Rational cuspidal plane curves of type $(d,\, d-3)$*

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Abstract

In the previous paper [FlZa 2] we classified the rational cuspidal plane curves C with a cusp of multiplicity $\deg C - 2$. In particular, we showed that any such curve can be transformed into a line by Cremona transformations. Here we do the same for the rational cuspidal plane curves C with a cusp of multiplicity $\deg C - 3$.

Introduction

Let $C \subset \mathbf{P}^2$ be a rational cuspidal curve; that is, it has only irreducible singularities (called *cusps*). We say that C is of type $(d,\, m)$ if $d = \deg C$ is the degree and $m = \max_{P \in \operatorname{Sing} C}\{\operatorname{mult}_P C\}$ is the maximal multiplicity of the singular points of C.

Topologically, C is a 2-sphere S^2 (non-smoothly) embedded into $\mathbf{P}^2$. Due to the Poincaré-Lefschetz dualities, the complement $X := \mathbf{P}^2 \setminus C$ to C is a $\mathbf{Q}-$acyclic affine algebraic surface, i.e. $\widetilde{H}_*(X;\, \mathbf{Q}) = 0$ (see e.g. [Ra, Fu, Za]). Furthermore, if C has at least three cusps, then X is of log-general type, i.e. $\overline{k}(X) = 2$, where $\overline{k}$ stands for the logarithmic Kodaira dimension [Wa].

In [FlZa 1] we conjectured that any $\mathbf{Q}-$acyclic affine algebraic surface X of log-general type is rigid in the following sense. Let V be a minimal smooth projective completion of X by a simple normal crossing (SNC for short) divisor D. We say that X is *rigid* (resp. *unobstructed*) if the pair $(V,\, D)$ has no nontrivial deformations (resp. if the infinitesimal deformations of the pair $(V,\, D)$ are unobstructed).

In the particular case when $X = \mathbf{P}^2 \setminus C$ with C as above, the rigidity conjecture would imply that the curve C itself is projectively rigid. This means that the only equisingular deformations of C in $\mathbf{P}^2$ are those provided by automorphisms of $\mathbf{P}^2$; in other words, all of them are projectively equivalent to C (see [FlZa 2, sect. 2]). In turn, this would imply that there is only a finite number of non-equivalent rational cuspidal plane curves of a given degree with at least three cusps. Therefore, one may hope to give a classification of such curves.

In [FlZa 2] we obtained a complete list of rational cuspidal plane curves of type $(d,\, d-2)$ with at least three cusps, and showed that all of them are projectively rigid and unobstructed.

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In the theorem below we do the same for rational cuspidal plane curves of type $(d,\, d-3)$ with at least three cusps.

The principal numerical invariant which characterizes a cusp up to equisingular deformation is its multiplicity sequence. Recall that, if

$$V_{n+1} \to V_n \to \ldots \to V_1 \to V_0 = \mathbf{C}^2$$

is a minimal resolution of an irreducible analytic plane curve germ $(C,\, \overline{0}) \subset (\mathbf{C}^2,\, \overline{0})$, and $(C_i,\, P_i)$ denotes the proper transform of $(C,\, \overline{0})$ in V_i, so that $(C_0,\, P_0) = (C,\, \overline{0})$, then $\underline{m} = (m^{(i)})_{i=0}^{n+1}$, where $m^{(i)} = \text{mult}_{P_i} C_i$, is called *the multiplicity sequence* of the germ $(C,\, \overline{0})$. Thus, $m^{(i+1)} \le m^{(i)}$, $m^{(n)} \ge 2$ and $m^{(n+1)} = 1$. A multiplicity sequence has the following characteristic property [FlZa 2, (1.2)]:

for any $i = 0, \ldots, n-1$ *either* $m^{(i)} = m^{(i+1)}$, *or there exists* $k > 0$ *such that* $i + k \le n$, *and*

$$m^{(i)} = m^{(i+1)} + \ldots + m^{(i+k)} + m^{(i+k+1)}, \qquad \text{where} \qquad m^{(i+1)} = \ldots = m^{(i+k)}\,.$$

We use the abbreviation (m_k) for a (sub)sequence $m^{(i+1)} = m^{(i+2)} = \ldots = m^{(i+k)} = m$. Thus, we present a multiplicity sequence as $(m^{(1)}_{k_1},\, \ldots,\, m^{(s)}_{k_s})$ with $m^{(i+1)} < m^{(i)}$; by abuse of notation, we assume here that $m^{(s)} \ge 2$. For instance, (2) means an ordinary cusp, and $(2_3) = (2,\, 2,\, 2,\, 1)$ corresponds to a ramphoid cusp. With this notation we can formulate our main result as follows.

Theorem. *(a) Let* $C \subset \mathbf{P}^2$ *be a rational cuspidal plane curve of type* $(d,\, d-3)$, $d \ge 6$, *with at least three cusps. Then* $d = 2k+3$, *where* $k \ge 2$, *and* C *has exactly three cusps, of types* $(2k,\, 2_k)$, (3_k), (2), *respectively.*

(b) For each $k \ge 1$ *there exists a rational cuspidal plane curve* C_k *of degree* $d = 2k+3$ *with three cusps of types* $(2k,\, 2_k)$, (3_k) *and* (2).

(c) Moreover, the curve C_k *as in (b) is unique up to projective equivalence. It can be defined over* $\mathbf{Q}$.

Remarks. (1) A classification of irreducible plane curves up to degree 5 can be found e.g. in [Nam]. In particular, there are, up to projective equivalence, only one rational cuspidal plane quartic with three cusps (*the Steiner quartic*) and only three rational cuspidal plane quintic curves with at least three cusps. Two of them have exactly three cusps, of types (3), (2_2), (2) resp. (2_2), (2_2), (2_2), and the third one has four cusps of types (2_3), (2), (2), (2) [Nam, Thm. 2.3.10].

(2) In his construction of $\mathbf{Q}-$acyclic surfaces (see e.g. [tD 1, tD 2]), T. tom Dieck found certain $(d,\, d-2)-$ and $(d,\, d-3)-$rational cuspidal curves, in particular, those listed in the theorem above, as well as some other series of rational cuspidal plane curves (a private communication[1]). Besides a finite number of sporadic examples, the curves with at least

[1] We are grateful to T. tom Dieck for communicating us the list of the multiplicity sequences of the constructed curves.

three cusps in the list of tom Dieck are organized in three series of $(d,\, d-2)-$, $(d,\, d-3)-$ and $(d,\, d-4)-$type, respectively. It can be checked that all those curves are rigid and unobstructed.

Following our methods, T. Fenske proved recently that the only possible numerical data of unobstructed rational cuspidal plane curves with at least three cusps and of type $(d, d-4)$ are those from the list of tom Dieck. He has also classified all rational cuspidal plane curves of degree 6 [Fe]. It turns out that the only examples with at least 3 cusps are those described in [FlZa 2].

(3) For a rational cuspidal plane curve C of type $(d,\, m)$ the inequality $m > d/3$ holds [MaSa]. Recently, S. Orevkov obtained a stronger one[2]: If the complement $\mathbf{P}^2 \setminus C$ has logarithmic Kodaira dimension 2, then $d < \alpha m + \beta$, where $\alpha := (3+\sqrt{5})/2 = 2.6180\ldots$ and $\beta := \alpha - 1/\sqrt{5} = 2.1708\ldots$.

(4) It was shown in [OrZa 1, OrZa 2] that a rational cuspidal plane curve with at least ten cusps cannot be projectively rigid.

Recall the Coolidge–Nagata Problem [Co, Nag]:

Which rational plane curves can be transformed into a line by means of Cremona transformations of $\mathbf{P}^2$?

It can be completed by the following question:

Is this possible, in particular, for any rational cuspidal plane curve?

Under certain restrictions, a positive answer was given in [Nag, MKM, MaSa, Ii 2, Ii 3]. It can be verified that the last question has a positive answer for the rational cuspidal plane curves of degree at most five. In [FlZa 2] we showed that any rational cuspidal plane curve of type $(d,\, d-2)$ with at least three cusps is rectifiable. Here we extend this result to $(d,\, d-3)-$curves. It will turn out to be an immediate consequence of our construction:

Corollary. *Any rational cuspidal plane curve of type* $(d,\, d-3)$ *with at least three cusps is rectifiable, i.e. it can be transformed into a line by means of Cremona transformations.*

1 Proofs

Let $C \subset \mathbf{P}^2$ be a plane curve, and let $V \to \mathbf{P}^2$ be the minimal embedded resolution of singularities of C, so that the reduced total transform D of C in V is an SNC–divisor. By [FlZa 1], the cohomology groups $H^i(\Theta_V\langle D\rangle)$ of the sheaf of germs of holomorphic vector fields on V tangent to D control the deformations of the pair $(V,\, D)$; more precisely, $H^0(\Theta_V\langle D\rangle)$ is the space of its infinitesimal automorphisms, $H^1(\Theta_V\langle D\rangle)$ is the space of infinitesimal deformations and $H^2(\Theta_V\langle D\rangle)$ gives the obstructions for extending infinitesimal deformations.

[2] We are grateful to S. Orevkov for providing us with a preliminary version of his paper.

The surface $X = V \setminus D = \mathbf{P}^2 \setminus C$ being of log-general type, the automorphism group $\mathrm{Aut}X$ is finite [Ii 1], and hence $h^0(\Theta_V\langle D\rangle) = 0$. Thus, the holomorphic Euler characteristic of the sheaf $\Theta_V\langle D\rangle$ is

$$\chi(\Theta_V\langle D\rangle) = h^2(\Theta_V\langle D\rangle) - h^1(\Theta_V\langle D\rangle).$$

Lemma 1.1. *If C is a rational cuspidal plane curve of type $(d,\, d-3)$ with at least three cusps, then $h^2(\Theta_V\langle D\rangle) = 0$, that is, C is unobstructed*[3], *and so $\chi = \chi(\Theta_V\langle D\rangle) \le 0$.*

Proof. Projecting from the cusp of multiplicity $d-3$ yields a fibration $V \to \mathbf{P}^1$, which is three–sheeted when restricted to the proper transform of C. Now [FlZa 1, (6.3)] shows that $h^2(\Theta_V\langle D\rangle) = 0$. Since $\overline{k}(V \setminus D) = 2$, we also have $h^0(\Theta_V\langle D\rangle) = 0$. Hence $\chi = -h^1(\Theta_V\langle D\rangle) \le 0$. □

The next proposition proves part (a) of our main theorem.

Proposition 1.1. *The only possible rational cuspidal plane curves C of degree $d \ge 6$ with a singular point Q of multiplicity $d-3$ and at least three cusps are those of degree $d = 2k+3,\ k = 1, \ldots,$ with three cusps of types $(2k,\, 2_k)$, (3_k) and (2). Furthermore, these curves are projectively rigid.*

Proof. By [FlZa 2, (2.5)] and Lemma 1.1 above, we have:

$$\chi = -3(d-3) + \sum_{P \in \mathrm{Sing}\, C} \chi_P \le 0\,, \qquad (R_1)$$

where

$$\chi_P := \eta_P + \omega_P - 1\,,$$

and where, for a singular point $P \in C$ with the multiplicity sequence $\underline{m}_P = (m^{(0)}, \ldots, m^{(k_P)})$,

$$\eta_P = \sum_{i=0}^{k_P} (m^{(i)} - 1) \qquad \text{and} \qquad \omega_P = \sum_{i=1}^{k_P} \left(\left\lceil \frac{m^{(i-1)}}{m^{(i)}} \right\rceil - 1\right)$$

(for $a \in \mathbf{R}$, $\lceil a \rceil$ denotes the smallest integer $\ge a$).

Observe that, by the Bezout theorem, $m_P^{(0)} + m_P^{(1)} \le d$ and $m_P^{(0)} + m_Q^{(0)} \le d$. Thus

$$\text{for} \quad P \ne Q \quad \text{we have} \quad m_P^{(0)} \le 3; \quad \text{moreover we have} \quad m_Q^{(1)} \le 2,$$

since otherwise the tangent line $T_Q C$ would have the only point Q in common with C, and so, $C \setminus T_Q C$ would be an affine rational cuspidal plane curve with one point at infinity and with two cusps. But by the Lin-Zaidenberg Theorem [LiZa], up to biregular automorphisms

[3] i.e. as a plane curve, it has unobstructed equisingular infinitesimal deformations.

of the affine plane $\mathbf{C}^2$, the only irreducible simply connected affine plane curves are the curves $\Gamma_{k,l} = \{x^k - y^l = 0\}$, where $1 \le k \le l$, and $(k, l) = 1$. Hence, such a curve cannot have two cusps. Using the above restriction and the characteristic property of a multiplicity sequence cited above we obtain the following possibilities for the multiplicity sequence $\underline{m}_P$ at a singular point P:

$$\begin{aligned} &\underline{m}_Q = (d-3) \text{ or } (d-3, 2), \\ &\underline{m}_P = (2_a) \text{ or } (3_a) \text{ or } (3_a, 2) \quad \text{for } P \neq Q. \end{aligned} \tag{R_2}$$

For different possible types of cusps of C we have:

(a) If $Q \in \operatorname{Sing} C$ has the multiplicity sequence $(d-3)$, then

$$\eta_Q = d - 4, \quad \omega_Q = d - 4 \quad \text{and so} \quad \chi_Q = 2d - 9.$$

(b) If $Q \in \operatorname{Sing} C$ has the multiplicity sequence $(d-3,\, 2_a)$ then, by the same characteristic property [FlZa 2, (1.2)],

$$(*) \qquad \text{either} \quad d - 3 \le 2a \quad \text{is even or} \quad d - 3 = 2a + 1.$$

In any case

$$\eta_Q = d - 4 + a, \quad \omega_Q = \lceil \frac{d-3}{2} \rceil \quad \text{and so} \quad \chi_Q = d - 5 + a + \lceil \frac{d-3}{2} \rceil.$$

(c) If $P \in \operatorname{Sing} C$ has the multiplicity sequence (2_a), then

$$\eta_P = a, \quad \omega_P = 1 \quad \text{and so} \quad \chi_P = a.$$

(d) If $P \in \operatorname{Sing} C$ has the multiplicity sequence (3_a), then

$$\eta_P = 2a, \quad \omega_P = 2 \quad \text{and so} \quad \chi_P = 2a + 1.$$

(e) If $P \in \operatorname{Sing} C$ has the multiplicity sequence $(3_a,\, 2)$, then

$$\eta_P = 2a + 1, \quad \omega_P = 2 \quad \text{and so} \quad \chi_P = 2a + 2.$$

Furthermore, since C rational, by the genus formula, we have

$$\binom{d-1}{2} = \sum_{P \in \operatorname{Sing} C} \delta_P \quad \text{where} \quad \delta_P := \sum_{i=1}^{k_P} \binom{m_P^{(i)}}{2}.$$

Since $m_Q^{(0)} = d - 3$, we get

$$\binom{d-1}{2} = \binom{d-3}{2} + \sum_{(P,i) \neq (Q,0)} \binom{m_P^{(i)}}{2},$$

or, equivalently,

$$2d-5=\sum_{(P,i)\neq(Q,0)}\frac{m_P^{(i)}(m_P^{(i)}-1)}{2}\,. \qquad (R_3)$$

At last, consider the projection $\pi_Q : C \to \mathbf{P}^1$ from the point Q. By the Riemann-Hurwitz Formula, it has at most four branching points. This gives the restriction (see [FlZa 2, (3.1)])

$$m_Q^{(1)}-1+\sum_{P\neq Q}(m_P^{(0)}-1)\leq 4\,. \qquad (R_4)$$

Thus, if the curve C has the numerical data

$$[(d-3,\,2_{a_1}),\,(2_{a_2}),\ldots,(2_{a_k}),(3_{b_1}),\ldots,(3_{b_l}),(3_{c_1},\,2),\ldots,(3_{c_m},\,2)]\,,$$

then $k+2(l+m)\leq 4$. Hence, either $l+m=0$ and $3\leq k\leq 4$, or $l+m=1$ and $k=2$, or $l+m=2$ and $k=0$.

Taking into account the above restrictions $(R_2)-(R_4)$ and $(*)$ from (b), the list of all possible data of rational cuspidal plane curves C of degree $d\geq 6$ with a point of multiplicity $d-3$ and at least 3 cusps is as follows, where $a,\,b,\,c,\,e>0$:

$$[(d-3),\ (2_a),\ (2_b)] \quad \text{where} \quad a+b=2d-5 \qquad (1)$$

$$[(d-3),\ (2_a),\ (3_b)] \quad \text{where} \quad a+3b=2d-5 \qquad (2)$$

$$[(d-3),\ (3_a),\ (3_b)] \quad \text{where} \quad 3a+3b=2d-5 \qquad (3)$$

$$[(d-3),\ (3_a,\,2),\ (2_b)] \quad \text{where} \quad 3a+b=2d-6 \qquad (4)$$

$$[(d-3),\ (3_a,\,2),\ (3_b)] \quad \text{where} \quad 3a+3b=2d-6 \qquad (5)$$

$$[(d-3),\ (3_a,\,2),\ (3_b,\,2)] \quad \text{where} \quad 3a+3b=2d-7 \qquad (6)$$

$$[(d-3,\,2_a),\ (2_b),\ (2_c)] \quad \text{where} \quad a+b+c=2d-5 \text{ and } (*) \text{ holds} \qquad (7)$$

$$[(d-3,\,2_a),\ (3_b),\ (2_c)] \quad \text{where} \quad a+3b+c=2d-5 \text{ and } (*) \text{ holds} \qquad (8)$$

$$[(d-3,\,2_a),\ (3_b,\,2),\ (2_c)] \quad \text{where} \quad a+3b+c=2d-6 \text{ and } (*) \text{ holds} \qquad (9)$$

$$[(d-3),\ (2_a),\ (2_b),\ (2_c)] \quad \text{where} \quad a+b+c=2d-5 \qquad (10)$$

$$[(d-3),\ (3_a),\ (2_b),\ (2_c)] \quad \text{where} \quad 3a+b+c=2d-5 \qquad (11)$$

$$[(d-3),\ (3_a,\,2),\ (2_b),\ (2_c)] \quad \text{where} \quad 3a+b+c=2d-6 \qquad (12)$$

$$[(d-3,\,2_a),\ (2_b),\ (2_c),\ (2_e)] \quad \text{where} \quad a+b+c+e=2d-5 \text{ and } (*) \text{ holds} \qquad (13)$$

$$[(d-3),\ (2_a),\ (2_b),\ (2_c),\ (2_e)] \quad \text{where} \quad a+b+c+e=2d-5. \qquad (14)$$

We will examine case by case, computing $\chi=\chi(\Theta_V\langle\, D\,\rangle)$. The genus formula and the restriction (R_1) $\chi\leq 0$ provided by Lemma 1.1 will allow to eliminate all the cases but one, namely, a subcase of (8).

Case (1): $[(d-3),\ (2_a),\ (2_b)]$ where $a+b=2d-5$. By (R_1), we have $\chi=(-3d+9)+(2d-9)+(a+b)=d-5\le 0$, a contradiction.

Case (2): $[(d-3),\ (3_b),\ (2_a)]$ where $a+3b=2d-5$. We have $\chi=(-3d+9)+(2d-9)+(a+2b+1)=d-4-b\le 0$, i.e. $b\ge d-4$. On the other hand, $2d-5=a+3b\ge 3b+1$, whence $b\le \frac{2}{3}d-2$. Therefore, $d-4\le\frac{2}{3}d-2$, i.e. $d\le 6$. In the case $d=6$ the only possibility would be $[(3),\ (2),\ (3_2)]$. Projecting from the cusp with the multiplicity sequence (3_2), we get a contradiction to the Hurwitz formula (see (R_4)).

Case (3): $[(d-3),\ (3_a),\ (3_b)]$ where $3a+3b=2d-5$. We have $\chi=(-3d+9)+(2d-9)+(2a+1+2b+1)=\frac{d-4}{3}\le 0$, i.e. $d\le 4$, and we are done.

Case (4): $[(d-3),\ (3_a,2),\ (2_b)]$ where $3a+b=2d-6$. We have $\chi=(-3d+9)+(2d-9)+(2a+2+b)=d-4-a\le 0$, i.e. $a\ge d-4$. But $2d-6=3a+b\ge 3a+1$, whence $a\le\frac{2}{3}d-\frac{7}{3}$, and thus $d-4\le\frac{2}{3}d-\frac{7}{3}$, or $d\le 5$, a contradiction.

Case (5): $[(d-3),\ (3_a,2),\ (3_b)]$ where $3a+3b=2d-6$. We have $\chi=(-3d+9)+(2d-9)+(2a+2+2b+1)=\frac{d}{3}-1\le 0$, i.e. $d\le 3$, which is impossible.

Case (6): $[(d-3),\ (3_a,2),\ (3_b,2)]$ where $3a+3b=2d-7$. We have $\chi=(-3d+9)+(2d-9)+(2a+2b+4)=\frac{d}{3}-\frac{2}{3}\le 0$, which is impossible.

Case (7): $[(d-3,2_a),\ (2_b),\ (2_c)]$ where $a+b+c=2d-5$ and $(*)$ holds. We have $\chi=(-3d+9)+(d-5+a+\lceil\frac{d-3}{2}\rceil)+(b+c)=\lceil\frac{d-3}{2}\rceil-1\le 0$, or $d\le 5$, and we are done.

Case (8): $[(d-3,2_a),\ (3_b),\ (2_c)]$ where $a+3b+c=2d-5$ and $(*)$ holds. We have $\chi=(-3d+9)+(d-5+a+\lceil\frac{d-3}{2}\rceil)+(2b+1+c)=\lceil\frac{d-3}{2}\rceil-b\le 0$, i.e. $b\ge\lceil\frac{d-3}{2}\rceil$.

If $d-3$ is odd, then we get $2d-5=a+3b+c\ge 3b+1+\frac{d-4}{2}$, as $a=\frac{d-4}{2}$ by $(*)$. Hence, $b\le\frac{d}{2}-\frac{4}{3}$. This leads to $\lceil\frac{d-3}{2}\rceil=\frac{d-2}{2}\le\frac{d}{2}-\frac{4}{3}$, which is a contradiction.

If $d-3$ is even, then by $(*)$ we get $2d-5=a+3b+c\ge 3b+1+\frac{d-3}{2}$, hence $b\le\frac{d}{2}-\frac{3}{2}$. Thus, $\lceil\frac{d-3}{2}\rceil=\frac{d-3}{2}\le b\le\frac{d-3}{2}$, which is only possible if $c=1,\ a=b=\frac{d-3}{2}$. With $k:=\frac{d-3}{2}$ we obtain that $d=2k+3,\ a=b=k$ and $c=1$; that is, C is as in the proposition. Observe that in this case $\chi=0$, and so $h^1(\Theta_V\langle D\rangle)=0$. Together with Lemma 1.1 this proves that the corresponding curve C is projectively rigid and unobstructed (see [FZ 2, Sect. 2]).

Case (9): $[(d-3,2_a),\ (3_b,2),\ (2_c)]$ where $a+3b+c=2d-6$ and $(*)$ holds. We have $\chi=(-3d+9)+(d-5+a+\lceil\frac{d-3}{2}\rceil)+(2b+2+c)=\lceil\frac{d-3}{2}\rceil-b\le 0$, which gives $b\ge\lceil\frac{d-3}{2}\rceil$.

If $d-3$ is odd, then we get $2d-6=a+3b+c\ge 3b+1+\frac{d-4}{2}$, as $a=\frac{d-4}{2}$ by $(*)$. Thus, $b\le\frac{d}{2}-\frac{5}{3}$, and so we have $\frac{d-2}{2}\le\frac{d}{2}-\frac{5}{3}$, which is a contradiction.

If $d-3$ is even, then we get $2d-6=a+3b+c\ge 3b+1+\frac{d-3}{2}$. Hence, $b\le\frac{d}{2}-\frac{11}{6}$. This yields $\frac{d-3}{2}\le\frac{d}{2}-\frac{11}{6}$, which again gives a contradiction.

Case (10): $[(d-3),\ (2_a),\ (2_b),\ (2_c)]$ where $a+b+c=2d-5$. We have $\chi=(-3d+9)+(2d-9)+(a+b+c)=d-5\le 0$, and we are done.

Case (11) resp. (12), (13), (14) can be ruled out by the same computations as in case (2) resp. (4), (7), (10). This completes the proof of Proposition 1.1. □

For the proof of part (b) and (c) the main theorem we need the following facts.

Lemma 1.2. *Let* $(C,\overline{0})$, $(D,\overline{0}) \subseteq (\mathbf{C}^2,\overline{0})$ *be two curve singularities which have no component in common. Then the following hold.*

(a) $(CD)_{\overline{0}} = \sum_P \operatorname{mult}_P C \operatorname{mult}_P D$, *where the sum is taken over* $\overline{0}$ *and all its infinitesimally near points.*

(b) Assume that $(D,\overline{0})$ *is a smooth germ and* $(C,\overline{0}=)$ *is a cusp with the multiplicity sequence* $\underline{m} = (m^{(0)},\ldots,m^{(n)})$. *Then* $(CD)_{\overline{0}} = m^{(0)} + \ldots + m^{(s)}$ *for some* $s \geq 0$, *where* $m^{(0)} = \ldots = m^{(s-1)}$.

(c) Let $\pi : X \to \mathbf{C}^2$ *be the blow up at* $\overline{0}$. *Denote by* $E \subseteq X$ *the exceptional curve, and by* C' *the proper transform of* C. *Then*

$$\operatorname{mult}_{\overline{0}} C = \sum_{P\in E} (EC')_P.$$

Proof. The statements (a) and (c) are well known (see e.g. [Co]), whereas (b) is shown in [FlZa 2, (1.4)]. □

The next result proves part (b) and (c) of the main theorem as well as the corollary from the introduction.

Proposition 1.2. *(a) For each* $k \geq 1$ *there exists a rational cuspidal plane curve* C_k *of degree* $d = 2k+3$ *with three cusps* $Q_k,\ P_k,\ R_k$ *of types* $(2k,\,2_k)$, (3_k) *and* (2), *respectively.*

(b) C_k *is unique up to a projective transformation of the plane.*

(c) C_k *is defined over* $\mathbf{Q}$.

(d) C_k *is rectifiable.*

Proof. We proceed by induction on k. Namely, given a curve C_k as in (a), we find a Cremona transformation $\psi_k : \mathbf{P}^2 \to \mathbf{P}^2$ such that the proper transform $C_{k+1} = \psi_k(C_k)$ of C_k under ψ_k is a cuspidal curve of degree $2k+5$ with three cusps of type $(2k+2,\,2_{k+1})$, (3_{k+1}), (2). Hence the existence follows. This construction will also show that (b)–(d) hold.

We start with the rational cuspidal cubic $C_0 \subseteq \mathbf{P}^2$ given by the equation $x^2z = y^3$. Observe that C_0 is rectifiable. It has a simple cusp at $R_0 := (0:0:1)$ and the only inflectional tangent line ℓ_0 at $P_0 := (1:0:0)$; that is, $\ell_0 \cdot C_0 = 3P_0$. Fix an arbitrary point[4] $Q_0 \in C_0 \setminus \{P_0,\,R_0\}$. Let t_0 be the tangent line to C_0 at Q_0; then we have $t_0 \cdot C_0 = 2Q_0 + S_0$, where, as it is easily seen, $S_0 \in C =_0$ is different from P_0, Q_0 and R_0. Let Q_0^* denote the intersection point $l_0 \cap t_0$; clearly, $Q_0^* \notin C_0$.

Let, for a given $k > 0$, C_k denotes a curve with the cusps $Q_k,\ P_k,\ R_k$ as in the proposition, and let C_0 be the rational cubic with the distinguished points $Q_0,\ P_0,\ R_0,\ S_0$ as described above. For $k > 0$ let t_k be the tangent line of C_k at Q_k, and ℓ_k be the line $\overline{P_kQ_k}$, whereas for $k = 0$ we choose t_0 and ℓ_0 as above. In any case, using Bezout's Theorem and Lemma 1.2, we have

$$\ell_k \cdot C_k = (d-3)Q_k + 3P_k, \quad \text{and} \quad t_k \cdot C_k = (d-1)Q_k + S_k\,,$$

[4] Observe that the projective transformation group $(x:y:z) \longmapsto (t^3x : t^2y : t^6z)$, $t \in \mathbf{C}^*$, acts transitively in $C_0 \setminus \{P_0,\,R_0\}$.

where $S_k \in C_k$ is different from P_k, Q_k and R_k. Indeed, the line t_k intersects C_k at the point Q_k with multiplicity $d-1$ if $k > 1$ (see Lemma 1.2 (b)) or $k = 0$. To show that this is also true for $k = 1$, assume that t_1 and C_1 only intersect in Q_1 with $(t_1 C_1)_{Q_1} = d = 5$. The linear projection from Q_1 yields a 3-sheeted covering of the normalization of C_1 onto $\mathbf{P}^1$. By the Riemann-Hurwitz formula, it must have four ramification points. But since $(t_1 C_1)_{Q_1} = d = 5$, the point Q_1 would be a ramification point of index ≥ 2 (see Lemma 1.2(a)), and so we would have three ramification points Q_1, P_1, R_1 of indices 2, 2, 1, respectively, which is a contradiction.

Hence, for any $k \geq 0$ there is exactly one further intersection point $S_k \in C_k \cap t_k$ with $(t_k C_k)_{S_k} = 1$.

Let $\sigma_k : X_k \to \mathbf{P}^2$ be the blow up at the point $t_k \cap \ell_k$, which is Q_k for $k > 0$ and Q_k^* for $k = 0$. Denote by C_k', ℓ_k', t_k' the proper transforms in X_k of the curves C_k, ℓ_k, t_k, respectively. Then $X_k \simeq \Sigma_1$ is a Hirzebruch surface with a ruling $\pi_k : X_k \to \mathbf{P}^1$ given by the pencil of lines through Q_k resp. Q_0^*, and with the exceptional section $E_k = \sigma_k^{-1}(Q_k)$, $k > 0$, resp. $E_0 = \sigma_0^{-1}(Q_0^*)$, where $E_k^2 = -1$. Thus, ℓ_k', $t_k =' $ are fibres of this ruling. By construction, the restriction $\pi_k \,|\, C_k' : C_k' \to \mathbf{P}^1$ is 3-sheeted, and we have

$$\ell_k' \cdot C_k' = 3P_k', \quad t_k' \cdot C_k' = 2Q_k' + S_k', \quad \text{and} \quad E_k' \cdot C_k' = (d-3)Q_k' = 2kQ_k' \,,$$

where P_k', Q_k', R_k' and S_k' are the points of C_k' infinitesimally near to P_k, Q_k, R_k and $S_k \in C_k$, respectively (indeed, by Lemma 1.2(c), we have $(E_k' C_k')_{Q_k'} = \text{mult}_{Q_k^*} C_k = d-3$, where for $k > 0$ we set $Q_k^* = Q_k$). Clearly, for $k > 0$ P_k', Q_k' and R_k' are cusps of C_k' of types (3_k), (2_k) and (2), respectively, whereas S_k' is a smooth point.

Next we perform two elementary transformations[5] of X_k, one at the point S_k' and the other one at the intersection point $T_k' := \{E_k \cap \ell_k'\}$. We arrive at a new Hirzebruch surface $X_{k+1} \simeq \Sigma_1$, with the exceptional section E_{k+1} being the proper transform of E_k (indeed, since we perform elementary transformations at the points $S_k \notin E_k$ and $T_k' \in E_k, =$ we have $E_{k+1}^2 = E_k^2 = -1$). Denote by C_{k+1}' the proper transform of C_k', and by t_{k+1}', ℓ_{k+1}' the fibres of the ruling $\pi_{k+1} : X_{k+1} \to \mathbf{P}^1$ which replace t_k' resp. ℓ_k'. Using formal properties of the blowing up/down process we obtain, once again, the relations

$$\ell_{k+1}' \cdot C_{k+1}' = 3P_{k+1}', \quad t_{k+1}' \cdot C_{k+1}' = 2Q_{k+1=}' + S_{k+1}', \quad \text{and} \quad E_{k+1}' \cdot C_{k+1}' = 2(k+1)Q_{k+1}' \,,$$

where P_{k+1}', Q_{k+1}', R_{k+1}' and S_{k+1}' are the points of C_{k+1}' infinitesimally near to P_k', Q_k', R_k' and $S_k' \in C_k'$, respectively. It is easily seen that P_{k+1}' resp. Q_{k+1}', R_{k+1}' are cusps of C_{k+1}' of types (3_{k+1}), (2_{k+1}) and (2), respectively, whereas S_{k+1}' is a smooth point.

Blowing down the exceptional curve $E_{k+1}' \subset X_{k+1}$ we arrive again at $\mathbf{P}^2$. Denote the images of $C_{k+1}', .Q_{k+1}', P_{k+1}', .R_{k+1}'$ resp. by C_{k+1}, Q_{k+1}, P_{k+1}, R_{k+1}. We have constructed a rational cuspidal plane curve C_{k+1} which has cusps at Q_{k+1}, P_{k+1}, R_{k+1} with multiplicity sequences $(2(k+1), 2_{k+1})$, (3_{k+1}), (2), respectively (see Lemma 1.2(c)). This completes the proof of existence.

[5]Recall that an elementary transformation of a ruled surface consists in blowing up at a point of a given irreducible fibre followed by the contraction of the proper transform of this fibre.

Note that the birational transformation $\psi_k : \mathbf{P}^2 \to \mathbf{P}^2$, by which we obtained $C_{k+1} = \psi_k(C_k)$ from C_k, is just the Cremona transformation in the points S_k, Q_k and the intersection point $E_k \cap \ell'_k$, which is infinitesimally near to Q_k. This transformation only depends upon Q_k, S_k and the line ℓ_k; we denote it by $\psi(S_k, Q_k, \ell_k) := \psi_k$. The inverse ψ_k^{-1} is the transformation $\varphi_k = \psi(P_{k+1}, Q_{k+1}, t_{k+1})$. Therefore, the curve C_k is always transformable into the cuspidal cubic, and thus also into a line, by means of Cremona transformations, proving (d). In order to show (c) we note that, moreover, so constructed C_k, as well as P_k, Q_k, R_k and S_k, are defined over $\mathbf{Q}$, as follows by an easy induction.

Finally, let us show that the curve C_k is uniquely determined up to a projective transformation of the plane. We will again proceed by induction on k. Clearly, the cuspidal cubic is uniquely determined up to a projective transformation. Assume that uniqueness is shown for the curve C_k, and consider two curves C_{k+1}, $\tilde{C}_{k+1}$ as in (a). Let $P_{k+1} \in C_{k+1}$, $Q_{k+1} \in C_{k+1}$ and the tangent line t_{k+1} of C_{k+1} at Q_{k+1} be as above; denote the corresponding data for $\tilde{C}_{k+1}$ by $\tilde{P}_{k+1}$, $\tilde{Q}_{k+1}$ and $\tilde{t}_{k+1}$. Consider the Cremona transformations $\varphi_k := \psi(P_{k+1}, Q_{k+1}, t_{k+1})$ and $\tilde{\varphi}_k := \psi(\tilde{P}_{k+1}, \tilde{Q}_{k+1}, \tilde{t}_{k+1})$, and also the proper transforms $C_k := \varphi_k(C_{k+1})$ and $\tilde{C}_k := \tilde{\varphi}_k(\tilde{C}_{k+1})$. Reversing the above arguments it i= s easily seen that the both curves C_k, $\tilde{C}_k$ are as in (a). By the induction hypothesis, they differ by a projective transformation $f : \mathbf{P}^2 \to \mathbf{P}^2$, i.e. $f(C_k) = \tilde{C}_k$. For $k > 0$ the points $Q_k \in C_k$, $S_k \in C_k$ and the line ℓ_k are intrinsically defined by the curve C_k, and so, f maps these data onto the corresponding data $\tilde{Q}_k$, $\tilde{S}_k$ and $\tilde{\ell}_k$ for the curve $\tilde{C}_k$. Moreover, in the case $k = 0$ it is easily seen that one can choose f in such a way that $f(Q_0) = \tilde{Q}_0$. Then again $f(S_k) = f(\tilde{S}_k)$ and $f(\ell_k) = \tilde{\ell}_k$. Hence, the map f is compatible with the Cremona transformations $\varphi_k^{-1} = \psi(S_k, Q_k, \ell_k)$ and $\tilde{\varphi}_k^{-1} = \psi(\tilde{S}_k, \tilde{Q}_k, \tilde{\ell}_k)$, i.e. there is a linear transformation g of $\mathbf{P}^2$ such that $\varphi_k \circ g = f \circ \tilde{\varphi}_k$. Clearly, g transforms C_{k+1} into $\tilde{C}_{k+1}$. □

Remarks. (1) By the same approach as in the proof of Proposition 1.2, it is possible to show the existence and uniqueness of the rational cuspidal curves of type $(d, d-2)$ with at least three cusps, which was done by a different method in [FlZa 2]. By the result of loc.cit such a curve C has exactly three cusps, say Q, P, R, with the multiplicity sequences $(d-2)$, (2_a), (2_b), respectively, where $a + b = d - 2$. Set $\ell_P := \overline{QP}$, $\ell_R := \overline{QR}$. and denote by t_Q the tangent line at Q. By Bezout's Theorem, t_Q intersects C in one further point S different from Q. Performing the Cremona transformation $\psi(S, Q, \ell_P)$ to the curve C, we obtain a curve of degree $d+1$ with the multiplicity sequences $(d-1)$, (2_{a+1}), (2_b) at the cusps. Similarly, under the Cremona transformation $\psi(P, Q, \ell_R)$ the curve C is transformed into a cuspidal curve of the same degree d with the multiplicity sequences $(d-2)$, (2_{a+1}), (2_{b-1}). Thus, starting from the rational cuspidal quartic with three cusps, we can construct all such curves. It follows from this construction that these curves are rectifiable.

(2) Using the above arguments, it is also possible to classify the rational cuspidal curves of degree five with at least three cusps, which was done by M. Namba by a different method, see [Nam, Thm.2.3.10].

Indeed, if the largest multiplicity of a cusp is 3, then projecting C from this point, say Q, gives a two-sheeted covering $C \to \mathbf{P}^1$ with two ramification points. Hence, in this case C

has three cusps, with multiplicity sequences (3) (at Q), (2_2), (2), respectively.

If all the cusps are of multiplicity 2, then C has singular points $P, Q, R, \ldots$ with multiplicity sequences (2_p), (2_q), $(2_r), \ldots$, where $p+q+r+\ldots=6$. We may assume that $p \geq q \geq r \ldots$. Projecting from P gives a three-sheeted covering $C \to \mathbf{P}^1$ with four ramification points. Hence, C has at most four cusps. The possibilities are as follows:

(1) C has 3 cusps of type $P=(2_2)$, $Q=(2_2)$, $R=(2_2)$.
(2) C has 3 cusps of type $P=(2_4)$, $Q=(2)$, $R=(2)$.
(3) C has 3 cusps of type $P=(2_3)$, $Q=(2_2)$, $R=(2)$.
(4) C has 4 cusps of type $P=(2_3)$, $Q=(2)$, $R=(2)$, $S=(2)$.
(5) C has 4 cusps of type $P=(2_2)$, $Q=(2_2)$, $R=(2)$, $S=(2)$.

Curves as in (1) and (4) do exist and can be constructed by Cremona transformations. The other cases are not possible, as can be seen by the following arguments.

(5) can be excluded since the dual curve would be a cubic with two cusps, which is impossible.

To exclude (3), denote by t_P the tangent line of C at P. By the Cremona transformation $\psi := \psi(Q, P, t_P)$ a curve C as in (3) is transformed into a quartic C' with three simple cusps P', Q', R'. It can be seen that there is a tangent line at a smooth point S' of C' passing through one of the cusps, say Q'. Projecting from Q' gives a two-sheeted covering $C' \to \mathbf{P}^1$ with three ramification points, namely P', R' and S'. This contradicts the Hurwitz formula.

In the case (2), consider the blow up at P, and perform an elementary transformation at the point of the proper transform of C over P. Then the image of P will be a point with the multiplicity sequence (2_2). Performing at this point another elementary transformation and blowing down to $\mathbf{P}^2$, we arrive at the same configuration as above. Hence, also (2) is impossible. (This last transformation may also be considered as a Cremona transformation, namely in the points P, P' and P'', where P' is infinitesimally near to P and P'' is infinitesimally near to P'.)

Similarly, using Cremona transformations for the cases 1 and 4, one can construct these curves and show that they are rectifiable and projectively unique. It is also possible to treat in the same way the rational cuspidal quintics with one or two cusps.

Finally, we give an alternative proof for the existence and uniqueness statements of Proposition 1.2. It provides a way of computing an explicit parameterization for these curves.

Alternative proof of Proposition 1.2 (a)-(c). For $k=1$ the result is known (see e.g. [Nam]). Let C_k ($k>1$) be a rational cuspidal plane curve of degree $d=2k+3$ with three cusps P, Q, R of types (3_k), $(2k,\, 2_k)$ and (2), respectively. Since, by Bezout's Theorem, they are not at the same line, we may chose them as $Q\,(0:0:1)$, $P\,(0:1:0)$, $R\,(1:0:0)$. We may also chose a parameterizatio= n $\mathbf{P}^1 \to C_k$ of C_k such that $(0:1) \mapsto Q$, $(1:0) \mapsto P$, $(1:1) \mapsto R$. Then, up to constant factors, this parameterization can be written as

$$(x,\, y,\, z) = \left(s^{2k}t^3, \quad s^{2k}(s-t)^2(as+bt), \quad t^3(s-t)^2 q_k(s,\, t)\right),$$

where $q_k \in \mathbf{C}[s,\, t]$ is a homogeneous polynomial of degree $2k-2$. Let Γ denotes a curve parameterized as above (with q instead of q_k). It is enough to prove the following

Claim. *There exists unique polynomials $as+bt$ and q with rational coefficients, where $q(1,\, 0) = 1$, such that the multiplicity sequences of Γ at the points $P,\, Q,\, R \in \Gamma =$ start, respectively, with (3_k), $(2k,\, 2_k)$ and (2).*

Indeed, if this is the case, then, by the genus formula, these multiplicity sequences actually coincide resp. with (3_k), $(2k,\, 2_k)$ and (2), and so, $C_k = \Gamma$ up to projective equivalence. This will prove the existence of the curves C_k defined over $\mathbf{Q}$ for all $k > 1$, as well as their uniqueness, up to projective equivalence.

Proof of the claim. It is easily seen that, after blowing up at Q, the infinitesimally near point Q' to Q at the proper transform Γ' of Γ will be a singular point of multiplicity 2 iff $as+bt = 2s+t$. By [FlZa 2, (1.2)], under this condition the multiplicity sequence of Γ at Q starts with $(2k,\, 2_k)$.

In the affine chart $(\widehat{x},\, \widehat{z}) := (x/y,\, z/y)$ centered at P we have

$$\widehat{x} = \frac{t^3}{(s-t)^2(2s+t)}, \quad \widehat{z} = \frac{t^3 q(s,\, t)}{s^{2k}(2s+t)}\,.$$

In the sequel we denote by the same letter t the affine coordinate t/s in $\mathbf{P}^1 \setminus \{(0:1)\}$. Thus, in this affine chart in $\mathbf{P}^1$ centered at $(1:0)$ we have

$$(\widehat{x},\, \widehat{z}) = \left(\frac{t^3}{(t-1)^2(t+2)},\, \frac{t^3}{(t+2)}\,\widehat{q}(t)\right)\,,$$

where $\widehat{q}(t) = \sum_{i=0}^{2k-2} c_i t^i$ and where, by the above assumption, $c_0 = 1$.

After blowing up at P, in the affine chart with the coordinates $(u,\, v)$, where $(\widehat{x},\, \widehat{z}) = (u,\, uv)$, we will have

$$(u,\, v) = (\widehat{x},\, \widehat{z}/\widehat{x}) = \left(\frac{t^3}{(t-1)^2(t+2)},\, \widehat{q}(t)(t-1)^2\right)\,.$$

To move the origin to the infinitesimally near point $P' \in \Gamma'$ of P, we set

$$(\widehat{u},\, \widehat{v}) = (u,\, v-1) = \left(\frac{t^3}{(t-1)^2(t+2)},\, \widehat{q}(t)(t-1)^2 - 1\right)\,.$$

The following conditions guarantee that the multiplicity of the curve $\Gamma' =$ at P' is at least 3:

$$t^3 \mid [\widehat{q}(t)(t-1)^2 - 1] \iff$$
$$[\widehat{q}(t)(t-1)^2 - 1]'_0 = [\widehat{q}(t)(t-1)^2 - 1]''_0 = 0 \iff$$
$$\widehat{q}'(0) = 2,\, \widehat{q}''(0) = 6 \iff c_1 = 2,\, c_2 = 3\,. \tag{15}$$

In the case when $k = 2$ this uniquely determines the polynomial q:

$$q(s,\, t) = s^2 + 2st + 3t^2 \,.$$

In what follows we suppose that $k > 2$. Assume that the conditions (15) are fulfilled. Then we have the following coordinate presentation of Γ':

$$(\widehat{u},\, \widehat{v}) = \left(\frac{t^3}{(t-1)^2(t+2)},\quad t^3 h(t) \right) \,,$$

where $h(t) := [\widehat{q}(t)(t-1)^2 - 1]/t^3$ is a polynomial of degree $2k-3$, which satisfies the conditions

$$(t-1)^2 \mid [t^3 h(t) + 1] \Longleftrightarrow h(1) = -1,\; h'(1) = 3 \,. \qquad (15')$$

Once (15′) are fulfilled, one can find $\widehat{q}$ as $\widehat{q} = [t^3 h(t) + 1]/(t-1)^2$, and we have $\widehat{q} \in \mathbf{Q}[t]$ iff $h \in \mathbf{Q}[t =]$.

Let $\xi \in \mathbf{C}[[t]]$ be such that $\xi^3 = \frac{t^3}{(t-1)^2(t+2)}$. By [FlZa 2, (3.4)], the multiplicity sequence of Γ' at P' starts with $(3)_{k-1}$ iff

$$t^3 h(t) \equiv \widehat{f}(\xi^3) \mod \xi^{3(k-1)} \,,$$

where $\widehat{f} = \sum_{i=0}^{k-1} \widehat{a}_i x^i \in \mathbf{C}[x]$ is a polynomial of degree $\le k-1$. Multiplying the both sides by the unit $[(t-1)^2(t+2)]^{k-1} \in \mathbf{C}[[t]]$, we will get

$$[(t-1)^2(t+2)]^{k-1} t^3 h(t) \equiv [(t-1)^2(t+2)]^{k-1} \sum_{i=0}^{k-1} \widehat{a}_i \xi^{3i} \equiv \sum_{i=0}^{k-1} \widehat{a}_i t^{3i} [(t-1)^2(t+2)]^{k-1-i} \mod t^{3(k-1)} \,.$$

Since, by our assumption, $k > 1$, we should have $\widehat{a}_0 = 0$, and after dividing out the factor t^3, we get

$$[(t-1)^2(t+2)]^{k-1} h(t) \equiv \sum_{i=0}^{k-2} \widehat{a}'_i t^{3i} [(t-1)^2(t+2)]^{k-2-i} \mod t^{3(k-2)} \,,$$

where $\widehat{a}'_i = \widehat{a}_{i-1},\; i = 1, \ldots, k-2$. In other words, we have

$$[(t-1)^2(t+2)]^{k-1} h(t) = \widehat{f}(t^3,\, (t-1)^2(t+2)) + \widehat{g}(t) t^{3(k-2)} \,,$$

where $\widehat{f}(x,\, y) = \widehat{f}_k(x,\, y) := \sum_{i=0}^{k-2} \widehat{a}'_i x^i y^{k-2-i=}$ is a homogeneous polynomial of degree $k-2$, and hence $\widehat{g}(t) = \widehat{g}_k(t) = \sum_{i=0}^{2k} \widehat{b}_i t^i$ should be a polynomial of degree $2k$. Denoting $\tau = t^3$ and $\lambda = (t-1)^2(t+2) = t^3 - 3t + 2$, we have

$$\lambda^{k-1} h = \widehat{f}(\tau,\, \lambda) + \tau^{k-2} \widehat{g} \,.$$

Observe that $\widehat{f}(\tau,\, \lambda)$ (resp. $\tau^{k-2}\widehat{g}$) contains the monomial $\widehat{a}'_0 \tau^{k+2}$ (resp. $\widehat{b}_0 \tau^{k+2}$). To avoid indeterminacy, we may assume, for instance, that $\widehat{a}'_0 = 0$. Then $\widehat{f} = \lambda f$, where $f(x,\, y) := \sum_{i=0}^{k-2} a_i x^i y^{k-3-i}$, $a_i := \widehat{a}'_{i-1}$, $i = 0, \ldots, k-3$, and so

$$\lambda^{k-1} h = \lambda f(\tau,\, \lambda) + \tau^{k-2} \widehat{g} \,.$$

Since $(\tau,\,\lambda)=1$, we have $\lambda\,|\,\widehat{g}$, that is, $\widehat{g}=\lambda g$, where $g(t):=\sum_{i=0}^{2k-3} b_i t^i$. Finally, we arrive at the relation

$$\lambda^{k-2}h(t)=f(\tau,\,\lambda)+\tau^{k-2}g(t)\,,$$

where $\deg f=k-3$, $\deg h=\deg g=2k-3$, and h should satisfy the conditions $(15')$. It follows that

$$\lambda^{k-2}\mid [f(\tau,\,\lambda)+\tau^{k-2}g]\,, \tag{16}$$

and

$$\tau^{k-2}\mid [f(\tau,\,\lambda)-\lambda^{k-2}h]\,. \tag{16$'$}$$

Each of these conditions together with $(15')$ determines the triple of polynomials $f,\,g,\,h$ as above in a unique way. Indeed, once f and g satisfy $(15')$ and (16), we can find h as $h=[f(\tau,\,\lambda)+\tau^{k-2}g]/\lambda^{k-2}$. Actually, (16) is equivalent to the vanishing of derivatives of the function $f(\tau,\,\lambda)+\tau^{k-2}g\in\mathbf{C}[t]$ at the point $t=1$ up to order $2k-5$ and at the point $t=-2$ up to order $k-3$. This yields a system of $3k-6$ linear equations in the $3k-4$ unknown coefficients of f and g; $(15')$ provides another two linear equations. That is, we have the following system:

$$\begin{aligned}
\left(f(\tau,\,\lambda)+\tau^{k-2}g\right)^{(m)}_{t=-2}&=0,\quad m=0,\dots,k-3\\
\left(f(\tau,\,\lambda)+\tau^{k-2}g\right)^{(m)}_{t=1}&=0,\quad m=0,\dots,2k-5\\
\left(f(\tau,\,\lambda)+\tau^{k-2}g\right)^{(2k-4)}_{t=1}&=-3^{k-2}(2k-2)!\\
\left(f(\tau,\,\lambda)+\tau^{k-2}g\right)^{(2k-3)}_{t=1}&=-3^{k-3}(k-11)(2k-1)!
\end{aligned} \tag{S}$$

(Indeed, put $u=t-1$; in view of $(15')$ we have

$$\lambda=(t-1)^2(t+2)=u^2(u+3),\quad h(t)=-1+3u+\dots\,,$$

and hence

$$f(\tau,\,\lambda)+\tau^{k-2}g(t)=\lambda^{k-2}h(t)=[u^2(u+3)]^{k-2}h(t)=$$
$$u^{2k-4}(3^{k-2}+(k-2)3^{k-3}u+\dots)(-1+3u+\dots)=u^{2k-4}(-3^{k-2}-3^{k-3}(k-11)u+\dots)\,.)$$

The system (S) has a unique solution iff it is so for the associated homogeneous system, say, (S_0). Passing from (S) to (S_0) actually corresponds to passing from h to a polynomial h_0 of degree $\le 2k-3$ which satisfies, instead of $(15')$, the conditions

$$h_0(1)=h_0'(1)=0\Longleftrightarrow (t-1)^2\mid h_0(t)\Longleftrightarrow h_0(t)=(t-1)^2\widetilde{h}(t),\ \ \deg\widetilde{h}\le 2k-5\,. \quad (15'')\,.$$

Thus, we have to prove that the equality

$$\lambda^{k-2}(t-1)^2\widetilde{h}(t)=f(\tau,\,\lambda)+\tau^{k-2}g(t)\,,$$

where $f = 0$ or $\deg f = k - 3$, $\deg g \le 2k - 3$, and $\deg \widetilde{h} \le 2k - 5$, is only possible for $f = g = \widetilde{h} = 0$. Or, equivalently, we have= to show that the $5k - 8$ polynomials in t in the union T of the three systems:

$$T_1 := \left\{\tau^i \lambda^{k-3-i}\right\}_{i=0,\dots,k-3}, \quad T_2 := \left\{t^i (t-1)^2 \lambda^{k-2}\right\}_{i=0,\dots,2k-5}, \quad T_3 := \left\{t^i \tau^{k-2}\right\}_{i=0,\dots,2k-3}$$

are linearly independent. After replacing the system T_2 by the equivalent one:

$$T_2' := \left\{(t-1)^{2k-2}(t+2)^{k-2+i}\right\}_{i=0,\dots,2k-5},$$

we will present these three systems as follows:

$$T_1 = \left\{p_i := \tau^{k-3-i}\lambda^i = t^{3(k-2-i)}(t-1)^{2i}(t+2)^i, \quad i = 0, \dots, k-3\right\}$$

$$T_2' = \left\{p_i := (t-1)^{2k-2}(t+2)^i, \quad i = k-2, \dots, 3k-7\right\}$$

$$T_3 = \left\{p_i := t^i, \quad i = 3k-6, \dots, 5k-9\right\}.$$

Denote $P = \operatorname{span}(T_1, T_2, T_3) = \operatorname{span}(T_1, T_2', T_3)$. Note that $\deg p \le 5k - 9$ for all $p \in P$, that is, $\dim P \le 5k - 8$. Consider the following system of $5k - 8$ linear functionals on P:

$$\varphi_i \,:\, p \longmapsto p^{(i)}(-2), \quad i = 0, \dots, 3k-7,$$

$$\varphi_i \,:\, p \longmapsto p^{(i)}(0), \quad i = 3k-6, \dots, 5k-9.$$

It is easily seen that the matrix $M := (\varphi_i(p_j))_{i,j=0,\dots,5k-9}$ is triangular with non-zero diagonal entries. This proves that, indeed, $\operatorname{rang} T = \dim P = 5k - 8$, as stated.

The coefficients of the system (S) being integers, its unique solution is rational, i.e. the polynomials f and g are defined over $\mathbf{Q}$. It follows as above that the polynomials h and q are also defined over $\mathbf{Q}$. This completes the alternative proof of Proposition 1.2. $\square$

Remarks. (1) In principle, the method used in the proof allows to compute explicitly parameterizations of the curves C_k. For instance, we saw above that for $k = 2$ a parameterization of C_2 is given by the choice

$$q_2(s,t) := s^2 + 2st + 3t^2, \quad a := 2, \quad b := 1.$$

(2) We have to apologize for a pity mistake in Lemma 4.1(b) [FlZa 2, Miscellaneous] (this does not affect the other results of [FlZa 2], besides only the immediate Corollary 4.2).

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