MALGRANGE'S VANISHING THEOREM IN 1-CONCAVE CR MANIFOLDS

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Abstract

We prove a vanishing theorem for the $\overline{\partial}_b$-cohomology in top degree on 1-concave CR generic manifolds.

The aim of this paper is an analogous in the CR setting of Malgrange's theorem [13] for the vanishing of the $\overline{\partial}$-cohomology in top degree in connected, non compact complex manifolds. We prove the following theorem

Theorem 0.1 *If M is a connected, $\mathcal{C}^{2+\ell}$-smooth, $\ell \in \mathbb{N}$, non compact, 1-concave, CR generic manifold of real codimension k in a complex manifold X of complex dimension n, $n \geq 3$, then for all p, $0 \leq p \leq n$,*

$$H_\ell^{p,n-k}(M) = 0,$$

where $H_\ell^{p,n-k}(M)$, $0 \leq p \leq n$, denote the $\overline{\partial}_M$-cohomology groups of top degree on M with coefficients of class $\mathcal{C}^\ell$.

If moreover M is assumed to be $\mathcal{C}^\infty$-smooth, then

$$H_\infty^{p,n-k}(M) = 0 \ .$$

We point out that this theorem holds without any global condition on M (1-concavity is a local condition, cf. Sect. 1). If, additional, certain global convexity condition is fulfilled then the vanishing of $H_\ell^{p,n-k}(M)$ is well-known. The first result of this type can be found in the paper [1] (th. 7.2.4) of Airapetjan and Henkin, where the vanishing of $H_\infty^{p,n-k}(M)$

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is obtained under the hypothesis that M is a closed submanifold of a Stein manifold. Generalizations of this result can be found in [9] and [12].

Note that in view of the lack of the Dolbeault isomorphism in top degree on 1-concave, CR-generic manifolds, one cannot deduce the vanishing of the groups $H^{p,n-k}_\ell(M)$, $0 \leq \ell \leq \infty$, from the vanishing of one of them.

The proof of the theorem is based on some local results on the solvability of the tangential Cauchy-Riemann equation in top degree and the approximation of $\overline{\partial}_M$-closed $\mathcal{C}^\ell$-forms of top degree minus one by $\mathcal{C}^{\ell+1}$-smooth, $\overline{\partial}_M$-closed forms in 1-concave, CR generic manifolds, on the unique continuation of CR functions and on the Grauert bumping method.

We may notice by looking precisely to the proof that the manifold M needs not to be a 1-concave CR-generic manifold embedded into a complex manifold but that Theorem 0.1 still holds under the following assumptions :

(i) The CR-manifold M is either locally embeddable and minimal in the sense of Tumanov [14] or abstract and 1-concave (this ensures in both cases the unique continuation of CR functions, see [14], [3]).

(ii) One can solve locally the tangential Cauchy-Riemann equation in top degree in the $\mathcal{C}^\ell$-class with an arbitrary small gain of regularity and approximate locally $\overline{\partial}_M$-closed $\mathcal{C}^\ell$-forms of top degree minus one by $\mathcal{C}^{\ell+1}$-smooth, $\overline{\partial}_M$-closed forms.

Note, moreover, that if E is a vector bundle over M, which locally extends as an holomorphic vector bundle, then Theorem 0.1 still holds for $H^{p,n-k}_\ell(M, E)$.

As a consequence of Theorem 0.1, we get a global approximation theorem.

Theorem 0.2 *If M is a connected, $\mathcal{C}^\infty$-smooth, non compact, 1-concave, CR-generic manifold of real codimension k in a complex manifold X of complex dimension n, $n \geq 3$, and p an integer, $0 \leq p \leq n$, then each continuous, $\overline{\partial}_M$-closed, $(p, n-k-1)$-form in M can be approximated uniformly on compact subsets of M by $\overline{\partial}_M$-closed, $(p, n-k-1)$-forms of class $\mathcal{C}^\infty$ in M.*

Again this theorem holds without any global condition on M. In the case when M is a closed submanifold of a Stein manifold, it was proved by Airapetjan and Henkin (cf. [1], Th. 7.2.3).

1 Notations and definitions

Let X be a complex manifold of complex dimension n. If M is a $\mathcal{C}^{2+\ell}$-smooth real submanifold of real codimension k in X, we denote by $T^{\mathbb{C}}_\tau(M)$ the complex tangent space to M at $\tau \in M$.

Such a manifold M can be represented locally in the form

$$M = \{z \in \Omega | \rho_1(z) = \cdots = \rho_k(z) = 0\} \tag{1.1}$$

where the ρ_ν's, $1 \leq \nu \leq k$, are real $\mathcal{C}^{2+\ell}$ functions in an open subset Ω of X. If M is $\mathcal{C}^\infty$ smooth the functions ρ_ν may be chosen of class $\mathcal{C}^\infty$.

In this representation we have

$$T_\tau^{\mathbb{C}}(M) = \left\{\zeta \in \mathbb{C}^n \mid \sum_{j=1}^n \frac{\partial \rho_\nu}{\partial z_j}(\tau)\zeta_j = 0, \quad \nu = 1, \ldots, k\right\} \tag{1.2}$$

and $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) \geq n-k$, for $\tau \in M \cap \Omega$, where $(z_1, \ldots, z_n)$ are local holomorphic coordinates in a neighborhood of τ.

Definition 1.1 *The submanifold M is called CR if the number $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M)$ is independent of the point $\tau \in M$. If moreover $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) = n-k$ for every $\tau \in M$, M is then called CR* generic.

In the local representation, M is CR generic if and only if

$$\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k \neq 0 \text{ on } M.$$

Definition 1.2 *Let M be a $\mathcal{C}^{2+\ell}$-smooth CR generic submanifold of X. M is* 1-concave, *if for each $\tau \in M$, each local representation of M of type (1.1) in a neighborhood of τ in X and each $x \in \mathbb{R}^k \setminus \{0\}$, the quadratic form on $T_\tau^{\mathbb{C}}(M)$ defined by $\sum\limits_{\alpha,\beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \overline{z}_\beta}(\tau)\zeta_\alpha \overline{\zeta}_\beta$, where $\rho_x = x_1\rho_1 + \cdots + x_k\rho_k$ and $\zeta \in T_\tau^{\mathbb{C}}(M)$, has at least one negative eigenvalue.*

The bundle of (p,q)-forms on M, denoted by $\Lambda^{p,q}|_M$, is, by definition, the restriction of the bundle $\Lambda^{p,q}$ of (p,q)-forms in X to the submanifold M. Thus a section f of $\Lambda^{p,q}|_M$ is obtained locally from an ambiant form by restriction of the coefficients of the (p,q)-form to M. We denote by $\mathcal{C}^\ell_{p,q}(M)$ (resp. $\mathcal{C}^\infty_{p,q}(M)$, if M is $\mathcal{C}^\infty$-smooth) the $\mathcal{C}^\ell$ (resp. $\mathcal{C}^\infty$) sections of the bundle $\Lambda^{p,q}|_M$.

Following Kohn and Rossi [10], two forms $f, g \in \mathcal{C}^\ell_{p,q}(M)$ (resp. $\mathcal{C}^\infty_{p,q}(M)$) are said to be equal if and only if $\int_M f \wedge \varphi = \int_M g \wedge \varphi$ for every form $\varphi \in \mathcal{C}^\infty_{n-p,n-k-q}(X)$ with compact support.

We set on $\mathcal{C}^\ell_{p,q}(M)$ the topology of uniform convergence of the coefficients and all their derivatives up to order ℓ on compact subsets of M. This topology will be called the $\mathcal{C}^\ell$-topology on M. The dual space of $\mathcal{C}^\ell_{p,q}(M)$ is denoted by $\mathcal{E}'^\ell_{n-p,n-k-q}(M)$, it is the space of $(n-p,n-k-q)$-currents of order ℓ with compact support on M. If M is of class $\mathcal{C}^\infty$, then the space $\mathcal{C}^\infty_{p,q}(M)$ is provided with the topology of uniform convergence of the coefficients and all their derivatives on compact subsets of M. Its dual $\mathcal{E}'_{n-p,n-k-q}(M)$ is the space of $(n-p,n-k-q)$-currents with compact support on M.

We denote by $\mathcal{D}'^\ell_{p,q}(M)$ the space of (p,q)-currents of order l on M, this space is the dual of the space $\mathcal{D}^\ell_{n-p,n-k-q}(M)$ of $\mathcal{C}^\ell$-smooth $(n-p,n-k-q)$-forms with compact support on M provided with its usual inductive limit topology. If M is of class $\mathcal{C}^\infty$, $\mathcal{D}'_{p,q}(M)$ denotes the space of (p,q)-currents on M, this space is the dual of the space $\mathcal{D}_{n-p,n-k-q}(M)$ of $\mathcal{C}^\infty$-smooth $(n-p,n-k-q)$-forms with compact support on M provided with its usual inductive limit topology.

We denote by $\overline{\partial}_M$ the tangential Cauchy-Riemann operator on M.

A current $f\in\mathcal{D}'^\ell_{p,q}(M)$ is called CR if and only if $\overline{\partial}_M f=0$.

If U is an open subset of M, then for $\ell\in\mathbb{N}\cup\{\infty\}$,

$Z^\ell_{p,q}(U)$ is the Frechet space of CR (p,q)-forms of class $\mathcal{C}^\ell$ on U;

$E^\ell_{p,q}(U)$ is the subspace of $Z^\ell_{p,q}(U)$ of the forms f such that $f=\overline{\partial}_M g$ with $g\in\mathcal{C}^\ell_{p,q-1}(U)$;

$H^{p,q}_\ell(U)$ denotes the quotient space $Z^\ell_{p,q}(U)/E^\ell_{p,q}(U)$;

If Ω is a relatively compact open subset in M, we denote by $\mathcal{C}^\ell_{p,q-1}(\overline{\Omega})$ the Banach space of (p,q)-forms of class $\mathcal{C}^\ell$ on $\overline{\Omega}$ and by $\mathcal{C}^{\ell+\alpha}_{p,q-1}(\overline{\Omega})$ the Banach space of (p,q)-forms whose coefficients are of class $\mathcal{C}^{\ell+\alpha}$, $0<\alpha<1$, on $\overline{\Omega}$.

If D is a relatively compact open subset in M, we denote by germ $\mathcal{C}^\ell_{p,q}(\overline{D})$ the space of germs of (p,q)-forms of class $\mathcal{C}^\ell$ in neighborhoods of $\overline{D}$. Then germ $Z^\ell_{p,q}(\overline{D})$ is the space of germs of CR (p,q)-forms of class $\mathcal{C}^\ell$ in neighborhoods of $\overline{D}$, germ $E^\ell_{p,q}(\overline{D})$ = germ $Z^\ell_{p,q}(\overline{D})\cap\overline{\partial}_M$ germ $C^\ell_{p,q-1}(\overline{D})$ and germ $H^{p,q}_\ell(\overline{D})$ = germ $Z^\ell_{p,q}(\overline{D})/$ germ $E^\ell_{p,q}(\overline{D})$.

2 Proof of Malgrange's theorem in the $\mathcal{C}^\ell$-case

Let X be a complex manifold of complex dimension n, $n\geq 3$, M a connected, $\mathcal{C}^{2+\ell}$-smooth, $\ell\in\mathbb{N}$, non compact, 1-concave, CR generic submanifold of real codimension k in X, and p an integer, $0\leq p\leq n$.

Local results We need first a result on the local solvability of the tangential Cauchy-Riemann equation in top degree on M.

Proposition 2.1 *For every point z_0 in M, one can find a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$, there exists a continuous linear operator K_Ω from $\mathcal{C}^\ell_{p,n-k}(\overline{\Omega})$ into $\mathcal{C}^{\ell+\frac{1}{2}}_{p,n-k-1}(\overline{\Omega})$ which satisfies $\overline{\partial}_M K_\Omega f = f$ for all differential forms f in $\mathcal{C}^\ell_{p,n-k}(\overline{\Omega})$.*

Proof. — This result can be easily deduced from Theorem 0.1 in [2]. Under the hypothesis $\ell > 0$, a slightly weaker result, also sufficient for our application, is given in Theorem 7.1.2 of [1]. □

We shall use also some approximation theorem for $\overline{\partial}_M$-closed $(p, n-k-1)$-differential forms.

Definition 2.2 *Let U and V be two open subsets of M such that $U \subset V$. We shall say that U* has no hole with respect to *V if for each compact subset K of U there exists a compact subset $\widetilde{K}$ of U such that $K \subset \widetilde{K}$ and $V \setminus \widetilde{K}$ has no connected component which is relatively compact in V.*

Proposition 2.3 *For every point z_0 in M, there exists a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$ without hole with respect to M_0 the image of the restriction map*

$$Z^\ell_{p,n-k-1}(M_0) \longrightarrow Z^\ell_{p,n-k-1}(\Omega)$$

is dense with respect to the uniform convergence of the coefficients and all their derivatives up to order ℓ on compact subsets of Ω.

Proof. — Let z_0 be a fixed point in M. By the Hahn-Banach theorem, it is sufficient to prove that there exists a neighborhood M_0 of z_0 in M such that for each open subset $\Omega \subset\subset M_0$ without hole with respect to M_0, if L is a continuous linear form on $\mathcal{C}^\ell_{p,n-k-1}(\Omega)$, whose restriction to $Z^\ell_{p,n-k-1}(M_0)$ vanishes, then the restriction of L to $Z^\ell_{p,n-k-1}(\Omega)$ is identically equal to zero. Note that such a linear form L is a $\overline{\partial}_M$-closed $(n-p,1)$-current of order ℓ on M_0, with compact support in Ω. By Theorem 1' in [7] (see also Theorem 2.4 in [11]) in the case $\ell = 0$ and their direct generalization, using Proposition 2.1, to the case $\ell > 0$, we can find a neighborhood M_0 of z_0 in M on which we can solve the $\overline{\partial}_M$-equation with compact support in M_0 in bidegree $(n-p,1)$ for currents of order ℓ. We choose such an M_0 and $\Omega \subset\subset M_0$, then for $L \in \mathcal{E}'^\ell_{p,n-k-1}(\Omega)$ with $L_{|Z^\ell_{p,n-k-1}(M_0)} \equiv 0$, there exists a $(p,0)$-form T with compact support in M_0 such that $\overline{\partial}_M T = L$. The $(p,0)$-form T is CR on $M_0 \setminus \operatorname{supp} L$ and vanishes on an open subset of $M_0 \setminus \operatorname{supp} L$. Since M is 1-concave, if Ω has no hole with respect to M_0, then T vanishes on a neighbordhood of $M_0 \setminus \Omega$ by analytic extension (cf. [6]). Consequently the support of T is contained in Ω. Let $f \in Z^\ell_{p,n-k-1}(\Omega)$, then by the Airapetjan-Henkin Theorem 7.2.1 in [1], f can be approximated locally by $\mathcal{C}^{\ell+1}$-smooth $\overline{\partial}_M$-closed $(p,n-k-1)$-differential forms. Let $(U_i)_{i\in I}$ be a finite open covering of the support of T by open subsets satisfying the Airapetjan-Henkin approximation theorem and for each $i \in I$, $(f^i_\nu)_{\nu\in\mathbb{N}}$ a sequence of $\mathcal{C}^\infty$-smooth $\overline{\partial}_M$-closed $(p,n-k-1)$-differential forms in U_i, which converges to f on U_i in the $\mathcal{C}^\ell$-topology. If $(\chi_i)_{i\in I}$ denotes a partition of unity subordinated to the covering $(U_i)_{i\in I}$, then setting $f_\nu = \sum_{i\in I} \chi_i f^i_\nu$ we get a sequence $(f_\nu)_{\nu\in\mathbb{N}}$ of $\mathcal{C}^{\ell+1}$-smooth $(p,n-k-1)$-differential forms which converges to f on Ω in the $\mathcal{C}^\ell$-topology and such that the sequence $(\overline{\partial}_M f_\nu)_{\nu\in\mathbb{N}}$ converges to zero on Ω in the $\mathcal{C}^\ell$-topology.We obtain

$$L(f) = \lim_{\nu\to\infty} L(f_\nu) = \lim_{\nu\to\infty} \langle \overline{\partial}_M T, f_\nu \rangle = \lim_{\nu\to\infty} \langle T, \overline{\partial}_M f_\nu \rangle = 0.$$

□

A first global consequence of the local results By standard arguments (see e.g. the proofs of Lemma 2.3.1 in [8] and Proposition 3 in Appendix 2 of [8]), it follows from Proposition 2.1 that, if D is a relatively compact open subset of M, $E^\ell_{p,n-k}(\overline{D})$ is closed and finite codimensional in $Z^\ell_{p,n-k}(\overline{D})$. Moreover we have

Proposition 2.4 *Let D be a relatively compact open subset of M. There exists a continuous linear operator $A : Z^\ell_{p,n-k}(\overline{D}) \to \mathcal{C}^\ell_{p,n-k-1}(\overline{D})$ such that $\overline{\partial}_M Af = f$ for all $f \in E^\ell_{p,n-k}(\overline{D})$.*

The bumping method

Definition 2.5 *A* bump *in M is an ordered collection $[M_0, \Omega_1, \Omega_2]$, where M_0, Ω_1 and Ω_2 are open subsets of M such that*

(i) *M_0 is as in Propositions 2.1 and 2.3.*

(ii) *Ω_1 and Ω_2 have $\mathcal{C}^2$-smooth boundary and $\Omega_1 \subset \Omega_2 \subset\subset M_0$.*

(iii) *$\overline{\Omega}_1$ admits a basis of neighborhoods without hole with respect to M_0.*

Note that $\Omega_1 = \emptyset$ is allowed in this definition.

Definition 2.6 *An* extension element *in M is an ordered pair $[D_1, D_2]$, where $D_1 \subset D_2$ are open subsets with $\mathcal{C}^2$-boundary in M such that there exists a bump $[M_0, \Omega_1, \Omega_2]$ in M with the following properties:*

$$D_2 = D_1 \cup \Omega_2, \quad \Omega_1 = D_1 \cap \Omega_2 \quad \textit{and} \quad \overline{(D_1 \setminus \Omega_2)} \cap \overline{(\Omega_2 \setminus \Omega_1)} = \emptyset.$$

Proposition 2.7 *Let $[D_1, D_2]$ be an extension element in M, then the restriction map*

$$\mathrm{germ}\, H_\ell^{p,n-k}(\overline{D}_2) \longrightarrow \mathrm{germ}\, H_\ell^{p,n-k}(\overline{D}_1)$$

is injective.

Proof. — Let $U_1 \subset U_2$ be open neighborhoods of $\overline{D}_1$ and $\overline{D}_2$ in M respectively and let $f \in Z^\ell_{p,n-k}(U_2)$ and $u_1 \in \mathcal{C}^\ell_{p,n-k-1}(U_1)$ be given such that $\overline{\partial}_M u_1 = f$ on U_1. We have to prove the existence of a neighborhood $W_2 \subset U_2$ of $\overline{D}_2$ in M and of a differential form $u_2 \in \mathcal{C}^\ell_{p,n-k-1}(W_2)$ with $\overline{\partial}_M u_2 = f$ on W_2.

Let $[M_0, \Omega_1, \Omega_2]$ be the bump associated to the extension element $[D_1, D_2]$ and $V_2 \subset\subset U_2 \cap M_0$ a neighborhood of $\overline{\Omega}_2$ in M. By Proposition 2.1, there exists $u \in \mathcal{C}^\ell_{p,n-k-1}(V_2)$ such that $\overline{\partial}_M u = f$ on V_2. Hence we get $\overline{\partial}_M(u_1 - u) = 0$ on $U_1 \cap V_2$. We choose a neighborhood $W_1 \subset U_1 \cap V_2$ of $\overline{\Omega}_1$ without hole with respect to M_0, then by Proposition 2.3, we can find a sequence $(\omega_\nu)_{\nu\in\mathbb{N}} \subset Z^\ell_{p,n-k-1}(M_0)$ which converges to $u_1 - u$ in the $\mathcal{C}^\ell$-topology on W_1. Let V be a neighborhood of $\overline{\Omega_2 \setminus \Omega_1}$ such that $V \subset V_2 \cap M_0$ and $V \cap \overline{(D_1 \setminus \Omega_2)} = \emptyset$, and χ a $\mathcal{C}^{\ell+1}$-smooth function with compact support in V equal to 1 on a neighborhood $\widetilde{V}$ of $\overline{\Omega_2 \setminus \Omega_1}$. Setting $v_\nu = (1-\chi)u_1 + \chi(u + w_\nu)$, we define a sequence $(v_\nu)_{\nu\in\mathbb{N}}$ in $\mathcal{C}^\ell_{p,n-k-1}(U_1 \cup V)$ such that the sequence $\overline{\partial}_M v_\nu = f - \overline{\partial}_M \chi \wedge (u_1 - u - w_\nu)$ converges to f in the $\mathcal{C}^\ell$-topology on the neighborhood $\widetilde{U}_1 \cup \widetilde{V}$ of $\overline{D}_2$ in M, where $\widetilde{U}_1$ is a neighborhood of $\overline{D}_1$ such that $\widetilde{U}_1 \subset U_1$ and $\widetilde{U}_1 \cap V = W_1 \cap V$. Let $W_2 \subset\subset \widetilde{U}_1 \cup \widetilde{V}$ be a neighborhood of $\overline{D}_2$. Then, using Proposition 2.4, we get a $(p, n-k-1)$-differential form u_2 of class $\mathcal{C}^\ell$ on W_2 such that $\overline{\partial}_M u_2 = f$ on W_2. □

Proposition 2.8 *Let $[D_1, D_2]$ be an extension element in M such that $D_1 \subset\subset M$, then the restriction map*

$$\mathrm{germ}\, Z^\ell_{p,n-k-1}(\overline{D}_2) \longrightarrow \mathrm{germ}\, Z^\ell_{p,n-k-1}(\overline{D}_1)$$

has dense image with respect to uniform convergence of the coefficients and their derivatives up to order ℓ on $\overline{D}_1$.

Proof. — Let U_1 be an open neighborhood of $\overline{D}_1$ in M and $[M_0, \Omega_1, \Omega_2]$ the bump associated to the extension element $[D_1, D_2]$. Let $f \in Z^{\ell}_{p,n-k-1}(U_1)$ be given and $W_1 \subset U_1$ a neighborhood of $\overline{\Omega}_1$ without hole with respect to M_0. By Proposition 2.3, there exists a sequence $(g_\nu)_{\nu\in\mathbb{N}} \subset Z^{\ell}_{p,n-k-1}(M_0)$ which converges to f in the $\mathcal{C}^\ell$-topology on W_1. Let V be a neighborhood of $\overline{\Omega_2 \setminus \Omega_1}$ such that $V \subset M_0$ and $V \cap \overline{(D_1 \setminus \Omega_2)} = \emptyset$, and χ a $\mathcal{C}^{\ell+1}$-smooth function with compact support in V equal to 1 on a neighborhood $\widetilde{V}$ of $\overline{\Omega_2 \setminus \Omega_1}$. Setting $\tilde{f}_\nu = (1-\chi)f + \chi g_\nu$, we define a sequence $(\tilde{f}_\nu)_{\nu\in\mathbb{N}}$ of forms of class $\mathcal{C}^\ell$ on the neighborhood $U_1 \cup V$ of $\overline{D}_2$, which converges to f in the $\mathcal{C}^\ell$-topology on $\overline{D}_1$. Moreover, since $\overline{\partial}_M \tilde{f}_\nu = \overline{\partial}_M \chi \wedge (f - g_\nu)$ the sequence $(\overline{\partial}_M \tilde{f}_\nu)_{\nu\in\mathbb{N}}$ converges to zero in the $\mathcal{C}^\ell$-topology on $U_2 = \widetilde{U}_1 \cup \widetilde{V}$, where $\widetilde{U}_1$ is a neighborhood of $\overline{D}_1$ such that $\widetilde{U}_1 \subset U_1$ and $\widetilde{U}_1 \cap V = W_1 \cap V$. As $D_1 \subset\subset M$, we can choose a relatively compact neighborhood W_2 of $\overline{D}_2$ in M and apply Proposition 2.4. Therefore, there exists a sequence $(u_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{C}^{\ell}_{p,n-k-1}(\overline{W}_2)$ which converges to zero in the $\mathcal{C}^\ell$-topology on $\overline{W}_2$ and satisfies $\overline{\partial}_M u_\nu = \overline{\partial}_M \tilde{f}_\nu$. If $f_\nu = \tilde{f}_\nu - u_\nu$, we get a sequence $(f_\nu)_{\nu\in\mathbb{N}} \subset Z^{\ell}_{p,n-k}(W_2)$ which converges to f in the $\mathcal{C}^\ell$-topology on $\overline{D}_1$. □

We need now two technical lemmas about the existence of extension elements to jump from one level of an exhausting function on M to another level.

Lemma 2.9 *Let φ be a function of class $\mathcal{C}^2$ on M and z_0 a non degenerate critical point for φ. Suppose $\varphi(z_0) = 0$, $\varphi^{-1}(0)$ is compact and z_0 is the only critical point on $\varphi^{-1}(0)$. Then there exists a neighborhood V_0 of z_0 in M such that for all neighborhood $V \subset\subset V_0$ of z_0 in M, we can find an extention element $[D_1, D_2]$ in M with the following properties:*

(i) $D_1 \supset \varphi^{-1}((-\infty, 0[) \setminus V$;

(ii) $z_0 \in D_2 \setminus \overline{D}_1 \subset V$.

Proof. — If z_0 is a point of local minimum, we choose V_0 so small that $V_0 \cap \varphi^{-1}((-\infty, 0[) = \emptyset$ and $M_0 \subset V \subset V_0$ a neighborhood of z_0 satisfying Propositions 2.1 and 2.3. Taking $\Omega_1 = \emptyset$, $\Omega_2 \subset\subset M_0$ a neighborhood of z_0 and setting $D_1 = \varphi^{-1}((-\infty, 0[)$ and $D_2 = D_1 \cup \Omega_2$, we get the required extension element.

Assume now that z_0 is not a point of local minimum. By the Morse lemma, there exist local real coordinates $(x_1, \dots, x_{2n})$ around z_0 in X such that $\varphi = x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_{2n-k}^2$. Let V_0 be a neighborhood of z_0 on which we are in the above situation and $M_0 \subset\subset V \subset V_0$ the intersection of M with a small ball centered in z_0 in holomorphic coordinates around z_0 as in Propositions 2.1 and 2.3. Let B be a ball centered in z_0 with respect to the Morse coordinates $(x_1, \dots, x_{2n-k})$ such that $B \subset M_0$, and U a small neighborhood of z_0 relatively compact in B. Let ε be equal to $\frac{1}{2} \min_{z \in \overline{U}} |\varphi(z)|$. We choose $\theta \in \mathcal{D}(U)$ such that $0 < \theta(z) < \varepsilon$, if $z \in U$, and we set $\Omega_1 = \{z \in B \mid \varphi(z) + \theta(z) < 0\}$ and $\Omega_2 = \{z \in B \mid \varphi(z) - \theta(z) < 0\}$. Then it is clear that Ω_1 has no hole with respect to B (it is sufficient to look at the picture in the Morse coordinates) and as the boundary of B is connected and M_0 has no compact connected component then Ω_1 has also no hole with respect to M_0. Smoothing the boundary of Ω_1 and Ω_2 we get a bump $[M_0, \Omega_1, \Omega_2]$ in M such that $D_2 = \varphi^{-1}((-\infty, 0[) \cup \Omega_2$ and $D_1 = D_2 \smallsetminus (\overline{\Omega}_2 \smallsetminus \Omega_1)$ have the required properties. □

From Lemma 2.9, one easily obtains the following lemma (cp. the proof of Theorem 7.10 in [12]).

Lemma 2.10 *Let φ be a function of class $\mathcal{C}^2$ on M all critical points of which are non degenerate such that the following conditions are fulfilled:*

(i) *no critical point of φ lies on $\varphi^{-1}(\{0,1\})$;*

(ii) *$\varphi^{-1}([0,1])$ is compact;*

(iii) *φ has no point of local maximum in $\varphi^{-1}(]0,1[)$.*

Then there exists a finite number of extension elements $[D_j, D_{j+1}]$, $j = 0, \dots, N$, such that $D_0 = \varphi^{-1}((-\infty, 0[)$ and $D_{N+1} = \varphi^{-1}((-\infty, 1[)$.

As an easy consequence of Propositions 2.7 and 2.8 and Lemma 2.10, we obtain the following result:

Proposition 2.11 *Let φ be a real exhausting function of class $\mathcal{C}^2$ on M without local maximum and such that all critical points of φ are non degenerate. Let $\alpha, \beta \in \varphi(M)$ with $\alpha < \beta$ and such that no critical point of φ lies on $\varphi^{-1}(\{\alpha, \beta\})$ and set $D_\alpha = \varphi^{-1}((-\infty, \alpha[)$ and $D_\beta = \varphi^{-1}((-\infty, \beta[)$.*

(i) *The restriction map*

$$\operatorname{germ} H_\ell^{p,n-k}(\overline{D}_\beta) \longrightarrow \operatorname{germ} H_\ell^{p,n-k}(\overline{D}_\alpha)$$

is injective

(ii) *The restriction map*

$$\text{germ}\, Z^{\ell}_{p,n-k-1}(\overline{D}_\beta) \longrightarrow \text{germ}\, Z^{\ell}_{p,n-k-1}(\overline{D}_\alpha)$$

has dense image with respect to uniform convergence of the coefficients and their derivatives up to order ℓ *on* $\overline{D}_\alpha$.

Proof of the first assertion of Theorem 0.1 We may now conclude the proof of our Malgrange type theorem in non compact, 1-concave CR manifolds in the $\mathcal{C}^\ell$ case, $\ell \in \mathbb{N}$.

Since M is connected and not compact, by a theorem of Green and Wu [4], M admits a real exhausting function φ of class $\mathcal{C}^2$ without local maximum and we may assume that all critical points of φ are non degenerate (cp. e.g. [5]). Let z_0 be a point where φ takes its minimum value. By Proposition 2.1, there exists a neighborhood Ω_0 of z_0 such that $H^{p,n-k}_\ell(D) = 0$ for all $D \subset\subset \Omega_0$. As φ is an exhausting function on M, it admits only a finite number of points where φ takes its minimum value. We denote by Ω the union of the previous neighborhoods associated to these points and we choose $\alpha_0 \in \varphi(M)$ such that $\varphi^{-1}((-\infty, \alpha_0[)$ is not empty and contained in Ω and $(\alpha_j)_{j\geq 1} \subset \varphi(M)$ such that no critical point of φ lies on $\varphi^{-1}(\alpha_j)$, $j{\geq}0$, and if $D_j = \varphi^{-1}((-\infty, \alpha_j[)$, $D_j \subset D_{j+1}$ for $j{\geq}0$ and $M = \bigcup\limits_{j\geq 0} D_j$. We deduce from Proposition 2.11 (i) and from the choice of D_0 that, for all $j{\geq}0$,

$$\text{germ}\, H^{p,n-k}_\ell(\overline{D}_j) = 0.$$

Let $f \in Z^\ell_{p,n-k}(M)$ be given. Then from Proposition 2.11 (ii) we obtain a sequence $(u_j)_{j\in\mathbb{N}}$ such that $u_j \in \text{germ}\, \mathcal{C}^\ell_{p,n-k}(\overline{D}_j)$, $\overline{\partial}_M u_j = f$ on a neighborhood of $\overline{D}_j$ and $\|u_{j+1} - u_j\|_{\ell,\overline{D}_j} \leq \frac{1}{2^j}$. Hence $u = \lim\limits_{j\to\infty} u_j$ exists, belongs to $\mathcal{C}^\ell_{p,n-k-1}(M)$, and solves the equation $\overline{\partial}_M u = f$ on M. □

3 Malgrange's theorem in the $\mathcal{C}^\infty$ case

We shall first prove an approximation theorem in 1-concave CR manifolds, which is a direct consequence of Malgrange's theorem in the $\mathcal{C}^\ell$-case. Then we shall use this theorem to get Malgrange's theorem in the $\mathcal{C}^\infty$-case.

Let X be a complex manifold of complex dimension n, $n{\geq}3$, M a connected, $\mathcal{C}^{2+\ell+1}$-smooth, $\ell \in \mathbb{N}$, non compact, 1-concave, CR generic submanifold of real codimension k in X and p an integer, $0 \leq p \leq n$.

Theorem 3.1 *The space $Z^{\ell+1}_{p,n-k-1}(M)$ is dense in the space $Z^{\ell}_{p,n-k-1}(M)$ for the topology of uniform convergence of the coefficients and their derivatives up to order ℓ on each compact subset of M.*

Proof. — By the Hahn-Banach theorem, it is sufficient to prove that for any $T \in \mathcal{E}'^{\ell}_{n-p,1}(M)$ such that $\langle T, \varphi \rangle = 0$ for all $\varphi \in Z^{\ell+1}_{p,n-k-1}(M)$ we have $\langle T, \psi \rangle = 0$ for all $\psi \in Z^{\ell}_{p,n-k-1}(M)$. Note that the hypothesis on T implies that T is $\overline{\partial}_M$-closed. We shall prove that T is $\overline{\partial}_M$-exact on M.

We define a linear form L on $\mathcal{C}^{\ell+1}_{p,n-k}(M)$ by setting $L(\varphi) = \langle T, \psi \rangle$ for $\varphi \in \mathcal{C}^{\ell+1}_{p,n-k}(M)$, where $\overline{\partial}_M \Psi = \varphi$. The application L is well defined since first $H^{p,n-k}_{\ell+1}(M) = 0$ and consequently all $\varphi \in \mathcal{C}^{\ell+1}_{p,n-k}(M)$ can be written in the form $\varphi = \overline{\partial}_M \psi$ with $\psi \in \mathcal{C}^{\ell+1}_{p,n-k-1}(M)$ and second $\langle T, \psi \rangle$ is independent of the choice of ψ satisfying $\overline{\partial}_M \psi = \varphi$ because $T_{|_{Z^{\ell+1}_{p,n-k-1}(M)}} = 0$.

Moreover $\overline{\partial}_M$ is a closed operator between $\mathcal{C}^{\ell+1}_{p,n-k-1}(M)$ and $\mathcal{C}^{\ell+1}_{p,n-k}(M)$ which is surjective since $H^{p,n-k}_{\ell+1}(M) = 0$, consequently by the open mapping theorem this implies the continuity of L. It follows that L can be represented by a current $S \in \mathcal{E}'^{\ell+1}_{n-p,0}$ which satisfies

$$\langle \overline{\partial}_M S, \varphi \rangle = \langle S, \overline{\partial}_M \varphi \rangle = \langle T, \varphi \rangle$$

for all $\varphi \in \mathcal{C}^{\infty}_{p,n-k-1}(M)$, *i.e.* $\overline{\partial}_M S = T$. By regularity of $\overline{\partial}_M$ in bidegree $(n-p, 1)$, the $(n-p, 0)$-current S is of order ℓ since T is of order ℓ.

It remains to prove that $\langle T, \psi \rangle = 0$ for all $\psi \in Z^{\ell}_{p,n-k-1}(M)$. Let $\psi \in Z^{\ell}_{p,n-k-1}(M)$.In the same way as at the end of the proof of Proposition 2.3, we can construct a sequence $(\psi_\nu)_{\nu \in \mathbb{N}}$ of $\mathcal{C}^{\ell+1}$-smooth $(p, n-k-1)$-differential forms which converges to ψ on M in the $\mathcal{C}^{\ell}$-topology and such that the sequence $(\overline{\partial}_M \psi_\nu)_{\nu \in \mathbb{N}}$ converges to zero on M in the $\mathcal{C}^{\ell}$-topology. It follows that

$$\langle T, \psi \rangle = \lim_{\nu \to \infty} \langle T, \psi_\nu \rangle = \lim_{\nu \to \infty} \langle \overline{\partial}_M S, \psi_\nu \rangle = \lim_{\nu \to \infty} \langle S, \overline{\partial}_M \psi_\nu \rangle = 0.$$

□

Assume now that M is $\mathcal{C}^{\infty}$-smooth, we shall prove that $H^{p,n-k}_{\infty}(M) = 0$.

Proof of Malgrange's theorem in the $\mathcal{C}^{\infty}$-case. — Since M is connected and not compact, by a theorem of Green and Wu [4], M admits a real exhausting function φ of class $\mathcal{C}^{\infty}$ without local maximum and we may assume that all critical points of φ are non degenerate. Following the proof of the $\mathcal{C}^{\ell}$-case we can construct a sequence $(D_j)_{j \in \mathbb{N}}$ of open subsets of M such that $D_j \subset D_{j+1}$ and $M = \bigcup_{j \geq 0} D_j$ and satisfying the following two conditions:

(i) germ $H_j^{p,n-k}(\overline{D}_j) = 0$.

(ii) The restriction map

$$\text{germ}\, Z^j_{p,n-k-1}(\overline{D}_{j+1}) \longrightarrow \text{germ}\, Z^j_{p,n-k-1}(\overline{D}_j)$$

has dense image with respect to the $\mathcal{C}^j$-topology.

Let $f \in Z^\infty_{p,n-k}(M)$ and $\varepsilon > 0$ be given. Then we can construct a sequence $(u_j)_{j\in\mathbb{N}}$ such that $u_j \in \text{germ}\, \mathcal{C}^j_{p,n-k-1}(\overline{D}_j)$, $\overline{\partial}_M u_j = f$ on a neighborhood of $\overline{D}_j$ and $\|u_{j+1} - u_j\|_{\overline{D}_j,j} < \frac{\varepsilon}{2^j}$. By (i) there exists $u_0 \in \text{germ}\, \mathcal{C}^0_{p,n-k-1}(\overline{D}_0)$ such that $\overline{\partial}_M u_0 = f$ on a neighborhood of $\overline{D}_0$. Assume now that we have already constructed $(u_j)_{0\leq j\leq j_0}$. By (i) there exists $\tilde{u}_{j_0+1} \in \text{germ}\, \mathcal{C}^{j_0+1}_{p,n-k-1}(\overline{D}_{j_0+1})$ such that $\overline{\partial}_M \tilde{u}_{j_0+1} = f$ on a neighborhood of $\overline{D}_{j_0+1}$. Then $\tilde{u}_{j_0+1} - u_{j_0} \in \text{germ}\, Z^{j_0}_{p,n-k-1}(\overline{D}_{j_0+1})$ and by (ii) we can find $v_{j_0+1} \in \text{germ}\, Z^{j_0}_{p,n-k-1}(\overline{D}_{j_0+1})$ such that $\|\tilde{u}_{j_0+1} - u_{j_0} - v_{j_0+1}\|_{\overline{D}_{j_0},j_0} < \frac{1}{2}\frac{\varepsilon}{2^{j_0}}$. Moreover by Theorem 3.1, we choose $\tilde{v}_{j_0+1} \in \text{germ}\, Z^{j_0+1}_{p,n-k-1}(\overline{D}_{j_0+1})$ with $\|\tilde{v}_{j_0+1} - v_{j_0+1}\|_{\overline{D}_{j_0+1},j_0} < \frac{1}{2}\frac{\varepsilon}{2^{j_0}}$. Setting $u_{j_0+1} = \tilde{u}_{j_0+1} - \tilde{v}_{j_0+1}$, then u_{j_0+1} has the required properties. It follows from the properties of the forms u_j that the sequence $(u_j)_{j\in\mathbb{N}}$ converges to a form u uniformly on each compact subset of M and moreover $u \in \mathcal{C}^\infty_{p,n-k-1}(M)$ and $\overline{\partial}_M u = f$ on M. □

As a consequence of Malgrange's theorem, we get an approximation theorem which generalizes Theorem 7.2.3 in [1].

Theorem 3.2 *Let M be a connected, $\mathcal{C}^\infty$-smooth, non compact, 1 concave, CR generic submanifold of real codimension k in a complex manifold of complex dimension n, and p an integer, $0 \leq p \leq n$, then the space $Z^\infty_{p,n-k-1}(M)$ is dense in the space $Z^0_{p,n-k-1}(M)$ with respect to uniform convergence on compact subsets of M.*

Proof. — The proof is analogous to the proof of Theorem 3.1, we have only to use that $H^{p,n-k}_\infty(M)$ vanishes instead of $H^{p,n-k}_{\ell+1}(M)$ and replace ℓ by zero and $\ell + 1$ by ∞. □

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