{j+1}^{j+k+1} \times \mathbb{R}_{>0}^{g-j-k} & \text{if } * \text{ is empty} \\ \sigma_{j+1,1}^k := \langle \delta_j^{j+k+1}, \delta_{j+1}^{j+k+1} \rangle \times \mathbb{R}_{\ge 0}^{g-j-k} & \text{if } * = 1 \\ \sigma_{j+1,2}^k := \langle \delta_{j+1}^{j+k+1}, \varepsilon_{j+1} \rangle \times \mathbb{R}_{\ge 0}^{g-j-k} & \text{if } * = 2 \end{cases}$$

where $\delta_\ell^{\ell'}$ is the vector obtained from δ_ℓ by replacing its $X_{\ell''}$-components, $\ell' < \ell'' \le g+1$, by 0. For $k = 1$, this is just what the proof of *b)* has established. Assume the claim for $k - 1 \ge 1$. For any $n = (\alpha_i)_{0 \le i \le g+1}$ in Δ,

$$h^{j+k}(n) = \min\Big(\alpha_{j+k+1}, \inf_{m \in \mathcal{N}_{j+k} \cap (X_{j+k+1} = \cdots = X_{g+1} = 0)} (m, n)\Big).$$

By using repeatedly the inequalities $n_i\overline{\beta}_i < \overline{\beta}_{i+1}$, $j \le i < j+k$, it is easily verified that the intersection point of the X_i-axis, $0 \le i \le j+k$, with any of the two hyperplanes $\langle \delta_j^{j+k}, ? \rangle = n_j \cdots n_{j+k}\overline{\beta}_j$ and $\langle \delta_{j+1}^{j+k}, ? \rangle = n_{j+1} \cdots n_{j+k}\overline{\beta}_{j+1}$ lies between 0 and its intersection point with H^{j+k}; moreover they coincide if and only if $i = j+k$ and this point is $n_{j+k}\varepsilon_{j+k}$. Therefore if n lies on $\overset{\circ}{\nu}{}_{j+1}^{k-1}$, $\overset{\circ}{\sigma}{}_{j+1,1}^{k-1}$ or $\overset{\circ}{\sigma}{}_{j+1,2}^{k-1}$, then $h^{j+k}(n) = \min(\alpha_{j+k+1}, n_{j+k}\alpha_{j+k})$ and $\mathcal{N}_{j+k} \cap H_n^{j+k} = \psi_{j+k}^1$. So $n \in \overset{\circ}{\sigma}_{\psi_{j+k}^1}$ if and only if $\alpha_{j+k+1} = n_{j+k}\alpha_{j+k}$, which completes the proof of *c)*.

At last, we get *iii)* and the following identities:

$$\Big(\bigcap_{1 \le i < j} \sigma_{\psi_i^0}\Big) \cap \sigma_{\psi_j^*} \cap \Big(\bigcap_{j < i \le g} \sigma_{\psi_i^1}\Big)$$

is

$$\begin{cases} \nu_j := \mathbb{R}_{>0}\delta_j & \text{if } * \text{ is empty and } 1 \le j \le g \\ \sigma_{j,1} := \langle \delta_{j-1}, \delta_j \rangle & \text{if } * = 1 \text{ and } 1 \le j \le g \\ \sigma_{j,2} := \langle \varepsilon_j, \delta_j \rangle & \text{if } * = 2 \text{ and } 1 \le j \le g \\ \sigma_{g+1} := \langle \delta_g, \varepsilon_{g+1} \rangle & \text{if } j = g+1 \end{cases}$$

iii) $\Longrightarrow$ *iv)* Set $m_i^0 = M_i^1 - M_i^0$, $m_i^1 = M_i^2 - M_i^1$, $1 \leq i \leq g$. In view of the g equations defining $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}}$ in $\mathbb{O}_\sigma$ previously identified and of the above identities, it is enough to check that $\{(m_i^0)_{1\leq i<j}\,;\,(m_i^1)_{j<i<g}\,;\,m_j^0, m_j^1,\ (\text{resp. } m_j^1),\ (\text{resp. } m_j^0 + m_j^1)\}$ is a $\mathbb{Z}$-basis of $\nu_j^\perp \cap \mathbb{Z}^{g+2}$, (resp. $\sigma_{j,1}^\perp \cap \mathbb{Z}^{g+2}$), (resp. $\sigma_{j,2}^\perp \cap \mathbb{Z}^{g+2}$) for $1 \leq j \leq g$, and that $\{m_i^0\}_{1\leq i\leq g}$ is a $\mathbb{Z}$-basis of $\sigma_{g+1}^\perp \cap \mathbb{Z}^{g+2}$. Verifications are left to the reader. ∎

Corollary 1.4. — *The system $(F_1, \ldots, F_g)$ is non-degenerated for $\mathcal{N}_1, \ldots, \mathcal{N}_g$.*

Proof. — The non degeneracy condition holds if and only if for any $\sigma \in \Sigma_{\mathcal{N}}$ which fulfills condition *i)* in 1.3, $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}}$ is the intersection of g non singular hypersurfaces in $\mathbb{O}_\sigma$ meeting transversally; we have just seen that it is defined by g coordinate functions on the torus $\mathbb{O}_\sigma$. ∎

We now derive explicit equivariant modifications of $\mathcal{Z}_{\Sigma_0}$ which "simplify" and ultimately desingularize $\widehat{S}$.

The following proposition is preparatory.

Proposition 1.5. — *Let Σ be a subdivision of the Newton fan $\Sigma_{\mathcal{N}}$ and let $\widehat{\mathfrak{X}}_\Sigma$ be the strict transform of $\widehat{\mathfrak{X}}_0$ by $\pi_\Sigma : \mathcal{Z}_\Sigma \to \mathcal{Z}_0$. The following three conditions on $\tau \in \Sigma$ such that $\tau \cap \mathring{\Delta} \neq \emptyset$ are equivalent:*

i) $\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_\Sigma \neq \emptyset$;

ii) *σ being the smallest cone in $\Sigma_{\mathcal{N}}$ containing τ,* $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}} \neq \emptyset$;

iii) *The scheme $\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_\Sigma$ is isomorphic to k^* or a simple point according to whether* $\dim \tau = 1$ *or* 2.

For any τ of dimension 1 (resp. 2) with the above properties, the surface $\widehat{\mathfrak{X}}_\Sigma$ is non singular along (resp. is analytically isomorphic to the germ at the closed orbit of the toric surface $\mathcal{Z}_{\tau,N_\tau}$ given by τ and the lattice N_τ induced by $\mathbb{Z}^{g+2}$ on the real vector space spanned by τ at) $\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_\Sigma$.

Proof. — First $\mathbb{O}_\sigma$ is the image of $\mathbb{O}_\tau$ by $\mathcal{Z}_\Sigma \to \mathcal{Z}_{\Sigma_{\mathcal{N}}}$, so *i)* $\Longrightarrow$ *ii)*. The proof of *ii)* $\Longrightarrow$ *iii)* is reduced to a mild adjustment of *ii)* $\Longrightarrow$ *iv)* in 1.3 once one observes that the smallest cone $\sigma_{\varphi_i} \in \Sigma_{\mathcal{N}_i}$ containing σ is also the smallest cone in $\Sigma_{\mathcal{N}_i}$ containing τ. Indeed since $\dim \sigma = 1$ or 2 and $\sigma^\perp \cap \mathbb{Z}^{g+2}$ is a direct summand in $\tau^\perp \cap \mathbb{Z}^{g+2}$, either these two lattices coincide; or $\tau^\perp \cap \mathbb{Z}^{g+2} \simeq (\sigma^\perp \cap \mathbb{Z}^{g+2}) \oplus \mathbb{Z}$, so $k[\mathbb{O}_\tau] \simeq k[\mathbb{O}_\sigma][U, U^{-1}]$ with U an indeterminate and the ideals defining $\mathbb{O}_\sigma \cap \widehat{\mathfrak{X}}_{\mathcal{N}}$ and $\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_\Sigma$ in $\mathbb{O}_\sigma$ and $\mathbb{O}_\tau$ respectively are generated by the same functions $(F'_{i,\varphi_i})_{1\leq i\leq g}$. Finally, recall that the affine open set $\mathcal{Z}_\tau$ in $\mathcal{Z}_\Sigma$ is isomorphic (not canonically) to $\mathcal{Z}_{\tau,N_\tau} \times \mathbb{O}_\tau$, that is $\mathbb{A}_k^1 \times k^{*g+1}$ if $\dim \tau = 1$, and $\mathcal{Z}_{\tau,N_\tau} \times k^{*g}$ if $\dim \tau = 2$. So the last part of the claim follows from the transversality property *iii)*. ∎

The first subdivision of $\Sigma_{\mathcal{N}}$ that we consider is the least fine subdivision $\Sigma_{\mathcal{T}}$ of $\Sigma_{\mathcal{N}}$ and Σ_0.

Proposition 1.6. — *Let $\widehat{\mathfrak{X}}_{\mathcal{T}}$ be the strict transform of $\widehat{\mathfrak{X}}_0$ by $\pi_{\mathcal{T}} : \mathcal{Z}_{\Sigma_{\mathcal{T}}} \to \mathcal{Z}_0$ and let $\widehat{q}_{\mathcal{T}} : \widehat{\mathfrak{X}}_{\mathcal{T}} \to \widehat{\mathfrak{X}}_{\Sigma_0}$ be the induced modification.*

i) The collection Θ of those $\tau \in \Sigma_{\mathcal{T}}$ such that $\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_{\mathcal{T}} \neq \emptyset$ is the fan consisting of the $2g+2$ cones of dimension two $\sigma_{i,1} = \langle \delta_{i-1}, \delta_i \rangle$, $\sigma_{i,2} = \langle \varepsilon_i, \delta_i \rangle$, $1 \leq i \leq g+1$, and of their faces.

ii) Let F_i be the Zariski closure of $\mathbb{O}_{\mathbb{R}_{\geq 0}\delta_i} \cap \widehat{\mathfrak{X}}_{\mathcal{T}}$, $1 \leq i \leq g+1$. Then, for $1 \leq i \leq g$, F_i is contracted to the closed orbit $\mathbb{O}_{\langle \varepsilon_0, \dots, \varepsilon_g, \delta_{g+1} \rangle}$ which is the singular point $\widehat{P}$ of $\widehat{\mathfrak{X}}_{\Sigma_0}$; while $\widehat{q}_{\mathcal{T}}$ induces an isomorphism from F_{g+1} to the exceptional curve $\widehat{p}_0^{-1}(O) \simeq \mathbb{P}^1$ on $\widehat{\mathfrak{X}}_{\Sigma_0}$.

iii) Let $(\tilde{\sigma}_{i,1}; \tilde{\sigma}_{i,2})_{1 \leq i \leq g+1}$ be the maximal cones in the elementary subdivision of $\mathbb{R}^2_{\geq 0}$ by $\mathbb{R}_{\geq 0}(e_{i-1}, \overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})$ ordered counterclockwise. Among these cones $\tilde{\sigma}_{g+1,2}$ is the only one to be regular w.r.t. $\mathbb{Z}^2$. The pair $(\sigma_{i,j}, N_{\sigma_{i,j}})$ is isomorphic to $(\tilde{\sigma}_{i,j}, \mathbb{Z}^2)$.

iv) Let $O_{i,j} := \mathbb{O}_{\sigma_{i,j}} \cap \widehat{\mathfrak{X}}_{\mathcal{T}}$ and let $\widetilde{O}_{i,j}$ be the closed orbit in the toric surface $\mathcal{Z}_{\tilde{\sigma}_{i,j}}$. The germs $(\widehat{\mathfrak{X}}_{\mathcal{T}}, O_{i,j})$ and $(\mathcal{Z}_{\tilde{\sigma}_{i,j}}, \widetilde{O}_{i,j})$ are analytically isomorphic.

Proof. — The smallest cone in Σ_0 containing δ_i, $0 \leq i \leq g$, being $\langle \varepsilon_0, \dots, \varepsilon_g, \delta_{g+1} \rangle$, the same holds for every $\sigma_{i,j}$ except $\sigma_{g+1,2}$. On the other hand $\delta_{g+1} \in \langle \delta_g, \varepsilon_{g+1} \rangle$; hence *i)* and *ii)* in view of 1.0, 1.3 and 1.5. Part *iii)* follows from explicit computations of $\mathbb{Z}$-basis of the lattices $N_{\sigma_{i,j}}$. Let $\gamma_i, \lambda_i, \rho_i$ be the primitive vectors on the half lines through δ_{i-1}, $\delta_i - \delta_{i-1}$, and $\delta_i - (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\varepsilon_i$ respectively. It is easily checked that (γ_i, λ_i) (resp. (ρ_i, ε_i)) is a $\mathbb{Z}$-basis of $N_{\sigma_{i,1}}$ (resp. $N_{\sigma_{i,2}}$), and that $\sigma_{i,1} = \langle \gamma_i, e_{i-1}\gamma_i + (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\lambda_i \rangle$ and $\sigma_{i,2} = \langle e_{i-1}\rho_i + (\overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1})\varepsilon_i, \varepsilon_i \rangle$. Part *iv)* follows from *iii)* and 1.5. ∎

Corollary 1.7. — *The collection Ξ of those $\tau \in \Sigma_{\mathcal{T}}$ such that $\widehat{P}$ lies in the Zariski closure of $\widehat{q}_{\mathcal{T}}(\mathbb{O}_\tau \cap \widehat{\mathfrak{X}}_{\mathcal{T}})$ is the fan obtained by removing $\sigma_{g+1,2}$ and $\mathbb{R}_{\geq 0}\varepsilon_{g+1}$ from Θ.*

The isomorphism between $\widehat{S}$ and the formal neighborhood of $\widehat{P}$ in $\widehat{\mathfrak{X}}_{\Sigma_{\mathcal{T}}}$ established in Prop. 1.1, allows us to give

Definition 1.8. — *We call Ξ the* skeleton *of the primitive singularity $\widehat{S}$.*

Note that Ξ does not depend on a particular choice of $\{x_0, \dots, x_{g+1}\}$.

1.9. — For any regular subdivision Σ of $\Sigma_{\mathcal{T}}$, the transversality property of $\widehat{\mathfrak{X}}_\Sigma$ with the orbits of the torus in $\mathcal{Z}_\Sigma$ observed in the proof of 1.5 may be reformulated by saying that $\mathcal{Z}_\Sigma \to \mathcal{Z}_{\Sigma_0}$ is an embedded desingularization of $\widehat{\mathfrak{X}}_{\Sigma_0}$. Note that by applying [T.E], Remark p. 35, to any subdivision of $\Sigma_{\mathcal{T}}$ whose trace on Θ is its minimal regular subdivision $\Theta_{\mathcal{R}}$, one gets regular subdivisions $\Sigma_{\mathcal{R}}$ of $\Sigma_{\mathcal{T}}$ with the same property; $\Sigma_{\mathcal{R}}$ is not uniquely determined.

Proposition 1.10. — *Let $\Sigma_{\mathcal{R}}$ be any regular subdivision of $\Sigma_{\mathcal{T}}$ whose trace on Θ is its minimal regular subdivision $\Theta_{\mathcal{R}}$, and let $\widehat{\mathfrak{X}}_{\mathcal{R}}$ be the strict transform of $\widehat{\mathfrak{X}}_{\Sigma_0}$ by $\mathcal{Z}_{\Sigma_{\mathcal{R}}} \to \mathcal{Z}_{\Sigma_0}$. Then $\widehat{\mathfrak{X}}_{\mathcal{R}}$ is the minimal desingularization of $\widehat{\mathfrak{X}}_{\Sigma_0}$.*

Proof. — The claim is a direct consequence of any of the two following independent remarks.

Remark 1.10.1. — Let $p : \mathfrak{X} \to \mathfrak{X}_0$ be the minimal sequence of point blowing-ups making $I\mathcal{O}_{\mathfrak{X}}$ invertible as in 0.3.1. One recovers the dual graph Γ of the exceptional curves for p from Proj $\Theta_{\mathcal{R}}$ by erasing its ends, namely Proj $\mathbb{R}_{\geq 0}\,\varepsilon_i$, and its adjacent edge, $0 \leq i \leq g+1$.

Proof. — It is based on the equivalence between the Zariski exponents $(\overline{\beta}_0, \ldots, \overline{\beta}_{g+1})$ and the *Puiseux exponents* $(\beta_1, \ldots, \beta_{g+1})$ of the valuation ν associated to the irreducible exceptional curve on S (see [Sp1], § 6).

Recall that the minimal desingularization of S is the map $q : \mathfrak{X} \to S$ factoring p, and that the only irreducible curve on $\mathfrak{X}$ to be exceptional for p and not for q is represented by v^{g+1} on Γ.

Let $w^1, \ldots, w^g$ be the stars of Γ as shown in 0.3.1. By erasing the edge adjacent to w^i on the linear subgraph of Γ between w^i and v^{g+1}, $1 \leq i \leq g$, one breaks up Γ into $g+1$ disjoint linear graphs Γ_i, $1 \leq i \leq g+1$. It follows from Enriques's theory that Γ_i has $\sum\limits_{1\leq j\leq s_i} a_j^{(i)}$ vertices, where

$$\frac{\beta_i - \beta_{i-1}}{e_{i-1}} = \left[a_1^{(i)}, \ldots, a_{s_i}^{(i)}\right] = a_1^{(i)} + \cfrac{1}{a_2^{(i)} + \cfrac{1}{\ddots + \cfrac{1}{a_{s_i}^{(i)}}}}$$

is the continued fraction expansion(*). (By convention $\beta_0 = 0$.)

The minimal regular subdivision $\widetilde{\Sigma}_{i\mathcal{R}}$ of the elementary subdivision $\widetilde{\Sigma}_i$ of $\mathbb{R}^2_{\geq 0}$ by $\mathbb{R}_{\geq 0}\,\tilde{\delta}_i := (e_{i-1}, \beta_i - \beta_{i-1})$ may also be described from this expansion. To be precise, Γ_i is

(*) Our notation does not coincide with that in [Sp1].

obtained from Proj $\widetilde{\Sigma}_{i\mathcal{R}}$ by erasing its ends, with $w^i = \mathrm{Proj}\,\tilde{\delta}_i$. The claim follows from the equality $\beta_i - \beta_{i-1} = \overline{\beta}_i - n_{i-1}\overline{\beta}_{i-1}$, $1 \le i \le g+1$, and 1.6, iii). ■

Remark 1.10.2. — Let $\Theta_{\mathcal{R}}(1)$ be the set of 1-dimensional cones in $\Theta_{\mathcal{R}}$ and let α_ℓ denote the primitive vector on $\ell \in \Theta_{\mathcal{R}}(1)$ (i.e. the generator of the semigroup $\mathbb{Z}^{g+2} \cap \ell$).

If $\ell \cap \overset{\circ}{\Delta} \neq \emptyset$ (resp. $\ell = \mathbb{R}_{\geq 0}\varepsilon_i$), the Zariski closure E_ℓ of $\mathbb{O}_\ell \cap \widehat{\mathfrak{X}}_{\mathcal{R}}$ is contracted to O by $\widehat{\mathfrak{X}}_{\mathcal{R}} \to \widehat{\mathfrak{X}}_0$ (resp. coincides with the strict transform of C_i).

Denote by (E_ℓ^2) the self-intersection of E_ℓ if contracted. Then

$$(E_\ell^2)\alpha_\ell + \sum_{\ell' \in \Lambda_\ell} \alpha_{\ell'} = 0 \qquad (*)$$

where $\Lambda_\ell = \{\ell' \in \Theta_{\mathcal{R}}(1)\,;\, \ell' \neq \ell \text{ and } \langle \ell', \ell \rangle \in \Theta_{\mathcal{R}}\}$.

One has $(E_\ell^2) \neq -1$ unless $\ell = \mathbb{R}_{\geq 0}\delta_{g+1}$ or equivalently E_ℓ is not contracted on $\widehat{P}$ in $\widehat{\mathfrak{X}}_{\Sigma_0}$.

Proof. — First recall that, for any 1-dimensional cone $\ell \in \Sigma_{\mathcal{R}}$ the vanishing order of X_i along $\mathbb{O}_\ell$ in $\mathcal{Z}_{\Sigma_{\mathcal{R}}}$ is $(\varepsilon_i, \alpha_\ell)$. Because of the transversality property of $\widehat{\mathfrak{X}}_{\mathcal{R}}$ with the orbits $\mathbb{O}_\ell, \ell \in \Theta_{\mathcal{R}}(1)$, it follows that

$$\mathrm{div}\, x_i = \sum_{\ell' \in \Theta_{\mathcal{R}}(1)} (\varepsilon_i, \alpha_{\ell'}) E_{\ell'}, \;\; 0 \le i \le g+1$$

hence the identity $(*)$ by intersecting with E_ℓ.

Because of 1.5 and 1.6, $\widehat{\mathfrak{X}}_{\mathcal{R}}$ is the minimal desingularization of $\widehat{\mathfrak{X}}_{\mathcal{T}}$. So $(E_\ell^2) \neq -1$, if $\ell \in \Theta_{\mathcal{R}} \setminus \Theta$. Assume now that $\ell = \mathbb{R}_{\geq 0}\delta_i$, $1 \le i \le g$; the set Λ_ℓ has 3 elements whose primitive vectors $\alpha_1, \alpha_2, \alpha_3$ lie respectively in $\overset{\circ}{\sigma}_{i+1,1}$, $\overset{\circ}{\sigma}_{i,1}$ and $\overset{\circ}{\sigma}_{i,2}$. Since $\delta_{i-1} \in \langle \varepsilon_0, \ldots, \varepsilon_i, \delta_i \rangle$, the identity $(*)$ implies that $\alpha_2 + \alpha_3$ lies in the intersection of $\langle \varepsilon_0, \ldots, \varepsilon_i, \delta_i \rangle$ with the plane generated by $\sigma_{i+1,1}$, which is a cone τ_{i+1}. The primitive vectors $\gamma_{i+1}, \lambda_{i+1}$ on $\mathbb{R}_{\geq 0}\delta_i$, and on $\mathbb{R}_{\geq 0}(\delta_{i+1} - \delta_i)$ respectively, form a $\mathbb{Z}$-basis of the lattice $N_{\sigma_{i+1,1}}$. In this basis $\sigma_{i+1,1}$ and τ_{i+1} are respectively the cones $\langle (1,0), (e_i, \overline{\beta}_{i+1} - n_i\overline{\beta}_i) \rangle$ and $\langle (1,0), (e_i, -n_i\overline{\beta}_i) \rangle$ and $\alpha_\ell - \alpha_1 = (1,0) - \alpha_1 \notin \tau_{i+1}$, so $(E_\ell^2) \neq -1$. Similarly, $\{\delta_g, \varepsilon_{g+1}\}$ is a $\mathbb{Z}$-basis of $N_{\sigma_{g+1,1}}$; in this basis $\delta_{g+1} = (1, \overline{\beta}_{g+1} - n_g\overline{\beta}_g)$, so, as expected, for $\ell = \mathbb{R}_{\geq 0}\delta_{g+1}$, one has $(E_\ell^2) = -1$. ■

1.11. — The non singular surface $\widehat{\mathfrak{X}}_{\mathcal{R}}$ together with the complement of the exceptional curve for $\widehat{\mathfrak{X}} \to \widehat{\mathfrak{X}}_0$ (resp. $\widehat{\mathfrak{X}}_{\mathcal{R}} \to \widehat{\mathfrak{X}}_{\Sigma_0}$) is a strict toroidal embedding (= without self-intersection). The corresponding rational conical polyhedral complex, as defined in [T.E], II.1, carries a natural extra structure of fan, namely the fan $\widetilde{\Theta}_{\mathcal{R}}$ (resp. $\widetilde{\Xi}_{\mathcal{R}}$), obtained from the minimal regular subdivision $\Theta_{\mathcal{R}}$ of Θ (resp. $\Xi_{\mathcal{R}}$ of Ξ) by removing the cones of which $\mathbb{R}_{\geq 0}\varepsilon_i$, $0 \leq i \leq g$, and $\mathbb{R}_{\geq 0}\varepsilon_{g+1}$ (resp. $\mathbb{R}_{\geq 0}\delta_{g+1}$) is a face. See also 2.5.1 below for an interpretation of the map “ord” of [T.E], p. 64.

2. Arcs and wedges on a primitive surface singularity

Before introducing arcs and relating them with the skeleton Ξ of the primitive singularity $\widehat{S}$, we draw a straightforward consequence of the description of Ξ in terms of the Newton fans of $(F_1, \ldots, F_g)$.

2.1. — Let $\widehat{A}$ be the local ring of $\widehat{S}$; in view of 1.1, $\widehat{A}$ is isomorphic to the completion of the local ring of $\widehat{\mathfrak{X}}_{\Sigma_0}$ at $\widehat{P}$ with respect to its maximal ideal.

By an *order function* on $\widehat{A}$, we mean a function $\omega : \widehat{A} \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ such that $\omega(k \setminus 0) = 0$, $\omega(0) = +\infty$, $\omega(xy) = \omega(x) + \omega(y)$ and $\omega(x+y) \geq \min(\omega(x), \omega(y))$ for all $x, y \in \widehat{A}$, with the convention that n (resp. $+\infty$) $+\infty = +\infty$ and $n < +\infty$ for any $n \in \mathbb{Z}_{\geq 0}$. We will only consider order functions such that $\omega(x_i) \neq +\infty$, $0 \leq i \leq g+1$.

Proposition 2.2. — *For any order function ω on $\widehat{A}$, $\alpha_\omega := (\omega(x_i))_{0 \leq i \leq g+1}$ lies on the skeleton Ξ of $\widehat{S}$.*

Proof. — The completion $\widehat{R}$ of $R := \mathcal{O}_{\mathfrak{X}_0, O}$ is a subring of $\widehat{A}$ and $Q := \{f \in \widehat{R}; \omega(f) > 0\}$ is a prime ideal in $\widehat{R}$. If $\alpha_\omega \neq 0$, there exists i, $0 \leq i \leq g+1$, such that $\omega(x_i) > 0$. Now $\widehat{R}$ is a two-dimensional regular local ring. Therefore, either Q is principal; $x_0, \ldots, x_{g+1}$ being irreducible elements in $\widehat{R}$, x_i generates Q and $x_j \notin Q$ for $j \neq i$, so $\alpha_\omega \in \mathbb{Z}_{>0}\varepsilon_i$.

Or Q is the maximal ideal of $\widehat{R}$, so $\alpha_\omega \in \mathbb{Z}_{>0}^{g+2}$. Because of the triangular inequality, the supporting hyperplane of the Newton polyhedron $\mathcal{N}_i$ of F_i in the direction α_ω intersects $\mathcal{N}_i$ along a compact face φ_i of dimension at least one and by 1.2.1, $\alpha_\omega \in \cap \mathring{\sigma}_{\varphi_i} \subset \sigma = \bigcap_{1 \leq i \leq g} \sigma_{\varphi_i}$. Therefore σ_{φ_i} is the smallest cone of $\Sigma_{\mathcal{N}_i}$ containing σ and any of the equivalent conditions of Prop. 1.3 holds for σ (for more details, go back to *ii)* $\Longrightarrow$ *iii)* in the proof of 1.3). Comparing 1.3 *iii)* with 1.6 *i)*, we get that α_ω lies on Θ.

On the other hand, by 1.6 *ii)* the singular point $\widehat{P}$ of $\widehat{\mathfrak{X}}_{\Sigma_0}$ is the closed orbit $\mathbb{O}_{\langle \varepsilon_0, \ldots, \varepsilon_g, \delta_{g+1} \rangle}$ on $\mathcal{Z}_{\Sigma_0}$. So for any $m = (m_0, \ldots, m_{g+1}) \in \langle \varepsilon_0, \ldots, \varepsilon_g, \delta_{g+1} \rangle^\vee$, $\underline{x}^m = \prod x_i^{m_i} \in \widehat{A}$, hence $\omega(\underline{x}^m) = (m, \alpha_\omega) \geq 0$ and $\alpha_\omega \in \langle \varepsilon_0, \ldots, \varepsilon_g, \delta_{g+1} \rangle$. In view of 1.7, α_ω lies on Ξ. ∎

We now come to the definitions of arcs and wedges on $\widehat{S}$ and of their characteristic vectors w.r.t. the generating system $\{x_0, \ldots, x_{g+1}\}$, via order functions.

Definition 2.3. — *An* arc *(resp. a* wedge*) on $\widehat{S}$ is a k-local morphism from the local ring $\widehat{A}$ of $\widehat{S}$ to the formal power series ring in one (resp. two) variables with coefficients in k whose kernel is a prime ideal of height one (resp. 0).*

Given an arc h (resp. a wedge φ) on $\widehat{S}$, to any discrete valuation of rank one ν, non negative on $k[[t]]$ (resp. $k[[u_1, u_2]]$), we associate the order function $\omega := \nu \circ h$ (resp. $\nu \circ \varphi$). By convention $\omega(f) = +\infty$ if and only if $f \in \operatorname{Ker} h$ (resp. $f = 0$). We denote by ω_t the t-adic order function; similarly for any irreducible p in $k[[u_1, u_2]]$, we denote by ω_p the p-adic order function.

Definition 2.4. — *Given an arc h on $\widehat{S}$ lying generically outside the total transform of C_i, $0 \leq i \leq g$, we define the* characteristic vector *of h, w.r.t. $\{x_0, \ldots, x_{g+1}\}$, to be $\alpha_h := (\omega_t(x_i))_{0 \leq i \leq g+1}$.*

By 2.2, $\alpha_h \in \Xi$. As a consequence, there exists a unique 2-dimensional cone in the minimal regular subdivision $\Xi_{\mathcal{R}}$ of Ξ, which contains α_h; hence α_h may be written uniquely as a linear combination with coefficients in $\mathbb{Z}_{\geq 0}$ of the primitive vectors lying on its 1-dimensional faces. Taking into account that the set of 1-dimensional cones $\Xi_{\mathcal{R}}(1)$ in $\Xi_{\mathcal{R}}$ is obtained by removing $\mathbb{R}_{\geq 0}\delta_{g+1}$ from $\Theta_{\mathcal{R}}(1)$, this has the following geometrical interpretation:

Proposition 2.5.1. — *Let h be an arc on $\widehat{S}$ as above and let $h_{\mathcal{R}}$ be its strict transform on its minimal desingularization identified with $\widehat{\mathfrak{X}}_{\mathcal{R}}$. With the notation introduced in 1.10.2, we have*

$$\alpha_h = \sum_{\ell \in \Xi_{\mathcal{R}}(1)} (E_\ell \cdot h_{\mathcal{R}}) \alpha_\ell$$

where $(E_\ell \cdot h_{\mathcal{R}})$ denotes the intersection multiplicity of E_ℓ with $h_{\mathcal{R}}$.

Proof. — Recall that the curve E_ℓ on $\widehat{\mathfrak{X}}_{\mathcal{R}}$ is not exceptional for $\widehat{q}_{\mathcal{R}} : \widehat{\mathfrak{X}}_{\mathcal{R}} \to \widehat{\mathfrak{X}}_{\Sigma_0} \simeq \widehat{S}$, if and only if $\ell = \mathbb{R}_{\geq 0}\varepsilon_i$, $0 \leq i \leq g+1$, $\ell = \mathbb{R}_{\geq 0}\delta_{g+1}$. These curves are respectively the strict transforms of C_i, $0 \leq i \leq g+1$, and the center of the valuation ν. Now the arc $h_{\mathcal{R}}$ intersects the exceptional curve of $\widehat{q}_{\mathcal{R}}$; so it does not meet the strict transform of C_{g+1}.

By definition $(E_\ell \cdot h_{\mathcal{R}})$ is the order in t of the image by $h_{\mathcal{R}}$ of the local equation of E_ℓ at the exceptional point of $h_{\mathcal{R}}$. The claim follows from the expression of the divisor of zeros, $\operatorname{div} x_i$, of x_i on $\widehat{\mathfrak{X}}_{\mathcal{R}}$ as a linear combination of E_ℓ, $\ell \in \Theta_{\mathcal{R}}(1)$, given in the proof of 1.10.2, since $\omega_t(x_i) = (\operatorname{div} x_i \cdot h_{\mathcal{R}})$. ∎

The arc $h_{\mathcal{R}}$ intersecting at least one exceptional E_ℓ for $\widehat{q}_{\mathcal{R}}$, we have that $\alpha_h \notin \mathbb{R}_{\geq 0}\varepsilon_i$, $0 \leq i \leq g$, and $\alpha_h \notin \mathbb{R}_{\geq 0}\delta_{g+1}$. In view of 0.4, we also get

Proposition-Definition 2.5.2. — *Let h be an arc on $\widehat{S}$ as above. Its strict transform on the minimal desingularization of $\widehat{S}$ is smooth and intersects transversally the exceptional curve E at a regular point of E other than the exceptional points of the center of the valuation ν and of the strict transforms of the C_i, $0 \leq i \leq g$, if and only if its characteristic vector α_h belongs to the minimal generating system G_σ of some $\sigma \in \Xi$.* An arc *with the equivalent above properties will be said to be* general *on* $\widehat{S}$ w.r.t. $\{x_0, \ldots, x_{g+1}\}$.

Proposition 2.2 leads naturally to the following definition of the characteristic vectors of a wedge.

Definition 2.6. — *Given a wedge φ on $\widehat{S}$, we define the* characteristic vectors *of φ, w.r.t. $\{x_0, \ldots, x_{g+1}\}$, to be the non zero $\alpha_{\varphi,p} := (\omega_p(x_i))_{0\leq i\leq g+1}$, where p runs over all irreducible elements in $k[[u_1, u_2]]$ up to multiplication by a unit.*

By 2.2, φ has only finitely many characteristic vectors which all lie on Ξ.

Proposition 2.7. — *The morphism* Spec $k[[u_1, u_2]] \to \widehat{S}$ *given by a wedge φ lifts to the minimal desingularization of $\widehat{S}$ if and only if there exists a cone in the minimal regular subdivision $\Xi_{\mathcal{R}}$ of Ξ which contains every characteristic vector of φ.*

Proof. — Let $\Sigma_{\mathcal{R}}$ be as in Prop. 1.10. The lifting property of 2.7 holds iff the morphism Spec $k[[u_1, u_2]] \to \widehat{S} \to \mathcal{Z}_{\Sigma_0}$ lifts to $\mathcal{Z}_{\Sigma_{\mathcal{R}}}$ by 1.10. This occurs iff there exists a cone $\tau \in \Sigma_{\mathcal{R}}$ which contains every characteristic vector of φ. Since they all lie in Ξ, the claim follows from the definition of $\Sigma_{\mathcal{R}}$. ∎

2.8. — There is an alternative way of proving that the characteristic vectors belong to Ξ. The emphasis is made on arcs rather than on functions. To compute the characteristic vectors of an arc, you have to control the infinitely near points of O shared by its image in the plane and by C_i, $0 \leq i \leq g+1$. This may be done with the help of Enriques diagrams or by using the notion of contact. See [Ab] or [Sp1], § 7 and 8. For wedges, you only have to observe that, either $\alpha_{\varphi,p} \in \mathbb{Z}_{>0}\varepsilon_i$ for some i, $0 \leq i \leq g+1$, or the p-adic valuation in $k[[u_1, u_2]]$ induces via $\widehat{R} \to \widehat{A} \xrightarrow{\varphi} k[[u_1, u_2]]$ a divisorial valuation centered in $\widehat{R}$. By the structure theorem for valuations in dimension 2, it is given by the intersection multiplicity with a general curve in a linear system with infinitely near base condition in the plane; hence $\alpha_{\varphi,p} \in \Theta$ as above. Using the same argument as in 2.2, you get that $\alpha_{\varphi,p} \in \Xi$.

2.9. — In fact the approach via Newton polyhedrons is needed to get finer restrictions, valid for all the characteristic vectors simultaneously. This is what we do in the rest of this section.

First, observe that the characteristic vectors of a wedge may not all lie on $\mathbb{R}_{>0}\delta_{g+1}$. Indeed with $b_{g+1,j}$, $0 \le j \le g$, as in 0.3.2, $\prod_j x_j^{b_{g+1,j}} x_{g+1}^{-1}$ is in the maximal ideal of $\widehat{A}$ and its image by φ would be a unit.

Besides, $\mathring{\Delta} = \mathbb{R}_{>0}^{g+2}$ must contain at least one of them, otherwise the induced morphism Spec $k[[u_1, u_2]] \to \widehat{\mathfrak{X}}_0$ would be finite. More directly setting $\alpha_{\varphi,p} = \alpha_p$ for simplicity, if $\alpha_{p,g+1} \neq 0$, then $\alpha_p \in \mathbb{Z}_{>0}^{g+2}$, because x_{g+1} generates $I\widehat{A}$, $x_i^{\overline{\beta}_{g+1}} \in I$, $0 \le i \le g$; but φ is injective.

The filtration $\mathcal{D} = (\Delta_i := \mathbb{R}_{>0}(\overline{\beta}_0, \ldots, \overline{\beta}_i) \times \mathbb{R}_{\geq 0}^{g-i+1})_{0 \le i \le g}$ of $\Delta = \Delta_0$ is strictly decreasing.

Proposition 2.10. — *Let Δ_i be the smallest cone in $\mathcal{D}$ containing a characteristic vector of a wedge φ on $\widehat{S}$. Then one of the following two conditions holds :*

i) $\sigma_{i+1,1}$ contains at least one characteristic vector of φ and they all lie in the union of $\sigma_{i+1,1}$ and of $\sigma_{k,j}$, $1 \le k \le i$, $j = 1, 2$.

ii) The only characteristic vectors outside $\sigma_{i+1,2}$ lie on $\mathbb{R}_{>0}\varepsilon_k$ with $0 \le k \le i$ and $b_{i+1,k} = 0$.

Proof. — The explicit computations carried out in the proof of 1.3 have shown that

$$\mathring{\sigma}_{k+1,j} \subset \Delta_k \smallsetminus \Delta_{k+1}, \ \ 0 \le k < g, \ \ j = 1, 2.$$

We also have that $\mathbb{R}_{>0}\delta_k$ and $\mathbb{R}_{>0}\varepsilon_{k+1} \subset \Delta_k \smallsetminus \Delta_{k+1}$, $0 \le k < g$, and that $\sigma_{g+1,1} \subset \Delta_g$.

Since in addition we know that every $\alpha_p \in \Xi$, *i)* holds if $i = g$. Assume now that $i \neq g$. We deduce from the above remarks that every α_p belongs to $\bigcup_{1 \le k \le i+1; j=1,2} \sigma_{k,j}$ and no one is on $\mathbb{R}_{>0}\delta_{i+1}$. Now recall that

$$\sigma_{k,j} = \Big(\bigcap_{1 \le \ell < k} \sigma_{\psi_\ell^0} \Big) \cap \sigma_{\psi_k^j} \cap \Big(\bigcap_{k < \ell \le g} \sigma_{\psi_\ell^1} \Big) .$$

As a consequence every α_p belongs to $\sigma_{\psi_{i+1}^1} \cup \sigma_{\psi_{i+1}^2}$. In view of the definitions of the faces $\sigma_{\psi_{i+1}^1}, \sigma_{\psi_{i+1}^2}$ of the Newton polygon $\mathcal{N}_{i+1}$ of F_{i+1} (see prop. 1.3, *ii)* $\Longrightarrow$ *iii)*), this means that

$$(*) \qquad h^{i+1}(\alpha_p) = (\varepsilon_{i+2}, \alpha_p) = \min \big\{ n_{i+1}(\varepsilon_{i+1}, \alpha_p), \sum_{s=0,\ldots,i} b_{i+1,s}(\varepsilon_s, \alpha_p) \big\}$$
$$\le (m, \alpha_p) \ \text{ for any } \ m \in \mathcal{N}_{i+1} \smallsetminus \psi_{i+1}$$

Moreover if $\alpha_p \in \overset{\circ}{\Delta}$, this last inequality is strict because the face of $\mathcal{N}_{i+1}$ in the supporting hyperplane of $\mathcal{N}_{i+1}$ in the direction α_p is ψ^1_{i+1}, ψ^2_{i+1} or ψ_{i+1}. Applying φ to the identity expressing x_{i+2} as a polynomial in $x_0, \ldots, x_{i+1}$ (0.3.2), we find that there exist units U_0, U_1, U_γ in $k[[u_1, u_2]]$ such that

$$\prod_p p^{h^{i+1}(\alpha_p)} = U_1 \prod_p p^{n_{i+1}(\varepsilon_{i+1},\alpha_p)} - c_{i+1} U_0 \prod_p p^{\sum_s b_{i+1,s}(\varepsilon_s,\alpha_p)} - \sum_\gamma c_{i+1,\gamma} U_\gamma \prod_p p^{(\gamma,\alpha_p)}.$$

Since there exists at least one $\alpha_p \in \overset{\circ}{\Delta}$, the minimum in $(*)$ must be achieved on the same side by every α_p; in other words, either $\sigma_{\psi^1_{i+1}}$ or $\sigma_{\psi^2_{i+1}}$ contains all the characteristic vectors of φ. (This is what will be used to prove the main theorem in section 3.)

To complete the proof in the first case, note that any $\alpha_p \in \overset{\circ}{\Delta}$ belongs to $\overset{\circ}{\sigma}_{\psi^1_{i+1}}$ or $\overset{\circ}{\sigma}_{\psi_{i+1}}$. This forces such an α_p to be on $\mathbb{R}_{>0}\delta_k$, $1 \le k \le i$, in $\overset{\circ}{\sigma}_{i+1,1}$ or in $\bigcup_{1\le k\le i, j=1,2} \overset{\circ}{\sigma}_{k,j}$. Since $\varepsilon_{i+1} \notin \sigma_{\psi^1_{i+1}}$ and $\varepsilon_k \in \Delta_{i+1}$, $i+2 \le k \le g$, every α_p is found in the cones listed in *i)*. Moreover one of them lies in $\overset{\circ}{\sigma}_{i+1,1} \cup \mathbb{R}_{>0}\delta_i$, hence *i)*.

In the second case, any $\alpha_p \in \overset{\circ}{\Delta}$ belongs to $\overset{\circ}{\sigma}_{\psi^2_{i+1}}$ or $\overset{\circ}{\sigma}_{\psi_{i+1}}$. This forces α_p to be in $\overset{\circ}{\sigma}_{i+1,2}$. Finally for $0 \le k \le i+1$, $\varepsilon_k \in \sigma_{\psi^2_{i+1}}$ if and only if $k = i+1$ or $0 \le k \le i$ and $b_{i+1,k} = 0$, hence *ii)*. ■

Remark that if $i \ne g$, either *i)* or *ii)* holds. If $i = g$, the only possible characteristic vectors of φ in $\sigma_{g+1,2}$ lie on $\mathbb{R}_{>0}\delta_{g+1}$ and *ii)* is a subcase of *i)*.

3. Lifting wedges centered at a general arc to the minimal desingularization

We give a formal definition of the notion of a wedge being centered at an arc, outlined in the introduction.

Definition 3.1. — *We will say that a wedge φ is centered at an arc h if there exists a local morphism h_0 factoring h through φ.*

Geometrically, this means that the arc h on the surface singularity is the image of an arc in the plane through the origin, by the morphism given by φ.

We also make precise what we will mean by a general arc on a sandwiched surface singularity.

Definition 3.2. — *Let $\widehat{S}$ be a sandwiched surface singularity and h an arc on $\widehat{S}$. We will say that h is general if its strict transform on the minimal desingularization of $\widehat{S}$ is smooth and intersects transversally the exceptional curve E at a regular point of E.*

We are now ready to prove:

Theorem 3.3. — *Let φ be a wedge on a sandwiched surface singularity $\widehat{S}$ centered at a general arc h. Then the morphism* $\operatorname{Spec} k[[u_1, u_2]] \to \widehat{S}$ *induced by φ lifts to the minimal desingularization of $\widehat{S}$.*

Proof. — Assume first $\widehat{S}$ to be primitive. We choose the non singular surface $\mathfrak{X}_0$ and the simple ideal I as in 0.1. Moreover by changing $\mathfrak{X}_0$ and I if necessary, we may assume that the strict transform of the arc h on the minimal desingularization of $\widehat{S}$ does not meet the strict transform of the exceptional curve for the blowing-up of $\widehat{I} = I\mathcal{O}_{\widehat{\mathfrak{X}}_0}$. Indeed, we may replace $\mathfrak{X}_0$ by the affine space $\mathbb{A}^2_k$ and I accordingly; using plumbing, we may construct a non singular algebraic surface $\mathfrak{X}'$ containing a neighborhood of the exceptional curve E on the minimal desingularization $\mathfrak{X}$ of the surface S obtained by blowing-up I and a (-1)-curve intersecting E transversally at any given regular point of E on its irreducible component meeting the (-1)-curve on $\mathfrak{X}$.

For almost any minimal generating sequence $\{x_0, \ldots, x_{g+1}\}$ as in 0.3.2, h will be general w.r.t. $\{x_0, \ldots, x_{g+1}\}$. So we are again in the situation discussed in sections 1 and 2, and we will use the notation introduced then.

Let $h_0 : k[[u_1, u_2]] \to k[[t]]$ be the local morphism factoring φ. We have $\alpha_h = \sum_p \omega_t(p)\alpha_p$ with $\omega_t = \operatorname{ord}_t h_0(p) \in \mathbb{Z}_{>0}$. We will see that φ has only one characteristic vector α_p and that $\omega_t(p) = 1$; in view of 2.7, this is enough to get the claim.

Let Δ_i, $0 \le i \le g$, be the smallest cone in the filtration $\mathcal{D}$ containing a characteristic vector of φ as in 2.10. If *i)* holds, then for any p, $\alpha_{\varphi,p} \in \sigma_{\psi^1_{i+1}}$. Being a convex cone, we get that

$$\alpha_h \in \sigma_{\psi^1_{i+1}} = \Big\{ n = (\alpha_0, \ldots, \alpha_{g+1}) \in \Delta ; \alpha_{i+2} = n_{i+1}\alpha_{i+1} \le (m, \alpha),\ \forall m \in \mathcal{N}_{i+1} \Big\}.$$

But $\alpha_h \in \mathring{\Delta}$, so either $\alpha_h \in \mathring{\sigma}_{\psi^1_{i+1}}$ or $\alpha_h \in \mathring{\sigma}_{\psi_{i+1}}$. As in 2.10, this forces α_h to be either in $\sigma_{i+1,1}$ or in $\bigcup_{1 \le k \le i, j=1,2} \sigma_{k,j}$; hence by 2.5 to be in the minimal generating system of one of these cones.

Now the coordinate on the X_0-axis of any $\alpha \in \mathring{\Delta}$ in the minimal generating system $G_{\sigma_{\ell,j}}$ of a cone $\sigma_{\ell,j}$ with $1 \le \ell \le g+1$, $j = 1, 2$, is bounded below by $n_1 \cdots n_{\ell-1}$ (1 if $\ell = 1$), and above by $n_1 \cdots n_\ell$ with equality if and only if $\alpha \in \mathbb{R}_{>0}\delta_\ell$. (This follows easily from the explicit isomorphism between $\sigma_{\ell,j}$ and $\tilde{\sigma}_{\ell,j}$ given in 1.6 *iii)*.) So φ having at least one characteristic vector in $\sigma_{i+1,1}$, we find that $\alpha_h \in G_{\sigma_{i+1,1}}$.

On the other hand, we derive from condition *i)* of 2.10 that $\alpha_h = \alpha_1 + \alpha_2$ with $\alpha_1 \in \sigma_{i+1,1}$, $\alpha_1 \neq 0$ and $\alpha_2 \in \langle \varepsilon_0, \dots, \varepsilon_i, \delta_i \rangle$. Therefore α_2 lies in the intersection of $\langle \varepsilon_0, \dots, \varepsilon_i, \delta_i \rangle$ with the plane generated by $\sigma_{i+1,1}$. We have named this cone τ_{i+1} in the proof of 1.10.2. Furthermore, we have seen there that, in a convenient $\mathbb{Z}$-basis of the lattice $N_{\sigma_{i+1,1}}$, $\sigma_{i+1,1}$ and τ_{i+1} are respectively the cones $\langle (1,0), (e_i, \overline{\beta}_{i+1} - n_i\overline{\beta}_i) \rangle$ and $\langle (1,0), (e_i, -n_i\overline{\beta}_i) \rangle$. So $\sigma_{i+1,1} \cup \tau_{i+1}$ is again a cone; obviously $(1,0)$ belongs to its minimal generating system G. From its characterization by convexity properties, we conclude that G is the union of $G_{\sigma_{i+1,1}}$ and $G_{\tau_{i+1}}$. Therefore $\alpha_2 = 0$, and as expected there exists a unique $\alpha_{\varphi,p} \neq 0$ with $\alpha_h = \alpha_{\varphi,p}$.

Assume now that *ii)* in 2.10 holds. We may assume that $i < g$, For any p, $\alpha_{\varphi,p} \in \sigma_{\psi^2_{i+1}}$, so $\alpha_h \in \sigma_{\psi^2_{i+1}}$. As above, since $\alpha_h \in \mathring{\Delta}$, either $\alpha_h \in \mathring{\sigma}_{\psi^2_{i+1}}$ or $\alpha_h \in \mathring{\sigma}_{\psi_{i+1}}$. In any case, α_h lies in $\sigma_{i+1,2}$ and by 2.5, $\alpha_h \in G_{\sigma_{i+1,2}}$. On the other hand, we derive from condition *ii)* of 2.10 that $\alpha_h = \alpha_1 + \alpha_2$ with $\alpha_1 \in \sigma_{i+1,2}$ and $\alpha_2 \in \langle \varepsilon_0, \dots, \varepsilon_i \rangle$. Therefore α_2 lies in the intersection of $\langle \varepsilon_0, \dots, \varepsilon_i \rangle$ with the plane generated by δ_{i+1} and ε_{i+1}; since $i + 1 < g + 1$, its coordinate on the X_{g+1}-axis may not be zero unless $\alpha_2 = 0$. Once again there exists a unique $\alpha_{\varphi,p} \neq 0$ with $\alpha_h = \alpha_{\varphi,p}$.

In the general case, the non singular surface $\mathfrak{X}_0$ and the complete ideal I being chosen as in 0.1, let $p_0 : S \to \mathfrak{X}_0$ (resp. $q : \mathfrak{X} \to S$) be the blowing-up of I (resp. the minimal desingularization of S). The irreducible exceptional curves $(F_i)_{i\in\Lambda}$ for p_0 are in 1-1 correspondence with the simple factors I_i of I and our assumption implies that for any $i \in \Lambda$, the unique singular point P of S lies on F_i and that its strict transform E_i on $\mathfrak{X}$ is a (-1)-curve. We have a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{X} & \xrightarrow{\ q\ } & S & \xrightarrow{\ p_0\ } & \mathfrak{X}_0 \\
{\scriptstyle g_i}\downarrow & & {\scriptstyle f_i}\downarrow & \nearrow{\scriptstyle p_{0i}} & \\
\mathfrak{X}_i & \xrightarrow[\ q_i\ ]{} & S_i & &
\end{array}$$

where p_{0i} (resp. q_i) is the blowing-up of I_i (resp. the minimal desingularization of S_i). The curves F_i and E_i project respectively onto $p_{0i}^{-1}(O)$ on S_i and the unique (-1)-curve on $\mathfrak{X}_i$. Since $g_i : \mathfrak{X} \to \mathfrak{X}_i$ is a composition of point blowing-ups and both curves E_i and $g_i(E_i)$ have self-intersection -1, g_i is a local isomorphism at any point of E_i, hence $P_i := f_i(P)$ is the unique singular point on S_i.

Consider now a regular point Q of $q^{-1}(P)$; there exists a unique irreducible curve $E \subset q^{-1}(P)$ containing Q. Two things may happen:

— either Q is the intersection point of E with some E_i; hence g_i is a local isomorphism at Q and $Q_i := g_i(Q)$ is a regular point of $q_i^{-1}(P_i)$.

— or Q is a regular point of $p^{-1}(O)$ with $p := p_0 \circ q$. The morphisms p and $p_i := p_{0i} \circ q_i$ are again composition of point blowing-ups. Now, I being the product of powers of I_i, I is made invertible as soon as each I_i is so. Hence there exists i, not necessarily unique, such that E projects onto an irreducible exceptional curve for p_i. Being contracted on P_i by q_i, this curve is also exceptional for q_i. Here also, g_i is a local isomorphism at Q, which sends Q on a regular point Q_i of $q_i^{-1}(P_i)$.

In both cases, a wedge on $\widehat{S}$ centered at a general arc gives rise by composition to a wedge centered at a general arc on the formal neighborhood $\widehat{S_i}$ of S_i at P_i, which is primitive. This reduces the proof of the theorem for arbitrary sandwiched singularities to the special case of primitive ones. ∎

References

[Ab] Abhyankar S.S. — *Irreducibility Criterion for Germs of Analytic Functions of Two Complex Variables*, Advances in Maths **74** (1989), 190–257.

[G.L1] Gonzalez-Sprinberg G., Lejeune-Jalabert M. — *Modèles canoniques plongés I*, Kodai Math. Journal **14** (2) (1991), 194–209.

[G.L2] Gonzalez-Sprinberg G., Lejeune-Jalabert M. — *Sur l'espace des courbes tracées sur une singularité*, Progress in Math **134** (1996), 8–32.

[G-S] Gonzalez-Sprinberg G. — *Éventails en dimension 2 et transformé de Nash*, Publications du Centre de Mathématiques de l'E.N.S., 45, rue d'Ulm, Paris 5e, 1977.

[G.T] Goldin R. and Teissier B. — *Resolving singularities of plane analytic branches with one toric morphism*, Preprint DMI, E.N.S., 45 rue d'Ulm, Paris 5e, 1996.

[T.E] Kempf G., Knudsen F., Mumford D and Saint Donat B. — *Toroidal embeddings I*, Lecture notes in Math., Springer, **339**, 1973.

[Kh] Khovanskii A.G. — *Newton polyhedra and toroidal varieties*, Funct. Anal. and its appl. **11** (4) (1977), 56–64, English translation 289–296.

[L-J] Lejeune-Jalabert M. — *Arcs analytiques et résolution minimale des singularités des surfaces quasi-homogènes*, Lecture notes in Math., Springer **777** (1980), 303–336.

[L.R] Lejeune-Jalabert M., Reguera A. — *Arcs and wedges on sandwiched surface singularities*, to appear at C. R. Acad. Sci. Paris Sér. I Math..

[N] Nash J. — *Arc structure of singularities*, Duke Math. Journal **81** (1) (1995), 31–38.

[Oda] Oda T. — *Convex bodies and algebraic geometry*, Springer-Verlag, Ergeb. der Math., 3. Folge. Band, **15**, 1988.

[Sp1] Spivakovsky M. — *Valuations in function fields of surface*, Amer. J. Math. **112** (1990), 107–156.

[Sp2] Spivakovsky M. — *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Annals of Maths **131** (1990), 411–491.

[Z] Zariski O. — *Le problème des modules pour les branches planes*, Hermann, Paris, 1986.

[Z.S] Zariski O., Samuel P. — *Commutative algebra II, App. 5*, Van Nostrand Princeton, 1960.

Monique LEJEUNE-JALABERT
CNRS, Institut Fourier,
Université de Grenoble I
UMR 5582 CNRS-UJF
38402 Saint-Martin-d'Hères (France)
E-mail : lejeune@fourier.ujf-grenoble.fr

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Ana REGUERA
Dep. de Algebra,
Geometría y Topología
Univ. de Valladolid
47005 Valladolid (España)
E-mail : areguera@cpd.uva.es