SOME NEW SEPARATION THEOREMS FOR THE DOLBEAULT COHOMOLOGY

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0. Introduction and statement of the results

If X is an n-dimensional complex manifold and E a holomorphic vector bundle over X, then we denote by $C^0_{s,r}(X,E)$ the Fréchet space of continuous E-valued (s,r)-forms on X, by $Z^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-closed forms, and by $E^0_{s,r}(X,E)$ the subspace of $\overline{\partial}$-exact forms ($E^0_{s,0}(X,E) := \{0\}$). As usual, the factor space

$$H^{s,r}(X,E) := Z^0_{s,r}(X,E)/E^0_{s,r}(X,E).$$

will be considered as topological vector space endowed with the factor topology. Recall that this topology is separated if and only if $E^0_{s,r}(X,E)$ is closed with respect to the topology of $C^0_{s,r}(X,E)$. If E is the trivial line bundle, then we write also $C^0_{s,r}(X)$ instead of $C^0_{s,r}(X,E)$ etc.

0.1. Definition. Let X be an n-dimensional complex manifold and let q, q^* be integers with $1 \leq q \leq n-1$ and $0 \leq q^* \leq n$. X will be called *q-concave-q^*-convex* if there exists a real C^2 function ρ on X such that, if $\inf\rho := \inf_{\zeta\in X}\rho(\zeta)$ and $\sup\rho := \sup_{\zeta\in X}\rho(\zeta)$, then $\inf\rho < \rho(\zeta)$ for all $\zeta \in X$, the sets $\{\alpha \leq \rho \leq \beta\}$, $\inf\rho < \alpha < \beta < \sup\rho$, are compact, and the following two conditions are fulfilled:

(i) There exists $\alpha \in]\inf\rho, \sup\rho[$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues everywhere on $\{\rho \leq \alpha\}$.

(ii) If $q^* = 0$, then, for all $\alpha \in]\inf\rho, \sup\rho[$, the set $\{\rho \geq \alpha\}$ is compact (and hence $\sup\rho = \max\rho$), i.e. X is q-concave in the sense of Andreotti-Grauert. If $1 \leq q^* \leq n-1$, then there exists $\beta \in]\inf\rho, \sup\rho[$ such that the Levi form of ρ has at least $n-q^*+1$ positive eigenvalues everywhere on $\{\rho \geq \beta\}$ (and hence $\sup\rho > \rho(\zeta)$ for all $\zeta \in X$).

The following separation theorem is well known:

0.2. Theorem. *Let X be an n-dimensional complex manifold which is q-concave-q^*-convex, where $1 \leq q \leq n-1$ and $0 \leq q^* \leq n-q-1$. Then, for any holomorphic vector bundle E over X, $H^{0,n-q}(X,E)$ is separated.*

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For $q^* = 0$ this theorem was proved by Andreotti and Vesentini [A-V]. The general case is contained in Theorem 2 of [R] of J.P. Ramis, where the more general situation of sheaves over complex spaces is studied. A simple direct proof of Theorem 0.2 is given in [La-L].

Consider the case

$$q^* = n - q, \qquad 1 \leq q \leq n-1.$$

First note that then it may happen that $H^{0,n-q}(X,E)$ is not separated. This follows from an example of Rossi [Ros] and Theorem 23.3 in [H-L] (for the details cp. the introduction of [La-L]). The example of Rossi is a 2-dimensional 1-concave-1-convex manifold such that $H^{0,1}(X)$ is not separated.[1]

However, in [H-L] it was proved that if ' the q-convex hole can be repaired', then nevertheless $H^{0,n-q}(X,E)$ is separated. More precisely, the following theorem holds:

0.3. Theorem. (cp. Theorem 19.1' in [H-L]) *Let X be an n-dimensional complex manifold which is q-concave-$(n-q)$-convex, $1 \leq q \leq n-1$, such that additional the following condition is fulfilled:*

(A) There exists a complex manifold Y with a relatively compact open subset H such that: X is an open subset of Y, $Y = X \cup H$ and if ρ is as in Definition 0.1, then, for certain γ with $\inf \rho < \gamma < \sup \rho$, *$X \cap H = \{\rho < \gamma\}$.*

Then, for any holomorphic vector bundle E over Y, $H^{0,n-q}(X,E)$ is separated.

The proof of Theorem 0.3 given in [H-L] is rather long and difficult and uses many estimates for integral operators of the Grauert-Henkin-Lieb type (not only the well known Hölder estimates). In Sect. 3 of the present paper we give a simple proof of Theorem 0.3 using only Andreotti-Grauert finiteness theorems and Serre duality.

In Sect. 4 we prove a finiteness and separation theorem for certain special families of support (compact with respect to a part of the boundary and arbitrary with respect to the other part).

Then in Sect. 5, using this result from Sect. 4 and the arguments of Sect. 3, we prove that the conclusion of Thorem 0.3 remains valid also without condition (A) if $q < n/2$, i.e. we prove the following

0.4. Theorem. *Let X be an n-dimensional complex manifold, $n \geq 3$, which is q-concave-$(n-q)$-convex, $1 \leq q \leq n-1$. If*

$$q < \frac{n}{2},$$

then, for any holomorphic vector bundle E over X, $H^{0,n-q}(X,E)$ is separated.

Note that for $q = 1$ the assertion of this theorem follows already from Theorem 0.3, because then, by a theorem of Rossi [Ros], the 1-concave 'hole' can be repaired.

[1]Note that there is a misprint in the formulation of Theorem 2 in [R] - by this formulation $H^{0,1}(X)$ should be separated also for the Rossi example.

In Sect. 6 we show that the arguments of Sect. 3 can be applied also to the situation considered in [Mi]. We prove the following theorem (see Corollary 6.3):

0.5 Theorem. *Let Y be a compact complex space of dimension n whose singular part S consists of a finite number of points. Set $X = Y \setminus S$, let some subset S_0 of S be fixed and denote by Φ the family of all closed subsets C of X such that $Y \setminus C$ is a neighborhood of S_0.*

Then, for any holomorphic vector bundle E over X, $H^{0,n-1}_{\Phi}(X, E)$ is separated.

In [Mi] this result was proved under the additional hypothesis that the bundle $K_X^{-1} \otimes E$ is extendable to S_0 as a holomorphic vector bundle. Note also that in the present paper the more general situation is admitted when the manifold X has arbitrary 1-convex 'holes' which, possibly, cannot be filled in by complex spaces.

In Sect. 7 we give some further applications of the separation theorem from Sect. 4. First we prove a version of Malgrange's vanishing theorem for the cohomology in top degree [M] (see also [O]) for forms vanishing around an '$(n-1)$-convex hole' (Theorem 7.2). Then we apply this to the Hartogs-Bochner extension problem (Theorem 7.3).

The basic tool of the present paper is Serre duality. Although Serre duality is well known [S,A-K,C-S], we could not find in the literature satisfactory proofs for all details which we need (cp., e.g., Problem 2.10). Therefore we begin the paper (Sects. 1 and 2) with a self contained presentation of this theory from our point of view, repeating also well known things, for the sake of completeness.

1. Families of supports and algebraic relations between corresponding Dolbeault groups

In this section X is an n-dimensional complex manifold countable at infinity.

By a *family of supports in* X we mean a collection Φ of closed subsets of X such that the following conditions are fulfilled (cf. [S]):

(S_1) if $C \in \Phi$, then each closed subset of C belongs to Φ;

(S_2) if $C_1, C_2 \in \Phi$, then $C_1 \cup C_2 \in \Phi$;

(S_3) for each $C \in X$ there exists an open neighborhood U of C with $\overline{U} \in \Phi$.

If Φ is a family of supports in X and $\Phi' \subseteq \Phi$ is a subfamily which is also a family of supports in X, then we denote by $\Phi * \Phi'$ the family of open sets $U \subseteq X$ such that $C \setminus U \in \Phi'$ for all $C \in \Phi$.

It is easy to see that every finite intersection of sets of $\Phi * \Phi'$ is in $\Phi * \Phi'$, and, unless $\Phi = \Phi'$, the empty set never belongs to $\Phi * \Phi'$. Furthermore, it is clear that $X \setminus C' \in \Phi * \Phi'$ if $C' \in \Phi'$. However it is not true in general that $X \setminus U \in \Phi'$ if $U \in \Phi * \Phi'$ (cf. Example II below).

1.2. Lemma-Definition. Let $\Phi' \subseteq \Phi$ be two families of supports in X. Then the follwing conditions are equivalent:

(i) There exists $C_0 \in \Phi$ with $\overline{C \setminus C_0} \in \Phi'$ for all $C \in \Phi$.

(ii) There exists $U \in \Phi * \Phi'$ with $\overline{U} \in \Phi$.

(iii) For each $U \in \Phi * \Phi'$ there exists $V \in \Phi * \Phi'$ such that $\overline{V} \subseteq U$ and $\overline{V} \in \Phi$.

If these equivalent conditions are fulfilled, then Φ' will be called *complete* in Φ.

Proof. (ii) $\Longrightarrow$ (i): Assume U is as in (ii). Set $C_0 = \overline{U}$ and consider $C \in \Phi$. Then, by definition of $\Phi * \Phi'$, $C \setminus U \in \Phi'$, and hence

$$C \setminus C_0 = C \setminus \overline{U} \subseteq C \setminus U \in \Phi'.$$

(i) $\Longrightarrow$ (ii): Let C_0 be as is (i). By condition (S_3) in the definition of a family of supports, there is an open neighborhood U of C_0 with $\overline{U} \in \Phi$. Then $C \setminus U \subseteq \overline{C \setminus C_0} \in \Phi'$ for all $C \in \Phi$. Hence $U \in \Phi * \Phi'$.

(ii) $\Longrightarrow$ (iii): Let $U \in \Phi * \Phi'$ be given. By (ii), we have $U_0 \in \Phi * \Phi'$ with $\overline{U_0} \in \Phi$. Set $W = U \cap U_0$. Then $W \in \Phi * \Phi'$ and $\overline{W} \in \Phi$. Hence the boundary $\partial W = \overline{W} \setminus W$ belongs to Φ'. Take an open neighborhood W' of ∂W with $\overline{W'} \in \Phi'$. Then $X \setminus \overline{W'} \in \Phi * \Phi'$ and hence

$$V := W \setminus \overline{W'} = W \cap (X \setminus \overline{W'}) \in \Phi * \Phi'.$$

Moreover it is clear that $\overline{V} \subseteq W \subseteq U$. Since $\overline{W} \in \Phi$ and $\overline{V} \subseteq \overline{W}$, we have also that $\overline{V} \in \Phi$.

(iii) $\Longrightarrow$ (ii) is trivial. □

It is easy to see that in each of the following examples, $\Phi' \subseteq \Phi$ are families of supports in X, where Φ' is complete in Φ.

Example I: Let Φ be the family of all closed subsets of X, and Φ' the family of the compact subsets of X. Then $\Phi * \Phi'$ consists of all complements of compact sets.

Example II: Let K be a fixed compact subset of X, Φ the family of the compact subsets of X, and Φ' the family of all $C \in \Phi$ with $K \cap C = \emptyset$. Then $\Phi * \Phi'$ is the family of neighborhoods of K.

Example III (cf. [Mi]): Let $X = \tilde{X} \setminus S$ where $\tilde{X}$ is a compact complex space whose singular points are isolated and S is the set of all singular points of $\tilde{X}$. Assume that S is divided into two non-empty subsets S_1 and S_2. Let Φ be the family of all closed subsets of X, and Φ' the family of all $C' \in \Phi$ such that $C' \cap U = \emptyset$ for some neighborhood U of S_1 in $\tilde{X}$. Then $\Phi * \Phi'$ is the family of open subsets U of X which are of the form $U = \tilde{U} \setminus S_1$ where $\tilde{U}$ is a neighborhood of S_1 in $\tilde{X}$.

Example IV: Let X be an open subset of $\mathbb{C}^n$, K a closed subset of the boundary of X in $\mathbb{C}^n$, Φ the family of all subsets of X which are closed in X, and Φ' the family of all $C' \in \Phi$ such that $C' \cap U = \emptyset$ for some $\mathbb{C}^n$-open neighborhood U of K. Then $\Phi * \Phi'$ consits of all sets of the form $U \cap X$ where U ranges over the $\mathbb{C}^n$-open neighborhoods of K.

In the forthcoming, let E be a holomorphic vector bundle over X.

We denote by $C^0_{s,r}(X, E)$ the space of E-valued continuous (s, r)-forms on X, by $Z^0_{s,r}(X, E)$ the subspace of $C^0_{s,r}(X, E)$ of $\overline{\partial}$-closed forms, and by $E^0_{s,r}(X, E)$ the subspace of $Z^0_{s,r}(X,E)$ of $\overline{\partial}$-exact forms ($E^0_{s,r}(X,E) := \{0\}$ if $r = 0$). As usual,

$$H^{s,r}(X,E) := Z^0_{s,r}(X,E)\big/E^0_{s,r}(X,E).$$

If Φ is a family of supports in X, then we use the following notations:

- $C^0_{s,r}(\Phi;X,E)$ is the space of all E-valued continuous (s,r)-forms f on X with $\operatorname{supp} f \in \Phi$.
- $Z^0_{s,r}(\Phi;X,E)$ is the subspace of all $\overline{\partial}$-closed forms in $C^0_{s,r}(\Phi;X,E)$.
- $E^0_{s,r}(\Phi;X,E) := C^0_{s,r}(\Phi;X,E) \cap \overline{\partial}C^0_{s,r-1}(\Phi;X,E)$ if $r \geq 1$;
- $E^0_{s,r}(\Phi;X,E) := \{0\}$ if $r = 0$;
- $H^{s,r}_{\Phi}(X,E) := Z^0_{s,r}(\Phi;X,E)\big/E^0_{s,r}(\Phi;X,E)$.

Note that $H^{s,r}_{\Phi}(X,E) = H^{s,r}(X,E)$ if Φ consists of all closed subsets of X. As usual, we write

$$H^{s,r}_c(X,E) := H^{s,r}_{\Phi}(X,E)$$

if Φ consists of the compact subsets of X.

Now let $\Phi' \subseteq \Phi$ be two families of supports in X. Then we use the following notations:

Two forms $f \in C^0_{s,r}(U,E)$, $g \in C^0_{s,r}(V,E)$ where $U,V \in \Phi * \Phi'$, will be called *equivalent* if there is an open $W \subseteq U \cap V$ with $f\big|_W = g\big|_W$. The corresponding space of equivalence classes of the disjoint union of all $C^0_{s,r}(U,E)$, $U \in \Phi * \Phi'$, will be denoted by $C^0_{s,r}(\Phi*\Phi',E)$. $Z^0_{s,r}(\Phi*\Phi',E)$ denotes the subspace of $C^0_{s,r}(\Phi*\Phi',E)$ defined by $\overline{\partial}$-closed forms, and $E^0_{s,r}(\Phi*\Phi',E)$ denotes the subspace of $Z^0_{s,r}(\Phi*\Phi',E)$ defined by $\overline{\partial}$-exact forms.

We set

$$H^{s,r}(\Phi*\Phi',E) = Z^0_{s,r}(\Phi*\Phi',E)\big/E^0_{s,r}(\Phi*\Phi',E).$$

Furthermore, we denote by $Z^0_{s,r}(\Phi;X,E)\big|\Phi*\Phi'$ the image of $Z^0_{s,r}(\Phi;X,E)$ in $Z^0_{s,r}(\Phi*\Phi',E)$ under the restriction map, and set

$$\hat{H}^{s,r}(\Phi*\Phi',E) = Z^0_{s,r}(\Phi*\Phi',E)\big/\big[Z^0_{s,r}(\Phi;X,E)\big|\Phi*\Phi'\big].$$

Finally we introduce the spaces

$$E^0_{s,r}(\Phi\to\Phi';X,E) := E^0_{s,r}(\Phi;X,E) \cap C^0_{s,r}(\Phi';X,E)$$

which are of special interest for our purpose.

1.3. Lemma. *If $\Phi' \subseteq \Phi$ are two families of supports in X such that Φ' is complete in Φ, then:*

(i) For all s,r with $0 \leq s,r \leq n$, we have the relation

$$E^0_{s,r}(\Phi*\Phi';X,E) \subseteq Z^0_{s,r}(\Phi;X,E)\big|\Phi*\Phi'. \tag{1.1}$$

and therefore the inequality

$$\dim \hat{H}^{s,r}(\Phi * \Phi', E) \leq \dim H^{s,r}(\Phi * \Phi', E). \tag{1.2}$$

(ii) For all s, r with $0 \leq s \leq n$ and $1 \leq r \leq n$, we have a natural isomorphism

$$\hat{\delta} : E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E) \longrightarrow \hat{H}^{s,r-1}(\Phi * \Phi', E), \tag{1.3}$$

and hence the equality

$$\dim \left[E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E)\right] = \dim \hat{H}^{s,r-1}(\Phi * \Phi', E). \tag{1.4}$$

(iii) For all s, r with $0 \leq s \leq n$ and $1 \leq r \leq n$, we have a natural linear epimorphism

$$\delta : E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E) \longrightarrow H^{s,r-1}(\Phi * \Phi', E), \tag{1.5}$$

and hence the inequality

$$\dim \left[E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E)\right] \leq \dim H^{s,r-1}(\Phi * \Phi', E). \tag{1.6}$$

Proof. (i): Let $f \in E^0_{s,r}(\Phi * \Phi'; X, E)$ be given. If $r = 0$, then $f = 0$ and hence $f \in Z^0_{s,r}(\Phi; X, E)\big|\Phi*\Phi'$. If $r \geq 1$, then there exists $U \in \Phi*\Phi'$ and $\varphi \in C^0_{s,r-1}(U, E)$ with continuous $\overline{\partial}\varphi$ such that f is defined by $\overline{\partial}\varphi$. By condition (iii) in 1.2, after shrinking U, we may assume that $\overline{U} \in \Phi$, and, by the same argument, we can find $V \in \Phi * \Phi'$ with $\overline{V} \subseteq U$. Take a real C^∞-function χ on X with $\operatorname{supp} \chi \subseteq U$ and $\chi \equiv 1$ on V. Let $\psi \in Z^0_{s,r}(X, E)$ be the form defined by

$$\psi = \overline{\partial}(\chi\varphi) = \overline{\partial}\chi \wedge \varphi + \chi\overline{\partial}\varphi$$

on U and $\psi \equiv 0$ outside U. Since $\overline{U} \in \Phi$, then $\psi \in Z^0_{s,r}(\Phi; X, E)$. Since $\psi = \overline{\partial}\varphi$ on V and therefore the germ f is defined by $\psi\big|V$, this implies that $f \in Z^0_{s,r}(\Phi; X, E)\big|\Phi*\Phi'$.

(ii): Let $f \in E^0_{s,r}(\Phi \to \Phi'; X, E)$. Take $u \in C^{s,r-1}(\Phi; X, E)$ with $\overline{\partial}u = f$. Then $u|_{(X \setminus \operatorname{supp} f)} \in Z^0_{s,r-1}(X \setminus \operatorname{supp} f, E)$. Therefore, since $X \setminus \operatorname{supp} f \in \Phi * \Phi'$, $u|_{(X \setminus \operatorname{supp} f)}$ defines an element in $\hat{H}^{s,r-1}(\Phi * \Phi', E)$. Denote this element by $\hat{\delta} f$. This element does not depend on the choice of u, for if $\tilde{u} \in C^{s,r-1}(\Phi; X, E)$ is another form with $\overline{\partial}\tilde{u} = f$, then $u - \tilde{u} \in Z^0_{s,r-1}(\Phi; X, E)$. Hence a linear map

$$\hat{\delta} : E^0_{s,r}(\Phi \to \Phi'; X, E) \longrightarrow \hat{H}^{s,r-1}(\Phi * \Phi', E)$$

is well defined. It remains to show that $\hat{\delta}$ is surjective and

$$\ker \hat{\delta} = E^0_{s,r}(\Phi'; X, E). \tag{1.7}$$

Proof of the surjectivity: Let $F \in \hat{H}^{s,r-1}(\Phi * \Phi', E)$ be given. Take $U \in \Phi * \Phi'$ and $f \in Z^0_{s,r-1}(U, E)$ such that F is defined by f. By condition (iii) in 1.2, we can find open sets $V, W \in \Phi * \Phi'$ such that $\overline{V} \subseteq W$, $\overline{W} \subseteq U$ and $\overline{W} \in \Phi$. Take a real C^∞-function χ on X with $\operatorname{supp} \chi \subseteq W$ and $\chi \equiv 1$ on $\overline{V}$ and let g be the form on X defined by $g = \overline{\partial}(\chi f)$ on W and by zero outside W. Since $\overline{W} \in \Phi$, then $g \in E^0_{s,r}(\Phi; X, E)$. Since $g \equiv 0$ outside $\overline{W} \setminus V$ and $\overline{W} \setminus V \in \Phi'$, we see that even $g \in E^0_{s,r}(\Phi \to \Phi'; X, E)$. As $(\chi f)|_V = f|_V$ defines F, we see that $\hat{\delta} g = F$.

Proof of (1.7): First let $f \in E^0_{s,r}(\Phi'; X, E)$ be given. Then there exists $u \in Z^0_{s,r-1}(X, E)$ with $\overline{\partial} u = f$ and $\operatorname{supp} u \in \Phi'$. Since $(X \setminus \operatorname{supp} u) \in \Phi * \Phi'$, then, by definition of $\hat{\delta}$, $\hat{\delta} f$ is defined by the form $u|_{(X \setminus \operatorname{supp} u)}$ which is zero.

Now let $f \in \ker \hat{\delta}$ be given, i.e. $f = \overline{\partial} u$ where u is a form from $C^0_{s,r-1}(\Phi; X, E)$ such that, for certain $v \in Z^0_{s,r-1}(\Phi; X, E)$ and some $U \in \Phi * \Phi'$, $u = v$ on U. Then $\operatorname{supp}(u - v) \in \Phi'$ and $\overline{\partial}(u - v) = f$, i.e. $f \in E^0_{s,r}(\Phi'; X, E)$.

(iii) follows from (i) and (ii). □

1.4. Corollary. *If $\Phi' \subseteq \Phi$ are two families of supports in X such that Φ' is complete in Φ, then*

$$\dim H^{s,r}_{\Phi'}(X, E) \le \dim H^{s,r}_{\Phi}(X, E) + \dim H^{s,r-1}(\Phi * \Phi', E) \tag{1.8}$$

for all s, r with $0 \le s \le n$ and $1 \le r \le n$.

Proof. From

$$E^0_{s,r}(\Phi'; X, E) \subseteq E^0_{s,r}(\Phi \to \Phi'; X, E) \subseteq Z^0_{s,r}(\Phi'; X, E)$$

it follows that

$$\begin{aligned}\dim H^0_{s,r}(\Phi'; X, E) = \dim &\left[E^0_{s,r}(\Phi \to \Phi'; X, E)/E^0_{s,r}(\Phi'; X, E)\right] \\ &+ \dim \left[Z^0_{s,r}(\Phi'; X, E)/E^0_{s,r}(\Phi \to \Phi'; X, E)\right].\end{aligned}$$

In view of Lemma 1.3 (iii) and the obvious inequality

$$\dim Z^0_{s,r}(\Phi'; X, E)/E^0_{s,r}(\Phi \to \Phi'; X, E) \le \dim H^{s,r}_{\Phi}(X, E),$$

this implies (1.8). □

2. Serre duality

In this section X is an n-dimensional complex manifold countable at infinity.

2.1. The topology of $C^0_{s,r}(\Phi; X, E)$ and $H^{s,r}_{\Phi}(X, E)$. Let E be a holomorphic vector bundle over X, and Φ a family of supports in X.

As usual, we consider $C^0_{s,r}(X, E)$ as Fréchet space with the topology of uniform convergence on the compact subsets of X. If C is a closed subset of X, then we denote by $C^0_{s,r}(C;X,E)$ the subspace of $C^0_{s,r}(X,E)$ of the forms f with $\operatorname{supp} f \subseteq C$, which we also consider as Fréchet space (with the topology induced by $C^0_{s,r}(X,E)$).

$C^0_{s,r}(\Phi;X,E)$ will be provided with the inductive limit topology of the Fréchet spaces $C^0_{s,r}(C;X,E)$, $C \in \Phi$, i.e. the finest locally convex topology such that, for each $C \in \Phi$, the natural injection of $C^0_{s,r}(C;X,E)$ in $C^0_{s,r}(\Phi;X,E)$ is continuous.

To ensure nice properties of this topology, we have to restrict ourselves to so-called *cofinal* families of supports (cf. [C-S]):

Definition: A family of supports Φ in X will be called *cofinal* if there exists a sequence $(C_j)_{j\in\mathbb{N}}$ of sets $C_j \in \Phi$ such that each $C \in \Phi$ is contained in certain C_j. In view of condition (S_3) in the definition of a family of supports, then this sequence always can be chosen so that each C_j is contained in the interior of C_{j+1}.

If Φ is a cofinal family of supports in X, then $C^0_{s,r}(\Phi;X,E)$ is an LF-space, i.e. a *countable strict inductive limit of Fréchet spaces* (cf., e.g., Chapter 13 in [T]) - if (C_j) is a sequence as in the definition above such that each C_j is contained in the interior of C_{j+1}, then the sequence of Fréchet spaces $C^0_{s,r}(C_j;X,E)$ may serve as defining sequence.

It is easy to see that all families of supports considered in Examples I - IV of Sect. 1 are cofinal.

We provide $Z^0_{s,r}(\Phi;X,E)$ and $E^0_{s,r}(\Phi;X,E)$ with the topology of $C^0_{s,r}(\Phi;X,E)$, and $H^{s,r}_\Phi(X,E)$ with the corresponding factor topology. The space of continuous linear forms on $H^{s,r}_\Phi(X,E)$ will be denoted by $\big(H^{s,r}_\Phi(X,E)\big)'$. Recall that the topology of $H^{s,r}_\Phi(X,E)$ is separated if and only if $E^0_{s,r}(\Phi;X,E)$ is topologically closed in $C^0_{s,r}(\Phi;X,E)$.

2.2. The dual family Φ^*. If Φ is a family of supports in X, then we denote by Φ^* the family of all closed subsets C^* of X such that, for all $C \in \Phi$, the intersection $C^* \cap C$ is compact. Φ^* will be called the *dual family of* Φ.

If Φ is a family of supports in X, then, obviously, conditions (S_1) and (S_2) in the definition of a family of supports are also fulfilled for Φ^*. However, condition (S_3) is not fulfilled in general for Φ^*. We thank Lee STOUT for submitting us by e-mail the following counterexample:

Let $\mathbb{R}_+$ be the nonnegative part of the real axis in $\mathbb{C}$. Denote by Φ the family of all closed subsets C of $\mathbb{C}$ for which $C \cap \mathbb{R}_+$ is compact. Then Φ is a family for supports in $\mathbb{C}$, but the dual family Φ^* does not satisfy condition (S_3). ($\mathbb{R}_+ \in \Phi^*$, but there is no neighborhood U of $\mathbb{R}_+$ with $\overline{U} \in \Phi^*$.)

It is easy to see that the family Φ in the example of STOUT is not cofinal. This is consistent with the following lemma.

2.3. Lemma. *If Φ is a cofinal family of supports in X, then Φ^* is a family of supports in X.*

Proof. Let $C^* \in \Phi^*$ be given. We have to find a neighborhood V of C^* such that $\overline{V} \in \Phi^*$. Since Φ is cofinal we have a sequence $C_j \in \Phi$, $j = 1,2,\dots$ such that each

$C \in \Phi$ is contained in some C_j and if U_j is the interior of C_j, then $C_j \subseteq U_{j+1}$. Set $C_1^* = C^* \cap C_1$ and $C_j^* = C^* \cap (C_j \setminus U_{j-1})$ if $j \geq 2$. Then all C_j^* are compact and $C_j^* \cap C_{j-2} = \emptyset$ if $j \geq 3$. Take for each $j \geq 1$ a relatively compact open set V_j with

$$C_j^* \subseteq V_j \qquad \text{and} \qquad \overline{V}_j \cap C_{j-2} = \emptyset \quad \text{if} \quad j \geq 3$$

Then

$$V := \bigcup_{j=1}^{\infty} V_j.$$

has the required properties. □

It is not true in general that the dual family of a cofinal family of supports is again cofinal. For example, the dual of the family Φ' in Example IV, Sect. 1, is not cofinal.

2.4. The properties (Cl_1) **-** (Cl_4) **of** $E^0_{s,r}(\Phi; X, E)$**.** Let Φ be a family of supports in X, E a holomorphic vector bundle on X, $0 \leq s \leq n$ and $1 \leq r \leq n$. We shall meet the following 4 conditions:

(Cl_1): For each $C \in \Phi$ there exists a finite dimensional linear subspace F of $C^0_{s,r}(C; X, E)$ such that $F + \big(C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E)\big)$ is topologically closed in $C^0_{s,r}(C; X, E)$ (with respect to uniform convergence on the compact subsets of C).

(Cl_2): For each $C \in \Phi$, $C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(C; X, E)$.

(Cl_3): $E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(\Phi; X, E)$.

(Cl_4): $E^0_{s,r}(\Phi; X, E) =$

$$\left\{ f \in Z^0_{s,r}(\Phi; X, E) \;\middle|\; \int_X f \wedge g = 0 \quad \text{for all} \quad g \in Z^0_{n-s,n-r}(\Phi^*; X, E^*) \right\}.$$

It is clear that $(Cl_4) \implies (Cl_3) \implies (Cl_2) \implies (Cl_1)$. Under certain additional conditions on Φ (see Theorem 2.9 below) these 4 conditions are even equivalent, but we do not know whether this is true in general.

2.5. Lemma. *Let Φ be a family of supports in X such that $\bigcup \Phi = X$. Further, let E be a holomorphic vector bundle over X and $0 \leq s, r \leq n$. Then $\operatorname{supp} f \in \Phi^*$ for all continuous linear forms f on $C^0_{s,r}(\Phi;, X, E)$.*

Proof. Take a sequence K_j $(j = 1, 2, \dots)$ of compact subsets of X such that each K_j is contained in the interior of K_{j+1} and

$$\bigcup_{j=1}^{\infty} K_j = X.$$

Assume now that, for certain $C \in \Phi$, the intersection $C \cap \operatorname{supp} f$ is not compact. Then we can find a sequence z_j $(j = 1, 2, \dots)$ of points $z_j \in C \cap \operatorname{supp} f$ such that

$$z_j \in X \setminus K_j \qquad \text{for all} \quad j. \tag{2.1}$$

Since $z_j \in C$ for all j and $C \in \Phi$, and Φ is a family of supports, there exists a neighborhood U of $\{z_1, z_2, \dots\}$ with $\overline{U} \in \Phi$. By (2.1), for each j, we can find a neighborhood $U_j \subseteq U$ of z_j so small that

$$U_j \subseteq X \setminus K_j \qquad \text{for all} \quad j. \tag{2.2}$$

Set

$$C_0 = \overline{\left(\bigcup_{j=1}^{\infty} U_j \right)}.$$

Then $C_0 \subseteq \overline{U}$ and therefore $C_0 \in \Phi$. Since $z_j \in \operatorname{supp} f$ for all j, we can find forms $\varphi_j \in C^0_{s,r}(X, E)$ with $\operatorname{supp} \varphi_j \subset\subset U_j$ and

$$f(\varphi_j) = 1 \qquad \text{for all} \quad j. \tag{2.3}$$

This sequence belongs to $C^0_{s,r}(C_0; X, E)$ and, by (2.2), it converges to zero uniformly on the compact subsets of C_0. This contradicts (2.3), for $C_0 \in \Phi$ and therefore f is continuous on $C^0_{s,r}(C_0; X, E)$. $\square$

2.6. Lemma. *Let Φ be a family of supports in X such that $\bigcup \Phi = X$ and Φ^* is also a family of supports. Further, let E be a holomorphic vector bundle over X and E^* the dual of E. Then, for all integers s, r with $0 \le s, r \le n$, there is a natural linear epimorphism*

$$h'_{s,r} : H^{n-s,n-r}_{\Phi^*}(X, E^*) \longrightarrow \big(H^{s,r}_{\Phi}(X, E) \big)' \tag{2.4}$$

which is an isomophism if and only if the space $E^0_{n-s,n-r}(\Phi^*; X, E^*)$ satisfies condition (Cl_4), i.e.

$$E^0_{n-s,n-r}(\Phi^*; X, E^*) = \left\{ f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*) \;\middle|\; \int_X f \wedge g = 0 \quad \textit{for all} \quad g \in Z^0_{s,r}(\Phi; X, E) \right\}. \tag{2.5}$$

Proof. Since, for $C \in \Phi$ and $C^* \in \Phi^*$, the intersection $C \cap C^*$ is compact, for each $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$, setting

$$f'(g) := \int_X f \wedge g \qquad \text{for} \quad g \in Z^0_{s,r}(\Phi; X, E),$$

we can define a continuous linear functional f' on $Z^0_{s,r}(\Phi; X, E)$, where, by Stokes' theorem, $f'(g) = 0$ if $f \in E^0_{n-s,n-r}(\Phi^*; X, E^*)$ or $g \in E^0_{s,r}(\Phi; X, E)$. Hence in this way we get a linear map from $H^{n-s,n-r}_{\Phi^*}(X,E^*)$ to $\big(H^{s,r}_{\Phi}(X,E)\big)'$ which we denote by $h'_{s,r}$.

Obviously, this map is injective if and only if (2.5) holds. Thus the only non-trivial assertion of the lemma is the surjectivity of $h'_{s,r}$.

To prove this, we consider an arbitrary functional $F \in \big(H^{s,r}_{\Phi}(X,E)\big)'$. Let

$$p : Z^0_{s,r}(\Phi; X, E) \longrightarrow H^{s,r}_{\Phi}(X,E)$$

be the canonical projection. We have to find $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$ with

$$\int_X f \wedge g = (F \circ p)(g) \qquad \text{for all} \quad g \in Z^0_{s,r}(\Phi; X, E). \tag{2.6}$$

First, by the Hahn-Banach theorem, we can find a continuous linear functional $\tilde{F}$ on $C^0_{s,r}(\Phi; X, E)$ with $\tilde{F} = F \circ p$ on $Z^0_{s,r}(\Phi; X, E)$. Since $\bigcup \Phi = X$, all compact subsets of X belong to Φ. Therefore, the continuity of $\tilde{F}$ on $C^0_{s,r}(\Phi; X, E)$ in particular means that $\tilde{F}$ is an E^*-valued current of bidegree $(n-s, n-r)$ on X. Since $\tilde{F}$ vanishes on $E^0_{s,r}(\Phi; X, E)$, $\tilde{F}$ is $\overline{\partial}$-closed, and it follows from Lemma 2.4 that $\operatorname{supp} \tilde{F} \in \Phi^*$.

If $r = n$, by regularity of $\overline{\partial}$, there exists $f \in Z^0_{n-s,0}(\Phi^*; X, E^*)$ (f is even holomorphic) such that

$$\int_X f \wedge g = \tilde{F}(g) = (F \circ p)(g) \qquad \text{for all} \quad g \in Z^0_{s,n}(\Phi; X, E),$$

i.e. such that (2.6) is fulfilled.

Now let $r \le n-1$. Since Φ^* is a family of supports, by means of the Dolbeault isomorphism (see, e.g., Corollary 2.15 (i) in [H-L]), we can find a current S on X with $\operatorname{supp} S \in \Phi^*$ such that the current $\tilde{F} - \overline{\partial} S$ is defined by a continuous form with support in Φ^*, i.e., we have a form $f \in Z^0_{n-s,n-r}(\Phi^*; X, E^*)$ such that

$$\int_X f \wedge \varphi = (\tilde{F} - \overline{\partial} S)(\varphi) \tag{2.7}$$

for all E-valued $C^\infty_{s,r}$-forms φ with compact support on X. It remains to prove (2.6).

First consider a form $g_\infty \in Z^0_{s,r}(\Phi; X, E)$ which is of class C^∞. Since

$$(\operatorname{supp} S \cup \operatorname{supp} f \cup \operatorname{supp} \tilde{F}) \cap \operatorname{supp} g_\infty$$

is compact, then it follows from (2.7) that

$$\int_X f \wedge g_\infty = (\tilde{F} - \overline{\partial} S)(g_\infty) = \tilde{F}(g_\infty). \tag{2.8}$$

Now let $g \in Z^0_{s,r}(\Phi; X, E)$ be arbitrary. If $r = 0$, then g is holomorphic and (2.6) follows from (2.8). Therefore we may assume that $r \geq 1$. Then, using that Φ is a family of supports, as above, again by means of the Dolbeault isomorphism, we can find a C^∞-form $g_\infty \in Z^0_{s,r}(\Phi; X, E)$ and a form $\psi \in C^0_{s,r-1}(\Phi; X, E)$ such that

$$g = g_\infty + \overline{\partial}\psi.$$

It follows from Stokes' theorem and (2.8) that

$$\int_X f \wedge g = \int_X f \wedge g_\infty + \int_X f \wedge \overline{\partial}\psi = \int_X f \wedge g_\infty = \tilde{F}(g_\infty).$$

Since $g - g_\infty = \overline{\partial}\psi \in E^0_{s,r}(\Phi; X, E)$ and therefore $\tilde{F}(g_\infty) = \tilde{F}(g)$, this implies (2.6). $\square$

2.7. Lemma. *Let Φ be a cofinal family of supports in X. Further, let E a holomorphic vector bundle over X, $C \in \Phi$, $0 \leq s \leq n$ and $1 \leq r \leq n$. Suppose there exists a finite dimensional linear subspace F of $C^0_{s,r}(C; X, E)$ such that the linear space*

$$F + \big(C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E)\big)$$

is topologically closed in $C^0_{s,r}(C; X, E)$. Then also $C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(C; X, E)$ and, moreover, there exists $C_0 \in \Phi$ with

$$C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E) = C^0_{s,r}(C; X, E) \cap \overline{\partial} C^0_{s,r-1}(C_0; X, E). \tag{2.9}$$

Proof. Set

$$H = C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E).$$

Since Φ is cofinal, we can find a sequence $C_j \in \Phi$ such that each $C \in \Phi$ is contained in some C_j. Then

$$F + H = \bigcup_{j=1}^{\infty} \Big(F + \big(C^0_{s,r}(C; X, E) \cap \overline{\partial} C^0_{s,r-1}(C_j; X, E)\big) \Big).$$

Since $F + H$ is a Fréchet space, this implies that for certain j_0, the space

$$F + \big(C^0_{s,r}(C; X, E) \cap \overline{\partial} C^0_{s,r-1}(C_{j_0}; X, E)\big) \tag{2.10}$$

is of second Baire categorie in $F + H$. Let D_0 be the linear subspace of all $\varphi \in C^0_{s,r-1}(C_{j_0}; X, E)$ with $\overline{\partial}\varphi \in C^0_{s,r}(C; X, E)$, and let $1_F \oplus \overline{\partial}_0$ be the linear operator with domain of definition $F \oplus D_0$ between the Fréchet spaces $F \oplus C^0_{s,r-1}(C_{j_0}; X, E)$ and $F + H$ defined by $(1_F \oplus \overline{\partial}_0)(f, \varphi) = f + \overline{\partial}\varphi$ for $(f, \varphi) \in F \oplus D_0$. Since this operator is closed, and its image (which is equal to (2.10)) is of second Baire categorie in $F + H$, it follows by the open mapping theorem that this operator is onto and open. Hence

$$(1_F \oplus \overline{\partial}_0)(F \oplus D_0) = F + H, \tag{2.11}$$

and

$$\overline{\partial}(D_0) = (1_F \oplus \overline{\partial}_0)(\{0\} \oplus D_0)$$

is finite codimensional in the topologically closed space $F + H$. Since $\overline{\partial}$ (with D_0 as domain of definition) is closed, it follows by the open mapping theorem that $\overline{\partial}(D_0)$ is moreover topologically closed in $F + H$. Finally, since

$$\overline{\partial}(D_0) \subseteq H \subseteq F + H,$$

we see that H is topologically closed. (2.9) follows from (2.11) repeating the first part of the proof with $F = \{0\}$. □

2.8. Lemma. *Let X be an n-dimensional complex manifold and Φ a cofinal family of supports in X such that $\bigcup \Phi = X$. Further, let E be a holomorphic vector bundle over X, E^* the dual bundle of E, $0 \le s \le n$ and $1 \le r \le n$.*

Then (by Lemma 2.3) Φ^* is a family of supports and the following conclusion holds: If $E^0_{s,r}(\Phi; X, E)$ satisfies condition (Cl_1), then $E^0_{n-s,n-r+1}(\Phi^*; X, E^*)$ satisfies condition (Cl_4).

Proof. Since the"$\subseteq$"-part of condition (Cl_4) holds always by Stokes' theorem, we only have to prove that any $f \in Z^0_{n-s,n-r+1}(\Phi^*; X, E^*)$ with

$$\int_X f \wedge g = 0 \quad \text{for all} \quad g \in Z^0_{s,r-1}(\Phi; X, E) \tag{2.12}$$

belongs to $E^0_{s,r}(\Phi^*; X, E^*)$. Let such f be given. Define a linear map

$$\tilde{u} : E^0_{s,r}(\Phi; X, E) \longrightarrow \mathbb{C}$$

as follows: If $\varphi \in E^0_{s,r}(\Phi; X, E)$, then we take a $\psi \in C^0_{s,r-1}(\Phi; X, E)$ with $\varphi = \overline{\partial}\psi$ and set

$$\tilde{u}(\varphi) = \int_X f \wedge \psi.$$

By (2.12) this definition is independent of the choice of ψ.

Now we show that $\tilde{u}$ is continuous with respect to the topology of $C^0_{s,r}(\Phi; X, E)$. For this it is sufficient to prove that, for each $C \in \Phi$, the restriction of $\tilde{u}$ to the space

$$H_C := C^0_{s,r}(C; X, E) \cap E^0_{s,r}(\Phi; X, E)$$

is continuous with respect to the topology of uniform convergence on the compact subsets of C.

Let $C \in \Phi$ be given, and let $(\varphi_j)_{j=1}^{\infty}$ be a sequence in H_C converging to zero uniformly on the compact subsets of C. Since condition (Cl_1) is fulfilled for Φ, it follows from Lemma 2.7 that H_C is closed in this topology. Applying again Lemma 2.7 (with $F = \{0\}$) we obtain a set $C_0 \in \Phi$ with

$$H_C = C^0_{s,r}(C; X, E) \cap \overline{\partial} C^0_{s,r-1}(C_0; X, E).$$

Since $\overline{\partial}$ is closed as an operator between the Fréchet spaces $C^0_{s,r-1}(C_0; X, E)$ and H_C, this implies by the open mapping theorem that this operator is open. Hence we can find a sequence $\psi_j \in C^0_{s,r-1}(C_0; X, E)$ which converges to zero uniformly on the compact subsets of C_0 such that $\overline{\partial}\psi_j = \varphi_j$ for all j, and it follows from the definition of $\tilde{u}$ that also $\tilde{u}(\varphi_j)$ converges to zero.

Hence it is proved that $\tilde{u}$ is continuous in the topology of $C^0_{s,r}(\Phi; X, E)$. Therefore, by the Hahn-Banach theorem, there is a continuous linear functional u on $C^0_{s,r}(\Phi; X, E)$ with $u = \tilde{u}$ on $E^0_{s,r}(\Phi; X, E)$. Since $\bigcup \Phi = X$, it follows from condition (S_3) in the definition of a family of supports, that all compact subsets of X belong to Φ. Therefore the continuity of u on $C^0_{s,r}(\Phi; X, E)$ in particular implies that u is an E^*-valued current of bidegree $(n-s, n-r)$ on X, where by definition of $\tilde{u}$ and Stokes' theorem,

$$\overline{\partial}u = \pm f.$$

From Lemma 2.5 it follows that $\operatorname{supp} u \in \Phi^*$. Since also Φ^* has property (S_3), in view of the regularity properties of $\overline{\partial}$ which follow from the Dolbeault isomorphism (cf, e.g., Corollary 2.15 (ii) in [H-L]), this implies the existence of a form $u_0 \in C^0_{n-s,n-r}(\Phi^*; X, E^*)$ with $\overline{\partial}u_0 = f$. □

2.9. Theorem. *Let Φ be a cofinal family of supports in X such that the dual family Φ^* (which is a family of supports, by Lemma 2.3) is also cofinal, and $\Phi^{**} = \Phi$. Further suppose E is a holomorphic vector bundle over X and E^* is the dual bundle of E. Then, for all integers s, r with $0 \le s \le n$ and $1 \le r \le n$, the following statements hold:*

(i) The 8 conditions which consist of conditions (Cl_1) - (Cl_4) for $E^0_{s,r}(\Phi; X, E)$ and the same conditions for $E^0_{n-s,n-r+1}(\Phi^*; X, E^*)$ are equivalent.

(ii) If $E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(\Phi; X, E)$ and, moreover, $E^0_{n-s,n-r}(\Phi^*; X, E^*)$ is topologically closed in $C^0_{n-s,n-r}(\Phi^*; X, E^*)$, then we have natural isomorphisms

$$H^{n-s,n-r}_{\Phi^*}(X, E^*) \cong \big(H^{s,r}_{\Phi}(X, E)\big)' \quad \textit{and} \quad H^{s,r}_{\Phi}(X, E) \cong \big(H^{n-s,n-r}_{\Phi^*}(X, E^*)\big)'.$$

In particular, then

$$\dim H^{n-s,n-r}_{\Phi^*}(X, E^*) = \dim H^{s,r}_{\Phi}(X, E).$$

Proof. (i): From Lemma 2.8 we obtain the two conclusions

$$(Cl_1) \text{ holds for } E^0_{s,r}(\Phi; X, E) \implies (Cl_4) \text{ holds for } E^0_{n-s,n-r+1}(\Phi^*; X, E^*)$$

and

$$(Cl_1) \text{ holds for } E^0_{n-s,n-r+1}(\Phi^*; X, E^*) \implies (Cl_4) \text{ holds for } E^0_{s,r}(\Phi; X, E).$$

Since the conclusions $(Cl_4) \implies (Cl_3) \implies (Cl_2) \implies (Cl_1)$ are always true, this shows the equivalence of all 8 conditions.

(ii): This follows immediately from Lemma 2.6. □

2.10. Problem. Let Φ be a cofinal family of supports in X. Under the additional hypothesis that Φ^* is also cofinal and $\Phi^{**} = \Phi$, Theorem 2.9 contains the equivalence of conditions (Cl_1) and (Cl_3). In particular then, for each holomorphic vector bundle E over X and for all integers s, r with $0 \le s \le n$ and $1 \le r \le n$, the following two conditions are equivalent:

(i) For each $C \in \Phi$ the intersection $E^0_{s,r}(\Phi; X, E) \cap C^0_{s,r}(C; X, E)$ is topologically closed in $C^0_{s,r}(C; X, E)$;

(ii) $E^0_{s,r}(\Phi; X, E)$ is topologically closed in $C^0_{s,r}(\Phi; X, E)$.

We do not know whether these conditions are equivalent also without this additional hypothesis.

3. A simple proof of Theorem 0.3

By Serre duality [S] (see also Theorem 2.9 (i) above), Theorem 0.3 is equivalent to the following

3.1. Theorem. *Let the hypothesis of Theorem 0.3 be fulfilled. Then, for any holomorphic vector bundle E over Y, $H^{0,q+1}_c(X, E)$ is separated.*

Proof of Theorem 3.1. Denote by Φ_Y and Φ_X the families of compact subsets of X resp. Y. Then Φ_X is complete in Φ_Y and therefore, by Lemma 1.3,

(3.1) $$\dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)/E^0_{0,q+1}(\Phi_X; Y, E)\right] \le \dim H^{0,q}(\Phi_Y * \Phi_X, E).$$

Let ρ be as in Definion 0.1 and let $\alpha_0 \in]\inf \rho, \sup \rho[$ such that the Levi form of ρ has at least $n - q + 1$ positive eigenvalues on $\{\rho \le \alpha_0\}$. Then the manifolds $U_\alpha := (Y \setminus X) \cup \{\rho < \alpha\}$, $\inf \rho < \alpha \le \alpha_0$, are q-convex in the sense of Andreotti-Grauert and hence, by the Andreotti-Grauert finiteness theorem [A-G],

$$\dim H^{0,q}(U_\alpha, E) < \infty, \qquad \text{if} \quad \inf \rho < \alpha \le \alpha_0.$$

Since $U_\alpha \in \Phi_Y * \Phi_X$ for all $\alpha \in]\inf \rho, \alpha_0]$ and, conversely, for each $U \in \Phi_Y * \Phi_X$, we can find $\alpha \in]\inf \rho, \alpha_0]$ with $U_\alpha \subseteq U$, it follows that

$$\dim H^{0,q}(\Phi_Y * \Phi_X; E) < \infty.$$

In view of (3.1) this implies that

$$\dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)/E^0_{0,q+1}(\Phi_X; Y, E)\right] < \infty,$$

i.e.

(3.2) $$\dim \left[E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)/E^0_{0,q+1}(\Phi_X; X, E)\right] < \infty$$

where $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ denotes the image of $E^0_{0,q+1}(\Phi_Y \to \Phi_X; Y, E)$ under the restriction map from Y to X. Since Y is $(n-q)$-convex, it follows from the Andreotti-Grauert finiteness theorem that

$$\dim H^{n,n-q}(Y, E^*) < \infty$$

where E^* denotes the dual of E. Hence, by Serre duality [S] (see also Theorem 2.9 (i) above), $E^0_{0,q+1}(\Phi_Y; Y, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_Y; Y, E)$. Since the map 'extending by zero'

$$C^0_{0,q+1}(\Phi_X; X, E) \longrightarrow C^0_{0,q+1}(\Phi_Y; Y, E)$$

is continuous and $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ is the preimage of $E^0_{0,q+1}(\Phi_Y; Y, E)$ with respect to this map, it follows that $E^0_{0,q+1}(\Phi_Y \to \Phi_X; X, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$. In view of (3.2) and since, by Theorem 2.9 (i), conditions (Cl_1) and (Cl_3) are equivalent for $E^0_{0,q+1}(\Phi_X; X, E)$, this implies that $E^0_{0,q+1}(\Phi_X; X, E)$ is toologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$, i.e. $H^{0,q+1}_{\Phi_X}(X, E)$ is separated. □

4. A finiteness and separation theorem on q-concave-q^*-convex manifolds for partially compact families of supports

In this section X is an n-dimensional complex manifold which is q-concave-q^*-convex where $1 \le q \le n-1$ and $1 \le q^* \le n$. Further we suppose that ρ, $\inf \rho$ and $\sup \rho$ are as in Definition 0.1, and E is a holomorphic vector bundle over X.

Let Φ be the family of all closed subsets C of X such that the sets $C \cap \{\rho \le \alpha\}$, $\inf \rho < \alpha < \sup \rho$, are compact. Φ is a cofinal family of supports in X. The dual family Φ^* of Φ consists of all closed subsets C of X such that the sets $C \cap \{\rho \ge \alpha\}$, $\inf \rho < \alpha < \sup \rho$, are compact. Φ^* is also a cofinal family of supports in X, and $\Phi^{**} = \Phi$ (hence the hypotheses of Theorem 2.9 are fulfilled).

4.1. Theorem. *(i) If* $\max(q+1, q^*) \le r \le n$, *then* $\dim H^{0,r}_{\Phi}(X, E) < \infty$.

(ii) If $0 \le r \le \min(n-q-1, n-q^*)$, *then* $\dim H^{0,r}_{\Phi^*}(X, E) < \infty$.

(iii) If $r = \min(n-q, n-q^*+1)$, *then* $H^{0,r}_{\Phi^*}(X, E)$ *is separated.*

By Serre duality (Theorem 2.9), parts (ii) and (iii) of this theorem follow from part (i). The remainder of this section is devoted to the proof of Theorem 4.1 (i).

If $\inf \rho < \alpha < \beta < \sup \rho$ and $d\rho(\zeta) \ne 0$ for all $\zeta \in X$ with $\rho(\zeta) = \beta$, then we denote by $C^0_{0,r}(\rho \ge \alpha; \rho \le \beta, E)$ the Banach space of continuous E-valued $(0,r)$-forms f on $\{\rho \le \beta\}$ with $\operatorname{supp} f \subseteq \{\rho \ge \alpha\}$ endowed with the topology of uniform convergence, and by $Z^0_{0,r}(\rho \ge \alpha; \rho \le \beta, E)$ then we denote the subspace of $\overline{\partial}$-closed forms in $C^0_{0,r}(\rho \ge \alpha; \rho \le \beta, E)$ endowed with the same topology.

Further, we fix some numbers α_0, β_0 with $\inf \rho < \alpha_0 < \beta_0 < \sup \rho$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $\{\rho \le \alpha_0\}$, and at least $n-q^*+1$ positive eigenvalues on $\{\rho \ge \beta_0\}$. We may assume that $d\rho(\zeta) \ne 0$ for all $\zeta \in X$ with $\rho(\zeta) = \beta_0$.

4.2. Lemma. *If* $\max(q+1, q^*) \le r \le n$, *then, for each* $\varepsilon > 0$, *the space*

$$C^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E) \cap \overline{\partial} C^0_{0,r-1}(\rho \ge \alpha_0 - \varepsilon; \rho \le \beta_0, E)$$

is of finite codimension and topologically closed in $Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E)$.

Proof. If $\inf \rho < \gamma < \beta_0$, then we denote by $C^{1/2}_{0,r-1}(\rho \geq \gamma; \rho \leq \beta_0, E)$ the Banach space of forms in $C^0_{0,r-1}(\rho \geq \gamma; \rho \leq \beta_0, E)$ which are Hölder continuous with exponent $1/2$ on $\{\gamma \leq \rho \leq \beta_0\}$. We shall prove that even the space

$$C^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E) \cap \overline{\partial} C^{1/2}_{0,r-1}(\rho \geq \alpha_0 - \varepsilon; \rho \leq \beta_0, E)$$

is of finite codimension and topologically closed in $Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$. By Ascoli's theorem and Fredholm theory, for this it is sufficient to construct continuous linear operators

$$A : Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E) \longrightarrow C^{1/2}_{0,r-1}(\rho \geq \alpha_0 - \varepsilon; \rho \leq \beta_0, E)$$

and

$$K : Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E) \longrightarrow C^{1/2}_{0,r-1}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$$

such that

$$\overline{\partial} Af = f + Kf \qquad \text{on} \quad \{\rho \leq \beta_0\}$$

for all $f \in Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$.

Take $\delta > 0$ so small that $\alpha_0 + \delta < \beta_0$ and the Levi form of ρ has at least $n - q + 1$ positive eigenvalues on $\{\rho \leq \alpha_0 + \delta\}$. Since $r \geq q + 1$, then by Lemma 1.2 in [La-L], there exists a continuous linear operator

$$A_0 : Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E) \longrightarrow C^{1/2}_{0,r-1}(\rho \geq \alpha_0 - \varepsilon; \rho \leq \beta_0, E)$$

such that $\overline{\partial} A_0 f = f$ on $\{\rho \leq \alpha_0 + \delta\}$ for all $f \in Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$. Since $r \geq q^*$ and the boundary $\{\rho = \beta_0\}$ is smooth, we can use the local integral operators of Fischer and Lieb [F-Li] (see also Sects. 7 and 9 in [H-L]) and obtain open sets $U_1, \ldots, U_N \subset\subset X$ with

$$\{\alpha_0 + \delta \leq \rho \leq \beta_0\} \subseteq U_1 \cup \ldots \cup U_N \subseteq \{\rho > \alpha_0\}$$

as well as continuous linear operators

$$A_j : Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E) \longrightarrow C^{1/2}_{0,r-1}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$$

such that $\overline{\partial} A_j f = f$ on $U_j \cap \{\rho \leq \beta_0\}$ for all $f \in Z^0_{0,r}(\rho \geq \alpha_0; \rho \leq \beta_0, E)$, $j = 1, \ldots, N$. Take real C^∞-functions $\chi_0, \ldots, \chi_N$ on X with $\operatorname{supp} \chi_0 \subset\subset \{\rho < \alpha_0 + \delta\}$, $\operatorname{supp} \chi_j \subset\subset U_j$ if $1 \leq j \leq N$, and $\chi_0 + \ldots + \chi_N \equiv 1$ on $\{\alpha_0 - \varepsilon \leq \rho \leq \beta_0\}$. Then the operators

$$A := \sum_{j=0}^{N} \chi_j A_j \qquad \text{and} \qquad K := \sum_{j=0}^{N} \overline{\partial} \chi_j \wedge A_j$$

have the required property. □

Lemma 4.3. *Let* $\max(q+1, q^*) \le r \le n$ *and* $\varepsilon > 0$. *Then the space*

$$Z^0_{0,r}(\rho \ge \alpha_0; X, E) \ \cap \ \overline{\partial}\, C^0_{0,r-1}(\rho \ge \alpha_0 - \varepsilon; X, E) \tag{4.1}$$

is the preimage of

$$Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E) \ \cap \ \overline{\partial}\, C^0_{0,r-1}(\rho \ge \alpha_0 - \varepsilon; \rho \le \beta_0, E) \tag{4.2}$$

with respect to the restriction map from $Z^0_{0,r}(\rho \ge \alpha_0; X, E)$ *to* $Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E)$.

Proof. Since by Lemma 4.2, the space

$$Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E) \ \cap \ \overline{\partial}\, C^0_{0,r-1}(\rho \ge \alpha_0 - \varepsilon; \rho \le \beta_0, E)$$

is topologiclally closed in $Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E)$, this can be proved in the same way as Theorem 12.13 (ii) in [H-L]. (The domain D in Theorem 12.13 in [H-L] is assumed to be relatively compact, but in the proof of Theorem 12.13 (ii) only the consequence is used that then $Z^0_{0,r}(\overline{D}, E) \cap \overline{\partial} C^0_{0,r-1}(\overline{D}, E)$ is topologically closed in $Z^0_{0,r}(\overline{D}, E)$.) □

End of the proof of Theorem 4.1 (i). By definition of Φ, for each $C \in \Phi$, there exists $\gamma \in]\inf \rho, \sup \rho[$ with $C \subseteq \{\rho \ge \gamma\}$. Therefore it follows from Lemma 1.2 (i) in [La-L] that

$$Z^0_{0,r}(\Phi; X, E) = E^0_{0,r}(\Phi; X, E) \ + \ Z^0_{0,r}(\rho \ge \alpha_0; X, E). \tag{4.3}$$

By Lemma 4.2, the space (4.2) is of finite codimension in $Z^0_{0,r}(\rho \ge \alpha_0; \rho \le \beta_0, E)$. Therefore it follows from Lemma 4.3 that the space (4.1) is of finite codimension in $Z^0_{0,r}(\rho \ge \alpha_0; X, E)$. Since (4.1) is a subspace of $E^0_{0,r}(\Phi; X, E)$, it follows that $Z^0_{0,r}(\rho \ge \alpha_0; X, E) \cap E^0_{0,r}(\Phi; X, E)$ is of finite codimension in $Z^0_{0,r}(\rho \ge \alpha_0; X, E)$. In view of (4.3), this implies that $E^0_{0,r}(\Phi; X, E)$ is of finite codimension in $Z^0_{0,r}(\Phi; X, E)$, i.e. $\dim H^{0,r}(\Phi; X, E) < \infty$. □

5. Proof of Theorem 0.4

By Serre duality [S] (see also Theorem 2.9), Theorem 0.4 is equivalent to the following theorem:

5.1. Theorem. *Let* X *be an* n*-dimensional complex manifold,* $n \ge 3$, *which is* q*-concave-*$(n-q)$*-convex,* $1 \le q \le n-1$. *If*

$$q < \frac{n}{2},$$

then, for any holomorphic vector bundle E *over* X, $H^{0,q+1}_c(X, E)$ *is separated.*

Proof. Let ρ be as in Definition 0.1 and let Φ be the family of closed subsets C of X such that the sets $C \cap \{\rho \ge \alpha\}$, $\inf \rho < \alpha < \sup \rho$, are compact. Denote by Φ_X

the family of all compact subsets of X. Then Φ_X is complete in Φ and it follows from Lemma 1.3 that

$$\dim \left[E^0_{0,q+1}(\Phi \to \Phi_X; X, E)/E^0_{0,q+1}(\Phi_X; X, E)\right] \le \dim H^{0,q}(\Phi * \Phi_X, E). \tag{5.1}$$

Take $\alpha_0 \in]\inf \rho, \sup \rho[$ such that the Levi form of ρ has at least $n-q+1$ positive eigenvalues on $\{\rho \le \alpha_0\}$. Then, by the Andreotti-Grauert finiteness theorem,

$$\dim H^{0,r}(\{\rho < \alpha\}, E) < \infty \qquad \text{if} \quad q \le r \le n-q-1 \quad \text{and} \quad \inf \rho < \alpha \le \alpha_0.$$

Since $q < n/2$ and hence $q \le n-q-1$, it follows that

$$\dim H^{0,q}(\{\rho < \alpha\}, E) < \infty \qquad \text{if} \quad \inf \rho < \alpha \le \alpha_0.$$

Since, for each $U \in \Phi * \Phi_X$, there exists $\alpha \in]\inf \rho, \alpha_0]$ with $\{\rho < \alpha\} \subseteq U$, this implies that

$$\dim H^{0,q}(\Phi * \Phi_X; X, E) < \infty$$

and hence, by (5.1),

$$\dim \left[E^0_{0,q+1}(\Phi \to \Phi_X; X, E)/E^0_{0,q+1}(\Phi_X; X, E)\right] < \infty. \tag{5.2}$$

Furthermore, the hypothesis $q < n/2$ implies that $q+1 \le \min(n-q, n-q^*+1)$ with $q^* := n-q$. Therefore it follows from Theorem 4.1 (ii) and (iii) that $E^0_{0,q+1}(\Phi; X, E)$ is topologically closed in $C^0_{0,q+1}(\Phi; X, E)$. Since the embedding map

$$C^0_{0,q+1}(\Phi_X; X, E) \longrightarrow C^0_{0,q+1}(\Phi; X, E)$$

is continuous and $E^0_{0,q+1}(\Phi \to \Phi_X; X, E)$ is the preimage of $E^0_{0,q+1}(\Phi; X, E)$ with respect to this map, it follows that $E^0_{0,q+1}(\Phi \to \Phi_X; X, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$. Hence, by (5.2), there is a finite dimensional linear subspace F of $C^0_{0,q+1}(\Phi_X; X, E)$ such that

$$F \; + \; E^0_{0,q+1}(\Phi_X; X, E)$$

is topologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$. Since, by Theorem 2.9, conditions (Cl_1) and (Cl_3) are equivalent for Φ_X, this means that $E^0_{0,q+1}(\Phi_X; X, E)$ is topologically closed in $C^0_{0,q+1}(\Phi_X; X, E)$, i.e. $H^{0,q+1}(\Phi_X; X, E)$ is separated. $\square$

6. Proof of Theorem 0.5 (and some generalization)

In this section we assume that X is a connected n-dimensional complex manifold which is q-concave in the sense of Andreotti-Grauert, $1 \le q \le n$, i.e. we have a real C^2 function ρ on X such that

(i) for all $\alpha > \inf \rho := \inf_{\zeta \in X} \rho(\zeta)$, the set $\{\rho \ge \alpha\} := \{\zeta \in X \mid \rho(\zeta) \ge \alpha\}$ is compact;

(ii) outside some compact set, the Levi form of ρ has at least $n-q+1$ positive eigenvalues.

Moreover we assume that, for certain $\alpha_0 > \inf \rho$, two open subsets H' and H of X are given such that

$$H' \cap H = \emptyset \quad \text{and} \quad H' \cup H = \{\rho < \alpha_0\}.$$

Denote by Φ the family of all closed subsets C of X such that $C \cap \overline{H}'$ is compact. Φ is a cofinal family of supports in X. The dual family Φ^* of Φ consists of all closed sets $C \subseteq X$ such that $C \cap H$ is compact. Φ^* is also a cofinal family of supports in X and $\Phi^{**} = \Phi$ (i.e. the hypothesis of Theorem 2.9 are fulfilled).

6.1. Theorem. *If $q = n-1$, then, for each holomorphic vector bundle E over X, $H_\Phi^{0,1}(X,E)$ is separated.*

Proof. If $\overline{H}'$ is compact, then the theorem coincides with the classical Andreotti-Vesentini theorem (Theorem 0.2 with $q^* = 0$). We therefore assume that $\overline{H}'$ is non compact.

By Theorem 2.9, conditions (Cl_1) and (Cl_3) are equivalent for $E_{0,1}^0(\Phi; X, E)$. Therefore it is sufficient to prove that, for all $C \in \Phi$, the space $C_{0,1}^0(C; X, E) \cap E_{0,1}^0(\Phi; X, E)$ is closed in $C_{0,1}^0(C; X, E)$.

Let $C_0 \in \Phi$ be fixed and let $f_j \in C_{0,1}^0(C_0; X, E) \cap E_{0,1}^0(\Phi; X, E)$ be a sequence converging to some $f \in C_{0,1}^0(C_0; X, E)$ in the topology of $C_{0,1}^0(C_0; X, E)$, i.e. uniformly on the compact subsets of C_0. We have to prove that $f \in E_{0,1}^0(\Phi; X, E)$.Set

$$H'_\alpha := H' \cap \{\rho < \alpha\} \qquad \text{for} \quad \alpha > \inf \rho.$$

Since $C_0 \cap \overline{H}'$ is compact, then we can find α_1 with $\inf \rho < \alpha_1 < \alpha_0$ such that $C_0 \cap \overline{H}'_{\alpha_1} = \emptyset$. Denote by $\hat{H}$ the union of $\overline{H}' \setminus H'_{\alpha_1}$ with all connected components of $\overline{H}' \setminus H'_{\alpha_1}$ which are relatively compact in H'. Then $\hat{H}$ is compact, $H'_{\alpha_1} \setminus \hat{H} \neq \emptyset$, and no connected component of $H'_{\alpha_1} \setminus \hat{H}$ is relatively compact in H'.

Since X is $(n-1)$-concave, it follows from the classical Andreotti-Vesentini theorem (Theorem 0.2 with $q^* = 0$), that $E_{0,1}^0(X, E)$ is topologically closed in $C_{0,1}^0(X, E)$. Since the sequence f_j converges to f also with respect to the topology of $C_{0,1}^0(X, E)$, it follows that $f \in E_{0,1}^0(X, E)$, i.e.

$$\overline{\partial} u = f \qquad \text{for some} \quad u \in C_{0,0}^0(X, E).$$

Moreover, by the open mapping theorem, we can find a sequence $u_j \in C_{0,0}^0(X, E)$ which converges to u in $C_{0,0}^0(X, E)$ such that

$$\overline{\partial} u_j = f_j \qquad \text{for all} \quad j.$$

On the other hand, since $f_j \in E_{0,1}^0(\Phi; X, E)$, we can solve the equations

$$\overline{\partial} v_j = f_j$$

also with $v_j \in C^0_{0,0}(\Phi; X, E)$. Consider the sequence of holomorphic sections

$$w_j := u_j - v_j \in Z^0_{0,0}(X, E)$$

and set

$$U = H'_{\alpha_1} \setminus \hat{H}.$$

Since $\operatorname{supp} f_j \subseteq C_0$ and $C_0 \cap H'_{\alpha_1} = \emptyset$, then $f_j \equiv 0$ on U for all j. Hence each v_j is holomorphic on U. Since no connected component of U is relatively compact, but $\operatorname{supp} v_j \in \Phi$, it follows that $v_j \equiv 0$ on U for all j. Hence the restricted sequence $w_j|_U$ converges to $u|_U$ with respect to the topology of $Z^0_{0,0}(U, E)$. Since the space $Z^0_{0,0}(X, E)$ is finite dimensional (X is $(n-1)$-concave) and the restriction map

$$Z^0_{0,0}(X, E) \longrightarrow Z^0_{0,0}(U, E)$$

is injective (by uniqueness of holomorphic functions), it follows that

$$h := \lim_{j \to \infty} w_j$$

exists in $Z^0_{0,0}(X, E)$, where $h = u$ on U. Set $\tilde{u} = u - h$. Then $\overline{\partial}\tilde{u} = f$ and $u \equiv 0$ on $U = H'_{\alpha_1} \setminus \hat{H}$. Hence $f \in E^0_{0,1}(\Phi; X, E)$. $\square$

6.2. Theorem. *If $q \leq n - q - 1$, then, for each holomorphic vector bundle E over X, $H^{0,n-q}_{\Phi}(X, E)$ is separated. If moreover $q \leq n - q - 2$, then*

(6.1) $$\text{dim } H^{0,r}_{\Phi}(X, E) < \infty \quad \text{if} \quad q + 1 \leq r \leq n - q - 1.$$

Proof. Set $H'_\alpha = H' \cap \{\rho < \alpha\}$ and take α_1 with $\inf \rho < \alpha_1 < \alpha_0$ such that the Levi form of ρ has at least $n - q + 1$ positive eigenvalues on H'_{α_1}. Then, by the Andreotti-Grauert finiteness theorem [A-G],

(6.2) $$\dim H^{0,r}(H'_\alpha, E) < \infty \quad \text{if } \inf \rho < \alpha < \alpha_1 > \quad \text{and } q \leq r \leq n - q - 1.$$

Let Ψ be the family of all closed subsets of X. Then $\Psi * \Phi$ consists of all open sets $U \subseteq H'$ such that $\overline{H}' \setminus U$ is compact, and Φ is complete in Ψ. Therefore, by Lemma 1.3 (iii),

$$\dim \left[E^0_{0,r}(\Psi \to \Phi; X, E)/E^0_{0,r}(\Phi; X, E)\right] \leq \dim H^{0,r-1}(\Psi * \Phi; E)$$

for all $r \geq 1$. Together with (6.2) this implies that

(6.3) $$\dim \left[E^0_{0,r}(\Psi \to \Phi; X, E)/E^0_{0,r}(\Phi; X, E)\right] < \infty \quad \text{if} \quad q + 1 \leq r \leq n - q.$$

By the classical Andreotti-Vesentini theorem, $E^0_{0,n-q}(X, E)$ is topologically closed in $C^0_{0,n-q}(X, E)$. Since the topology of $C^0_{0,n-q}(\Phi; X, E)$ is stronger than the topology of $C^0_{0,n-q}(X, E)$ and $E^0_{0,n-q}(\Psi \to \Phi; X, E) = E^0_{0,n-q}(X, E) \cap C^0_{0,n-q}(\Phi; X, E)$, it follows that $E^0_{0,n-q}(\Psi \to \Phi; X, E)$ is topologically closed in $C^0_{0,n-q}(\Phi; X, E)$. Since, by (6.3), there is a finite dimensional linear subspace F of $C^0_{0,n-q}(\Phi; X, E)$ with

$$F + E^0_{0,n-q}(\Phi; X, E) = E^0_{0,n-q}(\Psi \to \Phi; X, E),$$

this implies that condition (Cl_1) in Sect. 2 is fullfilled for $s = 0$ and $r = n - q$. Hence by Theorem 2.9, also condition (Cl_3) is fulfilled, i.e. $H^{0,n-q}(\Phi; X, E)$ is separated.

To prove (6.1), we observe that, by the Andreotti-Grauert finiteness theorem [A-G], $\dim H^{0,r}(X, E) < \infty$ if $0 \le r \le n - q - 1$ and therefore, in particular,

$$\dim \left[Z^0_{0,r}(\Phi; X, E) / E^0_{0,r}(\Psi \to \Phi; X, E) \right] < \infty \quad \text{if} \quad 0 \le r \le n - q - 1.$$

Together with (6.3) this implies (6.1). □

6.3. Corollary. *If $q = 1$, then, for each holomorphic vector bundle E over X, $H^{0,n-1}_\Phi(X, E)$ is separated.*

Proof. This follows from Theorem 6.1 if $n = 2$, and Theorem 6.2 if $n \ge 3$. □

Malgrange's vanishing theorem for forms vanishing around $(n-1)$-convex holes and an application to the Hartogs-Bochner extension phenomenon

The classical vanishing theorem of Malgrange for the cohomology in top degree [M] (see also [O]) says that, for any non compact connected complex manifold X of complex dimension n and each holomorphic vector bundle E over X, the equation $\bar\partial u = f$ can be solved for any E-valued $(0, n)$-form f on X. In this section we prove that if the form f vanishes around an '$(n-1)$-convex hole' of X, then also the solution u can be chosen vanishing around this hole.

7.1. Definition. Let X be a complex manifold. An open subset H of X will be called an $(n-1)$-*concave end of* X (or a *neighborhood of an* $(n-1)$-*convex hole*) if

(i) ∂H is compact, but $\overline{H}$ is not compact,

and there exists a real C^2-function ρ on X such that

(ii) $H = \{\rho < 0\}$ and $\partial H = \{\rho = 0\}$,

(iii) $\inf \rho < \rho(z)$ for all $z \in X$,

(iv) the sets $\{\alpha \le \rho \le 0\}$, $\inf \rho < \alpha \le 0$, are compact,

(v) the Levi form of ρ has at least 2 positive eigenvalues everywhere on $\overline{H}$.

Theorem 7.2 (Malgrange's theorem for forms vanishing around $(n-1)$-convex holes). *Let X be a connected complex manifold and let H be an $(n-1)$-concave end of X such that also $X \setminus H$ is not relatively compact in X. Denote by Φ the family of supports in X which consists of all closed subsets C such that $C \cap \overline{H}$ is compact. Then, for any holomorphic vector bundle E over X,*

$$H^{0,n}_\Phi(X, E) = 0 \qquad \text{where} \quad n = \dim_{\mathbb{C}} X.$$

Proof. First note that Φ is cofinal and the dual family Φ^* of Φ is also cofinal and $\Phi^{**} = \Phi$, i.e. we may apply Theorem 2.9.

Now recall that, by a theorem of Green and Wu [G-W] (see also [O]), any non-compact connected complex manifold of complex dimension n admits an exhausting function whose Levi form has at least one positive eigenvalue everywhere on the manifold. Since $X \setminus H$ is noncompact, this implies that the function ρ in Definition 7.1 can be chosen so that, moreover, $\sup \rho = \infty$ and the Levi form of ρ has at least one positive eigenvalue everywhere on $\{\rho > 1\}$.

Therefore X is $(n-1)$-concave-n-convex and it follows from Theorem 4.1 (i) that

$$\dim H^{0,n}_{\Phi}(X, E) < \infty.$$

Hence, by Theorem 2.9 (i), $E^0_{0,n}(\Phi; X, E)$ is topologically closed in $C^0_{0,n}(\Phi; X, E)$.

Since $X \setminus H$ is noncompact and belongs to Φ, we have $X \setminus C \neq \emptyset$ for all $C \in \Phi^*$. Since X is connected, this implies by uniqueness of holomorphic functions that $H^{n,0}_{\Phi^*}(X, E^*) = 0$. Since $E^0_{0,n}(\Phi; X, E)$ is topologically closed this implies by Theorem 2.9 (ii) that

$$\dim H^{0,n}_{\Phi}(X, E) = \dim H^{n,0}_{\Phi^*}(X, E^*) = 0. \quad \square$$

Using ideas of Lupacciolu [Lu] and Chirka-Stout (see Theorem 3.3.1 and its proof in [C-S]), Theorem 7.2 gives the following result on Hartogs-Bochner extension:

7.3. Theorem. *Let X be a connected n-dimensional complex manifold with an $(n-1)$-concave end H such that $X \setminus H$ is noncompact, and let D be an open subset of X with C^1-boundary such that $(X \setminus H) \cap \overline{D}$ is compact and $X \setminus \overline{D}$ has no more than a finite number of connected components U such that $(X \setminus H) \cap \overline{U}$ is compact.*

Then for any holomorphic vector bundle E over X and each CR-section $f : \partial D \to E$ the following two conditions are equivalent:

(i) There exists a continuous section $F : \overline{D} \to E$ which is holomorphic over D such that $F|\partial D = f$.

(ii) $\int_{\partial D} f\varphi = 0$ for any C^∞-smooth, $\overline{\partial}$-closed $(n, n-1)$-form φ with values in E^*, the dual bundle of E, defined in a neighborhood of $\overline{D}$ such that $\operatorname{supp} \varphi \cap \partial D$ is compact.

Proof. The conclusion (i)$\Rightarrow$(ii) follows from Stokes' theorem. Assume that condition (ii) is fulfilled.

Let $U_1, \dots, U_N$ be the connected components U of $X \setminus \overline{D}$ such that $(X \setminus H) \cap \overline{U}$ is compact. Take points $z_j \in (X \setminus \overline{H}) \cap U_j$, $1 \le j \le N$, and set $\tilde{X} = X \setminus \{z_1, \dots, z_N\}$. Then $D \subseteq \tilde{X}$ and, for each connected component U of $\tilde{X} \setminus \overline{D}$, $(\tilde{X} \setminus H) \cap \overline{U}$ is noncompact.

Let ρ be the function from Definition 7.1 for H. Take $\alpha \in [\inf \rho, 0]$ so small that $\rho > \alpha$ on $\overline{U}_1 \cup \ldots \cup \overline{U}_N$ and set $H_\alpha = \{\rho \le \alpha\}$. Then H_α is an $(n-1)$-concave end of $\tilde{X}$. Denote by Φ the family of supports in $\tilde{X}$ which consists of all closed subsets

C of $\tilde{X}$ such that $C \cap \overline{H}_\alpha$ is compact, and let Φ^* be the dual family of Φ. Then $\partial D \in \Phi^*$.

By Theorem 7.2, $H^{n,n}_\Phi(\tilde{X}, E^*) = 0$. This implies by Theorem 2.9 (i) that

$$E^0_{0,1}(\Phi^*; \tilde{X}, E) = \left\{ \psi \in Z^0_{0,1}(\Phi^*; \tilde{X}, E) \;\middle|\; \int_{\tilde{X}} \psi \wedge \varphi = 0 \quad \text{for all} \quad \varphi \in Z^0_{n,n-1}(\Phi; \tilde{X}, E^*) \right\}. \tag{7.1}$$

Consider the E-valued $(0,1)$-current $[i_* f]^{0,1}$ on $\tilde{X}$ defined by

$$\langle [i_* f]^{0,1}, \varphi \rangle = \int_{\partial D} f \varphi$$

for all E^*-valued $C^\infty_{n,n-1}$-forms φ with compact support in $\tilde{X}$. Then $\operatorname{supp} [i_* f]^{0,1} = \partial D \in \Phi^*$ and, since f is CR, $\overline{\partial}[i_* f]^{0,1} = 0$. Therefore, by regularity of $\overline{\partial}$ (see, e.g., Corollary 2.15 (i) in [H-L]) and property (S_3) in the definition of a family of supports, we can find an E-valued current S of $\tilde{X}$ with

$$\operatorname{supp} S \in \Phi^* \qquad \text{and} \qquad [i_* f]^{0,1} - \overline{\partial} S \in Z^0_{0,1}(\Phi^*; \tilde{X}, E) \cap C^\infty_{0,1}(\tilde{X}, E).$$

Since (again by regularity of $\overline{\partial}$) any $\varphi \in Z^0_{n,n-1}(\Phi; \tilde{X}, E^*)$ is of the form $\varphi = \overline{\partial}\alpha + \beta$ where α is continuous and β of class C^∞, it follows from condition (ii) of the theorem that

$$\int_{\tilde{X}} ([i_* f]^{0,1} - \overline{\partial} S) \wedge \varphi = 0 \qquad \text{for all} \quad \varphi \in Z^0_{n,n-1}(\Phi; \tilde{X}, E^*).$$

Therefore it follows from (7.1) that

$$[i_* f]^{0,1} - \overline{\partial} S \;\in\; E^0_{0,1}(\Phi^*; \tilde{X}, E).$$

Hence there exists an E-valued $(0,0)$-current F on $\tilde{X}$ with $\operatorname{supp} F \in \Phi^*$ and

$$\overline{\partial} F = [i_* f]^{0,1}. \tag{7.2}$$

Since $F \in \Phi^*$, $(X \setminus H_\alpha) \cap \operatorname{supp} F$ is compact. On the other hand, any connected component of $\tilde{X} \setminus \overline{D}$ has a noncompact intersection with $X \setminus H_\alpha$. Since F is holomorphic outside $\operatorname{supp} F$, it follows that $F = 0$ outside D.

Hence F is a function on $\tilde{X}$ which is holomorphic over D and zero outside $\overline{D}$. Together with (7.2) this implies by standard arguments (using local properties of the Bocher-Martinelli transform - see, e.g., the proof of Theorem 5.1 in [La]) that F extends continuously from D to $\overline{D}$ and that this extension is equal to f of ∂D. □

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