Cubulations of compact manifolds and a problem of Habegger

Louis FUNAR
Institut Fourier, BP 74, Univ.Grenoble I, Mathématiques,
38402 Saint-Martin-d'Hères cedex, France,
e-mail: funar@fourier.ujf-grenoble.fr [1]

Abstract

The aim of this paper is to consider the set of cubical decompositions of the compact manifolds mod a set of combinatorial moves analog to the bistellar moves considered by Pachner. We prove that, in general there are obstructions for two cubulations of the same PL-manifold be related by such moves, which we called bump moves, answering in the negative a question of Habegger. Therefore we consider the quotient set of cubulations for a sphere and prove that it inherits a natural group structure. On the other side, we prove that, when restricting to a special class of cubulations, called mappable, these moves act transitively on the set of all cubulations. The mappable cubulations are those which map combinatorially on the standard decomposition of $\mathbf{R}^n$ for some large enough n. Finally a detailed description of the 2-dimensional cubulation groups is given.

AMS MOS Subj.Classification(1991): 57 N 15, 57 R 45, 58 E 05.
Keywords and phrases: Cubulation, bump/bubble move, f-vector, mappable, embeddable, simple, standard, cubulation group.

Contents

[1] First version: December 20, 1996. This version: October 7, 1997. This preprint is available electronically at http://www-fourier.ujf-grenoble.fr/~funar

1 Introduction

1.1 Outline

In the twenties Alexander proved that two triangulations of the same polyhedra (or equivalently of two PL-homeomorphic manifolds) are related by a set of combinatorial moves, called stellar moves. After seventy years Alexander's moves were refined to a set of finite local moves on the triangulations of manifolds which were used to prove that certain state-sums associated to a triangulation provide topological invariants of 3-manifolds, the so-called Turaev-Viro invariants. The new moves are the so-called bistellar moves and Pachner ([37]) has proved that they relate any two triangulations of the same polyhedra so settling a long standing conjecture in combinatorial topology. Basically such a move in dimension n corresponds to replace a ball B by another ball B', where B and B' are complementary balls, unions of simplexes in the boundary of the standard $(n+1)$-simplex. For a nice exposition of Pachner's result and various extensions, see [31].

On the other hand Alexander's theorem becomes trivial in the context of some more general cell decompositions ("cellulation régulière") considered by Siebenmann [38] where the analog moves are called "bisections". Here the cells are convex subsets in some Euclidean space, with an arbitrary number of vertices.

When restricting ourselves to the 3-dimensional context we notice that the topological invariants of Turaev-Viro, deduced from triangulations, are carrying slightly less information than their Reshetikhin-Turaev counterpart, which are defined using some Dehn surgery presentation of the manifolds. Actually the latter has a strong 4-dimensional flavor, as explained by the theory of shadows developed by Turaev (see [42]).

In this setting it is naturally to ask whether some other intrinsic definitions of state-sum invariants would get a refined version of the Turaev-Viro invariants, containing also the phase factor. It is naturally then to look towards some more general decompositions of manifolds into standard pieces, which encodes in their combinatorics more information. The first example we sought is the decompositions into cubes, also known under the names of cubulations, cubications, cubillages, etc. In order to apply the state-sum machinery to these decompositions we have to search for an analog of the Alexander's theorem (or Pachner's theorem) to cubulations. There is a natural candidate for the bistellar moves, which was considered by Habegger. Specifically, the problem 5.13 from Kirby's book ([26]), proposed by Habegger states that:

Problem 1 *Suppose M and N are PL-homeomorphic cubulated n-manifolds. Are they related by the following set of moves: excise B and replace it by B', where B and B' are complementary balls (union of n-cubes which are homeomorphic to B^n) in the boundary of the standard $(n+1)$-cube?*

These moves will be called *bump* moves in the sequel. A complete list of bump moves will be given in section 2. A more restrictive family of moves is obtained by requiring that one of the balls B or B' have no parallel faces. We call them *bubble* moves. There are $n+1$ distinct bubble moves b_k, $k = 1, 2, ..., n+1$ and their inverses; here b_k replace B which is the union of exactly k cubes by B' which is the union of $2(n+1)-k$ cubes (the $(n+1)$-cube has $2(n+1)$ faces). We drawn in picture 1 the bubble moves for $n=1$ and for $n=2$. In case $n=1$ the move b_2 is the identity, and it is not figured. The last bump move, denoted by $b_{3,1}$, is also figured in picture 1.

Set $C(M)$ for the the set of cubulations of a closed manifold M, $CBB(M)$ for the equivalence classes of cubulations modulo bubble moves and $CB(M)$ for the equivalence classes of cubulations modulo bump moves. The case when M has non-void boundary needs some additional conditions. For example two simplicial triangulations of a manifold with boundary are bistellar equivalent if the restrictions of the two triangulations at the boundary coincide (see [8]). We restrict ourselves in this sequel to the case without boundary.

Look first at the case $n=1$. The move b_1 is dividing an edge into three other. It follows that the number of edges in the cubulation, when considered modulo 2, is invariant with respect to the bubble moves. Conversely if we have a cubulated circle, we can use b_1^{-1} to obtain either a bigon or else a 1-edge cubulation. This means that $CB(\bigsqcup_n S^1) = CBB(\bigsqcup_n S^1) = (\mathbf{Z}/2\mathbf{Z})^n$, where n is the number of

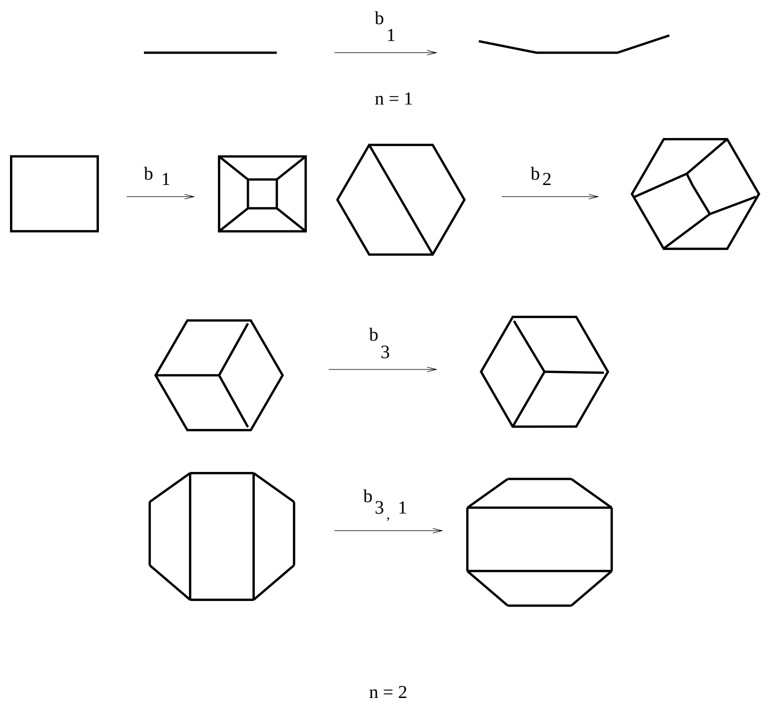


Figure 1: Bubble moves for $n = 1$ and $n = 2$

components. We observe that we have non-trivial obstructions for two cubulations be bubble (bump) equivalent. Assume from now on, that the manifolds considered are connected unless the contrary will be specified.

In the following section we will describe the analog obstructions in higher dimensions. These are defined in an elementary way: we look for the transforms of the f-vector under bubble/bump moves. There are further some other arithmetic obstructions coming from the fact that two f-vectors can be transformed one into the other by bubble/bump moves if and only if some linear system of equations has integers solutions. All these together define a finite Abelian group of obstructions.

Notice that there exist some other obstructions as the index of non-simplicity, which has a geometric definition.

We did not find a good invariant using the state-sum method already used in quantum topology. Maybe here it is much more complicated than the case of triangulations, where only topological invariants can appear.

This numerical invariant shows that the number of obstructions is infinite. And this suggests that the study of bubble equivalent cubulations should be restricted (with respect to the the problem above) to the class of simple, standard cubulations. A smaller but related class is that of embeddable cubulations.

Let us explain the motivation for looking to embeddable cubulations: the index of non-simplicity is related by the results of [16] to embeddability and mappability. On the other hand, the most natural way to get an affirmative answer in a restricted case using the geometry, would be the following: to each triangulation C (or cubulation) a Morse function f_C (or a class of a Morse function, up to isotopy) is associated. For instance we could ask for the critical points of index k of the function f_C be precisely the barycenters of the k-cells. Then, in a suitable space of functions D, larger than the space of Morse function, (in order to become path connected) we get a path f_t between the two functions f_0 and f_1 associated to two cell decompositions C_0 and C_1, which we wish to compare to each other. We would like that the space of Morse functions arising from triangulations, form an open dense subset of D. Also by general position (Cerf's theory) arguments the complementary (of positive codimension) would be made transverse to the path f_t, by a small homotopy of the later. Around a critical value t_c of the parameter (i.e. f_{t_c} is not a Morse function) we follow how the cell decompositions changes, when passing from $f_{t_c-\varepsilon}$ to $f_{t_c+\varepsilon}$. This idea of associating spaces of functions to topological objects and to study their discriminant has been revealed very powerful in the theory of singularities developed by Arnold as well as the recent Vassiliev-type invariants for knots, planar immersions etc. (see for a very interesting account the lectures given in [1]).

However if for triangulations such an approach could succeed, for cubulations it definitely fails. The reason is that the Morse functions associated to cubulations in the space of all Morse functions cannot be generic: small deformations change the cubical decomposition into a more general cell decomposition.

Another way to overcome this difficulty would be to use embedded Morse theory. Equivalently we look at manifolds as sub-manifolds of the Euclidean space. Morse functions correspond to hypersurface levels having only few singularities. The simplest case is the behavior with respect to pencils of hyperplanes. In a discrete model this is equivalent to an embedding of the given cubulation into the standard cubulation of some Euclidean space. In this context we will use the idea from above: each manifold has a suitable approximation by a cubical structure coming from a sufficiently fine lattice. When an isotopy is perturbing the manifold the approximation is locally constant unless for a finite number of critical values of the parameter, when a jump occurs. The jumps will be precisely bump moves. This way we give a partial affirmative answer for the problem we stated at the beginning: two embeddable cubulations C, C' are bump equivalent. Notice that using bubble/bump moves we could exit the class of embeddable cubulations. A larger class is that of mappable cubulations: we don't require that the cubulation be embedded but only combinatorially mapped on an Euclidean cubulation. Then our main result below states that two mappable cubulations are bump equivalent.

Acknowledgements: Part of this work was done during the author's visit at University of Palermo and University of Columbia, whose support and hospitality are gratefully acknowledged. I'm thankful to Sergei Matveev for the discussions we had on this subject and the suggestions he gave me, Maria-Rita Casali for sending me her paper [8], Alexis Marin for a careful reading of the manuscript and correcting some mistakes, to Eric Babson and Clara Chan for sending me their paper [3] and to Joan Birman for her interest.

1.2 Elementary obstructions

We want to identify obstructions, similar to that for the 1-dimensional case, in higher dimensions: set for $x \in C(M)$, and M of dimension n, $f_i(x) = card\{$ the set of i-dimensional cubes in $x\}$. Putting all these together we get a map $f : C(M) \longrightarrow \mathbf{Z}^{n+1}$ whose components are f_i, and it is usually called the f-vector in the theory of polytopes.

Proposition 1.1 *There exist some natural numbers $a_i(n) \in \mathbf{Z}_+$ such that:*

- *All the $a_i(n)$ are non-trivial, and divisible by 2.*
- *The map f induces a well-defined map $fb : CBB(M) \longrightarrow \prod_{i=0}^{n} \mathbf{Z}/a_i(n)\mathbf{Z}$ by*

$$fb(x) = (f_i(x)(modulo\ a_i(n))_{i=0,1,\ldots,n}.$$

- *The greatest such numbers $a_i(n)$ verify*

$$a_n(n) = 2, a_{n-1}(n) = 2n, a_{n-2}(n) = 2, a_0(n) = 2, a_1(n) = 3 + (-1)^n, (n > 2)$$

This means that we have arithmetical obstructions for two cubulations be bubble equivalent as in the 1-dimensional situation. We compute easily for $n = 2$ and $n = 3$ the vector $\mathbf{a}(n)$ whose components are $a_i(n)$: $\mathbf{a}(2) = (2, 4, 2)$, and $\mathbf{a}(3) = (2, 2, 6, 2)$. After some messy computations we obtain $\mathbf{a}(4) = (2, 4, 2, 8, 2)$, $\mathbf{a}(5) = (2, 2, 4, 2, 10, 2)$, and $a_2(6) = 6$. We denote by $fb^{(2)}$ the values of fb reduced modulo $(2, 2, 2, ..., 2, 2n, 2)$.

Next we can extend this result on obstructions to bump moves:

Proposition 1.2 *There exist some natural numbers $\tilde{a}_i(n) \in \mathbf{Z}_+$ such that:*

- *All the $\tilde{a}_i(n)$ are non-trivial, and divisible by 2.*
- *The map f induces a well-defined map $fb : CB(M) \longrightarrow \prod_{i=0}^{n} \mathbf{Z}/\tilde{a}_i(n)\mathbf{Z}$ by*

$$fb(x) = (f_i(x)(\text{modulo } \tilde{a}_i(n))_{i=0,1,\ldots,n}.$$

- *The greatest such numbers $\tilde{a}_i(n)$ verify*

$$\tilde{a}_n(n) = 2, \tilde{a}_{n-1}(n) = 2n, \tilde{a}_{n-2}(n) = 2, \tilde{a}_0(n) = 2, \tilde{a}_1(n) = 3 + (-1)^n, (n > 2)$$

The two sequences a_j and $\tilde{a}_j$ are not identical since we have:

$$\tilde{a}_2(6) = 2 \neq a_2(6),\ \tilde{a}_2(4) = 2 \neq a_2(4).$$

Notice that $f_{n-1}(x) = nf_n(x)$ for all x. This means that the image of fb_{n-1} is $\mathbf{Z}/2\mathbf{Z} \cong \{0, n\} \subset \mathbf{Z}/2n\mathbf{Z}$. By the way the component fb_{n-1} is determined by fb_n.

Remark also that the Euler-Poincaré equation holds:

$$\chi(M) = \sum_{i=0}^{n} (-1)^i f_i(x), \text{ for any cubulation } x.$$

Thus, fixing M, there at most $(n-1)$ independent components of fb: for $n = 2$ everything is determined by the number of vertices, for $n = 3$ we have the number of vertices and the number of edges etc.

The most natural question now would be to know which are the images $fb(CB(M))$, $fbb(CBB(M))$, or at least their mod 2 reduction. This will be necessary, in order to know how powerful these invariants are. We are far from having a complete answer now, and this problem is more difficult than it seems at first glance. There are some partial results for the case of the mod 2 reductions $fb^{(2)}(CB(M))$ and $fbb^{(2)}(CBB(M))$. Actually this is equivalent to characterize those f-vectors mod 2 which can be realized by cubulations of the manifold M. Obviously there are constraints for the existence of a simplicial polyhedra with a given f-vector and fixed topological type. For convex simplicial polytopes we have for instance the McMullen conditions (conjectured by McMullen in [32] and proved in [6, 7, 39]; the reader may consult also other proofs and results in [32, 5, 33, 34]). The complete characterization of the f-vectors of simplicial polytopes (and PL-spheres) was obtained by Stanley in [40]. The analogous problem of the realization of f-vectors by cubical polytopes has also been addressed in some recent papers, for example [4, 3, 23, 24] and references therein. The Dehn-Sommerville equations have a counterpart for cubical polytopes as in [21]. The lower bound conjecture and the upper bound conjecture have also analogous statements in the cubical case. The new feature is that, unlike in the simplicial case, there are parity restrictions on the f-vectors. This was firstly observed in [4]. Remark that it is exactly these restrictions in which we are interested. We have, as a simple application of the Dehn-Sommerville equation, a first constraint on the mod 2 image:

Proposition 1.3 *The rank of the affine module rank $fb^{(2)}(CB(M)) \leq \left[\frac{n+1}{2}\right]$.*

The relationship between cubical PL n-spheres and the immersions was described in the following beautiful result of Babson and Chan (see [3]):

Proposition 1.4 *Let $\varphi : M \longrightarrow S^n$ a codimension 1 normal crossing immersion. Then there exists a PL cubical n-sphere K, such that*

$$f_i(K) = \chi(X_i(M, \varphi))(\text{mod } 2),$$

where χ denotes the Euler characteristics, and $X_i(M, \varphi) = \{x \in S^n;\ \text{card } \varphi^{-1}(x) = i\}$.

As a consequence, there exists a PL cubical n-sphere K with given $f_i(\text{mod } 2)$ if and only if there exists a codimension 1 immersion (M, φ) in S^n, for which the Euler characteristic of the multiple point loci $X_i(M, \varphi)$ of degree i equals f_i mod 2.

Remark that this result extends immediately to other varieties than the spheres. We have only to consider immersions $\varphi : M \longrightarrow N$ such that the image $\varphi(M)$ is a spine of N, which means that $N - \varphi(M)$ is an union of balls.

There is a wide literature on immersions, and especially on the following function θ_n, considered first by Freedman ([20]), where $\theta_n(\varphi)$ is the number of multiple n-points mod 2. The beginning of this theory was the result of Banchoff [2] saying that the number of normally triple points of a closed surface immersed in $\mathbf{R}^3$ is congruent mod 2 with its Euler characteristic. The function θ_n is easily seen to be well-defined as a function on the Abelian group B_n of bordism classes of immersions of $(n-1)$-manifolds in S^n. We have therefore an induced homomorphism:

$$\theta_n : B_n \longrightarrow \mathbf{Z}/2\mathbf{Z}.$$

Remark that the question on whether θ_n is surjective (i.e. nontrivial) is equivalent to find the image of $fb^{(2)}_{n-1}(S^n)$. From the results concerning the function θ_n obtained in [20, 17, 18, 19, 22, 28, 29, 30, 9, 10, 11] we deduce that the f-vectors of a n-sphere have the following properties (see also [3]):

- For $n = 2$ we have $f_0 = f_2 (mod\, 2)$ and $f_1 = 0 (mod\, 2)$ and thus $fb^{(2)}(CB(S^2)) = fb^{(2)}(CBB(S^2)) = \mathbf{Z}/2\mathbf{Z}$.
- For $n = 3$, $f_0 = f_1 = 0(mod\, 2)$, $f_2 = f_3 (mod\, 2)$. The existence of Boy's immersion $j : \mathbf{R}P^2 \longrightarrow S^2$, with a single degree 3 intersection point furnish a 3-sphere with an odd number of facets. Therefore $fb^{(2)}(CB(S^3)) = fb^{(2)}(CBB(S^3)) = \mathbf{Z}/2\mathbf{Z}$.
- The problem of characterizing the image $fb^{(2)}_{n-1}(S^n)$ is reduced to a homotopy problem. Namely, the image is $\mathbf{Z}/2\mathbf{Z}$ if and only if
  - either n is $1, 3, 4$ or 7.
  - or else $n = 2^a - 2$, with $a \in \mathbf{Z}_+$, and there exists a framed n-manifold with Kervaire invariant 1. The latter is known to be true for $n = 2, 6, 14, 30, 62$.
- If we consider only the class of edge orientable cubulations in the sense of [23] the problem of characterizing the image $fb^{(2)}_{n-1}(S^n)$ is also reduced to a homotopy problem, which is completely solved. In fact the condition of edge orientability is equivalent to ask that the associated manifold M immersed in S^n be orientable. Thus we have to consider only the restriction of the map θ_n at the subgroup of oriented bordism classes of immersions, as originally considered by Freedman [20]. Away from the trivial cases $n = 1, 2$ the only case when the restriction of θ_n remains surjective in the orientable context is $n = 4$. Thus $f_{n-1} = 0(mod\, 2)$ if $n \neq 1, 2, 4$.

Thus, at this time only a finite number of n is known, for which the last component of $fb^{(2)}$ is nontrivial. However the remarks from above show that the bump/bubble invariants fb and fbb are interesting and nontrivial. One may also wonder if these are all the obstructions for bump/bubble equivalence. There are some other arithmetical obstructions coming from the linear equations satisfied by the two f-vectors must satisfy if the polyhedra are related by a sequence of bump/bubble moves. Basically these amount to a series of additional congruences, which are rather complicated to write down explicitly, but they can be worked out in concrete examples. This will be briefly explained in 2.3.

On the other side there are another, more subtle, obstructions taking values in $\mathbf{Z}$ and showing that there are infinitely many of bubble non equivalent cubulations with the same f-vectors mod 2. Such an invariant is described for $n = 2$ in section 5.

1.3 The 2-dimensional case

The *derivative complex* of a cubulation M was introduced abstractly in [3]. We only say now that it is endowed with an immersion into M, whose image is the $(n-1)$ skeleton of the dual of the cubical structure M. In the 2-dimensional case we have therefore an union of circles K_i immersed in M, with normal crossings.

An easy observation is that

Proposition 1.5 *The collection of homology/homotopy classes of the circles K_i is bubble invariant.*

Notice that we have a collection, and not a set, since some elements can be multiple, i.e. can appear several times in the collection. It follows, as a simple corollary, that $CBB(M)$ is infinite provided that the genus of M is at least 1. This generalizes also in higher dimensions for manifolds with infinite $\pi_1(M)$. As a simple consequence we derive a strong rigidity of the bubble equivalence, which makes it unsuitable for a complete classification: consider an embeddable cubulation M, and λM be a homothetic copy embedded in $\mathbf{R}^N$, with integer λ. Then, if $\pi_1(M) \neq 0$, there exists at least one non-trivial homotopy class among the collection of K_i. However the homothetic cubulation λM will contain λ times more the same homotopy classes, so λM and M cannot be bubble equivalent. In particular the theorem (see the next section) on the bump equivalence of mappable cubulations cannot be improved to bubble equivalence, in general. However it is possible that two mappable cubulations of the sphere are bubble equivalent. We will prove that this holds in dimension 2. We say that a two dimensional cubulation is simple if the circles K_i are individually embedded in the respective surface. It is known that for the cubulations of S^2 the simplicity is equivalent to mappable. A cubulation is called now semi-simple if each image circle $\varphi(K_i)$ has an even number of double points, which form cancelling pairs. A cancelling pair is a set of two distinct double points connected by two distinct and disjoint arcs.

The main result of this section is the characterization of $CB(S^2)$.

Theorem 1.6 *The bubble moves act transitively on the set of simple cubulations of S^2. The orbit of the cubulation ∂C^{n+1} is the set of semi-simple cubulations.*

The map $fb^{(2)}$ (or equivalently F_b) is an isomorphism between $CB(S^2)$ and $\mathbf{Z}/2\mathbf{Z}$.

The proof is given in section 3. It is very tempting to claim that $fb^{(2)}$ is an isomorphism between $CB(M)$ and $\mathbf{Z}/2\mathbf{Z}$ for a general surface M.

1.4 Embeddable and mappable cubulations

Cubical complexes, as objects of study from a topological point of view, were introduced by Novikov ([35], p.42). One of the first question was to answer whenever a cubical complex of dimension n embeds (or can be mapped to) the n-skeleton of the standard cubic lattice of dimension N? These will be called *embeddable* and respectively *mappable* cubulations. By the standard cubic lattice (or the standard cubical decomposition) is meant the usual partition of $\mathbf{R}^N$ into cubes whose vertices are identified to $\mathbf{Z}^N$. When speaking of the decomposition into cubes we specify $\mathbf{R}^N_c$, and when speaking of the whole cubic complex we use the lattice terminology, so we can specify the k-dimensional skeleton etc. We will use both terms in the sequel.

Our main result (see also the theorems 4.1 and 4.7) is the following:

Theorem 1.7 *Two mappable cubulations M_1 and M_2 of a DIFF manifold M are bump equivalent.*

The proof will be given in section 4.

In order to have a deeper understanding of this results we should know how far are these embeddable/mappable cubulations from arbitrary cubulations, and what single them out.

Let us say that a cubulation is *simple* if any circuit in which consecutive points correspond to edges which are opposite sides of some square of the cubulation does not contain two orthogonal edges from the same cube. The cubulation is *standard* if any two cubes (of dimensions running from 0 to the top dimension) of the cubulation are either disjoint, or have exactly one common face. An immediate observation is that embeddable cubulations are standard and simple and mappable cubulations are simple. On the other hand the simplicity is very close to the mappability, at least for manifolds with small fundamental group. We have for instance the following results of Karalashvili ([25]) and Dolbilin, Shtanko and Shtogrin ([15]) stating that:

- The double of a simple cubulation is mappable. Recall that the double of a cubulation is the result of dividing each k-dimensional cube in 2^k equal cubes.
- If the manifold M underlying the simple cubulation has the fundamental group $\pi_1(M)$, which does not admit nontrivial homomorphisms into a free Abelian group, then the cubulation is mappable. This was strenghted in [15], where the mappability is obtained only under the condition that the homology group $H_1(M, \mathbf{Z}/2\mathbf{Z}) = 0$, or the cohomology group $H^1(M, \mathbf{Z}/2\mathbf{Z}) = 0$.
- A simplicial decomposition S can be subdivided in a standard manner into a cubical decomposition $C(S)$: divide any n-simplex into $n+1$ cubes of the same dimension. Then, it is shown in [15] that the associated cubical decomposition $C(S)$ is always embeddable.

There is then a close connection between triangulations and cubulations, between Pachner's result and the theorem from above. The last remark shows that the cubulations of type $C(S)$ associated to simplicial complexes are always bump equivalent. It would be interesting to find a direct combinatorial proof for this corollary of the theorem. Remark that it fits well with Pachner's theorem. We think that conversely, an indirect proof for Pachner's theorem may be derived from these results, in the case of DIFF manifolds.

We already saw that M and N mappable implies N is bump equivalent to M. In particular, for cubulations of the sphere two simple cubulations are bump equivalent. Notice however that the set of simple (or mappable) cubulations is not closed to arbitrary bump moves. In general the simplicity is not preserved even by the bubble moves: it may be destroyed using the move b_2.

However it should be stressed that the bubble equivalence is interesting only for the PL-spheres. We think that any two simple cubulations of the sphere are bubble equivalent, likewise to the two dimensional case. However in the presence of non-trivial topology, the mappability alone does not suffice for the bubble equivalence. This is explained in the last part of the section 4.

1.5 Cubulation groups

We have previously seen that the obstructions induced by fb are lying in some finite Abelian groups which are rather complicated to find out explicitly. However we are able to prove now that the obstruction map fb factors through a product of $\mathbf{Z}/2\mathbf{Z}$'s. This makes sense because all $\tilde{a}_i(n)$ are even. This will be derived as a corollary of the following:

Theorem 1.8 *The set $CB(S^n)$ has a natural group structure induced by the connected sum of cubulations. Moreover $CB(S^n)$ is a direct product of $\mathbf{Z}/2\mathbf{Z}$'s.*

The proof will be given in section 4.

Therefore each element of the cubulation group has order 2. On the other hand it is not difficult to see that fb becomes a group homomorphism. Therefore the congruence

$$2f_i(x) = 0(\text{ modulo } \tilde{a}_i(n))$$

holds for all i, n. We derive that the true obstructions are in $\mathbf{Z}/2\mathbf{Z}$:

Corollary 1.9 *The formula*

$$F_b(x) = (2f_i(x)/\tilde{a}_i(n)(\textit{modulo } 2))_{i=0,1,\ldots,n}$$

defines a well-defined map

$$F_b : CB(S^n) \longrightarrow (\mathbf{Z}/2\mathbf{Z})^{n+1}.$$

Now, a characterization of the image $F_b(CB(S^n))$ is an useful intermediate step towards the computation of $CB(S^n)$. We have an immediate result modeled on proposition 1.3:

Corollary 1.10 *For all $n \geq 1$ we have rank $F_b(CB(M)) \leq \left[\frac{n+1}{2}\right]$.*

Proof : Set $f_i = f_i' a_i(n)/2$ and introduce the new variables f_i' in the Dehn-Sommerville equations. We can therefore reduce mod 2 all the linear equations in f_i'. □
On the other hand notice that we already obtained some new obstructions for the f-vector of a cubical n-spheres:

Corollary 1.11 *The f vector of a PL n-sphere satisfies:*

$$2f_i(x) = 0(mod\, \tilde{a}_i(n)),$$

which are meaningful only for those i, n, for which $\tilde{a}_i(n) > 2$.

Remark that the corollary 3.2 of [3] states that the only nontrivial modular relations which hold among the f vectors of all cubical polytopes (and hence spheres) are modulo 2. This agrees with the previous corollary, since the order of the image of $f_i(mod\, \tilde{a}_i(n))$ is in fact 2. Thus it seems natural to ask whether there are some other numbers $b_L(n)$ such that for a linear functional L we have

$$2L(f_0(x), ..., f_n(x)) = 0(mod\, b_L(n)),$$

so that the nontrivial modular relations induced are always modulo 2. This will give informations about the realization of $f_i(mod\, 2)$, as well as about the image group $F_b(CB(S^n))$.

2 The obstructions

2.1 The proof of Proposition 1.1

At the beginning we make some notations: D_k^+ is the union of k cubes (of dimension n) which are the faces of a $(n+1)$-dimensional cube, and no two of them are parallel faces, for $k = 1, 2, ..., n+1$. The complementary union of $2(n+1) - k$ cubes is denoted D_k^-. So a bubble move b_k replaces D_k^+ by D_k^-. Now $f_p(D_k^+)$ is the number of p-dimensional cubes in D_k^+. The number of interior p-cubes is $f_p(int(D_k^+)) = f_p(D_k^+) - f_p(\partial D_k^+)$. Notice that this is a notational convention because it does not count the number of p-cubes in the interior of D_k^+ but the number of open such p-cubes sitting in the interior of D_k^+.

In order to find the corresponding $a_i(n)$ we have to compute the numbers $f_p(b_k(x)) - f_p(x) = f_p(D_k^+) - f_p(D_k^-) = a(k, n, p)$; then $a_i(n) = gcd\{f_p(b_k(x)) - f_p(x), k = 1, 2, ..., n+1\}$.

We use the method of generating functions: set $F_X(T) = \sum_{p=0}^n f_p(X)T$. It is well-known that $f_p(C_n) = \binom{n}{p} 2^{n-p}$ for the n-cube C_n so that $F_{C_n}(T) = (2+T)^n$.

Let e_i, $i = 1, ..n+1$ be the vectors spanning C_{n+1} and L_i be the n-cube spanned by $e_1, e_2, ..., e_{i-1}, e_{i+1}, ..., e_{n+1}$, where e_i is omitted. Then a combinatorial model for D_k^+ is $\bigsqcup_{i=1}^k L_i$. The inclusion-exclusion principle states that:

$$\begin{aligned} f_p(\bigsqcup_{i=1}^k L_i) &= \sum_{i=1}^k f_p(L_i) - \sum_{i<j}^k f_p(L_i \cap L_j) + ... \\ &+(-1)^{l+1} \sum_{i_1<i_2<...<i_l}^k f_p(L_{i_1} \cap L_{i_2} \cap ... \cap L_{i_l}) + ... \\ &+(-1)^{k+1} f_p(L_1 \cap L_2 \cap ... \cap L_k). \end{aligned}$$

We observe that $L_{i_1} \cap L_{i_2} \cap ... \cap L_{i_l}$ is combinatorially the cube C_{n+1-l} and we derive

$$f_p(\bigsqcup_{i=1}^k L_i) = \sum_{i=1}^k (-1)^{i+1} \binom{n-i+1}{p} \binom{k}{i} 2^{n+1-i-p}.$$

It follows that, at the level of generating functions, we have:

$$\begin{aligned} F_{D_k^+}(T) &= \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}F_{C_{n+1-i}}(T) = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}(2+T)^{n+1-i} = \\ & (2+T)^{n+1-i}\left((2+T)^k - \sum_{i=1}^{k}(-1)^i\binom{k}{i}(2+T)^{k-i}\right) = \\ & (2+T)^{n+1} - (2+T)^{n+1-k}(1+T)^k. \end{aligned}$$

Therefore the generating function counting the interior cubes in D_k^- is simply

$$F_{int(D_k^-)}(T) = (2+T)^{n+1-k}(1+T)^k - T^{n+1} - kT^n.$$

In fact the total number of p-cubes in D_k^+ and D_k^- is $f_p(C_{n+1})$, but there are no $(n+1)$-dimensional faces and also the n-dimensional cubes are not interior in D_k^-, so k of them have to be removed from the total.

It remains to compute the number of interior p-cubes in D_k^+: all of them come as intersections $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_{n-p+1}}$ with the additional condition that $n-p+1 \geq 2$. It follows that

$$f_p(int(D_k^+)) = \begin{cases} \binom{k}{n-p+1} & \text{if} \quad n-1 \geq p \geq n-k+1 \\ 0 & \text{elsewhere} \end{cases}$$

The generating function is therefore

$$F_{int(D_k^+)}(T) = \sum_{p=n-k+1}^{n-1}\binom{k}{n-p+1}T^p = T^{n-k+1}[(1+T)^k - kT^{k-1} - T^k].$$

We find that the series associated to the jumps of the f-vector is

$$-F_{int(D_k^+)}(T) + F_{int(D_k^-)}(T) = \left((2+T)^{n+1-k} - T^{n+1-k}\right)(T+1)^k.$$

so that

$$\sum_{p=0}^{n} a(k,n,p)T^p = \left((2+T)^{n+1-k} - T^{n+1-k}\right)(T+1)^k.$$

As an immediate corollary we derive that all $a(k,n,p)$ are divisible by 2 since $(2+T)^{n+1-k} - T^{n+1-k}$ has even coefficients. This prove the first two claims. Developing the terms we obtain by a simple computation $a(k,n,n) = 2(n+1-k)$, $a(k,n,n-1) = 2n(n+1-k)$, and

$$a(k,n,n-2) = \frac{n+1-k}{3}\left(3n(n-1) + (n-k)(n-1-k)\right).$$

Also

$$a(k,n,0) = \begin{cases} 2^{n-k+1}, & if \quad k < n+1 \\ 0 & elsewhere \end{cases}$$

$$a(k,n,1) = \begin{cases} 2^{n-k}(n+1+k), & if \quad k \leq n-1 \\ 2n, & if \quad k = n \\ 0 & elsewhere \end{cases}$$

$$a(k,n,2) = \begin{cases} 2^{n-k-2}\left((n+1-k)(n+k) + 4k(k-1)\right), & if \quad k < n-1 \\ 2n(n-1), & if \quad k = n-1 \\ n(n-1), & if \quad k = n \\ 0 & elsewhere \end{cases}$$

A tedious verification ends the proof. □

2.2 The proof of Proposition 1.2.

Set more generally $D^+_{k,r}$ for the union of k cubes (of dimension n) which are the faces of a $(n+1)$-dimensional cube, and exactly $2r$ of them arise in pairs of parallel faces, for $k = 1, 2, ..., n+1$, and $2r < k$. The complementary set is denoted by $D^-_{k,r}$. Denote by $b_{k,r}$ the bump move which replaces $D^+_{k,r}$ by $D^-_{k,r}$.

Lemma 2.1 *The set of all combinatorially distinct unions of k cubes, $k \le n+1$, of dimension n, which are faces of the $n+1$-dimensional cube, and are topologically disks, is exactly the set of all $D^+_{k,r}$ with $2r < k$. Therefore the bump moves are the $b_{k,r}$ and their inverses.*

Proof: Obvious. □
We have the following:

Lemma 2.2 *For all k, r, p we have the identity:*

$$f_p(D^+_{k,r}) - f_p(D^-_{k,r}) = f_p(D^+_k) - f_p(D^-_k)(\text{ modulo } 2).$$

Proof: Set $k = 2r + q$. We denote the faces of C_{n+1} by L_i^ε, where $L_i^0 = L_i$ and L_i^1 is the opposite face. Then $D_{k,r}$ is combinatorially equivalent to some

$$D_{k,r} \cong \bigsqcup_{i=1}^{r} L_i^0 \bigsqcup_{i=1}^{r} L_i^1 \bigsqcup_{i=r+1}^{r+q} L_i^{\varepsilon_i},$$

for some ε_i. But now the choice of the ε_i is irrelevant because there is an isometry of C_{n+1} which transforms an ε_j into $1-\varepsilon_j$ and preserve the others. In terms of coordinates x_i this is defined by $x_i \longrightarrow x_i$ for $i \neq j$, and $x_j \longrightarrow x_j$. Therefore we can choose a combinatorial model for $D_{k,r}$ with all $\varepsilon_i = 0$.

The only difference with the previous case is that

$$\bigcap_{s=1}^{m} L_{i_s}^{\varepsilon_s} = \begin{cases} C_{n+1-m} & \text{if there are no } s_1, s_2 \text{ such that } i_{s_1} = i_{s_2}, \varepsilon_{s_1} \neq \varepsilon_{s_2} \\ \emptyset & \text{otherwise} \end{cases}$$

Therefore, with respect to the computation we made previously we have to take into account that some of intersections are void. The void intersections correspond to the combinations of h-tuples $L_i^0, ..., L_i^1, ...$ (the remaining $h-2$ faces being arbitrary) which has been counted as cubes C_{n+1-h} previously. In the calculation of $f_p(\bigsqcup_{i=1}^r L_i^0 \bigsqcup_{i=1}^r L_i^1 \bigsqcup_{i=r+1}^{r+q} L_i^{\varepsilon_i})$ we have to see how each term arising in the inclusion-exclusion principle, namely $X_h = \sum_{L_{i_s}^{\varepsilon_s} \in A} f_p(\cap_{s=1}^h L_{i_s}^{\varepsilon_s})$, has changed. Here A is the set of faces of $D_{k,r}$.

For $h = 1$ there are no intersections so that this factor is conserved. For $h = 2$ all the factors $L_i^0 \cap L_i^1$ have been counted before, but now their contribution is zero. There are r such factors which implies that:

$$X_2 = \binom{n-1}{p} 2^{n-p-1} \left(\binom{k}{2} - r \right)$$

For $h = 3$ the now vanishing combinations are $r(k-2)$ since a couple can be chosen among the r pairs and the third can be chosen in $k-2$ ways. It follows that :

$$X_3 = \binom{n-2}{p} 2^{n-p-2} \left(\binom{k}{3} - r(k-2) \right)$$

We continue with $h = 4$: we have $r\binom{k-2}{2}$ possibilities to get at least one couple of parallel faces but the couples $L_i^0, L_i^1, L_j^1, L_j^0$ are counted two times. Applying again the inclusion-exclusion principle we derive that

$$X_4 = \binom{n-3}{p} 2^{n-p-3} \left(\binom{k}{4} - r\binom{k-2}{2} + \binom{r}{2} \right)$$

This generalizes easily by induction to:

$$X_s = \binom{n+1-s}{p} 2^{n+1-p-s} \left(\sum_{j=1}^{2j \le s} (-1)^j \binom{k-2j}{s-2j} \binom{r}{j} \right)$$

We are able now to write:

$$f_p(D^+_{k,r}) = f_p(D^+_k) + \sum_{s=2}^{n-p+1} (-1)^s \binom{n+1-s}{p} 2^{n+1-p-s} \left(\sum_{j=1}^{2j \le s} (-1)^j \binom{k-2j}{s-2j} \binom{r}{j} \right)$$

Now we need to know how the number of interior p-cubes $f_p(int(D^+_{k,r}))$ has been changed. Of course these interior p-cubes are always coming as intersections

$$\bigcap_{s=1; L^{\varepsilon_s}_{i_s} \in A}^{n-p+1} L^{\varepsilon_s}_{i_s}$$

Again some of these intersections are void because parallel faces are allowed to be in A. But the inclusion-exclusion principle gives (in the non-trivial case $n-1 \ge p \ge n-k+1$) :

$$f_p(int(D^+_{k,r})) = f_p(int(D^+_{k,0})) \sum_{j=1}^{2j \le n-p+1} (-1)^j \binom{k-2j}{n-p+1-2j} \binom{r}{j}$$

It suffices now to see that the difference $f_p(D^+_{k,r}) - f_p(int(D^+_{k,r}))$ has the same parity for all r. In the previous formula for $f_p(D^+_{k,r})$ only the term for $s = n-p+1$ has not a coefficient divisible by 2. But for $s = n-p+1$ the contributing term in $f_p(D^+_{k,r})$ is exactly the same as the total contributing term in $f_p(int(D^+_{k,r}))$ and they cancel each other. □

Remark that in fact the greater common divisors of $f_p(D^+_{k,r}) - f_p(D^-_{k,r})$ are the analogs $\tilde{a}_i(n)$ of $a_i(n)$. It is clear than $\tilde{a}_i(n)$ are divisors of $a_i(n)$. However the explicit computations are more difficult. We omit the annoying details for checking that $\tilde{a}_0(n), \tilde{a}_1(n), \tilde{a}_{n-1}(n)$ are exactly those claimed. This ends the proof of the Proposition 1.2. □

2.3 The proof of Proposition 1.3.

The proof of our claim has two steps: we show firstly that the affine $\mathbf{Z}/2\mathbf{Z}$-module generated by $fb^{(2)}(C(B(M))$ has dimension less or equal than $\left[\frac{n+1}{2}\right]$. This is a consequence of the Dehn-Sommerville equations.

For the first step we concentrate firstly on the case $M = S^n$. Let P^{n+1}_c denotes the family of convex cubical polytopes. Some of the cubulations of the sphere corresponds to the boundaries ∂P^{n+1}_c of elements from P^{n+1}_c. The last component of the f-vector of elements in P^{n+1}_c is trivial. Consider next A^{n+1}_c be the affine $\mathbf{Z}$-submodule (or $\mathbf{Z}$-submodule coset) generated by all the f-vectors of elements of P^{n+1}_c , viewed in $\mathbf{Z}^{n+1}$.

Lemma 2.3 (Dehn-Sommerville Equations) *The affine space $A^{n+1}_c \otimes \mathbf{Q} \subset \mathbf{Q}^{n+1}$ is of dimension $\left[\frac{n+1}{2}\right]$. A set of defining equation is obtained from the Euler-Poincaré equation*

$$\sum_{i=0}^{n} (-1)^i f_i(x) = 1 - (-1)^{d+1}$$

together with

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} 2^{j-k} f_j = (-1)^{d-1} f_k, k = 0, 1, 2, ..., n-1.$$

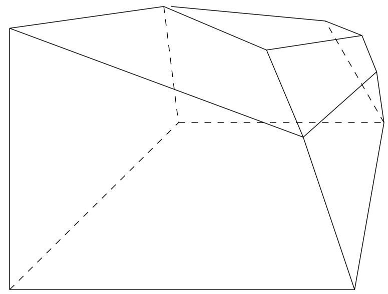


Figure 2: A cubulation with odd f_2

Equivalently we have the Euler-Poincaré equation and the set of independent equations

$$\sum_{j=k}^{n}(-1)^j\binom{j}{k}f_j = 0, k \equiv n+1 (mod\ 2), 1 \leq k \leq n-1.$$

Proof : See [21], p. 156-159. □

In the general case of an arbitrary manifold M, not necessary S^n, we have to replace the Euler-Poincaré equation by the corresponding:

Lemma 2.4 (Dehn-Sommerville Equations for a manifold) *The affine space $A_c^{n+1}(M)\otimes\mathbf{Q} \subset \mathbf{Q}^{n+1}$ is of dimension at most $\left[\frac{n+1}{2}\right]$. A set of equations which define a flat containing $A_c^{n+1}(M)\otimes\mathbf{Q}$ is obtained from the Euler-Poincaré equation*

$$\sum_{i=0}^{n}(-1)^i f_i(x) = \chi(M)$$

together with

$$\sum_{j=k}^{n}(-1)^j\binom{j}{k}2^{j-k}f_j = (-1)^{d-1}f_k, k = 0,1,2,...,n-1.$$

Equivalently, we have the Euler-Poincaré equation and the set of independent equations

$$\sum_{j=k}^{n}(-1)^j\binom{j}{k}f_j = 0, k \equiv n+1 (mod\ 2), 1 \leq k \leq n-1.$$

This was proved by Klee for simplicial complexes in [27], see also [21], p. 152. The case of cubulations is entirely analog, and we omit the proof. □

As a corollary we derive that the affine $\mathbf{Z}/2\mathbf{Z}$-submodule of $(\mathbf{Z}/2\mathbf{Z})^{n+1}$ generated by $fb^{(2)}(CB(M))$ has rank at most $\left[\frac{n+1}{2}\right]$. In fact, the system of independent equations written above has the determinant 1 mod 2.

In the case $M = S^n$ the affine $\mathbf{Z}/2\mathbf{Z}$-submodule is a $\mathbf{Z}/2\mathbf{Z}$-submodule because the only hyperplane which was not incident to the origin was the (Euler-Poincaré)-hyperplane, but $1+(-1)^d \equiv 0(mod\ 2)$, so that also this hyperplane pass through the origin, when tensorizing with $\mathbf{Z}/2\mathbf{Z}$.

Let us say few words about the image of $fb^{(2)}$, from an elementary point of view. For $n = 2$ we have to prove that there exist cubulations of the sphere S^2 having an odd number of faces (or vertices). An example of a such cubulation of the sphere S^2 is drawn in picture 2: it has the f-vector (11,18,9). This proves that the image is in fact $\mathbf{Z}/2\mathbf{Z}$. Remark that our example is not an isolated one. There exist polyhedrons having $3k$ 4-gonal faces, for each $k \geq 3$, which are constructed likewise. Topologically, some of these can be obtained by sewing the three regions from picture 3.

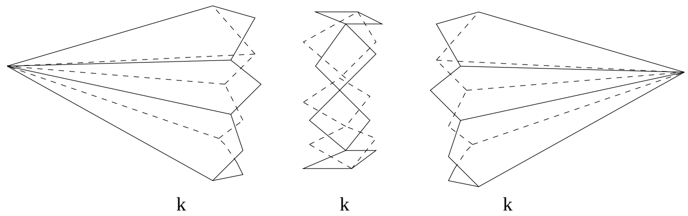


Figure 3: More general cubulations with odd f_2

2.4 Other arithmetic obstructions

We have seen that $\mathbf{f}(b_k(x)) = \mathbf{f}(x) + a(k,n)$, where $a(k,n) = (a(k,n,p)_{p=0,\dots,n})$. Therefore, starting from the f-vector α we will obtain β after some bubble moves if and only if the system of linear equations

$$\sum_{k=1}^{n+1} x_k a(k,n) = \beta - \alpha,$$

has integer solutions $x_k \in \mathbf{Z}$. Here x_k is the algebraic number of b_k moves used: an inverse move b_k^{-1} counts as -1. Of course for a linear system

$$\sum_{k=1}^{n+1} x_k a(k,n) = b,$$

has integer solutions, a necessary condition is that $gcd(a(k,n,p), k = 1, n+1)$ divides b_p, for each $p = 0, 1, ..., n$. These obstructions on $\alpha - \beta$ are exactly those described in the first part, namely $f_i(mod\ a_i(n))$. But these are only necessary conditions for solving this linear system, and they are not sufficient.

For the over-determined systems there are several conditions on the compatibility (as for usual systems) and another conditions for the integrality of the solutions. Written in an explicit form terms the latter give rise to another series of congruences.

On the other side the proposition 1.2. says that, in our case, the linear system has rank less than $\left[\frac{n+1}{2}\right]$. In fact it suffices to look at the first $\left[\frac{n+1}{2}\right]$ components of the f-vector, the other one being determined by the these. For this type of under-determined linear system the situation is a little more complicated. There are some new compatibility relations obtained as follows: an equation $ax + by = c$ has as solutions x and y in some arithmetical progressions. Let us add a new equation $a'x + b'z = c'$. Then the two arithmetical progressions which describe the set of solutions for x, for each equation taken alone, must have a non-trivial intersection. Basically a short computation for this particular example shows that the system above has solutions x, y, z if and only if (the conditions given above being satisfied altogether) we have $gcd(a, a')$ divides $bc' - b'c$. The new obtained obstructions will be called secondary arithmetic obstructions. These can be explicitly got by hand as follows: we consider the same number of unknowns as the rank, the other unknowns being considered parameters. We saw that such a system has secondary obstructions by setting that the (unique) solutions be integral. This is a system of congruences in the parameters.The compatibility conditions for the latter give another conditions only in the coefficients. The concrete realization is elementary and algorithmic but rather cumbersome.

In the case of our system modeling the transformations of the f-vector by bump/bubble moves, for $n = 1, 2$, we do not find any new obstructions.

For $n = 3$ there is one new obstruction. There are at most two independent f_i, let choose for instance f_0 and f_1. Then the bubble moves b_i act as follows:

$$b_1(f_0, f_1) = (f_0, f_1) + (8, 20),$$

$$b_2(f_0, f_1) = (f_0, f_1) + (4, 12),$$

$$b_3(f_0, f_1) = (f_0, f_1) + (2, 6),$$

$$b_4(f_0, f_1) = (f_0, f_1).$$

Suppose now that (f_0, f_1) is transformed into (f_0', f_1'), and $f_i' - f_i = u_i$, $i = 0, 1$. Then, if we used x times b_1, y times b_2, z times b_3, we have the following system:

$$8x + 4y + 2z = u_0,$$

$$20x + 12y + 6z = u_1.$$

Therefore u_i are even numbers, thus the obstructions $f_i(mod\ 2)$ are obtained. Set now $u_i = 2v_i$, with $v_i \in \mathbf{Z}$. We find then that $2x = 3v_0 - v_1$ holds, which implies that $v_0 + v_1$ must be even. Equivalently, this amounts to say that $(f_0 + f_1)(mod\ 4)$ is an invariant. This is the new obstruction we got, and it is not a consequence of the previous obtained congruences modulo $a_i(3)$. Therefore for $n = 3$ we have three essentially independent obstructions from the f-vector for the bubble equivalence: $f_0(mod\ 2)$, $f_1(mod\ 2)$ and $(f_0 + f_1)(mod\ 4)$.

In the case we have to do with the bump moves, there are two more moves to add to the previous list, namely $b_{3,1}$ and $b_{4,1}$. Then an explicit computation shows that

$$b_{3,1}(f_0, f_1) = (f_0, f_1) + (0, 4),$$

$$b_{4,1}(f_0, f_1) = (f_0, f_1).$$

Thus $(f_0 + f_1)(mod\ 4)$ is also an invariant for bump moves.

Let observe that the standard cubulation of S^3 (the boundary of the 4-cube) has trivial $(f_0 + f_1)(mod\ 4)$. On the other hand, $CB(S^3)$ is a group whose elements are of order two, hence $2(f_0 + f_1) = 0(mod\ 4)$ for any cubulation of the sphere. This implies that $f_0 = f_1(mod\ 2)$, which agrees with the results of [3] for the polytopes. Remark however that the obstructions $f_0(mod\ 2)$, $f_1(mod\ 2)$ and $(f_0+f_1)(mod\ 4)$. are defined on $CB(M)$ for all surfaces M, not only for the sphere.

3 Cubulations of surfaces

3.1 Non-simplicity and non-standardness for $n = 2$

The exact form of all arithmetic obstructions in the case $n > 3$ is hard to make explicit but there are algorithmically defined. So far the obstructions live in finite groups (also the secondary ones). But we will express now a different kind of obstructions.

We begin by looking at the case $n = 2$. Here the arithmetic obstruction we get is only f_0(mod 2), or equivalently f_2(mod 2). This means that two cubulations with the the same number mod 2 of faces can be transformed by bubble moves into cubulations with the same number of faces. Let us look at the dual 4-valent graph associated to the cubulations of S^2. These are planar graphs with f_2 vertices. The four examples drawn in picture 4 fulfill the following properties. They are combinatorially distinct.

All of them are non-standard. Here *standard* means according [15] that the intersection of any two cubes is either empty or a face common to both. Of course the non-standardness implies that none of A, B, C, D is embeddable. But remark that A and B can be mapped into the standard cubulation of $\mathbf{R}^2$. Also C and D are not simple and A and B are simple. Since A and B are mappable they are bump equivalent; actually it can be shown they are bubble equivalent. Moreover the bubble moves preserve the mappability so A and C cannot be bubble equivalent. Thus there are at least two different classes of bubble equivalence among A, B, C, D.

This suggests that it should exist some other obstructions. We define the index of non-simplicity $ns(C)$ of a cubulation in dimension 2 as follows. First observe that the dual graph is canonically embedded in the cubulated surface. Therefore there is a cyclic order on the edges incident to each vertex. Define a path p in the dual graph be *straight* if at any vertex the incoming and outcoming edges are not consecutive. A $\frac{\pi}{2}$*-loop* in the graph is a closed path which is straight away from the start and end point, where is not

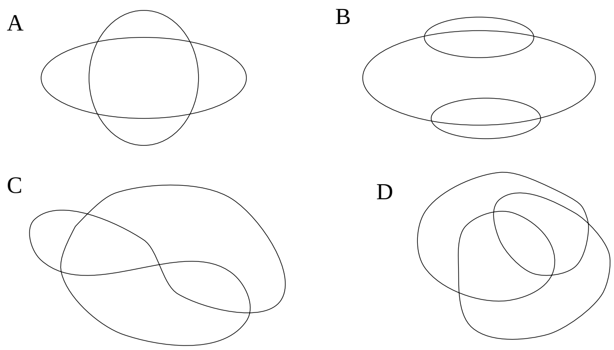


Figure 4: Examples of cubulations with $f_2 = 4$

straight. Consider $ns(C)$ be half the number of such $\frac{\pi}{2}$ -loops in the graph (some other possible variants would be to consider some additional properties, like being minimal, or pairwise non-homotopic etc). Alternatively, we can define $ns(C)$ as the number of corners in the straight lines.

Now we claim that:

Proposition 3.1 *$ns(C)$ is invariant by all bubble moves except b_2. Also $ns(C)(mod\ 2)$ is both bump and bubble invariant, and in fact it coincide with $f_0(mod\ 2)$.*

Proof: We prove that $ns(b_k(x)) = ns(x)$. The transformations b_k at the level of dual graphs are shown in picture 5. To prove that $ns(b_1(x)) \geq ns(x)$ it suffices to see that the $\frac{\pi}{2}$ -loops of x survive in some sense in $b_1(x)$. It suffices to see when the corner of the $\frac{\pi}{2}$ -loop, which is the vertex where is not straight, is not in the region changed by the bubble move. In fact otherwise there is a canonical extension, because any straight line transforms through a bubble move to another straight line. This may be easily visualized on the picture. For example see the picture 5 to see such a transformation. But now case by case it can be shown that also corners given by the edges a and b inside a small region where the b_j acts transform into corners passing through the images of a and b, and living already in the region considered.

For instance if we look at b_1 there is only one possibility for a corner occurs, up to symmetry, as can be seen in figure 7. For b_2 we have essentially two possibilities and for b_3 as well, up to the obvious symmetries. We figured out the transformation of corners.

Further we need to prove the converse inequality, namely $ns(b_1(x)) \leq ns(x)$. It suffices now to see that the corners which could appear in a region after bubbling can be collapsed into a corner which existed before. Alternatively, the corners from the right hand side of the pictures 7 are the only possible corners in the corresponding regions. In fact a new corner for b_1 (different from the center) would determine by straight continuation a closed loop so that we would not have a $\frac{\pi}{2}$ -loop. And up to symmetry there is no other possible corner not figured in 7, and this proves the claim.

We remark that there exists only one situation where the bubble move b_2 modify ns. This arises precisely when we have two *twin* cells. This means that we have two adjacent cells e and e' having a common edge f. Let x and x' the two edges of the dual graph which are parallel to f. We assume furthermore that x and x' are on the same connected component, or equivalently, that there is a straight line containing both of them. Then a b_2 move with the support on the union of e and e' changes the number of $\frac{\pi}{2}$-loops by 4. Thus ns reduced mod 2 is invariant. On the other hand, the circuits made of straight lines are topologically circles, which are immersed in the surface, with only ordinary double points. Then the number $ns(C)$ is half the sum, over all connected components, of the number of self-intersections of each circle. It follows now that $f_0 = ns(mod\ 2)$. □

So far $ns(C)$ could have been defined for arbitrary graphs embedded in surfaces, whose complement is an union of disks. The bubble/bump moves are local, so that the proposition from above has an immediate reformulation in this more general context. A priori we saw that $ns(C)$ is not bubble invariant, because a b_2-move could create/annihilate 2 more self-intersections, if the support is a pair of twin cells. Thus the non-invariance of $ns(C)$ is conditioned by the existence of twin cells in the cubulation. We will see

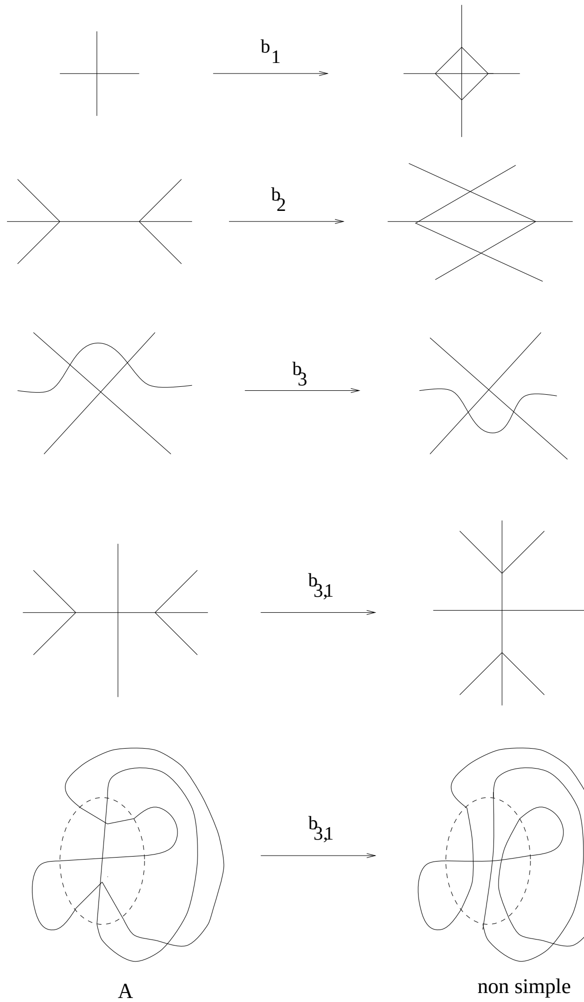


Figure 5: Dual graph bubble

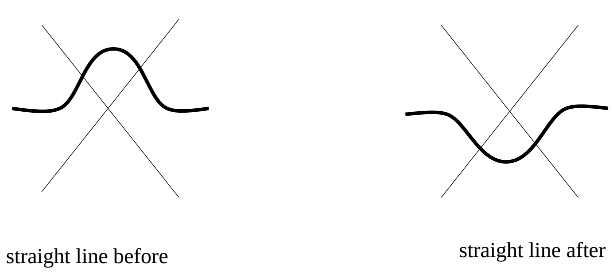


Figure 6: Straight lines and bubbles

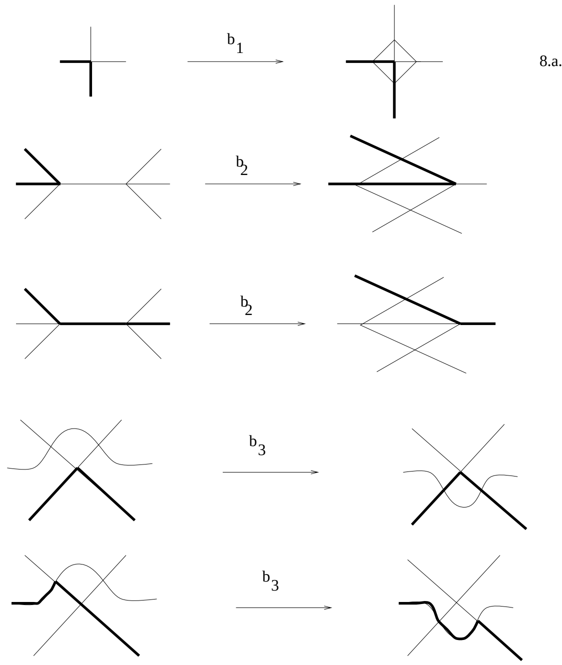


Figure 7: The various corners

in the proof of the lemma 4.9 that embeddable cubulations have no twin cells so that the simplicity (i.e. $ns(C) = 0$) is preserved by bubble moves, in this particular case.

Remark 3.2

- *The standardness is not invariant to bubble moves: for instance apply a b_1^{-1} to the standard ∂C_{n+1}. We get a two cubes cubulation of the sphere which is not standard.*
- *Also for $n \geq 2$ the simplicity, and ns are not invariant at bump moves. For example apply the bump move $b_{3,1}$ (the dual graph changes like in figure 5) to the cubulation represented by A. The image is no more simple, as can be seen in picture 5.*
- *Observe that ns is always an integer because any straight closed curve has an even number of $\frac{\pi}{2}$ -loop components.*

The discussion above shows that the most optimistic conjecture (however not true without additional hypothesis) would be that two cubulations with the same f-vector, both of them being standard and simple, are bubble equivalent. We will prove a weaker statement by restricting ourselves to the mappable cubulations and allowing all bump moves. The first condition is intuitively not a strong constraint because there are sufficiently many mappable cubulation. The second is stronger, but it seems more realistic to work with all bump moves, because the bubble equivalence is too finer, as it will be seen below.

3.2 The simple cubulations of S^2

We will prove in the next section that simple cubulations of the sphere are bump equivalent. The aim of this paragraph is to prove the analogous affirmation about the bubble equivalence. The first result is:

Proposition 3.1 *The set $CBB(S^2)$ is infinite.*

Proof: The derivative complex associated to the cubulation C in [3] is topologically the desingularization of the dual graph. The maximal straight lines (see 3.2) are connected 1-manifolds K_i, hence circles which are immersed in the surface underlying C. Consider the disjoint union of these circles K_i, which is

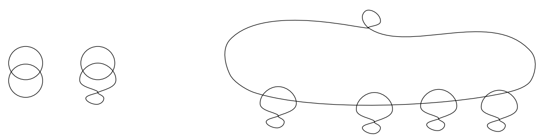


Figure 8: Conjectured representatives for elements of $CBB(S^2)$

naturally endowed with an immersion $\varphi : \cup_i K_i \longrightarrow C$. Recall that the non-simplicity index $ns(C)$ is the sum of the numbers of self-intersections of the components $\varphi(K_i)$. In particular C is simple if all K_i are individually embedded.

We can define then an invariant of the bubble class by setting

$$ns_1(C) = \inf\{ns(x);\ x \text{ is bubble equivalent to } C\},$$

but a priori it is hard to compute it effectively.

We claim that ns_1, as function on $CBB(S^2)$, is not bounded. It suffices to show that there exist cubulations C whose derivative complex has an arbitrary large number k of components K_i, and each $ns(K_i)$ is odd. Indeed, a bubble move b_2 may modify $ns(C)$ by 2 units, and the other ones keep it invariant. Meantime the number of components is not invariant, but the number of components with odd self-intersections is an invariant, because the only move creating new components is b_1, and each new created component must be an embedded circle. Remark that the image of all K_i is always a connected graph; this is a necessary and sufficient condition for the graph to be associated to a cubulation of the sphere. In fact, we identified before the moves b_2 and b_3 with the second and the third Reidemester moves, in the context where some additional strings exist in the picture (even if they are not changing during the operation). This proves that the number of components is invariant under b_2 and b_3.

It follows that $ns_1(C) \geq k$, for a cubulation C fulfilling the previous conditions. For instance consider a cubulation C whose dual graph is the union of k kinks (or figure eight), which form a connected planar graph. This proves the claim.□

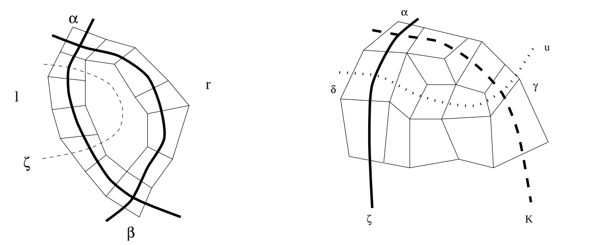
The similarity with the Reidemester moves in the plane suggests that

$CBB(S^2)$ is closed to the set of planar curves mod Reidemester moves, which is equivalent to the set of framed circles in the plane. Framed here means that we have to count the number of double points of each component. An immediate observation is that unlike the case of Reidemester moves, the dual graph here must remain connected, so that the various components cannot be completely separated. This is due of course, to the fact that Reidemester moves can be applied only if some additional strings exist in the configuration, conditions which avoid the possibility of separating into disjoint pieces. On the other hand the move b_2 can create/annihilate a pair of self-intersections. However it is not at all clear that the singularities can be paired such that suitable bubble moves destroy all pairs of singularities and each circle will have only $ns(K_i)(mod\ 2) \in \{0, 1\}$ remaining singularities. Moreover if this is true it should be also proved that all configurations of circles among which there are m circles, each of them with one singularity and the others being embedded, are bubble equivalent. This would establish a semi-group isomorphism between $CBB(S^2)$ and $\mathbf{Z}$. Then the elements of $CBB(S^2)$ would have canonical representatives as in the figure 8.

We are able to prove this statement for the case $m = 0$. Alternatively, this can be stated as follows:

Theorem 3.2 *Two simple cubulations of S^2 are bubble equivalent.*

Proof: The only way to shrink a cubulation C is to get rid of the self-intersections of one component K of its derivative complex. We do not assume for the moment that the cubulations are simple, because we do not know whenever two equivalent simple cubulations are always equivalent among simple cubulations. More much we think this is not the case and in our proof below we will use paths of equivalences which are outside of the realm of simple cubulations. Suppose that we have at least two self-intersections of K which fulfill the following conditions: there exist two points α and β on K which are joined by two arcs,



The monocolor biangle with the arc ζ   The bicolored biangle and the transversal u

Figure 9: Biangles

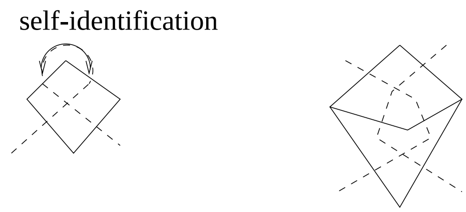


Figure 10: Smashed kinks and smashed biangles

say l and r, of K. The arcs l and r bound together a disk in the plane, and the intersection of this disk with the complementary arc to l and r is void. Up to a topological equivalence the situation is that shown in the figure 9. Observe that two different embedded components are always intersecting each other like in this description. A cubulation whose dual graph components have only this kind of self-intersections is called a *semi-simple* cubulation of S^2. Also K is immersed in C, and α, β are corresponding to cells of C (see the picture). The disk on S^2, which is also cubulated, will be called a *biangle*. Another type of self-intersection is that of a component where $\alpha = \beta$. This means that we have only one arc $r = l$. Most times such singularities arise with a biangle, but in the case when the ns is odd there is at least one which cannot be paired. The first step is to see how biangles can be destroyed using bubble moves.

Biangles are of two types: either there is some transversal line in the graph cutting one of the two arcs l or r, or else there is no room for other transversals. In the second case the biangle is called a smashed biangle. By default a biangle, if not explicitly specified otherwise is like in the first case. We have an explicit model for a smashed biangle, which is drawn below. Notice that an analogous notion of smashed self-intersection exists in the case of $\alpha = \beta$, and will be called a smashed kink.

Such biangles can appear from two arcs with common endpoints in the dual graph which are not necessary in the image of the same connected component of the derivative complex. In this case it will be called a bicolored biangle, and in the former case a monocolored biangle. That is equivalent with coloring the arcs of each component of the derivative complex with a different color.

Proposition 3.3 *The dual graph of a semi-simple cubulation can be reduced using bubble moves to a picture where all minimal biangles are smashed biangles.*

Proof : We have actually to see that in this context we can apply Reidemester moves as long as the self-intersection are not smashed biangles. Assume now we have a minimal monocolor biangle, as above. The first cell near α, on l, has an inward normal arc ζ. If the arc ζ enters the disk and exits (Jordan's theorem) by cutting again the arc l we obtain either a bicolored biangle - made up from ζ and l - if ζ and l are differently colored, or else a smaller monocolor biangle, otherwise. Using the minimality hypothesis the second alternative is not viable.
Claim 1: We can get rid of the bicolored biangles using bubble moves which do not increase ns.

We can again suppose we have a bicolored biangle which is minimal, with respect to the inclusion of the associated disks. Also, using some b_1 bubble moves we can assume the cubulation C is standard. Let ζ and K the two arcs intersecting in α and β, which are also in different components. Consider the arc

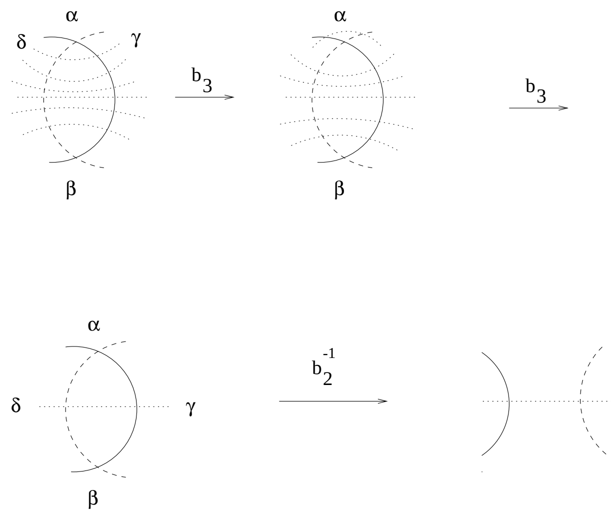


Figure 11: Destructive moves getting rid of transversals

u which is, like before, the normal inward arc u from the first cell near the cell α sitting on ζ. Then the arc u is forced to exit by cutting first the arc K, otherwise it should define a smaller bicolored biangle. Let γ be this new intersection point,and δ the first intersection point, between u and ζ. The length of the sub-arc $\alpha\gamma \subset K$ is defined as one plus the number of cells between the cells α and γ. The area of a triangle like $\alpha\delta\gamma$ is defined as the number of cells contained.

Claim2: Using some b_3 moves we can reduce the area of the triangle $\alpha\delta\gamma$ and the length of $\alpha\gamma$, preserving the length of $\alpha\delta$ (supposed to be 1 unit).

Let consider the trace of all other components of the dual graph on the triangle $\alpha\delta\gamma$. Then we have some other arcs which, from minimality, they cut only the arcs u and K. They may however intersect each other. This gives a partition into smaller polygonal domains. A triangle cannot be partitioned into polygonal domains each of them having more than 3 edges. Thus we find a smaller triangle inside, and we continue until we obtain a triangle T, whose edge length are all 1 unit. Moreover, since this partition is dual to a partition in squares (the original cells) there exists such a T on the boundary. This means that, either one vertex is γ, or else one edge is on $\delta\gamma$. A b_3 move acts like the Reidemester move on the dual graph (see picture 5), so assume we apply it on this triangle T. Then either the length of $\alpha\gamma$ diminish in the former case, or else the area of the triangle $\alpha\delta\gamma$ decreases, because the role of the b_3 move is to push up the arc $\delta\gamma$. This proves the second claim. □

This procedure stops when the triangle $\alpha\delta\gamma$ has all its edges of length 1. One more b_3-move as before, will push the transversal u outside the biangle. By the way the area of the biangle decreased and we can continue with the next transversal u' near α. We stop when the biangle has only one transversal u, of length 1, and then a move b_2^{-1} destroys the biangle.

This proves the claim 1. □

Notice that ns is not affected by these transformations, because the considered biangle is bicolored.

Now, exactly the same procedure permits to get rid of monocolor biangles. Unlike the previous case the application of the last move b_2^{-1} drops the index ns by 2.

We need now some preparations in order to attack the proof the theorem. The previous proposition shows that the analogy with Reidemester moves can be pursued outside of a local picture. In what concerns the third Reidemester move this is even simpler, since we have no constraints. The idea now is to establish a recipe for dealing with smashed biangles. The possible configurations of smashed biangles can be rather complicated, even if the cubulation is simple. We are looking first to a local picture with a single disk D embedded in the plane which is in general position with respect to X. Here X is the union of two orthogonal segments in the plane, which are the midsections of a square. We suppose then that D is contained in the interior of the square. The square is divided by X into 4 sectors. Consider two points