On the TQFT representations of the mapping class groups *

Louis Funar

Institut Fourier BP 74, Univ. Grenoble I

38402 Saint-Martin-d'Hères cedex, France

e-mail: funar@fourier.ujf-grenoble.fr

Abstract

We prove that the image of the mapping class groups by the representations arising in the $sl_2(\mathbf{C})$-TQFT is infinite, provided that the genus $g \geq 2$ and the level of the theory $r \neq 2, 3, 4, 6$ (and $r \neq 10$ for $g = 2$). In particular it follows that the quotient groups $\mathcal{M}_g/N(t^r)$ by the normalizer of the r^{th} power of a Dehn twist t are infinite if $g \geq 3$ and $r > 3$.

AMS Classification: 20 F 36, 20 F 38, 57 N 10, 57 M 25, 16 A 46.

Keywords: Mapping class group, TQFT, Braid group, Hecke algebra.

Contents

1 Introduction

Witten [55] constructed a TQFT in dimension 3 using path integrals and afterwards several rigorous constructions arose, like those using the quantum group approach ([44, 29]), the Temperley-Lieb algebra ([34, 35]), the theory based on the Kauffman bracket ([4, 5]) or that obtained from the mapping class group representations and the conformal field theory ([31]).

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Any TQFT gives rise to a tower of representations of the mapping class groups $\mathcal{M}_g$ in all genera g and this tower determines in fact the theory, up to the choice of the vacuum vector (see [13, 5, 53]). The aim of this paper is to answer whether the image of the mapping class groups is finite or not under such representations.

There is some evidence supporting the finiteness of this image group. First, in the Abelian $U(1)$-theory the representations can be identified with the monodromy of a system of theta functions. The latter is explicitly computed (see e.g. [14, 20, 40]) and it is easy to see that it factors through a finite extension (due to the projective ambiguity) of $Sp(2g, \mathbf{Z}/r\mathbf{Z})$, where r is the level of the theory. For a general Lie group G, the monodromy associated to the genus 1 surfaces may also be determined (see [22, 14]) using some formulas of Kac (see [25, 26]) and again it factors through a finite extension of $Sp(2g, \mathbf{Z}/r\mathbf{Z})$, where r is now the shifted level. This has already been suggested by the fact that the Reshetikhin-Turaev invariant for lens spaces $L_{a,b}$ ($a \leq b$ running over the positive integers) takes only a finite number of distinct values, namely for the cosets mod r of a and b (see for instance [33]). For low levels $r = 4, 6$ the whole tower of representations was described by Blanchet and Masbaum in [6] and by Wright ([56, 57]) and in particular the images are finite groups. On the other hand all TQFTs are associated to conformal field theories (abb. CFT) in dimension 2 (see e.g. [13, 55]) and the finiteness question appeared also in the context of classification of rational conformal field theories. For instance in [41] one asks whether the algebraic CFT have finite monodromy (which is equivalent to our problem for some classes of TQFTs, as for example the $sl_2(\mathbf{C})$-TQFT). Some of the irreducible representations of $SL(2, \mathbf{Z}/r\mathbf{Z})$ which admit extensions as monodromies of some CFT in all genera were discussed in [11, 12]. Also in [28] the action of $SL(2, \mathbf{Z})$ on the conformal blocks was computed for all quantum doubles, and it could be proved that the image is always finite. Gilmer obtained in another way (see [18]) the finiteness of the image for $g = 1$, in the $SU(2)$ theory, result which seems to be known in the conformal field theory community, and noticed also by M.Kontsevich. Meantime Stanev and Todorov [46] have a partial answer to this question in the case of the 4-punctured sphere, as we will explain bellow.

This is the motivation for our main result:

Theorem 1.1 *The image $\rho(\mathcal{M}_g)$ of the mapping class group $\mathcal{M}_g$ under the representation ρ arising in the $sl_2(\mathbf{C})$-TQFT (in both the BHMV and RT versions) is infinite provided that $g \geq 2$, $r \neq 2, 3, 4, 6$, and if $g = 2$ also $r \neq 10$.*

To explain briefly what means the two versions BHMV and RT we recall that the $sl_2(\mathbf{C})$-TQFT was constructed either using the Kauffman polynomial - and this is the BHMV version from [5] - or else using the Jones polynomial - and this is the RT version from [44, 29]. The invariants obtained for closed 3-manifolds are "almost" the same, but their TQFT extensions are different. That is the reason for considering here both of them, though as it should be very unlikely that the mapping class group representations do not share the same properties, in the two related cases.

Before we proceed let us outline the relationship with the results from [46], where the Schwarz problem is considered for the $\widehat{su(2)}$ Knizhnik-Zamolodchikov equation. The authors determined whether the image of the mapping class group of the 4-punctured sphere is finite, thereby solving a particular case of our problem, however in a slightly different context. It would remain to identify the following two representations of the mapping class groups (in arbitrary genus):

- one is that arising from the conformal field theory based on the $\widehat{su(2)}$ Knizhnik-Zamolodchikov equation. Tsuchiya, Ueno and Yamada [48] constructed this using tools from algebraic geometry, for all Riemann surfaces.
- the other one is that arising in the RT-version of the $sl_2(\mathbf{C})$-TQFT.

There are some naturally induced representations of braid groups in both approaches, which can be proved to be the same by the explicit computations of Tsuchyia and Kanie [47].

Presumably the two representations of the mapping class groups are also equivalent, but a complete proof of this fact does not exist, on author's knowledge. First it should be established that the conformal field theory extends to a TQFT in 3 dimensions, which is equivalent to know exactly the behaviour of conformal blocks sheaves over the compactification divisor on the moduli space of curves. Observe that a different and direct construction of the asociated TQFT can be given ([31, 32, 16]) if we assume the conformal field theory has all the properties claimed by the physicists. Notice that a complete solution of that problem would furnish an entirely algebraico-geometric description of the TQFT following Witten's prescriptions, in which the mapping class group representation is the monodromy of a projectively flat connection on some vector bundle of non-abelian theta functions over the Teichmuller space.

Thus we cannot deduce directly from [46] the finiteness of the mapping class group representation without assuming the previous unproved claim. Our purpose is to use instead the BHMV approach which has a simple and firmly established construction. Then from the mapping class groups we descend to braid group representations using basically the monodromy of the holed spheres. The data we obtain is similar to that obtained by all the other means, hence also to that from [46, 47]. Speciffically, the idea of the proof of the main theorem is to identify a certain subspace of the space on which $\mathcal{M}_g$ acts, which is invariant to the action of a subgroup of $\mathcal{M}_g$, the last being a quotient of a pure braid group P_n, $n \geq 3$. Next we observe that the action of P_n extends naturally to an action of the whole braid group B_n, and this it turns to factor through the Hecke algebra $H_n(q)$ of type A_{n-1} at a root of unity q. This was inspired by the computations done by Tsuchyia and Kanie ([47], see also [46]) of the monodromy in the conformal field theory on $\mathbf{P}^1$. Now the precise identification of the Hecke algebra representation among those constructed by Wenzl in [54], and an easy modification of the Jones theorem ([24]), about the generic infiniteness of the image of B_n in Hecke algebra representations, will settle our question.

For fixing the notations, we denote by r the level, which is supposed to be in this sequel exactly the order of the roots of unity which appear in the definition of the invariants.

The groups $\mathcal{M}_g/N(t^r)$, quotients of $\mathcal{M}_g$ by the normalizer of a power of a Dehn twist, were previously considered for $r = 2, 3$ by Humphreis in [21], and it is shown these are finite groups for $r = 2$ and arbitrary g, and infinite for $g = 2$ and $r \geq 3$. This solved the problem 28 asked by Birman in [2], p.219. We derive a generalization of that, to all other genera g, namely:

Corollary 1.2 *The quotient groups $\mathcal{M}_g/N(t^r)$ are infinite for $g \geq 3$, $r \geq 4$.*

Proof: It is well-known (see [31]) that the image of a Dehn twist $\rho(t)$, in some nice basis, is a diagonal matrix whose entries are $exp(2\pi\sqrt{-1}(\frac{j(j+1)}{4r} - \frac{r-2}{8r}))$. This is for the RT-version of the invariant at level r, or equivalently, for the BHMV representation at some $4r$-th root of unity. It follows that $\rho(t)^{2r}$ is a scalar matrix in this particular basis, and furthermore it is a scalar matrix in any other basis. Therefore, the image group $\rho(\mathcal{M}_g)$(modulo multiplication by roots of unity of order $8r$) is a quotient of $\mathcal{M}_g/N(t^{2r})$. For $4r > 6$, and $g \geq 3$, the theorem says that image group $\rho(\mathcal{M}_g)$ is infinite, and now the claim follows.□
Notice that the proof given by Humphreis used the Jones representation [23] of $\mathcal{M}_2$ which arises as follows: the group $\mathcal{M}_2$ is a quotient of the braid group B_6, and then some Hecke algebra representation factors through $\mathcal{M}_2$. In general, $\mathcal{M}_g$ has only a subgroup which is a quotient of B_{2n}, so that it is complicated to extend Jones representation to higher genus. However the $sl_2(\mathbf{C})$-TQFT representation seems to be suitable for this purpose.

In the last section we consider more general representations of the braid group induced by the TQFT in the same manner. In particular these factor through generalized Hecke algebras, which are quotients of the group algebra of the braid group by polynomial relations. We identify those which are of finite dimension like Coxeter [8, 9] did for the quotients of the braid groups.

It seems that not only the representations have infinite image, but the set of values taken by the $sl_2(\mathbf{C})$-invariant (at a given level r) on the set of closed 3-manifolds, of fixed Heegaard genus g, is also infinite. Our result does not imply this stronger statement, because the infinite image we found comes from a subgroup of $K \subset \mathcal{M}(F)$ of homeomorphisms of the surface extending to the handlebody. But by twisting the gluing map of a Heegaard splitting by an element of K yields a homeomorphic manifold. However it is very likely that the same method could be refined to yield this stronger statement.

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2 Preliminaries

2.1 Representations of Hecke algebras

We will outline briefly, for the sake of completeness, some basic notions concerning the representations of Hecke algebras, following Wenzl [54].

Recall that the Hecke algebra of type A_{n-1} is the algebra over $\mathbf{C}$ generated by $1, g_1, ..., g_{n-1}$ and the following relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ i = 1, 2, ..., n-2,$$
$$g_i g_j = g_j g_i,\ \mid i - j \mid > 1,$$
$$g_i^2 = (q-1) g_i + q,\ i = 1, 2, ..., n-1,$$

where $q \in \mathbf{C} - \{0\}$ is a complex parameter. Denote this algebra by $H_n(q)$. It is known (see e.g. [7], p.54-55) that $H_n(q)$ is isomorphic to the group algebra $\mathbf{C}S_n$ of the symmetric group S_n, provided that q is not a root of unity.

Notice that $H_n(q)$ is the quotient of the group algebra CB_n of the braid group B_n. The braid group is usually presented as generated by $g_1, ..., g_{n-1}$, together with the first two relations from above. In particular there is a natural representation of B_n in $H_n(q)$.

From the quadratic relation satisfied by g_i it follows that g_i has at most two spectral values. For $q \neq -1$ set e_i for the spectral projection corresponding to the eigenvalue -1; then $g_i = q - (1+q)e_i$, and another presentation of $H_n(q)$ can be obtained in terms of the generators $1, e_1, ..., e_{n-1}$, as follows:

$$e_i e_{i+1} e_i - q(1+q)^{-2} e_i = e_{i+1} e_i e_{i+1} - q(1+q)^{-2} e_{i+1},\ i = 1, 2, ..., n-2,$$
$$e_i e_j = e_j e_i,\ \mid i - j \mid > 1,$$
$$e_i^2 = e_i,\ i = 1, 2, ..., n-1.$$

Let $\lambda = [\lambda_1, \lambda_2, ..., \lambda_k]$ be a Young diagram consisting of an array with n boxes, from which λ_1 are on the first row, λ_2 are in the second row and so on. Set Λ_n for the set of all Young diagrams with n boxes. For $\lambda \in \Lambda_n$ let T_λ be the set of tableaux having the shape λ and ST_λ be the set of standard tableaux belonging to λ. The symmetric group acts on T_λ, by permuting the numbers (or symbols, e.g. in [37]) in the boxes. The transform by the transposition $\sigma_i = (i, i+1) \in S_n$ of a standard tableau is again standard if and only if i and $i+1$ are not in the same row or the same column.

Let $\tilde{V}_\lambda$ (and respectively V_λ) be the $\mathbf{C}$-vector space generated by the elements of T_λ (respectively ST_λ). Denote by $c_{t,s}$ and respectively $r_{t,s}$ the numbers for which the symbol s is contained in the $c_{t,s}$-th box from the left to the right (or the $c_{t,s}$-th column) and in the $r_{t,s}$-th row of the tableaux t. Set then:

$$d(t, s, m) = c_{t,s} - c_{t,m} + r_{t,s} - r_{t,m}.$$

For $q \neq 1$ denote also

$$\alpha_d(q) = (1 - q^{d+1})(1+q)^{-1}(1-q)^{-d},$$

and

$$\alpha(t, s, m) = \alpha_{d(t,s,m)}(q).$$

We are ready now to define representations of $H_n(q)$ on V_λ by means of the following formula:

$$\pi_\lambda(e_i)(v_t) = \alpha(t, i, i+1)(q) v_t + \sqrt{\alpha(t, i, i+1)\alpha(t, i+1, i)(q)}\, v_{\sigma_i t},$$

where $v_t \in V_\lambda$ is the vector corresponding to the standard tableau $t \in ST_\lambda$, $\sigma(g_i) = \sigma_i = (i, i+1) \in S_n$ is the transposition interchanging i and $i+1$. Notice that in the case when $v_{\sigma_i t}$ is no more a standard tableau, its coefficient is zero, so that π_λ is well-defined.

Then one of the results of [54] states that, for n-regular q (i.e. $q^j \neq 1$, for $j \leq n$), π_λ is an irreducible representation of $H_n(q)$.

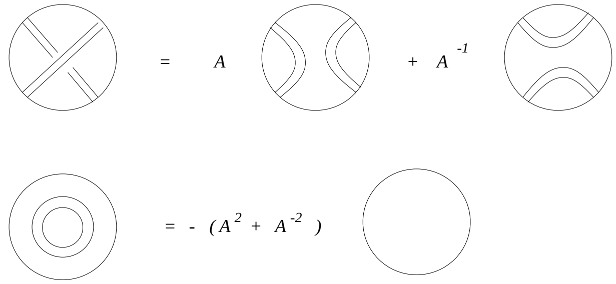


Figure 1: Skein relations

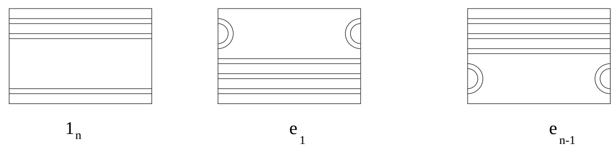


Figure 2: Generators of the Temperley-Lieb algebra

Consider now the subset $T_\lambda^{(k,l)} \subset T_\lambda$ of tableaux t such that t', which is the tableau t with the box containing n removed, belongs to $T_{\lambda'}^{(k,l)}$, for some (k,l)-diagram λ'. Here, a diagram λ is a (k,l)-diagram if it has at most k rows and $\lambda_1 - \lambda_k \leq l - k$. Let $ST_\lambda^{(k,l)} = ST_\lambda \cap T_\lambda^{(k,l)}$, and $V_\lambda^{(k,l)}$ be the subspace of V_λ spanned by the vectors v_t with $t \in ST_\lambda^{(k,l)}$.

Then it is shown in [54] that for q a primitive root of unity of order l the previous formula for π_λ defines actually a representation of $H_n(q)$ into $V_\lambda^{(k,l)}$. Moreover, if λ is not only a (k,l)-diagram, but also a $(k+1,l)$-diagram, then the associated representations coincide and so we may denote it by π_λ^l.

2.2 Mapping class group representations - BHMV version

Most of the material presented here comes from [34, 45, 38]. Let A be a fixed complex number and M be a compact oriented 3-manifold. The skein module $S(M)$ is the vector space generated by the isotopy classes (rel ∂M) of framed links, quotiented by the (skein) relations from figure 1.

For example $S(S^3)$ is one dimensional (as a module over $\mathbf{Z}[A, A^{-1}]$), with basis the empty link; the image of the framed link $L \subset S^3$ in $S(S^3)$ is the value of the Kauffman bracket evaluated at A.

The skein space for the 3-ball with $2n$ boundary (framed) points has an algebra structure, by representing the framed link in a planar projection sitting into a rectangle, and separating the points into two groups of n on opposite sides. The multiplication is given by the juxtaposition of diagrams, and the algebra TL_n thus obtained is called the Temperley-Lieb algebra. A system of generators for TL_n is provided by the elements $1_n, e_1, ..., e_{n-1}$ pictured in figure 2.

Now the Jones-Wenzl idempotents $f^{(n)} \in TL_n$ are uniquely determined by the conditions $f^{(n)2} = f^{(n)}$, $f^{(n)}e_i = e_i f^{(n)} = 0$, for $i = 1, 2, ..., n-1$, whenever A is such that all $\Delta_i = (-1)^i \frac{A^{2i+1} - A^{-2i-1}}{A^2 - A^{-2}}$ for $i = 0, 1, ..., n-1$ are non-zero. This implies that $f^{(n)}x = xf^{(n)} = \lambda_x f^{(n)}$, for all x, with a suitable chosen complex number λ_x.

Denote in a planar diagram by a line labeled with n (in a small rectangle) the element $1_n \in TL_n$, and by a line with a dash labeled n the insertion of the element $f^{(n)} \in TL_n$. This will give a convenient description of elements of skein modules.

One construction for the SU(2) invariants (or $sl_2(\mathbf{C})$-invariants) via skein modules, was given in [34, 35] and latter extended to a TQFT in [5], and to higher SU(n)-invariants recently in [36].

Let us outline first the construction of the conformal blocks, which are the vector spaces associated to surfaces via the TQFT. Decompose the sphere S^3 as the union of two handlebodies H of genus g, and

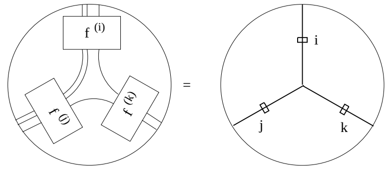


Figure 3: Vertex elements in the skein modules

H' with a small cylinder $F \times I$ over the surface $F = \partial H = \partial H'$ inserted between them. There is a map

$$<,>: S(H) \times S(H') \longrightarrow S(S^3) = \mathbf{C},$$

induced by the Kauffman bracket and the union of links. In [5] it was shown that, if A is a primitive $4r$-th root of unity, then

$$W(F) = S(H)/\ker <,>$$

is the space associated to the surface F by the $sl_2(\mathbf{C})$-TQFT at level r. Here ker denotes the left kernel of the bilinear form $<,>$. This space has however a more concrete description. If i, j, k satisfy the following conditions:

$$0 \leq i, j, k \leq r-1, \mid i-j \mid \leq k \leq i+j, \; i+j+k \text{ is even},$$

then we can define an element of the skein space of the 3-ball with $i+j+k$ boundary points, given by inserting $f^{(i)}, f^{(j)}, f^{(k)}$ in the diagram, and therefore connecting up with no crossings (see figure 3).

Now the triple (i, j, k) is called admissible if, additionally to the previous conditions, it satisfies $0 \leq i, j, k \leq r-2$ and $i+j+k \leq 2(r-2)$. Furthermore let consider the standard 3-valent graph in H which is the standard spine of the handlebody H, and label its edges with integers $i_1, i_2, ..., i_{3g-3}$, such that all labels incident to a vertex form an admissible triple. We form an element of $S(H)$ by inserting idempotents $f^{(i)}$ along the edges of the graph and triads, like we did above at vertices. It is shown in [5, 45] that the vectors we obtain this way form a basis of the quotient space $W(F)$.

For a 3-valent graph Γ, possibly with leaves and some of the edges already carrying a label, we denote by $W(\Gamma)$ the space generated by the set of labelings of (non labeled) edges which have the property that all triples from incident edges are admissible. An easy extension of the arguments in [5, 45] shows that $W(\Gamma)$ is isomorphic to $W(F)$, provided that Γ is some closed 3-valent graph of genus g.

If K and K' are the subgroups of the mapping class group $\mathcal{M}(F)$ of F consisting of the classes of those homeomorphisms which extend to the handlebodies H and H' respectively, then we have natural actions of K on $S(H)$, and K' on $S(H')$. Moreover these actions descend to actions on the quotient $W(F)$. One of these two actions, say that of K on H, has a simple meaning: consider an element $x \in S(H)$, which is a representative of the class $[x] \in W(F)$, $u \in K$, then $u(x) = [\varphi(x)] \in W(F)$, where φ is a homeomorphism of H whose restriction at F, modulo isotopy, is u. The other action, that of K' on $W(F)$ can be described in a similar manner, using the non-degenerate bilinear form $<,>$ on $W(F)$. Namely, $u(x)$, for $u \in K'$, $x \in W(F)$ is defined by the equality:

$$< ux, y >=< x, [u(y')] >,$$

holding for any $y \in W(F)$; on the right hand side $y' \in S(H')$ is a lift of y, and the action of K' on $S(H')$ is the obvious one.

Moreover we have an induced action of the free group generated by K and K' on $W(F)$. It is shown in [45, 38] that this action descends to the mapping class group $\mathcal{M}(F)$. This is the representation coming from TQFT. Actually we can build up the TQFT starting from that representation. The main idea is that, if we cut a closed 3-manifold M along a (closed embedded) surface F into two pieces M_1 and M_2, then the invariant $Z(M)$ can be recovered from the invariants $Z(M_i)$ associated to M_i (which are vectors in the space $W(F)$) as follows:

$$Z(M) =< Z(M_1), Z(M_2) > .$$

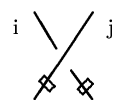
If we want to glue back now M_1 to M_2 using an additional twist $\varphi \in \mathcal{M}(F)$ then we can compute also the invariant of the resulting manifold $M_1 \cup_\varphi M_2$ using the representation $\rho : \mathcal{M}(F) \longrightarrow GL(W(F))$, defined above:

$$Z(M_1 \cup_\varphi M_2) = < \rho(\varphi) Z(M_1), Z(M_2) > .$$

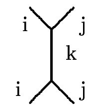
This gives a simple formula for the invariant in terms of Heegaard splittings. In fact the vector $Z(H) = Z(H') \in W(F)$, associated to the handlebody is corresponding to the graph of genus g whose labels are all 0 (up to a normalization factor, which we skip for simplicity). Then $Z(H \cup_\varphi H')$, the invariant of the closed manifold obtained by gluing two handlebodies along their common surface F using the homeomorphism φ, is now $< \rho(\varphi) Z(H), Z(H') >$.

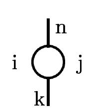
2.3 Transformation rules for planar diagrams in the skein modules

In order to make explicit computations we will freely use the recipes from [39] which allows us to transform planar diagrams representing elements in the skein module of the 3-ball (with some boundary points) into simpler planar diagrams, eventually arriving to linear combinations of the elements of a fixed basis. For completeness we include these rules below.

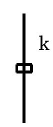


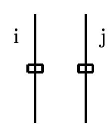
$$= \sum_k \delta(k,i,j) \frac{<k>}{<i,j,k>}$$



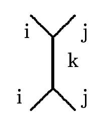


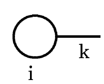
$$= \delta_n^k \frac{<k>}{<i,j,k>}$$



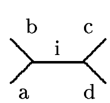


$$= \frac{<k>}{<i,j,k>}$$

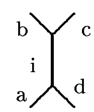


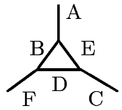


$= 0$ for $k \geq 1$

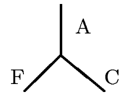


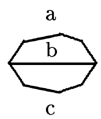
$$= \sum_i \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$$



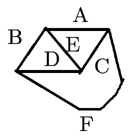


$$= \frac{\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle}{<A,F,C>}$$





$$= < a, b, c >$$



$$= \left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle$$

where

$$< k >= (-1)^k [k+1] = (-1)^k \frac{A^{2k+2} - A^{-2k-2}}{A^2 - A^{-2}},$$

$$\delta(c; a, b) = (-1)^k A^{ij - k(i+j+k+2)},$$

$$< a, b, c >= (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![i+k]![j+k]!}$$

. Here i, j, k are the internal colors given by

$$i = \frac{b+c-a}{2},\ j = \frac{a+c-b}{2},\ k = \frac{b+a-c}{2},$$

and $[n]! = [1][2]...[n]$.

Consider now A, B, C, D, E, F such that (A, B, E), (B, D, F), (E, D, C), (A, C, F) are admissible triples and make some notations: $\Sigma = A+B+C+D+E+F$, $a_1 = \frac{A+B+E}{2}$, $a_2 = \frac{B+D+F}{2}$, $a_3 = \frac{E+D+C}{2}$, $a_4 = \frac{A+C+F}{2}$, $b_1 = \frac{\Sigma-A-F}{2}$, $b_2 = \frac{\Sigma-B-C}{2}$, $a_1 = \frac{\Sigma-A-D}{2}$.

The tetrahedron coefficient is defined as:

$$\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle = \frac{\prod_i \prod_j [b_i - a_j]!}{[A]![B]![C]![D]![E]![F]!} \left(\begin{array}{ccccccc} a_1 & & a_2 & & a_3 & & a_4 \\ & b_1 & & b_2 & & b_3 & \end{array} \right),$$

where

$$\left(\begin{array}{ccccccc} a_1 & & a_2 & & a_3 & & a_4 \\ & b_1 & & b_2 & & b_3 & \end{array} \right) = \sum_{\max a_j \le \zeta \le \min a_j} \frac{(-1)^\zeta [\zeta+1]!}{\prod_i [b_i - \zeta]! \prod_j [\zeta - a_i]!}.$$

The quantum 6j-symbol of [39] is given by the formula:

$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{< i > \left\langle \begin{array}{ccc} i & b & c \\ j & d & a \end{array} \right\rangle}{< i, a, d >< i, b, c >}.$$

Figure 4: The graphs $\Gamma'(n,m)$

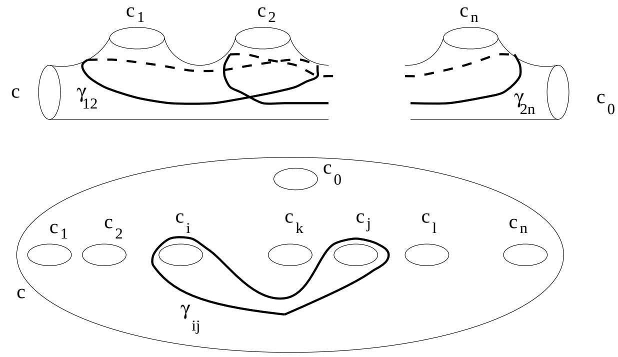


Figure 5: The curves γ_{ij}

3 Proof of the theorem

3.1 Outline

Consider a surface F of genus g and let $\Gamma \subset H$ be a 3-valent graph embedded in the handlebody H. Suppose that the graph Γ' shown in figure 4 is a subgraph of Γ. Then Γ' can be viewed as the spine of the $(n+2)$-holed sphere $F' \subset F$, which is the intersection of a regular neighborhood of Γ' (in $\mathbf{R}^3$) with F. Consider a partial labeling $\Gamma'(n,m))$ of Γ' as shown in the figure. Notice that the leaf with label 0 can be removed without affecting the space $W(\Gamma'(n,m)$.

Lemma 3.1 *For a suitably chosen Γ, of genus $g \geq 4$, there exist $m \geq 0, n \geq 5$, such that $W(\Gamma'(n,m)) \subset W(\Gamma)$ and $\dim W(\Gamma'(n,m)) \geq 2$. For $g = 3$ we have $W(\Gamma'(5,1)) \subset W(\Gamma)$, and for $g = 2$ $W(\Gamma'(3,1)) \subset W(\Gamma)$.*

The proof is obvious: Consider the leaves of Γ' are connected to each other, by some new edges in order to obtain a closed graph of minimal genus. □

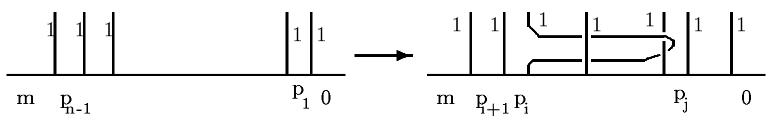
Fix now once for all the embedding of graphs $\Gamma' \subset \Gamma$ as in the lemma, and denote by $V \subset W(\Gamma)$ the image of $W(\Gamma'(n,m)$. Consider the curves $\gamma_{ij} \subset F'$, $1 \leq i,j \leq n$, drawn on the $(n+2)$-holed sphere F' which encircle the holes i and j like in the picture 5. The Dehn twists $T_{\gamma_{ij}}$ generate a subgroup S of the mapping class group $\mathcal{M}(F)$.

Proposition 3.1 *The subspace $V \subset W(F)$ is $\rho(S)$-invariant. Moreover the image $\rho\mid_V(S) \subset GL(V)$ is an infinite group, provided that $g \geq 3$ and the level $r \neq 2,3,4,6$, or $g = 2$ and $r \neq 2,3,4,6,10$.*

The first part of the proposition is easy to check: there is a more general fact concerning sub-surfaces $F' \subset F$ and a subgroup S of $\mathcal{M}(F)$ of classes of homeomorphisms which keep F' invariant up to a isotopy, and send each boundary component into itself. Assume we fix a labeling of the boundary components, of F': this amounts to fix a labeling of the leaves of the subgraph Γ', the spine of F'. Then the subspace $W(\Gamma') \subset W(\Gamma)$ is invariant by the action of S on $W(F)$. Moreover, consider now that F' may be sent by a larger group S' into a subsurface F'' which is isotopic to F, but the boundary components may be permuted among themselves. We claim now, that a subspace $W(\Gamma')$, associated to a labeling of the boundary components, is sent by such a homeomorphism into the subspace $W(\Gamma')$ associated to the permuted labeling on the boundary components. In particular the space V from the proposition is not only invariant under $\rho(S)$, but also under larger groups which could permute the n boundary components c_i, $i = 1, 2, ..., n$, since all their labels are identical.

Another observation is that the action of $\mathcal{M}(F')$ on the space $W(\Gamma')$, where Γ' has one external edge e (corresponding to the boundary component $c_e \subset \partial F'$) is the same as the action of $\mathcal{M}(F' \cup_{c_e} D^2)$ on the space $W(\Gamma'')$; here $F' \cup_{c_e} D^2$ is the result of gluing a disk on the circle c_e, and Γ'' is Γ with the edge e removed from it. This way we see that $\rho(S)$ is the image of a pure braid group P_n, acting like $\mathcal{M}(F' \cup_{c_0} D^2)$. This will help to find out the corresponding extension to the braid group.

Before we proceed in explaining this action, remark that all Dehn twists along γ_{ij} are elements from the subgroup $K \subset \mathcal{M}(F)$ of classes of homeomorphisms extending to the handlebody H. Therefore, according to the discussion in the previous section, the action of $T_{\gamma_{ij}}$ on V has a simple expression in the skein module of the 3-ball with $(n+2)$-boundary points: just perform the Dehn twist on the 3-ball which is a regular neighborhood of the graph Γ', viewed as part of the handlebody H, whose spine is Γ. This is equivalent to twist the i-th and j-th legs of the graph Γ', and further to apply the skein relations, in order to compute this element in terms of the basis of V, where the legs are straight.



But now the extension of the representation of P_n to B_n is obvious: consider that the i-th and $i+1$-th legs are only half-twisted. This defines the action of the i-th generator g_i of the braid group B_n. In fact, looking at the generators A_{ij} of P_n as elements of B_n, their action on V consists in twisting the corresponding legs of Γ', modulo Reidemester moves in plane. On the other hand the fact that we obtained a representation of B_n is checked the same manner: the relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$ translates into the third Reidemester move, which is obviously satisfied in the skein module. We continue to denote by ρ the representation of B_n on V. In fact this should enter in the computation of the action of elements in the larger group $\mathcal{M}(F)$, so actually is "part" of ρ but for bigger genus.

The main ingredient of the proof of proposition 3.1 is to establish:

Proposition 3.2 *The representation*

$$(-A)\rho : B_n \longrightarrow End(V)$$

factors through a representation of the Hecke algebra $H_n(q)$, which is equivalent to Wenzl irreducible representation π^r_λ associated to the $(2, r)$-diagram $\lambda = [\frac{n+m}{2}, \frac{n-m}{2}]$, and the parameter $q = A^{-4}$.

In order to end the proof of the proposition 3.1 it will suffice to prove that the image $\pi^r_\lambda(B_n)$ of the braid group B_n is infinite. This was done by Jones in [24] for one value of A, but the proof extends to an arbitrary primitive root of unity, and we state the result as:

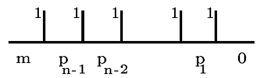
Proposition 3.3 *The image group $\pi^r_\lambda(B_n)$ is infinite provided that $n \geq 4$ and $r \neq 2, 3, 4, 6$ or $n = 2, 3$ and $r \neq 2, 3, 4, 6, 10$, for q a primitive root of unity of order r.*

This will establish the proposition 3.1, because P_n is of finite index in B_n, henceforth the claim of theorem 1 follows.

3.2 Proof of proposition 3.2

The TQFT considered here is the one constructed in [5], for A a primitive $4r$-th root of unity. This is called also the $SU(2)$-TQFT. The same should work for the representation associated to the invariants of [4] at a $2r$-th primitive root of unity, i.e. the $SO(3)$-TQFT, with only minor modifications.

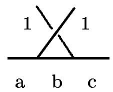
Lemma 3.2.1 *A basis for V is provided by the labeled graphs $L(\mathbf{p})$ below,*



whose labels are in one-to-one correspondence with

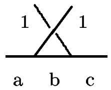
$$B(V) = \{\mathbf{p} = (p_0, p_1, ..., p_n); p_i \in \mathbf{Z}_+, p_0 = 0, p_n = m, p_i \leq 2r-2, \mid p_i - p_{i+1} \mid = 1, i = 0, ..., n\}.$$

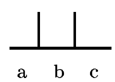
Proof: It follows immediately from the admissibility conditions on the triples $(p_i, p_{i+1}, 1)$. □
Then the computation of $\rho(g_i)$ is reduced to that of $g_i L(\mathbf{p})$, in the skein module. Observe now that the only values of the labels p_j which may change when g_i is applied are p_{i-1}, p_i, p_{i+1}. This will be also seen during the explicit computation.

Actually we have to compute 

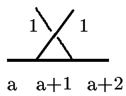
and according to the lemma it suffices to consider that $\mid a-b \mid = \mid b-c \mid = 1$.

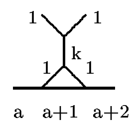
Lemma 3.2.2 *Suppose that* $\mid a-b \mid = \mid b-c \mid = 1$ *and* $\mid a-c \mid = 2$*. Then*



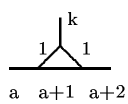
$= A$


Proof: Suppose for simplicity that $c = a+2, b = a+1$. Then according to [39] and the appendix we have

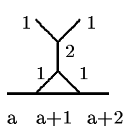


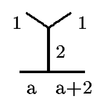
$= \sum_k \delta(k,1,1) \frac{<k>}{<1,1,k>}$


Therefore the triple $(1,1,k)$ has to be admissible, so that $k \in \{0, 2\}$. Also the open tetrahedron

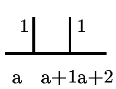


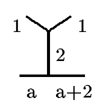
vanishes if $(k, a, a+2)$ is not admissible, so that there is only one possibility left, namely $k = 2$. We get rid of the triangular face by the formula:



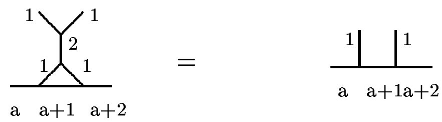
$= \dfrac{\left\langle \begin{matrix} 2 & 1 & 1 \\ a+1 & a+2 & a \end{matrix} \right\rangle}{<2,a,a+2>}$


We eventually perform a fusion in order to express the right hand side in terms of the the usual basis $L(\mathbf{p})$. The formula of the fusing is:



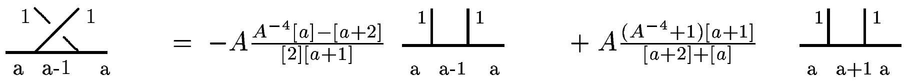
$= \left\{ \begin{matrix} a & 1 & 2 \\ 1 & a+2 & a+1 \end{matrix} \right\}$


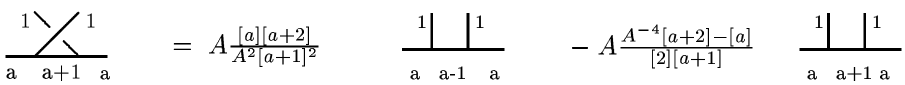
where the quantum 6-j symbol involved, namely $\left\{ \begin{matrix} a & 1 & 2 \\ 1 & a+2 & a+1 \end{matrix} \right\}$, it turns out to be equal to 1, for all a. This implies that

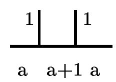


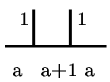
and the lemma follows. □

Lemma 3.2.3 *The following identities hold:*



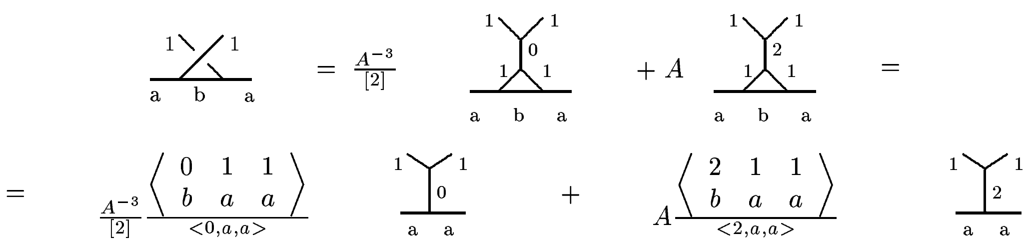


Here is to be understood that for $a = 2r-2$ we have: 

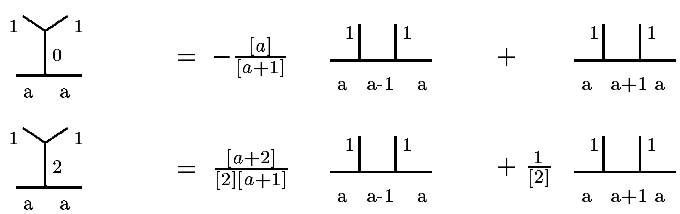
$= 0$ *, and for $a = 0$, also the corresponding equality:* 

$= 0$ *, holds.*

Proof: We have, like in the previous lemma, the following formula:



We have to perform a fusing, in order to arrive to the standard basis of V. Using the computations of 6j-symbols appearing in this particular fusing we obtain that:



The convention is that diagrams whose labels form non-admissible triples are vanishing. Explicit computations now yield our claim. □

Lemma 3.2.4 *The following numerical identities are satisfied:*

$$[a] + [a+2] = [2][a+1],$$

$$\frac{[a+1](1+A^{-4})}{[a]+[a+2]} = \frac{1}{A^2}, \frac{[a][a+2](1+A^{-4})}{[2][a+1]^2} = \beta^2 A^2,$$

where $\beta^2 = \alpha_{a+1}\alpha_{-a-1}$, *with* α *defined in the first section of preliminaries.*

$$\frac{[a+2]A^{-4} - [a]}{[2][a+1]} = \frac{A^{-4}-1}{1-A^{4+4a}},$$

$$\frac{[a]A^{-4} - [a+2]}{[2][a+1]} = \frac{A^{-4}-1}{1-A^{-4-4a}}.$$

The proof is a mere computation.□

Lemma 3.2.5 *Let* λ *be the* $(2,r)$*-diagram* $[\frac{n+m}{2}, \frac{n-m}{2}]$, *with* $m \leq r-2$. *Then there is a natural bijection* ϕ *between* $B(V)$ *and* ST_λ.

Proof: Choose some $\mathbf{p} \in B(V)$. We define recurrently the position to be assigned to the number i in the boxes of the Young diagram λ:

1. 1 sits in the first position, with $c_1 = d_1 = 1$.
2. $i+1$ sits in the first place not yet occupied from the left to the right
   - in the first line, if $p_{i+1} - p_i = 1$.
   - in the second line if $p_{i+1} - p_i = -1$.

   Here i ranges from 1 to $n-1$.

Notice that $m = (p_n - p_{n-1}) + (p_{n-1} - p_{n-2}) + ... + (p_1 - p_0)$, so that the number of those i with the property that $p_{i+1} - p_i = 1$, is exactly $\frac{n+m}{2}$. As we have $p_1 - p_0 = 1$, we deduce that the tableau so defined, say $\phi(\mathbf{p})$ has the shape λ, and is a standard tableau by construction. Furthermore the map $\phi : B(V) \longrightarrow ST_\lambda$ is obviously injective and therefore it is a bijection because the two sets have the same cardinality. □

Lemma 3.2.6 *The function* $c_{t,i}$ *and* $r_{t,i}$ *have the following expressions:*

$$r_{\phi(\mathbf{p}),i} = \frac{1+p_{i-1}+p_i}{2} \in \{0,1\},$$

$$c_{\phi(\mathbf{p}),i} = \frac{i+(p_i-p_{i-1})p_i}{2} \in \{1, ..., \frac{n+m}{2}\},$$

Proof: We know that i sits on the first row if $p_i - p_{i-1} = 1$, and on the second row, otherwise. On the other hand, among the first $i-1$ occupied boxes there are $\frac{i-1+p_{i-1}}{2}$ on the first row, and the remaining $\frac{i-1-p_{i-1}}{2}$ on the second row. Therefore, if $p_i - p_{i-1} = 1$ then the position of i is on the $\frac{i+1+p_{i-1}}{2}$-th column; otherwise it will sit on the lower row, on the $\frac{i+1-p_{i-1}}{2}$-th box. Since $i+1+(p_i-p_{i-1})p_{i-1} = i+(p_i-p_{i-1})p_i$ holds, the lemma is proved.□

Lemma 3.2.7 *The action of the symmetric group* S_n *on tableaux can be expressed as follows:*

1. *if* $\mid p_{i+1} - p_i \mid= 2$, *then* $\sigma_i\phi(\mathbf{p}) = \phi(\mathbf{p})$.
2. *if* $p_{i+1} = p_{i-1}$ *then*

$$(\sigma_i\phi(\mathbf{p}))_j = \begin{cases} p_j & \text{if } j \neq i \\ 2p_{i+1} - p_i & \text{for } j = i \end{cases}$$

Proof: The transposition σ_i acts on the tableaux by interchanging the positions of the two numbers i and $i+1$ in their respective boxes. The resulting tableau is no more standard if i and $i+1$ are in the same line (henceforth in adjacent boxes), so the associated action of g_i is trivial. If i and $i+1$ are in different rows the switch in the pair i, $i+1$, corresponds to replace $p_i - p_{i-1}$ (and respectively $p_{i+1} - p_i$) by $-(p_i - p_{i-1})$ (and respectively $-(p_{i+1} - p_i)$). This eventually amounts to replace p_i by $2p_{i+1} - p_i$. □

We are able now to end the proof of proposition 3.2. We have to compare the actions of $\pi_\lambda^{(r)}$ and ρ which both act on V_λ from the previous identifications.

Let first consider $\mathbf{p} \in B(V)$, with $| p_{i+1} - p_{i-1} |= 2$. From lemmas 5 and 6 we derive that

$$d_{\phi(\mathbf{p}),i,i+1} \in \{-1, 1\},$$

so that

$$\alpha_{\phi(\mathbf{p}),i,i+1}\alpha_{\phi(\mathbf{p}),i+1,i} = 0.$$

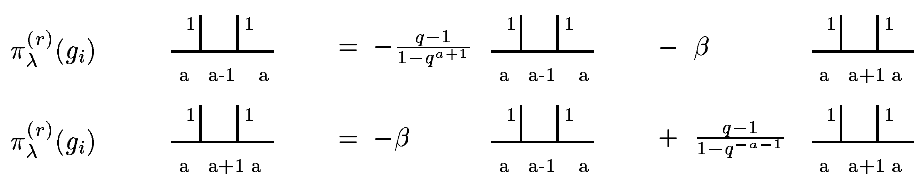
Therefore we obtain:

$$\rho(g_i)v_{\phi(\mathbf{p})} = Av_{\phi(\mathbf{p})} = -A\pi_\lambda^{(r)}(g_i)v_{\phi(\mathbf{p})}.$$

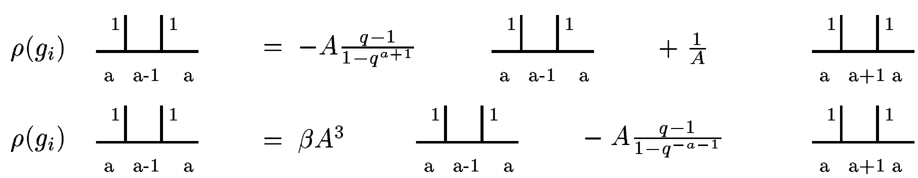
Consider now an element $\mathbf{p} \in B(V)$, with $p_{i+1} = p_{i-1} = a$. Again from lemmas 5 and 6 we find that

$$d_{\phi(\mathbf{p}),i,i+1} = (a - p_i)(a+1).$$

Therefore the representation $\pi_\lambda^{(r)}$ acts (via lemma 7) as follows:



Set now $q = A^{-4}$. Then lemma 3 can be reformulated as



We can prove now that $-A\rho$ and $\pi_\lambda^{(r)}$ are equivalent representations. Choose for instance $x, y, z, t \in \mathbf{C}$ such that the following conditions be fulfilled:

$$y = A^{-2}\beta^{-1}\gamma^{-1}z,$$

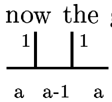
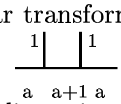
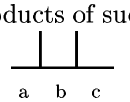
$$x = \beta^{-2}A^{-2}\gamma^{-1}(1 - \delta\gamma^{-1})z + A^{-2}\beta^{-1}\gamma^{-1}t,$$

where

$$\gamma = \frac{q-1}{1-q^{a+1}}, \quad \delta = \frac{q-1}{1-q^{-a-1}},$$

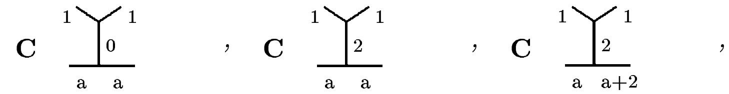
and t satisfies

$$A\beta\gamma(t^2 - z^2 + \beta^{-1}(1 - \delta\gamma^{-1})) = 1.$$

Consider now the global linear transform $\Omega : V \longrightarrow V$, having the property that, on each subspace of type $\mathbf{C}$  $\bigoplus \mathbf{C}$ , Ω acts as $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$. But V is a direct sum of tensor products of such subspaces of dimension 2, and 1-dimensional subspaces spanned by elements of the type $\mathbf{C}$ , with $| a - c |= 2$. This implies that such a global Ω is well-defined globally, as a sum of tensor products of 2-by-2 matrices and identity. Now it is a mere calculation to show that Ω is an interwinner i.e. $-A\rho \circ \Omega = \Omega \circ \pi_\lambda^{(r)}$. This ends the proof of proposition 3.1.□

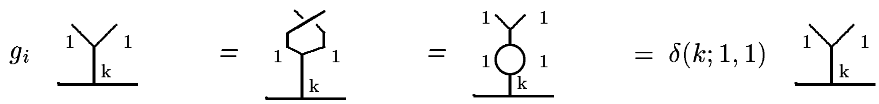
Remarks 3.2.8 *1. Although they are equivalent, the representations $\pi_\lambda^{(r)}$ and ρ are distinct under the identification coming from lemma 5.*

2. We could show from the very beginning of the proof that the representation of B_n factors through the Hecke algebra $H_n(q)$ with $q = A^{-4}$. Observe first that the vectors



and the corresponding ones with $a+2$ replaced by $a-2$, span all of V. Indeed using the fusing matrices (which are invertible) we can relate this system to the standard basis $L(\mathbf{p})$.

But now these are precisely the eigenvectors for g_i, because we have the following relations:



This implies that the eigenvalues are $-A^{-3}$ and A, so that shifting ρ by a factor of $-A$ will change them into -1 and A^{-4}, as in the usual presentation of $H_n(q)$ with $q = A^{-4}$.

3.3 Proof of proposition 3.3

In [24] a proof of proposition 3.3 is given for the case when $q = \exp(\frac{2\pi\sqrt{-1}}{r})$, but the argument generalizes easily to all primitive roots of unity. We outline it below, for the sake of completeness.

Define first the algebras $A_{\beta,n}$ (following the convention from [19], section 2.8), which are generated over $\mathbf{C}$ by $1, f_1, ..., f_{n-1}$ and the relations:

$$f_i f_{i+1} f_i = f_i f_{i-1} f_i = \beta^{-1} f_i, \ i = 1, 2, ..., n-2,$$

$$f_i f_j = f_j f_i, \ | i - j | > 1,$$

$$f_i^2 = f_i, \ i = 1, 2, ..., n-1.$$

Remark that $A_{\beta,n}$ is a quotient of the Hecke algebra $H_n(q)$, for $\beta = 2 + q + q^{-1}$. In fact, the image of the projector e_i generating $H_n(q)$ (see the section 2.1) is $1 - f_i$.

Lemma 3.3.1 *The representation π_λ of B_n associated to the 2-rows tableau $\lambda = [\frac{n+m}{2}, \frac{n-m}{2}]$ factors through $A_{\beta,n}$, where $\beta = 2 + q + q^{-1}$.*

Proof: It is known that the condition that λ has at most two rows implies that the images of $g_i' = \pi_\lambda(g_i)$ satisfy the following (S) relation:

$$g_i' g_{i+1}' g_i' + g_i' g_{i+1}' + g_{i+1}' g_i' + g_i' + g_{i+1}' + 1 = 0,$$

according to [24], p.261. Otherwise, it can be checked explicitly by direct computation. Furthermore from lemma 4.1., p.260 of [24], we derive from (S) that the defining relations of $A_{\beta,n}$ are satisfied by the images of $1 - e_i$. This proves the claim. □

Therefore the restriction of π_λ (a priori defined on B_n) to B_3 yields a representation of $A_{\beta,3}$. It is known that $A_{\beta,3}$ is semi simple and splits as $A_{\beta,3} = M_2(\mathbf{C}) \oplus \mathbf{C}$, for all $\beta \neq 1$ (see the theorem 2.8.5, p.98 from [19]). It suffices now to see that the images of g_1 and g_2 generate an infinite group.

Observe first that $\pi_\lambda(g_1)$ and $\pi_\lambda(g_2)$ do not commute each other, since the associated representation of the Hecke algebra is irreducible and of dimension bigger than 1. Hence there is at least one summand in the associated $A_{\beta,3}$-module which corresponds to the simple non-trivial factor $M_2(\mathbf{C})$. This holds because the abelianization of $A_{\beta,3}$ is the other factor. As a consequence it suffices to see what happens with the images of these two generators, when restricted to this summand. The representation π_λ is also unitarizable when q is a root of unity according to proposition 3.2, p.257 from [24]. Thus it makes sense to consider the images $\iota(g_1)$ and $\iota(g_2)$ in $SO(3) = U(2)/\mathbf{C}^*$. We have then the following decomposition in orthogonal projectors:

$$\iota(g_i) = qf_i - (1 - f_i),$$

so that the order of $\iota(g_i)$ in $SO(3)$ is $2r$ if r is odd, $r/2$ if $r = 2(4)$ and r if $r = 0(4)$, because q is a primitive root of unity of order r. As $r \neq 1$ these two elements cannot belong to a cyclic or dihedral subgroup of $SO(3)$. But no other subgroups have elements of order bigger than 5. Thus for $r = 5, 7, 8, 9$ or $r \geq 11$, the images of the subgroup generated by g_1 and g_2 is infinite.

The proof for the case of B_4 and $r = 10$ is the same as that given in [24], p.269: it works as long as the associated algebra $A_{\beta,4}$ remains semi simple. This is equivalent to the condition $1 - 3\beta^{-1} + \beta^{-2} \neq 0$. This will single out the case $g = 2$ and $r = 10$ in the theorem.

We may wonder whether an element of infinite order in the image can be explicitly found out. Since we have to consider only the matrices $\iota(B_n)$ in $SO(3)$, it is very likely that the element $\iota(g_1^{-1}g_2)$ has infinite order.

Remark 3.3.2 *Once we obtained the fact that the image of $\mathcal{M}_g$ is infinite at a particular primitive root of unity, we may argue also as follows: the Galois group $Gal(\overline{\mathbf{Q}};\mathbf{Q})$ acts on the set of roots of unity, as well as on the entries of the matrices $\rho(x)$, with $x \in \mathcal{M}_g$. It suffices to prove that the two actions of $Gal(\overline{\mathbf{Q}};\mathbf{Q})$ are compatible to each other, in order to conclude that the image group is infinite at all roots of unity. This argument was pointed to me by Gregor Masbaum.*

3.4 The RT version

Lickorish [35] established the relationship among the invariants obtained via the Temperley-Lieb algebra (basically those from [4]) $I(M, A)$ and the Reshetikhin-Turaev invariant $\tau_r(M)$ (see [29]), for closed oriented 3-manifolds M:

$$I(M, -\exp\frac{\pi\sqrt{-1}}{2r}) = \exp(\frac{(6-3r)b_1(M)\pi\sqrt{-1}}{4r})\tau_r(M),$$

where b_1 is the first Betti number. Roughly speaking the two invariants are the same up to a normalization factor. There are however two associated TQFTs, still very close to each other:

1. The TQFT based on the Kauffman bracket, as described in [5], which arises in a somewhat canonical way; in fact any invariant of closed 3-manifolds extends to a TQFT via this procedure (see [5, 13] for details. The associated mapping class group representation we denote it by ρ^K.
2. The TQFT based on the Jones polynomial, as described in [29] (see also [17]). The associated mapping class group representation we denote it by ρ^J, and may be computed using the definitions from conformal field theory like in [42]. A derivation of this representation, and the reconstruction of the invariant from it was first given by Kohno [31] (see also [49, 50, 15]).

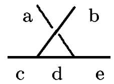
The two representations are similar: the associated spaces on which they act are naturally isomorphic. This means that in both theories $W(F)$ has a distinguished basis given by labelings of 3-valent graphs, with the same set of labels. Basically both theories are built up using some variants of the quantum 6j-symbols:

1. in [39] these are identified with the tetrahedron coefficients, (see also [27]); the relationship with the usual 6j-symbol (coming from representation theory) was outlined in [43].
2. in the case of ρ^J the 6j-symbols are coming from the representation theory of $U_q(sl_2)$ and where described in [30].

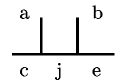
Consider now the analog subspace $V = W(\Gamma'(n, m)$ of $W(F)$, as in 3.1.We have again an action of the braid group B_n on V, but this time the interpretation is no longer related to skein modules of the ball. Here the graph Γ is considered to be embedded in the surface F, giving a rigid structure on F [13, 53]. This means that there is a pants decomposition c of F with the property that all circles in c

are transversal to Γ, the intersection of Γ with every trinion is the suspension of 3 points (topologically, the space underlying the figure Y). Remark that c and Γ determine uniquely an identification of F with a fixed and decomposed surface, up to an isotopy.

This time twisting the legs of the labeled graphs in $L(\mathbf{p})$ may be expressed in terms of the data of conformal field theory (see [31]). Specifically, we have:



$$= \sum_j B_{dj} \left[\begin{array}{cc} a & b \\ c & e \end{array} \right]$$



where the matrix B is the so-called braiding matrix. The braiding matrix can be expressed in terms of the fusing matrix F (see [31, 42] by the following formula:

$$B_{ij} \left[\begin{array}{cc} j_2 & j_3 \\ j_1 & j_4 \end{array} \right] = (-1)^{j_1+j_4-i-j}/2 \exp(\pi\sqrt{-1}(\Delta_{j_1} + \Delta_{j_4} - \Delta_i - \Delta_j))F_{ij} \left[\begin{array}{cc} j_1 & j_3 \\ j_2 & j_4 \end{array} \right],$$

where

$$\Delta_j = \frac{j(j+1)}{4r}.$$

We use the same set of labels for the graphs, namely integers running from 1 to $2r-2$ as before, instead of the traditional half-integer labels from [30, 31, 27]. Set also $q = \exp \frac{2\pi\sqrt{-1}}{r}$, and $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

The natural choice for the fusing matrix F is (see [31],p.213-214,[51]):

$$F_{ij} \left[\begin{array}{cc} j_1 & j_3 \\ j_2 & j_4 \end{array} \right] = \left\{ \begin{array}{ccc} j_1 & j_3 & i \\ j_2 & j_4 & j \end{array} \right\}_{KR},$$

where $\{,\}_{KR}$ denotes the quantum 6j-symbols of Kirillov and Reshetikhin.

Using the computations from [30], and the appendix, we find that the only non-trivial braiding matrix for $a = b = 1$ is that with $c = d$, and its value is therefore:

$$B \left[\begin{array}{cc} 1 & 1 \\ a & a \end{array} \right] = \left(\begin{array}{cc} -q^{a+\frac{1}{4}} \left(\frac{[a]}{[2][a+1]} \right)^{1/2} & -q^{-\frac{1}{4}} \left(\frac{[a+2]}{[2][a+1]} \right)^{1/2} \\ -q^{-\frac{1}{4}} \left(\frac{[a+2]}{[2][a+1]} \right)^{1/2} & -q^{-a-\frac{3}{4}} \left(\frac{[a]}{[2][a+1]} \right)^{1/2} \end{array} \right).$$

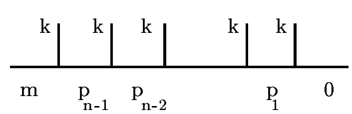
Notice that the braiding matrices arising in conformal field theory were previously computed by Tsuchyia and Kanie in [47]. Their result, used however a different normalization and the matrices are not identical, but equivalent up to a power of q. In fact, in our case, the representation $q^{1/4}\rho^J$ is also equivalent to $\pi_\lambda^{(r)}$. As an immediate consequence the representation ρ^J has an infinite image, too, under the same condition as ρ^K. This ends the proof of the main theorem.

4 Other representations of B_n coming from TQFT

4.1 The higher modules $V(k)$

If we are looking for invariant subspaces of conformal blocks we have some natural choices generalizing those from the previous section.

Consider for instance the vector space $V(k)$ (to avoid the confusions we should put also the subscripts n and m) spanned by the labeled trees



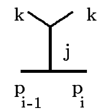
where the labels $\mathbf{p}$ are subjected to the following conditions:

1. $p_0 = 0, p_n = m$.
2. $\mid p_{i+1} - p_{i-1} \mid \in \{2k-2, 2k\}$, for all i.
3. (k, p_i, p_{i+1}) is an admissible triple.

We have therefore:

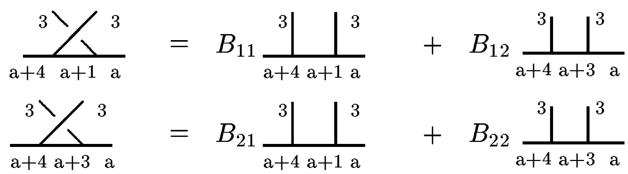
Proposition 4.1 *The space $V(k)$ is B_n-invariant, and the representation ρ_n^k of B_n so obtained, normalized by the scalar $\delta(2k-2;k,k)^{-1}$, factors through the Hecke algebra $H_n(q)$, with $q = A^{-4k}$.*

Proof: This follows from the fact that the span of the graphs



with $j \in \{2k-2, 2k\}$ is isomorphic via the obvious fusion, to $V(k)$. Likewise in the Remark 3.2.8 g_i acts diagonally on these vectors as the scalar $\delta(j;k,k)$, and the claim follows. □

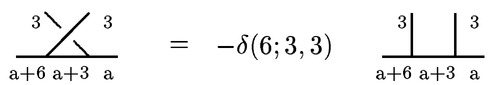
The next interesting case after $k=1$, which was discussed above, is $k=3$. Using the same ideas as in the previous case we derive that:



where the matrix B is given by

$$\begin{pmatrix} (-1)^{a+1}A^{-3}\frac{[a+5]}{[6]} + A^9\frac{[a+5]}{[6][a][a+6]} & (-1)^a A^{-3}\frac{[a+5][a+6][3]}{[6]^2[a+3]} + (-1)^a A^9 \frac{[a+5]}{[6][a+3]} \\ (-1)^{a+1}A^{-3}\frac{[a+5][a+6]}{[6][a+3]} + (-1)^{a+1}A^9\frac{[a+5]}{[6][a+3]} & (-1)^{a+1}A^{-3}\frac{[a+5][a+6]^2[3]}{[6]^2[a+3]^2} + (-1)^{a+1}A^9\frac{[a][a+5][a+6]}{[6][a+3]^2} \end{pmatrix}$$

Also we have:



As a consequence we find another representation of the braid groups, not equivalent to the previously considered $V(1)$. We have a precise description of this representation. We will use the subscripts $_{m,n}$ in order to specify that we have a representation of B_n on $V(k)_{m,n}$ and the label on the left side of the considered graphs is m. Observe that there is an injection of $V(3)_{n,m} \subset V(1)_{n+1,m-2n}$, given, at the level of standard basis by the map

$$\psi(\mathbf{p})_i = \begin{cases} 3 + p_{i-1} - 2i & \text{if } i \geq 1 \\ 0 & \text{if } i = 0 \end{cases}$$

It is easy to see that the restriction of

$$\rho_{n+1} : B_{n+1} \longrightarrow End(V(1)_{n+1,m-2n})$$

at the subgroup B_n keeps invariant the subspace $\psi V(3)_{n,m}$.

Proposition 4.2 *The two representations thus obtained,*

$$\rho^1_{n+1} \mid_{B_n} : B_n \longrightarrow End(\psi V(3)_{n,m}),$$

and

$$\rho^3_n : B_n \longrightarrow End(V(3)_{n,m}),$$

are equivalent.

The proof is a mere computation.

4.2 Generalized Hecke algebras

If one tries to generalize $V(1)$ and $V(k)$ to allow the labels $\mathbf{p}$ and those associated to the external legs to take any admissible values, we obtain a larger space $W(k)$ on which the braid group acts. This is no longer an irreducible representation, and also the spectral values of the g_i's run over the all possible values of $\delta(m;k,k)$, with $m \leq 2k$. Then we do not derive a representation of the Hecke algebra, but of a more general quotient of the group algebra $\mathbf{C}B_n$. Specifically set:

$H(Q,n) = \mathbf{C} < g_1, g_2, ..., g_{n-1} \mid g_i g_j = g_j g_i, g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i, \text{ if } \mid i-j \mid > 1; Q(g_i) = 0 >$

where Q is a polynomial in one variable. The Hecke algebras are quotients of the group algebra of the braid group by quadratic relations. A study of these generalized Hecke algebras was started in [15] for the cubic case.

It is simply to check that $W(k)$ is an image of the algebra $H(Q,n)$, for the polynomial Q having precisely the roots $\delta(m;k,k)$. One may wonder whether these algebras $H(Q,n)$ are finite dimensional, semi-simple etc. We will prove below that, in general, $H(Q,n)$ are infinite dimensional. It would be nice to find a finite dimensional algebra HH_n, presented by generators and relations, such that the representation of B_n on $W(k)$ factors through HH_n. One such algebra exists: the generators are $1, g_1, ..., g_{n-1}$ and the relations are those satisfied by the elements $g_i L(\mathbf{p})$, when viewed in the skein module. However it is very difficult to find out explicitly the complete set of relations: they correspond to the complete set of skein relations satisfied by the colored Jones polynomial.

Let us outline first what is known at the level of group quotient of B_n, because the generalized Hecke algebras should be deformations of the group algebras of such quotients. It is well-known that $B_n/(g_i^2)$ is the permutation group S_n. Define after Coxeter [8] the factor groups

$B_n(d) = B_n/(g_i^d)$.

Coxeter gave an exhaustive list of those factor groups from the list $B_n(p)$ which are finite, together with their respective description (see also [9, 10]). They are exactly those for which $(d-2)(n-2) < 4$. Away from the trivial case $d = 2$ we have another five groups:

- n=3
  - For d=3, the group $B_3(3)$ is the binary tetrahedral group $< 2,3,3 >$, isomorphic to $SL(2, \mathbf{Z}_3)$, and has order 24.
  - For d=4, the group $B_3(4)$ has order 96, and is the group $< -2, 3 \mid 4 >$.
  - For p=5,the group $B_3(5)$ has order 600, it is isomorphic to $< 2,3,5 > \times \mathbf{Z}_5 \cong GL(2, \mathbf{Z}_5)$.
- For n=4, d=3, the group $B_4(3)$ has order 648, it is the central extension of the Hessian group by $\mathbf{Z}_3$.
- For n=5, d=3, the group $B_5(3)$ has order 155 520 and it is the central extension of the simple group of order 25 920 by $\mathbf{Z}_6$.

Our result is the extension of the previous Coxeter finiteness theorem for these generalized Hecke algebras:

Proposition 4.3 *1. The vector spaces $H(Q,n)$ are of finite dimension if and only if $(d-2)(n-2) < 4$, in which case the dimension does not depend on the particular Q with $Q(0) \neq 0$ chosen.*

2. Assume $Q(0) = 0$. Then the vector spaces $H(Q,n)$ are finite dimensional if and only if $(d-2)(n-2) = 0$.

Proof: The result obviously holds generically. In fact, $H(Q,n)$ is a deformation of the group algebra $CB_n(d)$. In the case the group $B_n(d)$ is finite its group algebra is semi-simple, and therefore a multi-matrix algebra. Generic deformations therefore will preserve the semi-simplicity and the finite dimensionality. Meantime the precise description of the discriminant should be difficult to obtain. Another proof of the finiteness result is via Bergmann's diamond lemma, as in [15]. However, the computations are cumbersome, and a computer verification was used. We will explain this method in the simpler case, when $Q(0) = 0$, $n = 3$.

The second part consists in proving that for $(d-2)(n-2) \geq 4$ the algebras are infinite dimensional. In the corresponding group case Coxeter used a very nice trick. The group $B_n(d)$ was represented in $U(n)$ as a specific group of complex reflections and the image was shown to be infinite since it has no positive definite invariant hermitian form. In our case the same approach does not apply anymore. On the other side, the direct computational procedure used for the finiteness cannot handle all the cases. Our strategy is to show first the following:

Lemma 4.4 *1. For* $Q(0) = 0$, *we have* $\dim H(Q,3) = \infty$.

2. For $Q(0) \neq 0$, *deg* $Q = 3$, *we have* $\dim H(Q,6) = \infty$.

3. For $Q(0) \neq 0$, *deg* $Q = 4$, *we have* $\dim H(Q,4) = \infty$.

4. For $Q(0) \neq 0$, *deg* $Q = 6$, *we have* $\dim H(Q,3) = \infty$.

Proof: In this specific cases we can still apply the method of Bergmann. Recall first what a complete system of relations is. We have thus a system of relations, $R_i = S_i$, where all the words in the generators are lexicographically ordered; we ask that $R_i > S_i$. Further it is asked for all relations must be obtainable from the given system. That will provide a normal form for all the words in the generators. One way to get such a complete system (under suitable conditions on the relations: for instance the algebra should be the quotient of the tensor algebra on the generators) is to consider a ascending sequence of relation systems: once $R_1 = S_1$ and $R_2 = S_2$, are two relations at some level, we have to look at the possible interactions between them. An interaction is a ambiguous word T, having two reductions; it suffices to check what happens if T is containing R_1 as a prefix and R_2 as a suffix. Then using the two existing relations we may reduce T to lower expressions, in two different ways as $T = T_1$, and $T = T_2$. If the expressions T_1 and T_2 are not equivalent, using the already existing relations, then a new relation is to be added: $x = T_1 - T_2 - x$, where x is the highest monomial (word) in $T_1 - T_2$. We have to throw away the redundant relations now from the enlarged system: these are the relations $R_1 = S_1$ for which there is another relation in the system, say $R_2 = S_2$, such that R_2 is a sub-word in R_1. We continue this procedure until we get a complete system: all interactions give no new relations, and we say that the ambiguities are solvable. This will provide a basis as a vector space for the algebra. The basis contains the words in the generators, which do not contain as a sub-word any of the left hand side words R_i of a complete system of relations.

Let consider the example of $H(Q,3)$, with $Q(x) = x^3 - \alpha x^2 - \beta x$.

We claim that a complete system of relations presenting $H(Q,3)$ is given by:

$$Q(g_i) = 0, i = 1,2,$$

$$g_2 g_1 g_2 = g_1 g_2 g_1,$$

$$g_2 g_1^2 g_2 g_1 = g_1 g_2 g_1^2 g_2,$$

$$g_1 g_2 g_1^2 g_2^2 = \alpha g_1 g_2 g_1^2 g_2 + \beta g_1 g_2 g_1^2.$$

We proceed with 3 relations :

(1) $g_2 g_1 g_2 = g_1 g_2 g_1$

(2) $g_1^3 = \alpha g_1^2 + \beta g_1$

(3) $g_2^3 = \alpha g_2^2 + \beta g_2$

and the system of generators S containing all words in g_1 and g_2 without sub words appearing in the left hand of some relation, i.e. upon now without containing a $g_2 g_1 g_2$, g_1^3, g_2^3. We develop each ambiguity word (i.e. which has two resolutions) by underlining the sub-word replaced in each case. Away from the starting point the computations, even messy, became canonical, which means that the words involved have unique reduction, and we will write only the final result. Also if an ambiguity is solvable, so no new relation appear we mark by a $\Box$ in the final.

The interactions (2-2), (3-3), (1-2), (1-3) give only identities. Further

(1-1) $g_2g_1\underline{g_2g_1g_2} = g_2g_1^2g_2g_1$ and $\underline{g_2g_1g_2}g_1g_2 = g_1g_2g_1^2g_2$

so we obtain a new relation

(4) $g_2g_1^2g_2g_1 = g_1g_2g_1^2g_2$.

Next we have an interaction

(1-4) $\underline{g_2g_1^2g_2g_1}g_2 = g_1g_2g_1^2g_2^2$ and

$g_2g_1^2\underline{g_2g_1g_2} = \alpha g_1g_2g_1^2g_2 + \beta g_1g_2g_1^2$

and a new relation is obtained

(5) $g_1g_2g_1^2g_2^2 = \alpha g_1g_2g_1^2g_2 + \beta g_1g_2g_1^2$.

(4-2) $\underline{g_2g_1^2g_2g_1}g_1^2 = \alpha g_1^2g_2g_1^2g_2 + \beta g_1g_2g_1^2g_2$

$g_2g_1^2g_2\underline{g_1^3} = \alpha g_1^2g_2g_1^2g_2 + \beta g_1g_2g_1^2g_2$. $\Box$

(1-4) $\underline{g_2g_1g_2}g_1^2g_2g_1 = \alpha g_1^2g_2g_1^2g_2 + \beta g_1^2g_2g_1^2$

$g_2g_1\underline{g_2g_1^2g_2g_1} = \alpha g_1^2g_2g_1^2g_2 + \beta g_1^2g_2g_1^2$. $\Box$

(3-4) $\underline{g_2^3}g_1^2g_2g_1 = (\alpha^2+\beta)g_1g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1$

$g_2^2\underline{g_2g_1^2g_2g_1} = (\alpha^2+\beta)g_1g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1$. $\Box$

(2-5) $\underline{g_1^3}g_2g_1^2g_2^2 = \alpha^2g_1^2g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1^2 + \alpha\beta g_1g_2g_1^2g_2 + \beta^2g_1g_2g_1^2$

$g_1^2\underline{g_1g_2g_1^2g_2^2} = \alpha^2g_1^2g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1^2 + +\alpha\beta g_1g_2g_1^2g_2 + \beta^2g_1g_2g_1^2$. $\Box$

(4-4) $\underline{g_2g_1^2g_2g_1}g_1g_2g_1 = \alpha^2g_1^2g_2g_1^2g_2 + \alpha\beta g_1g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1^2 + \beta^2g_1^2g_2g_1$

$g_2g_1^2\underline{g_2g_1^2g_2g_1} = \alpha^2g_1^2g_2g_1^2g_2 + \alpha\beta g_1g_2g_1^2g_2 + \alpha\beta g_1^2g_2g_1^2 + \beta^2g_1^2g_2g_1$. $\Box$

(3-5) $\underline{g_1g_2g_1^2g_2^2}g_2 = \alpha g_1g_2g_1^2g_2^2 + \beta g_1g_2g_1^2g_2$

$g_1g_2g_1^2\underline{g_2^3} = \alpha g_1g_2g_1^2g_2^2 + \beta g_1g_2g_1^2g_2$. $\Box$

$(4\text{-}5)_1$ $\underline{g_2g_1^2g_2g_1}g_1g_2 = g_1g_2g_1^2g_2g_1g_2^2 = g_1g_2g_1^2g_1g_2g_1g_2 = g_1g_2g_1^3g_1g_2g_1 = (\alpha^2+\beta)g_1^2g_2g_1^2 + \alpha\beta g_1^2g_2g_1^2$

$g_2g_1\underline{g_1g_2g_1^2g_2^2} = \alpha g_2g_1^2g_2g_1^2g_2 + \beta g_2g_1^2g_2g_1^2 = \alpha g_1g_2g_1^2g_2g_1g_2 + \beta g_1^2g_2g_1^2g_2 = (\alpha^2+\beta)g_1^2g_2g_1^2 + \alpha\beta g_1^2g_2g_1^2$.

$\Box$

and $(4\text{-}5)_2$ comes from $g_2g_1^2g_2g_1g_2g_1^2g_2^2$ where again the ambiguity is solvable. This proves our claim.

As a consequence, the infinite set $(g_1^2g_2^2)^k$, $k \in \mathbf{Z}_+$ is linearly independent in $H(Q,3)$ and therefore $\dim H(Q,3) = \infty$.

The proof is similar for the other cases. $\Box$

It follows now that $H(Q,n)$, for n bigger than those considered in the lemma 4.4, are infinite dimensional too. Actually we have an injection $H(Q,n) \hookrightarrow H(Q,n+1)$, but the proof is somewhat laborious. Some ad hoc arguments will suffice: for instance, assume that *deg* $Q = 6$, or *deg* $Q = 3$ and $Q(0) = 0$. It is known that there are only a finite number of conjugacy classes (at most 3) in B_3, whose associated Artin closures, as oriented links, are the same (see [3]). If we quotient the algebra $H(Q,3)$ by the Lie submodule generated by the combinations $ab - ba$ we get again an infinite dimensional module. It follows that the vector space generated by links of braid index at most 3, modulo the cubic skein relation induced by Q, is still infinite dimensional. Furthermore the vector space generated by the links of braid index less than n ($n \geq 4$), modulo the skein relations induced by Q, will be also infinite dimensional. But the latter is a quotient of $H(Q,n)$, and the claim follows.

When we consider some arbitrary polynomial P of degree bigger than those considered in 4.4, we have a surjection $H(P,n) \longrightarrow H(Q,n)$, for any divisor Q of P. This proves the proposition. $\Box$

References

[1] L.C. Biederharn and J.D. Louck. *Angular momentum in Quantum Physics*, Encycloped.Math.and Its Appl., no.8, Addison-Wesley 1981.

[2] J. Birman. *Braids, links and mapping class groups*, Ann.Math.Studies, Princeton Univ.Press,no.82, 1974.

[3] J. Birman and W.Menasco. Studying links via closed braids III.Classifying links which are closed 3-braids, *Pacific J.Math.*, 161(1993), 25–115.

[4] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel. Three-manifold invariants derived from the Kauffman bracket, *Topology*, 31(1992),685-699.

[5] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel. Topological Quantum Field Theories derived from the Kauffman bracket, *Topology*, 34(1995), 883-927.

[6] C. Blanchet, G. Masbaum. Topological quantum field theories for surfaces with spin structures, *Duke Math.J.*, 82(1996), 229–267.

[7] N. Bourbaki. *Algèbres de Lie, IV, V, VI.* Masson, 1981.

[8] H.S.M.Coxeter. On factors of braid groups, *Proc.4-th Canadian Math.Congress, Banff 1957, Univ.Toronto Press 1959*, 95-122.

[9] H.S.M.Coxeter and W.O.J.Moser. *Generators and relations for discrete groups*, 3-rd edition, Springer, Berlin.

[10] H.S.M.Coxeter. *Regular complex polytopes*, Cambridge Univ.Press 1974.

[11] W. Eholzer. Fusion algebras induced by representations of the modular group, *Internat. J.Modern Phys. A*,8(1993), no.20, 3495-3507.

[12] W. Eholzer and N.P.Skoruppa. Conformal characters and theta series, *Lett.Math.Phys.*, 35(1995), 197–211.

[13] L. Funar. 2+1-D Topological Quantum Field Theory and 2-D Conformal Field Theory, *Commun.Math.Phys.*, 171(1995), 405–458.

[14] L. Funar. Thêta functions, root systems and 3-manifold invariants, *J.Geom.Phys.*,17(1995), 61–82.

[15] L. Funar. On the quotients of cubic Hecke algebras, *Commun.Math.Phys.* 173(1995), 407-453.

[16] L. Funar. TQFT for general Lie algebras and open 3-manifolds, *J.Math.Sci., Univ.of Tokyo*, 4(1997), 121-181.

[17] R. Gelca. The SL(2,C) Topological Quantum Field Theory with corners, *Preprint q-alg/9604006*, 1996.

[18] P. Gilmer. On the WRT representations of the mapping class groups, *Preprint*, 1997.

[19] F.M. Goodman, P. de la Harpe, V.F.R. Jones. *Coxeter graphs and tower of algebras*, MSRI Series, Berkeley, Springer, New-York, 1989.

[20] T. Gocho. The topological invariant of three manifolds based on the U(1) gauge theory, *J.Fac.Sci.Univ.Tokyo,Sect.1A,Math.*, 39(1992), 169–184.

[21] S.P. Humphreis. Normal closures of powers of Dehn twists in mapping class groups, Glasgow J.Math. 34(1992), 313–317.

[22] L. Jeffrey. Chern-Simons-Witten Invariants of Lens Spaces and Torus Bundles, *Commun.Math.Phys.*, 147(1992), 563–604.

[23] V.F.R.Jones: Hecke algebras and a polynomial invariant for knots, *Ann. of Math.*, 126(1987), 335-388.

[24] V.F.R. Jones. Braid groups, Hecke algebras and type II_1-factors *in "Geometric methods in operator algebras, Proc.US-Japan Seminar,* H.Araki, E.G.Effros Ed., 1986, 242-271.

[25] V. Kac. *Infinite dimensional Lie algebras*, Progress in Math., 1983.

[26] V.G. Kac and D.H. Petersen. Infinite dimensional Lie algebras, theta functions and modular forms, *Adv.Math.*, 53(1984), 125–264.

[27] L. Kauffman and S.L. Lins. *Temperley-Lieb Recoupling Theory and Invariants of 3-manifolds*, Ann.Math.Studies 134, Princeton Univ.Press, 1994.

[28] T.Kerler. Mapping class group actions on quantum doubles, *Commun.Math.Phys.*, 168(1995), 353–388.

[29] R. Kirby and P. Melvin. The 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C), *Invent.Math.*, 105(1991), 473–545.

[30] A.N. Kirillov and N.Yu.Reshetikhin. Representation of the algebra of $U_q(sl_2)$, q-orthogonal polynomials and invariants for links, *in "Infinite dimensional Lie algebras and groups", Adv.Ser.Math.Phys.,* V.Kac, Editor, 1988, 285–339.

[31] T. Kohno. Topological invariants for 3-manifolds using representations of mapping class groups I, *Topology*, 31(1992), 203–230.

[32] T. Kohno. Topological invariants for 3-manifolds using representations of mapping class groups II, *Contemporary Math.*, vol. 175, 1994, 193–217.

[33] B.H. Li and T.J. Li. Generalized Gaussian sums and Chern-Simons-Witten-Jones invariants of lens spaces, *J.Knot Theory and Its Ramif.* , 5(1996), 183–224.

[34] R. Lickorish. Three-manifold invariants and the Temperley-Lieb algebra, *Math.Annalen.*, 290(1991), 657–670.

[35] R. Lickorish. Calculations with the Temperley-Lieb algebra, *Comment.Math.Helv.*, 67(1992), 571–591.

[36] R. Lickorish. Sampling the $SU(n)$-invariants, *J.Knot Theory and Its Ramif.*, 6(1997), 45-60.

[37] D. Littlewood. *The theory of group characters,* Oxford Univ.Press, 1940.

[38] G. Masbaum and J.Roberts. On central extensions of mapping class groups, *Math.Annalen.*, 302(1995), 131–150.

[39] G. Masbaum and P.Vogel. 3-valent graphs and the Kauffman bracket, *Pacific J.Math.*, 164(1994), 361–381.

[40] M. Manoliu. Abelian Chern-Simons Theory, *preprint dg-ga/9610001*, 1996.

[41] S.D. Mathur and A.Sen. Group theoretic classification of rational conformal field theories with algebraic characters, *Nucl.Phys. B*, 327(1989), 725-743.

[42] G. Moore and N. Seiberg. Classical and quantum field theory, *Commun.Math.Phys.*, 123(1989), 177–254.

[43] S. Piunikhin. Reshetikhin-Turaev and Kontsevich-Kohno-Crane invariants for 3-manifolds coincide, *J. Knot Theory and its Ramif.*, 2(1993), 65–95.

[44] N. Reshetikhin and V. Turaev. Invariants of three-manifolds via link polynomials and quantum groups, *Invent.Math.*, 103(1991), 547–597.

[45] J. Roberts. Skeins and mapping class groups, *Math.Proc.Cambridge Phil.Soc.*, 115(1994), 53–77.

[46] Y.S. Stanev and I.T. Todorov. On the Schwarz problem for the $\widehat{su(2)}$ Knizhnik-Zamolodchikov equation, *Lett.Math.Phys.*, 35(1995), 123–134.

[47] A. Tsuchyia and Y. Kanie. Vertex operators in two dimensional conformal field theory on P^1 and monodromy representations of braid groups, *Adv.Studies Pure Math.,Tokyo*, vol.16, 1988, 297–372.

[48] A. Tsuchyia, K. Ueno and Y. Yamada. Conformal field theory on universal family of stable curves with gauge symmetries, *Adv.Studies Pure Math.,Tokyo*, vol.19, 1989, 459–566.

[49] V. Turaev. Quantum invariants of links and 3-valent graphs in 3-manifolds. *Publ.Math.I.H.E.S.*, 77(1993), 121–171.

[50] V. Turaev. *Quantum invariants of knots and 3-manifolds*, vol.18. de Gruyter Studies in Math., 1994.

[51] V. Turaev and O.Viro. State-sum invariants of 3-manifolds and quantum 6j-symbols, *Topology*, 31(1992), 865–902.

[52] V. Turaev and H. Wenzl. Quantum invariants of 3-manifolds associated with classical simple Lie algebras, *International Journal of Math.*, 4:323–358, 1993,

[53] K. Walker. Chern-Simons-Witten invariants of 3-manifolds, *preprint*, 1993.

[54] H. Wenzl. Hecke algebras of type A_n and subfactors, *Invent.Math.*, 92(1988), 349–388.

[55] E. Witten. Quantum field theory and the Jones polynomial, *Commun.Math.Phys.*, 121(1989), 351–399.

[56] G. Wright. The Reshetikhin-Turaev representation of the mapping class groups, *J.Knot Theory and Its Ramif.*, 3(1994), 547–574.

[57] G. Wright. The Reshetikhin-Turaev representation of the mapping class groups at sixth roots of unity, *J.Knot Theory and Its Ramif.*, 5(1996), 721–741.