LECTURES on EXOTIC ALGEBRAIC STRUCTURES ON AFFINE SPACES

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Abstract

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Contents

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1 Introduction

Exotic algebraic structures on the affine spaces appeared in the general framework of the Affine Algebraic Geometry. Within the traditional algebraic geometry of (quasi)projective varieties, the affine geometry occupies a special place, being known as a source of highly difficult problems. Let us recall the most famous ones[1].

1. The Zariski Cancellation Problem:

Is it true that an isomorphism $X \times \mathbf{C}^n \approx \mathbf{C}^{n+k}$ *is only possible if* $X \approx \mathbf{C}^k$ *?*

2. The structure of the Automorphism Group:

Given a polynomial automorphism of $\mathbf{C}^n$ *, can it be presented as a product of linear and triangular ones?*

3. The Linearization Problem:

Is any regular $\mathbf{C}^*$ *-action on* $\mathbf{C}^n$ *conjugate with a linear one?*

4. The Embedding Problem (Abhyankar, Sathaye):

Is any regular embedding $\mathbf{C}^k \hookrightarrow \mathbf{C}^n$ *equivalent to a linear one?*

5. The Jacobian Problem:

Given a regular mapping $\mathbf{C}^n \hookrightarrow \mathbf{C}^n$ *with a constant non-zero Jacobian, is it necessarily an automorphism of* $\mathbf{C}^n$ *?*

To clarify the present day situation, some commentaries are in order.

1. The affirmative answer to (1) for $k = 2$ was the result of a series of papers by Miaynishi, Sugie and Fujita [MiySu, Fu 1] (see also [Kam 1]). In higher dimensions $k \geq 3$, there is no significant progress.

In the birational setting, the analog of the Zariski Cancellation Problem was answered in negative by Beauville, Colliot-Thelene, Sansuc and Swinnerton-Dyer [BCTSSD].

As for the following more general Cancellation Problem (see e.g. [AEH, EH, Ho]):

Given an isomorphism of polynomial rings $A[x] \simeq B[x]$ *over two rings* A *and* B *, does it follow that* $A \simeq B$ *?*

its geometric counterpart is also answered in negative (Danielewski [Dan]; see also Fieseler [Fi], tom Dieck [tD 3]). In the corresponding example of Danielewski, A and B are the

[1] Hereafter we restrict the consideration to the varieties defined over $\mathbf{C}$.

rings of regular functions on any two of the smooth affine surfaces $\{x^n y + z^2 = 1\} \subset \mathbf{C}^3, \quad n \in \mathbf{N}$.

2. The structure of the automorphism group $\mathrm{Aut}\,\mathbf{C}^n$ for $n = 2$ is classically known (Jung [Ju], van der Kulk [vdK]). Starting with $n = 3$, it is completely mysterious (see e.g. [AAS, Na]).

3. To answer to (2), it would be rather useful to describe the one-parameter subgroups of the automorphism group $\mathrm{Aut}\,\mathbf{C}^n$; that is, the regular $\mathbf{C}_+$-actions and $\mathbf{C}^*$-actions on $\mathbf{C}^n$, where $\mathbf{C}_+$ resp. $\mathbf{C}^*$ denotes the additive resp. the multiplicative group of the complex number field. It was natural to expect that any $\mathbf{C}_+$-action resp. any $\mathbf{C}^*$-action on $\mathbf{C}^n$ is conjugate with a triangular one resp. with a linear one. The former one was shown to be false starting with $n = 3$ (Bass [Ba]), while it is true for $n = 2$ (Rentschler [Re]). It is worthwhile noticing that giving a $\mathbf{C}_+$-action on an affine variety X is the same as giving a locally nilpotent derivation (LND for short) of the algebra $\mathbf{C}[X]$ of regular functions on X [Re].

As for the latter one, i.e. for the Linearization Problem, the positive answer is known for $n = 2$ (Gutwirth [Gut]). Being answered in affirmative at least for $n = 3$, this would lead to linearization of any connected, reductive group action on the affine space (Kraft, Popov [KrPo, Po]). An example of a non-linearizable action of a semi-direct product of $\mathbf{C}^*$ and $\mathbf{Z}/2\mathbf{Z}$ on $\mathbf{C}^4$ was constructed by G. Schwarz [Sch]. It is known that a certain restricted form of the Linearization Problem is equivalent to the Zariski Cancellation Problem (Kambayashi-Russell [KamRu]; cf. also Bass-Haboush [BaHa]). Below we say more about the recent positive solution for $n = 3$ and the role of exotic $\mathbf{C}^3$-s in this solution (see Koras–Russell [KoRu 2, KoRu 3], Makar-Limanov [ML], Kaliman and Makar-Limanov [KaML 3], Kaliman, Koras, Makar-Limanov, Russell [KaKoMLRu]).

4. Whereas any regular embedding $\mathbf{C} \hookrightarrow \mathbf{C}^2$ is equivalent to a linear one (Abhyankar-Moh, Suzuki [AM, Suz 1]), this is unknown already for the embeddings $\mathbf{C} \hookrightarrow \mathbf{C}^3$ and $\mathbf{C}^2 \hookrightarrow \mathbf{C}^3$. But the embeddings $\mathbf{C}^k \hookrightarrow \mathbf{C}^n$ are linearizable as soon as $n \geq 2(k+1)$ (Jelonek [Je], Kaliman [Ka 4, Ka 5], Nori, Srinivas [Sr]).

5. There is a number of equivalent reductions of the Jacobian Problem and partial results; see e.g. [An, AAS, BCW, Dr, Kam 2, Or 1]. It is also famous for a lot of false proofs, which appear regularly. Definitely, this certifies its difficulty.

At the last decade, new unusual objects in the Affine Algebraic Geometry appeared and attracted some attention. They were called *exotic* $\mathbf{C}^n$. These are smooth affine varieties diffeomorphic, but non-isomorphic to the affine spaces. Their existence alluded to (as a remark) in a deep paper of Ramanujam [Ram], where the non–existence of exotic $\mathbf{C}^2$ was proven. Later on, many exotic $\mathbf{C}^n$ -s for all $n \geq 3$ were constructed (Choudary–Dimca [ChoDi], Dimca [Di 1], Kaliman [Ka 1, Ka 2], Koras–Russell [KoRu 2], Petrie–tom Dieck [PtD 2], Russell [Ru 1], tom Dieck [tD 1, tD 2], the author [Za 2, Za 3, Za 4], e.a.).

In the work of Koras and Russell on the Linearization Problem (see [KoRu 1, KoRu 2]) it was reduced, in the particular case of $n = 3$, to classification problems for a certain series of smooth contractible threefolds $X \subset \mathbf{C}^4$ (the *Koras–Russell threefolds*) and for a certain series of affine singular quotient surfaces. As for the latter one, it was recently settled completely [KoRu 3].

As for the former one, it consists of to clarify whether or not all the Koras-Russel threefolds $X \subset \mathbf{C}^4$ are exotic $\mathbf{C}^3$ -s. The first partial results were obtained by Kaliman and Russell [Ka 1, Ru 1], who succeeded to show that the logarithmic Kodaira dimension is non-negative for at least some of these threefolds.

A methods invented by Kaliman and Makar-Limanov [KaML 1] allowed them to enlarge this class. Namely, it was shown that, under certain restrictions on X, there is no dominant regular mapping $\mathbf{C}^3 \to X$.

But all the above methods failed to distinguish from $\mathbf{C}^3$ a certain subseries of the Koras-Russel threefolds. The Russell cubic threefold $X \subset \mathbf{C}^4$, given by the equation $x + x^2y + z^2 + t^3 = 0$, is one of them. It looks especially simple, but in fact, it is this one the most difficult to analyze. Its geometric structure can be described as follows. It contains 'the book-surface' $B := \{x = 0\} \subset X$ isomorphic to the product $\mathbf{C} \times \Gamma_{2,3}$, where $\Gamma_{2,3} \subset \mathbf{C}^2$ is the affine cuspidal cubic $z^2 + t^3 = 0$. The complement $X \setminus B$ is isomorphic to $\mathbf{C}^* \times \mathbf{C}^2$. Thus, X is obtained from $\mathbf{C}^3$ after replacing $\mathbf{C}^2 \subset \mathbf{C}^3$ by the book-surface B; notice also that there exists a dominant morphism $\mathbf{C}^3 \to X$. Using the fact that B is contractible, one can show that X is contractible, too. It follows from the Smale h-Cobordism Theorem that, actually, X is diffeomorphic to $\mathbf{R}^6$.

Finally, Makar-Limanov [ML] succeeded to prove that the Russell cubic is an exotic $\mathbf{C}^3$. Soon after, Kaliman and Makar-Limanov [KaML 3], along the same approach, completed the classification of the Koras-Russel threefolds, showing, in particular, that all of them are exotic. Thus, the Linearization Problem was answered in positive, at least for $\mathbf{C}^3$ [KoRu 2, KoRu 3].

The proof of Makar-Limanov [ML] is based on the use of locally nilpotent derivations (LND), - the idea that was (tentatively) known to the specialists. The principal new ingredients suggested in [ML] provide powerful tools to work with LND-s. The kern of this approach consists in

* using jacobian derivations; in particular,
* reducing the study of general LND-s to study of jacobian LND-s;
* introducing and systematically using generalized degree functions;
* reducing the study of the LND-s of a filtered ring to those of the associated graded ring.

In section 9 below we present a simplified proof of the Makar-Limanov Theorem due to Derksen [De]. In sections 2 and 3 we deal with contractible and, more generally, acyclic surfaces; they serve as a base for constructing exotic $\mathbf{C}^n$ –s, but certainly merit being studied on their own right. Sections 4–8 are devoted to several known constructions which lead to exotic $\mathbf{C}^n$ –s. Besides, in section 8 some computations of the logarithmic Kodaira dimension are done.

We have tried to simplify presentation as much as possible, restricting it to particularly interesting examples. We do not touch at all, or say very little on some related subjects, such as analytically exotic structures (see e.g. [Ka 2, Za 3, Za 5]), deformations of exotic structures (see [FlZa 1, Za 5]), $\mathbf{Q}$ -acyclic surfaces (see e.g. [FlZa 1, Fu 2, Miy 2, Or 2]), the positive characteristic case, etc. Whereas, we provide a rather extended list of literature, although by no means complete. The interested reader can find certain additional information and some open questions in [OPOV, Za 5].

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2 Acyclic surfaces (an introduction)

By the Hironaka Resolution of Singularities Theorem, any smooth quasiprojective variety X admits a smooth projective completion V by a divisor D with simple normal crossings $X = V \setminus D$. We call (V, D) an *SNC-completion* of X (or an *SNC-pair*). A variety X is *acyclic* if $H_*(X, \mathbf{Z}) \simeq \mathbf{Z}$.

Lemma 2.1. (Fujita [Fu 2]) *Let X be a surface. If X is acyclic, then it is affine.*

Proof. Assume that X is acyclic. Let V be a smooth completion of X by a reduced divisor D (not necessarily SNC). Let $D = \sum_{i=1}^k D_i$, where each D_i is an irreducible component of X. We will show that there exists an effective ample divisor $A = \sum_{i=1}^k a_i D_i$ supported by D, i.e. such that $a_i > 0 \, \forall i = 1, \dots, k$. Thus, mA for m large enough is a hyperplane section (for the embedding $\Phi_{\mid mA \mid} : V \hookrightarrow \mathbf{P}^N$). Hence,

$$X = V \setminus D = V \setminus \text{supp }(mA) \hookrightarrow \mathbf{P}^N \setminus H \simeq \mathbf{C}^N$$

is affine. By the Nakai-Moišezon criterion, it suffices to choose any A as above such that $A^2 > 0$ and $AC > 0$ for any irreducible curve C in V.

In view of acyclicity of X , from the standard topological dualities (see the proof of Proposition 2.1 below) it follows that the natural homomorphism $H_2(D) \longrightarrow H_2(V)$ is surjective, and D is connected[2]. Set $\sum = \{A = \sum_{i \in I} a_i D_i \mid a_i > 0 \quad \forall i \in I, AD_i > 0 \quad \forall i \in I\}, \ I \subset \{1, \dots, k\}$. First, we show that $\sum$ is non-empty. Indeed, let $H \in \text{Div } V$ be any ample divisor. The classes of $D_i, \quad i = 1, \dots, k,$ (which we denote by the same letters) generate the group $H_2(V, \mathbf{Z})$, and so $H = \sum_{i=1}^k h_i D_i = \sum_{i \in I} a_i D_i - \sum_{j \in J} a_j D_j = A_0 - B_0$, where $I, J \subseteq \{1, \dots, k\}, \ I \neq \emptyset, \ I \bigcap J = \emptyset$, and $a_i > 0 \quad \forall i \in I \cup J$. For any irreducible curve C in V, we have $A_0 C - B_0 C = HC > 0$, whence $A_0 C > B_0 C$. Given $C = D_i, \quad i \in I,$ this implies $A_0 C > B_0 C \geq 0$. Therefore, $A_0 \in \sum$.

Suppose that $A \in \sum, \quad \text{supp } A \neq D, \quad D_j$ does not lie in $\text{supp } A$, and $D_j A > 0$. Then $m_j A + D_j \in \sum$ for some $m_j > 0$. Indeed, $(m_j A + D_j) D_i > 0$ for all $D_i \subset \text{supp } A$, and $(m_j A + D_j) D_j > 0$ when $m_j > -D_j^2 / D_j A$.

Recall that D is connected. Therefore, starting with A_0 and applying the procedure as above, in a finite number of steps one can find a divisor $A \in \sum$ with $\text{supp } A = D$. Clearly, $A^2 > 0, \quad AD_i > 0$ for all $i = 1, \dots, k,$ and $AC \geq DC$ for any irreducible curve C such that C is not contained in D. Since $A_0 C \geq HC > 0,$ we have $DC > 0,$ whence also $AC > 0$. Thus, A is ample, and $\text{supp } A = D$. □

Remark. In higher dimensions the analogous statement is not true, in general, as an example of Winkelmann [Win] shows. In this example $X = Q \setminus E$ is a contractible non-affine (and even non-Stein) quasi-affine variety, where Q is a smooth projective quadric of dimension 4, and $E \subset Q$ is a codimension 2 smooth subvariety.

[2] in fact, to prove that X is affine we use only these two conditions. It is well known that the boundary of an irreducible affine variety is connected, so, the second one is necessary.

Proposition 2.1. *Let* $X = V \setminus D$, *where* V *is a smooth projective surface, and* D *is a curve in* V. *Then* X *is acyclic if and only if the following conditions hold:*

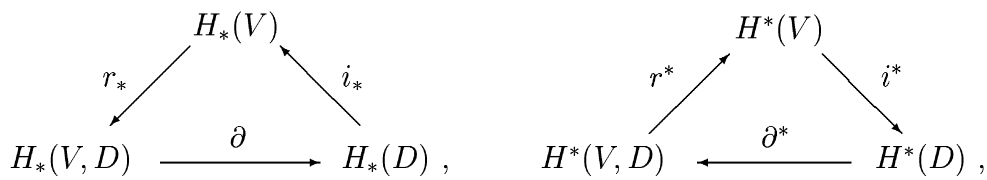
(*i*) $\pi_0(D) = \pi_1(V) = \pi_1(D) = \mathbf{1}$.

(*ii*) $i_* : H_2(D, \mathbf{Z}) \longrightarrow H_2(V, \mathbf{Z})$ *is an isomorphism.*

Proof. By the Lefschetz duality,[3] we have

$$H^i(V, D) \simeq H_{n-i}(X)\,, \quad H_i(V, D) \simeq H^{n-i}(X)\,, \quad i = 0, \ldots, 4\,.$$

Assume that X is acyclic; then the above groups are zero for $i = 0, \ldots, 3$. From the standard exact sequences of a pair (all the homology groups have coefficients in $\mathbf{Z}$):



where $\deg i_* = \deg i^* = \deg r_* = \deg r^* = 0$, $\deg \partial = -1$, and $\deg \partial^* = 1$, it follows that $H_i(D) \simeq H_i(V)$, $H^i(D) \simeq H^i(V)$ for all i, $0 \le i \le 3$. In particular, $H_2(D) \simeq H_2(V)$, which proves (ii). Also, $H^3(V) \simeq H^3(D) = 0$. By the Poincaré duality, $H_1(V) \simeq H_1(D) = 0$, whence $\pi_1(D) = \mathbf{1}$. Since $H_0(D) \simeq H_0(V) \simeq \mathbf{Z}$, $\pi_0(D) = \mathbf{1}$.

The proof of Lemma 2.1 shows that D is a hyperplane section. By the Lefschetz Theorem on hyperplane sections [Lef, AF], [Mil 2, Thm. 7.4], $i_* : \pi_1(D) \longrightarrow \pi_1(V)$ is a surjection. This implies that $\pi_1(V) = \mathbf{1}$, as claimed.

Conversely, assume that the conditions (i) and (ii) are satisfied. Then $H_1(V) = H_1(D) = 0$, whence $H^3(V) = 0$, by the Poincaré duality. Furthermore, the group $H_2(V) \simeq H^2(V) \simeq H^2(D) \simeq \mathbf{Z}^d$ is free (here d stands for the number of irreducible components of D). Since $\text{Tors}\, H^j = \text{Tors}\, H_{j-1}$, $H^1(V)$ is also a free group. Hence $H^1(V) \simeq H_1(V) = 0$, and so $H_3(V) = 0$. Then $H_i(D) \simeq H_i(V)$, $i = 0, \ldots, 3$, whence $H_i(V, D) = 0$, $i = 0, \ldots, 3$. Also, we have the same equalities for cohomologies.

By the Lefschetz duality, $0 = H^i(V, D) = H_{n-i}(X)$, $i = 0, \ldots, 3$. Therefore, X is acyclic. □

Corollary 2.1. *All the irreducible components of* D *are rational curves without self-intersections, and are arranged as a tree.*

Definition 2.1. Let $D = \sum_{i=1}^d D_i$ be an SNC-curve on a projective surface V. *The dual graph* Γ_D of D is the graph which possesses the irreducible components $\{D_i\}$ of

[3] As for a reference book on algebraic topology, we address e.g. to Dold [Do].

D as vertices, and $[D_i, D_j]$ ($i \neq j$) is an edge of Γ_D iff $D_i D_j > 0$. Each vertex D_i of Γ_D is weighted by D_i^2.

If $X = V \setminus D$ is acyclic, then, by Proposition 2.1, Γ_D is a tree.

Theorem 2.1. (Gurjar-Schastri [GuSha]) *Every acyclic surface is rational.*

Corollary 2.2. *An SNC-pair* (V, D) *is a completion of an acyclic surface* $X = V \setminus D$ *iff* D *is a rational tree on a smooth rational surface* V *such that the Picard group* $\mathrm{Pic}\, V \simeq G(D) \simeq \mathbf{Z}^d$ *is freely generated over* $\mathbf{Z}$ *by the irreducible components of* D.

Proof. Indeed, in the case of a rational surface V, we have $\mathrm{Pic}\, V \simeq H_2(V)$. □

Definition 2.2. An SNC-pair (V, D) is called *minimal* if no contraction of a component of D leads to a new SNC-pair. (Equivalently, Γ_D has neither linear nor end vertices weighted by -1; recall the Castelnuovo criterion.)

Theorem 2.2. (Ramanujam [Ram]) *(a) Assume that* (V, D) *is a minimal SNC-completion of a smooth acyclic surface* $X = V \setminus D$. *Then* $X \simeq \mathbf{C}^2$ *iff the dual graph* Γ_D *is linear.*

(b) Furthermore, a smooth contractible surface X *is isomorphic to* $\mathbf{C}^2$ *iff it is simply connected at infinity*[4].

Example 2.1. A *Hirzebruch surface* $\sum_n$ is a $\mathbf{P}^1-$ bundle over $\mathbf{P}^1$: $\sum_n \stackrel{p}{\longrightarrow} \mathbf{P}^1$ such that there exists a section $E_n \subset \sum_n$ with $E_n^2 = -n \quad (n \geq 0)$. If F_∞ is a fiber over the point $\infty = (1:0) \in \mathbf{P}^1$, then $\sum_n \setminus (E_n \bigcup F_\infty) \simeq \mathbf{C}^2$, and the dual graph of this completion of $X = \mathbf{C}^2$ looks like

Note that the standard completion $(\mathbf{P}^2, \mathbf{P}^1)$ of $\mathbf{C}^2$ has the dual graph

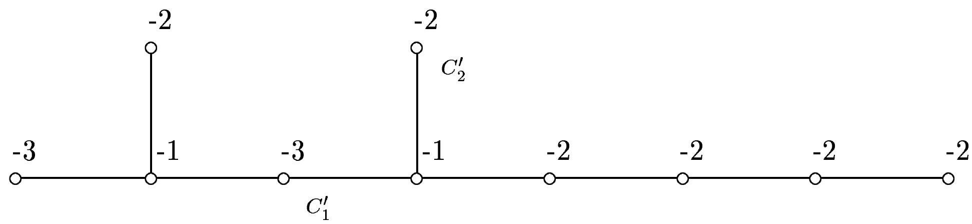
Example 2.2. *The Ramanujam surface* [Ram]. There exists an arrangement of a smooth conic C_2 and a cuspidal cubic C_1, $\{zx^2 - y^3 = 0\}$ say, in $\mathbf{P}^2$ such that $(C_1 C_2)_A = 1$, and $(C_1 C_2)_B = 5$, where A, B are the smooth intersection points.

[4]i.e. $\pi_1^\infty(S) = \mathbf{1}$, where $\pi_1^\infty(S) := \varinjlim \pi_1(S \setminus K)$ is the direct limit over the system of all compacts $K \subset S$. In fact, the latter condition is equivalent to the former one of linearity of Γ_D .

Let $\sigma_a : V \longrightarrow \mathbf{P}^2$ be the blow-up of $\mathbf{P}^2$ at A with the exceptional (-1)-curve $E \subset V$, and let $C'_1, C'_2 \subset V$ be the proper transforms of C_1, C_2. Set $D = C_1 \bigcup C_2$. We have $H_2(V) \simeq \text{Pic } V \simeq \mathbf{Z}H' + \mathbf{Z}E$, where H' is the proper transform of a generic line H in $\mathbf{P}^2$. Since $C_1 \sim 3H$, and $C_2 \sim 2H$ in Pic $\mathbf{P}^2 \simeq \mathbf{Z}$, we get $(C'_1, C'_2) = T(H', E)$, where T is the unimodular matrix

$$\begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}$$

Thus, $H_2(V) \simeq H_2(D)$, and so it follows from Proposition 2.1 that the surface $X = V \backslash D$ is acyclic. By Fujita's Lemma 2.1, X is affine. The resolution graph of $D \subset V$ looks as follows:



This graph is minimal and non-linear, so the Ramanujam Theorem 2.2(a) yields that X is not isomorphic to $\mathbf{C}^2$.

Exercises (**2.1**) Show that $\pi_1(X) = \mathbf{1}$, and so X is contractible.

(**2.2**) The boundary S of a ('tubular') neighborhood of D in V (= an attached boundary of X) is not simply connected (what is[5] $\pi_1(S)$?), and so X is not homeomorphic to $\mathbf{R}^4$.

Next we give some more examples of contractible surfaces, following [Za 1, Za 4].

Example 2.3. Let T be a matrix of non-negative integers of the form

$$T = \begin{pmatrix} m_{00} & 0 & n_{00} & 0 \\ m_{10} & 0 & 0 & n_{10} \\ 0 & m_{01} & n_{01} & 0 \\ 0 & m_{11} & 0 & n_{11} \end{pmatrix}$$

Consider the lines $l_{i,j} \simeq \mathbf{P}^1$ in $Q := \mathbf{P}^1 \times \mathbf{P}^1$, $l_{i,j} = \{ix + (1-i)y = j\}_{i,j=0,1}$ (i.e. $x = 0, 1$, $y = 0, 1$). Blow up over the points $z_{i,j} = (i, j)$, $i, j = 0, 1$, until the four functions

$$\frac{(x-i)^{m_{ij}}}{(y-j)^{n_{ij}}}, \quad i, j = 0, 1,$$

[5]see e.g. [Mu, Hir] for an algorithm of computing $\pi_1(S)$.

become regular; denote the resulting surface by $V_T \xrightarrow{\pi} Q$. Set $D_0 = e_0 \bigcup e_1 \bigcup \{l_{ij}\} \subset Q$, where $e_0 = \{y = \infty\}$, $e_1 = \{x = \infty\} \subset Q$ (see Figure 1).

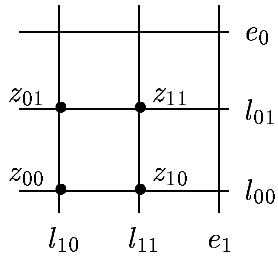


Figure 1

Set $\pi^{-1}(D_0) = D_T \bigcup \{v_{ij}\}$, where v_{ij} are the only (-1) curves in the exceptional divisor of π. Then D_T is a rational tree (deleting v_{ij} is called 'cutting cycles' by Petrie and tom Dieck [PtD 1, PtD 3]). The proper transforms e_0', e_1' of e_0, e_1 and the components of the exceptional divisor of π form a natural basis in Pic $V_T = H_2(V_T, \mathbf{Z})$, a new one given by the components of D_T , assuming that the decomposition matrix

$$\begin{pmatrix} T & 0 \\ B & I \end{pmatrix}$$

is unimodular, i.e. that T is unimodular. Now Proposition 2.1 asserts that $X_T := V_T \backslash D_T$ is an acyclic surface iff $\det T = \pm 1$.

Remark. Petrie and tom Dieck [PtD 1, PtD 3] found all the basic line arrangements in $\mathbf{P}^2$ that lead to acyclic surfaces in a process as above; there are seven of them. The first one depends on discrete and continuous parameters (see Figure 2; cf. the Classification Theorem 3.3(d) below); the other six are projectively rigid.

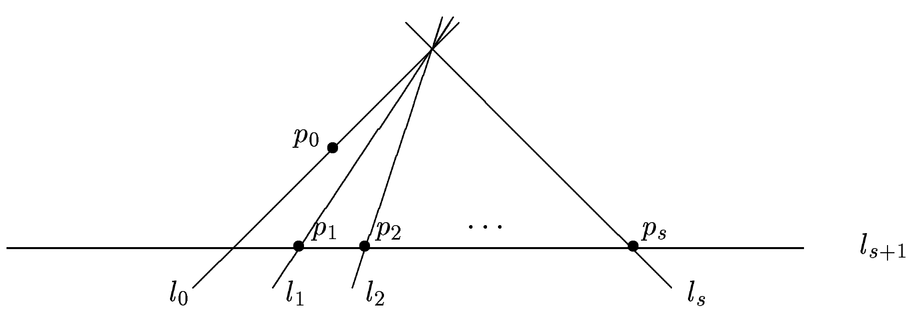


Figure 2

Lemma 2.2. *If T is unimodular, then X_T is a contractible surface.*

Proof. We should check that $\det T = \pm 1$ implies that $\pi_1(X) = \mathbf{1}$. Denote

$$X_0 := X_T \setminus \{v_{i,j}\} \simeq Q \setminus \{6 \quad \text{lines}\}.$$

Choose four generators $a_i,\, b_j \in \pi_1(X_0),\, i,j = 0,1$, which are 'vanishing loops' [6] of the lines $l_{i,j}$ under the embedding $X_0 \hookrightarrow Q$, and so

$$\pi_1(X_0) = \mathbf{F}_2 \times \mathbf{F}_2 =< a_0, a_1, b_0, b_1 \mid [a_i, b_j] = 1,\ i,j = 0,1 > .$$

We will see in Lemma 2.3 below that $\pi_1(X) = \pi_1(X_0)/N$, where

$$N =<< a_i^{m_{ij}} b_j^{n_{ij}} \mid i,j = 0,1 >>$$

is the minimal normal subgroup generated by the four products. Thus, we have the following relations in $\pi_1(X)$: $a_0^{m_{00}} b_0^{n_{00}} = 1,\quad a_1^{m_{10}} b_0^{n_{10}} = 1,$ and so $a_0^{m_{00}n_{10}} = a_1^{m_{10}n_{00}}$. Also, $a_0^{m_{01}} b_1^{n_{01}} = 1,\quad a_1^{m_{11}} b_1^{n_{11}} = 1,$ and hence $a_0^{m_{01}n_{11}} = a_1^{m_{11}n_{01}}$. It follows that

$$a_0^{m_{00}n_{10}m_{11}n_{01}} = a_0^{m_{01}n_{11}m_{10}n_{00}},$$

whence $a_0^{\det T} = 1,$ that is $a_0 = 1$. In the same way we obtain $a_1 = b_0 = b_1 = 1,$ and therefore $\pi_1(X) = \mathbf{1}$. $\square$

Lemma 2.3. (Fujita [Fu 2]) *(a) Let D be a closed hypersurface in a complex manifold M, $\dim M \geq 2$. Then the group Ker $\{i_* : \pi_1(M \setminus D) \to \pi_1(M)\}$ is generated by the vanishing loops* [7] *of D. In particular, if D is irreducible, this kernel is generated, as a normal subgroup, by any of these loops.*

(b) Let M be a surface, D_1, D_2 be two curves in M, and p be an intersection point which is an ordinary double point of $D_1 \bigcup D_2$. Let $\sigma_p : M' \longrightarrow M$ be the blow-up at p. Then (the class of) a vanishing loop α_E of the exceptional $(-1)-$ curve $E \subset M'$ of σ_p in the group $\pi_1(M \setminus (D_1 \cup D_2)) = \pi_1(M' \setminus (E \cup D_1' \cup D_2'))$ can be represented as $\alpha_E = \alpha_{D_1'} \alpha_{D_2'}$, where $\alpha_{D_1'}, \alpha_{D_2'}$ are vanishing loops of the proper transforms D_i' of D_i, $i = 1, 2$.

(c) In the notation as in (b), let $\sigma : \widehat{M} \longrightarrow M$ be a sequence of blow-ups over p such that $D_1^* = mE + \ldots,\quad D_2^* = nE + \ldots$ for E being the exceptional $(-1)-$ curve of the last blow-up. Then we have $\alpha_E = \alpha_{D_1'}^m \alpha_{D_2'}^n$.

[6] see the next footnote.

[7] By *a vanishing loop* of D at a smooth point $e \in D$ we mean any loop δ in $M \setminus D$ consisting of a path α which joins a base point $e_0 \in M \setminus D$ with a point $e' \in \omega \setminus D$ of a small complex disc $\omega \subset M$ transversal to D at e , and a simple loop β in positive direction in $\omega \setminus D$ with the base point e' (i.e. e is in the interior of β in ω).

Proof. (a) Denote Δ the unit disc in $\mathbf{C}$. Let $\gamma : \partial\Delta = \mathbf{S}^1 \longrightarrow M \backslash D$. Observe that $\gamma_* \in \operatorname{Ker} i_*$ iff there exists $\tilde{\gamma} : \Delta \longrightarrow M$ such that $\tilde{\gamma} \mid_{\partial\Delta} = \gamma$. After a small deformation, we may assume that $\tilde{\gamma}(\Delta)$ meets D transversally at smooth points $p_1, \dots, p_k$; let $q_1, \dots, q_k$ be the corresponding disc points.

Choose disjoint vanishing loops of $q_1, \dots, q_k$ in Δ, and contract the circle $\mathbf{S}^1 = \partial\Delta$ onto their union. Being composed with $\tilde{\gamma}$ this yields a desired homotopy of γ to a product of vanishing loops of D.

(b) Let $D_i = \{z_i = 0\}$, $i = 1, 2$, in a local chart (z_1, z_2) in M centered at p. Representing $\alpha_{D_1}\alpha_{D_2}$ on the torus $\mid z_1 \mid = 1, \quad \mid z_2 \mid = 1$ as its diagonal section, after blowing up this loop becomes a vanishing one α_E.

(c) Apply induction on the number of blow-ups. □

Exercises (**2.3**) (after Fujita [Fu 2]) Let X be an acyclic (resp. contractible) surface, and let $C \subset X$ be an irreducible simply connected curve. Consider the blow-up $\sigma_p : \widehat{X} \longrightarrow X$ at a smooth point $p \in C$, and set $X' = \widehat{X} \setminus C'$, where $C' \subset \widehat{X}$ is the proper transform of C. Show that X' is also acyclic (contractible).

(**2.4**) Draw the dual graph Γ_{D_T}, where D_T is as in Example 2.3 above. Deduce that in many cases X_T is not isomorphic to $\mathbf{C}^2$.

3 Elements of classification: the logarithmic Kodaira dimension

Definition 3.1. Let $L \longrightarrow V$ be a line bundle over a smooth projective variety V, and let $H^0(V, L)$ be the space of its regular sections. By the Cartan-Serre Theorem, $h^0(V, L) = \dim H^0(V, L) < \infty$. Suppose that $h^0(V, L) > 0$, and fix a basis $s_0, \dots, s_n$ of $H^0(V, L)$, where $n = h^0(V, L) - 1$. Set $Z = \{z \in V \mid s_0(z) = \dots = s_n(z) = 0\}$; Z is a proper subvariety of V. For $z \in V \setminus Z$, fix an affine structure in the fiber $L_z \cong \mathbf{C}$; then the point $\Phi_L(z) := (s_0(z) : \dots : s_n(z)) \in \mathbf{P}^n$ is well-defined, and the rational map $\Phi_L : V \longrightarrow \mathbf{P}^n$ is regular in $V \setminus Z$. L is called *very ample* if Φ_L is an embedding (assuming $Z = \emptyset$); *ample* if mL is very ample for some $m > 0$; *big* if[8] $\mathrm{L-dim}\,(V) := \overline{\lim}_{m\to\infty} \dim \Phi_{mL} = \dim_{\mathbf{C}} V$. Put $\mathrm{L-dim}\,(V) = -\infty$ if $h^0(mL) = 0$ $\forall m$.

[8] In section 8.2 below we also use the notation $k(V, L) := \mathrm{L-dim}\,(V)$.

Theorem 3.1. (Serre-Siegel-Kodaira; see e.g. [Ii 3, Thm. 10.2]) *For some* $m_0 > 0$ *we have* $h^0(mm_0L) \sim m^l$, *where* $l = \mathrm{L} - \dim\,(V)$.

Definition 3.2. If $L = K_V$ is the canonical line bundle (i.e. $K_V = \Lambda^n T^* V$, where $n = \dim_{\mathbf{C}} V$), then $k(V) := \mathrm{K} - \dim V$ is called the *Kodaira dimension* of V ; $k(V) \in \{-\infty, 0, 1, \ldots, \dim V\}$. If $k(V) = \dim V,$ then V is said to be of *general type.*

Thus, V is of general type if K is big, i.e. for some $m > 0,$ $\Phi_{mK} : V \hookrightarrow \mathbf{P}^n$ is a birational embedding.

Exercise (3.1) A smooth irreducible projective curve V is of general type (i.e. $k(V) = 1$) iff $g(V) \geq 2$; $k(V) = 0$ iff $g(V) = 1$, i.e. if $V = \mathbf{T}_\Lambda := \mathbf{C}/\Lambda$ is an elliptic curve, where $\Lambda = \mathbf{Z} + \tau\mathbf{Z} \subset \mathbf{C}$ is a lattice; $k(V) = -\infty$ iff $g(V) = 0$, i.e. if $V \cong \mathbf{P}^1$ is a rational curve.

Definition 3.3. For an SNC-completion (V, D) of a smooth quasi-projective variety $X = V \setminus D,$ its *log-Kodaira dimension* is $\overline{k}(X) := \mathrm{L} - \dim\,(V)$, where $L = K + D$ (Iitaka [Ii 1]). X is said to be *of log-general type* if $\overline{k}(X) = \dim X.$

Sometimes $K + D$ is called *the log-canonical divisor*; the sections of $O(K + D)$ correspond to the meromorphic forms regular in $X,$ which can be written as

$$a\frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_k}}{z_{i_k}} \wedge dz_{i_{k+1}} \wedge \ldots \wedge z_{i_n}$$

in local coordinates in $V,$ where $D = \{z_{i_1} = \ldots = z_{i_k} = 0\}.$

Theorem 3.2. (Iitaka [Ii 1], [Ii 3, Ch. 11]) $\overline{k}(X)$ *is an invariant of* X *which does not depend on the choice of an SNC-completion* (V, D) *of* $X.$

Exercise (3.2) If X is a smooth irreducible quasi-projective curve, then $\overline{k}(X) = -\infty$ iff $X = \mathbf{C}$ or $X = \mathbf{P}^1$; $\overline{k}(X) = 0$ iff $X = \mathbf{T}_\Lambda$ or $X = \mathbf{C}^* := \mathbf{C} \setminus \{0\}$; $\overline{k}(X) = 1$ otherwise.

Proposition 3.1. (a) ([Ii 1], [Ii 3, Thm. 11.3]) $\overline{k}(X \times Y) = \overline{k}(X) + \overline{k}(Y).$

(b) ([Ii 1], [Ii 3, Prop. 11.5]) *If* Y *is a Zariski open subset of* $X,$ *then* $\overline{k}(Y) \geq \overline{k}(X),$ *and* $\overline{k}(Y) = \overline{k}(X)$ *if* $\mathrm{codim}_X(X \setminus Y) \geq 2.$

(c) (the Iitaka Easy Addition Theorem [Ii 1, Thm. 4], [Ii 3, Thm. 11.9]) *If* $\pi : Y \longrightarrow X$ *is a surjective morphism of smooth quasi-projective varieties with a connected generic fiber* $F,$ *then* $\overline{k}(Y) \leq \overline{k}(F) + \dim X$.

(d) (the Kawamata-Viehweg Addition Theorem [Kaw 1, Vie]) *If, in addition,* $\dim F = 1$, *then* $\overline{k}(Y) \geq \overline{k}(F) + \overline{k}(X)$.

(e) ([Ii 1, Thm. 3], [Ii 3, Thm. 11.10]) *If* $f : Y \longrightarrow X$ *is an étale (i.e. non-ramified) covering, then* $\overline{k}(Y) = \overline{k}(X)$.

(f) (the Logarithmic Ramification Formula) [Ii 1], [Ii 3, Thm. 11.3]) *Let* $\dim X = \dim Y$, *and let* $f : Y \longrightarrow X$ *be a dominant morphism*[9]. *By the Hironaka Resolution of Singularities Theorem,* f *can be extended to a morphism* $\overline{f} : V_Y \longrightarrow V_X$, *where* (V_X, D_X) *(resp.* $V_Y, D_Y)$ *) is an appropriate SNC-completion of* X *(resp. of* Y *). Then there exists an effective divisor* $R_{\overline{f}} \subset V_Y$ *(which is called the* logarithmic ramification divisor*) such that*

$$K_{V_Y} + D_{V_Y} = \overline{f}^*(K_{V_X} + D_{V_X}) + R_{\overline{f}}. \qquad (R)$$

In particular,

$$H^0(V_X, m(K_{V_X} + D_{V_X})) \hookrightarrow H^0(V_Y, \overline{f}^* m(K_{V_X} + D_{V_X})) \subset$$

$$H^0(V_Y, m\overline{f}^*(K_{V_X} + D_{V_X}) + mR_{\overline{f}}) = H^0(V_Y, m(K_{V_Y} + D_{V_Y})).$$

Therefore, $\overline{k}(X) \leq \overline{k}(Y)$.

(g) ([Ii 1, Prop. 1, Thm. 3], [Ii 3, Thms. 10.5, 11.10]) *If, in addition,* f *is either a proper birational morphism, or an étale covering, then we may assume* $R_{\overline{f}}$ *being an* $f-$ *exceptional divisor, i.e.* $\operatorname{codim} \overline{f}(R_{\overline{f}}) \geq 2$, *and we have* $\overline{k}(Y) = \overline{k}(X)$.

Classification Theorem 3.3. *Let* X *be an acyclic surface. Then*

(a) (Miyanishi-Sugie-Fujita [MiySu, Fu 1]) $\overline{k}(X) = -\infty$ *iff* $X \simeq \mathbf{C}^2$.

(b) (Fujita [Fu 2]) *If* X *is non-isomorphic to* $\mathbf{C}^2$, *then* $\overline{k}(X) \geq 1$.

(c) (Iitaka-Kawamata [Ii 1, Thm. 5], [Kaw 2]) *If* $\overline{k}(X) = 1$, *then there exists a morphism* $X \longrightarrow \Gamma$ *onto a smooth curve* Γ *with generic fibers isomorphic to* $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ *(called a* $\mathbf{C}^*-$ fibration*)*[10].

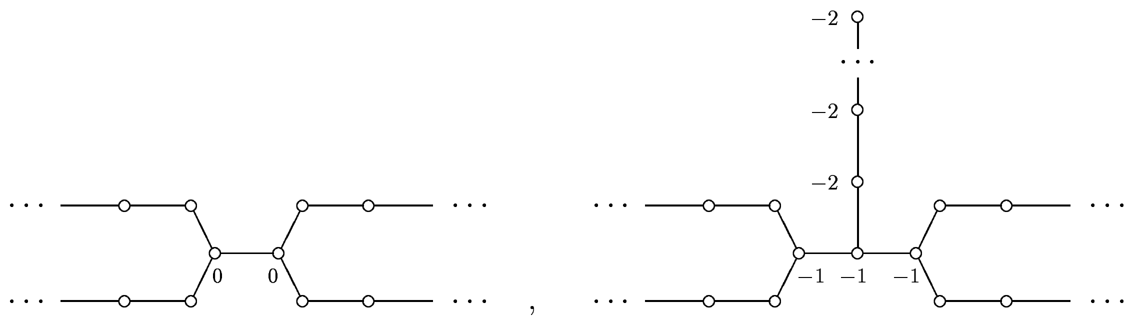
(d) (Gurjar-Miyanishi [GuMiy 1]; cf. also [PtD 1, FlZa 1]) *There exists a complete list of acyclic surfaces with* $\overline{k}(X) = 1$. *Any such surface can be obtained from a Petrie–tom Dieck line configuration in* $\mathbf{P}^2$ *of the first kind (see Figure 2 above) by blowing up over the points* $p_0, \ldots, p_s$ *to get, as* $\sigma^{-1}(p_i)$, $i = 1, \ldots, s$, *a linear chain of rational curves with only one* $(-1)-$ *curve, and leaving this last exceptional* $(-1)-$ *curve of each chain*

[9] i.e. $f(Y)$ contains a Zariski open subset of X .

[10] Actually, any (not necessarily acyclic) affine surface X with $\overline{k}(X) = 1$ possesses a $\mathbf{C}^*-$ fibration.

in the affine part[11]. *For each* $i = 1, \dots, s$, *we fix a rational number* $\frac{m_i}{n_i}$; *the numbers* $(m_i,\, n_i)_{i=1,\dots,s}$ *must satisfy a diophantine equation of unimodularity. The blow-up process over* p_i *is done according to the data* (m_i, n_i); *this means that locally at* p_i *it resolves the point of indeterminacy of the rational function* x^{m_i}/y^{n_i}, $i = 1, \dots, s$.

In particular, for $s = 2$ *and* $m_1 n_2 + m_2 n_1 - m_1 m_2 = \pm 1$, $m_i > n_i$, $i = 1, 2$, *we obtain in this way all the contractible surfaces with* $\overline{k}(X) = 1$. *Their minimal dual graphs look as follows:*



The choice of the centers of blow-ups over p_0 *(besides the first one), and the positions of the points* $p_3, \dots, p_s$ *on* l_{s+1} *(once the first three intersection points on* l_{s+1} *are fixed) give rise to the deformation parameters* [FlZa 1].

Theorem 3.4. [AM, LiZa, GuMiy 2, Suz 1, Za 1] *Let* X *be a smooth acyclic surface, and let* Γ *be an irreducible simply connected curve in* X . *Then either*

$*$ $(X,\, \Gamma) \simeq (\mathbf{C}^2,\, \Gamma_{k,l})$, *where* $\Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbf{C}^2$, $k \ge l \ge 1$, $(k,\, l) = 1$, *or*

$*$ $\overline{k}(X) = 1$ *and* $\Gamma = E \setminus D \simeq \mathbf{C}$, *where* $E \subset V$ *is the last* $(-1)-$ *curve over the point* p_0 *in the reconstruction process as in Theorem 3.3(d) above, and* D *is the boundary divisor of the corresponding SNC-completion of* X .

This theorem shows, in particular, that, up to automorphisms of the affine plane, there is only a sequence (namely, $\{\Gamma_{k,l}\}$) of irreducible simply connected curves in $\mathbf{C}^2$; each smooth acyclic surface of logarithmic Kodaira dimension 1 contains exactly one such curve, and this curve is smooth; at last, there is no simply connected curves at all on acyclic surfaces of log-general type.

Example 3.1. (Petrie, tom Dieck [PtD 2]) The surface $X_{k,l} \subset \mathbf{C}^3$ given by the

[11]In addition, all the blow ups over p_0 are *outer*, i.e. each of them is done at a smooth point of the total transform of l_0 , whereas under the *cutting cycle* procedure over p_i, $i = 1, \dots, s$, the blow ups are *inner*, i.e. they are done at double points only.

equation

$$\frac{(xz+1)^k - (yz+1)^l}{z} = 1,$$

where $k > l \geq 2,\ (k,\, l) = 1$, is a smooth contractible one with $\overline{k}(X_{k,\,l}) = 1$ (see Example 6.1 below). The only simply connected curve in $X_{k,\,l}$ is given by the equation $z = 0$.

In a similar way, any smooth contractible surface with $\overline{k} = 1$ can be realized in $\mathbf{C}^3$ (Kaliman, Makar-Limanov [KaML 2]).

Example 3.2. It can be shown (see [Za 3, Za 4]) that $\overline{k}(X_T) = 1$ for a surface X_T as in Example 2.3, iff $m_{ij} = n_{ij} = 1$ for a pair of diagonal points from the square vertices $(z_{ij} = (i,\, j))_{i,\,j=0,\,1}$ (see Figure 1 above). If so, then the only simply connected curve in X_T is the proper transform of the corresponding diagonal line. Otherwise, $\overline{k}(X_T) = 2$, i.e. X_T is of log-general type.

Remark. There is a number of examples of acyclic or even contractible surfaces of log-general type (see e.g. [tD 2, FlZa 1, GuMiy 1, Sug]), but no classification is known. While acyclic surfaces of log-Kodaira dimension 1 admit deformations (see the Classification Theorem 3.3(d)), those of log-general type are rigid in all known examples [FlZa 1, FlZa 2]. So, the problem arises whether or not all of them are rigid (on this and other problems on acyclic surfaces, see e.g. the problem list [OPOV]).

4 Exotic product structures

We begin this section by recalling

The Zariski Cancellation Problem. *Given an isomorphism* $X \times \mathbf{C}^k \overset{\Phi}{\simeq} \mathbf{C}^{n+k}$, *does it follow that* $X \simeq \mathbf{C}^n$ *?*

Take $\mathbf{C}^n$ generic in $\mathbf{C}^{n+k}$, and combine Φ with the first projection. This yields a surjective morphism $\mathbf{C}^n \longrightarrow X$. Thus, by Proposition 3.1.(f), $\overline{k}(X) = -\infty$. Also, X is homotopically trivial; in particular, for $n = 2$ X is an acyclic surface. By the Miyanishi-Sugie-Fujita Theorem 2.3.(a), $X \simeq \mathbf{C}^2$. This provides the positive answer to the Zariski Cancellation Problem for $n = 1, 2$. For $n \geq 3$ the problem is open.

Iitaka-Fujita Strong Cancellation Theorem 4.1. [IiFu] *Let* X, Y *be smooth quasi-projective varieties of the same dimension, and let* $\Phi : Y \times \mathbf{C}^n \longrightarrow X \times \mathbf{C}^n$ *be an isomorphism. Assume that* $\overline{k}(X) \geq 0$. *Then there is a commutative diagram*

$$
\begin{array}{ccc}
Y \times \mathbf{C}^k & \xrightarrow{\ \Phi\ } & X \times \mathbf{C}^k \\
\Big\downarrow \mathrm{pr} & & \Big\downarrow \mathrm{pr} \\
Y & \xrightarrow{\ \varphi\ } & X\,,
\end{array}
$$

where φ *is an isomorphism.*

We use below the following well known corollary of the Smale h-cobordism Theorem.

Proposition 4.1. (see [Mil 1, §9]) *Let* D^n *be a smooth simply connected manifold of (real) dimension* $n \geq 5$ *with a simply connected boundary. Then the following conditions are equivalent:*

1) D^n *is diffeomorphic to the closed unit* $n-$ *ball* $\overline{B}^n$.
2) D^n *is homeomorphic to* $\overline{B}^n$.
3) D^n *is contractible.*
4) D^n *is acyclic.*

Theorem 4.2. (Dimca-Ramanujam [Di 1, Ram]) *Let* X *be a contractible smooth affine algebraic variety. If* $\dim_{\mathbf{C}} X = n \geq 3$ *, then* X *is diffeomorphic to* $\mathbf{R}^{2n}$.

Proof. [12] Fix a closed embedding $X \hookrightarrow \mathbf{C}^N$ such that the smooth function $\varphi := ||z||^2 \,|\, X$ on X is a *Morse function*, i.e. it has only non-degenerate critical points (see [Mil 2, Thm. 6.6]). Since $\varphi : X \to \mathbf{R}$ is proper and has only finite number of critical values, for $R > 0$ large enough X is diffeomorphic to $X_R := \{\varphi < R\}$. Denote $S_R = \partial \overline{X_R}$; that is, $\overline{X_R}$ is a smooth manifold with the boundary S_R . By the Morse Theory applied to the Morse function $\psi := R - \varphi$ on X , the manifold $\overline{X_R}$ can be obtained, starting with the boundary S_R , by successively gluing handles of indices equal to those of the critical points of ψ on X_R .

If $p \in X_R$ is a critical point of ψ , then $\mathrm{ind}_p \psi = 2n - \mathrm{ind}_p \varphi$. But $\mathrm{ind}_p \varphi \leq n$ [Mil 2, the proof of Thm. 7.2]. Hence, $\mathrm{ind}_p \psi \geq n \geq 3$. Therefore, $\overline{X_R}$ is obtained from $S_R = \partial \overline{X_R}$ by attaching handles of indices at least 3 . Consequently, $\overline{X_R}$ is homotopically equivalent to a cell complex obtained from S_R by successively attaching cells of dimension at least 3 . It follows that the first two relative homotopy groups $\pi_i(\overline{X_R}, S_R)$, $i = 1, 2$, are trivial. Since $\overline{X_R}$ is contractible, applying the exact homotopy

[12] Cf. the proof of the Lefschetz Hyperplane Section Theorem in [Mil 2].

sequence of a pair

$$\mathbf{1} = \pi_2(\overline{X_R},\, S_R) \xrightarrow{\partial_*} \pi_1(S_R) \xrightarrow{i_*} \pi_1(\overline{X_R}) = \mathbf{1}$$

we conclude that $\pi_1(S_R) = \mathbf{1}$. Now the theorem follows from Proposition 4.1. $\square$

Remark. In section 2 above we have seen examples of contractible smooth affine surfaces S with non-simply connected attached boundaries ∂S (in other words, S is non-simply connected at infinity: $\pi_1^\infty(S) \neq \mathbf{1}$). Therefore, these contractible surfaces are not homeomorphic to $\mathbf{R}^4$. This shows that the restriction $n \geq 3$ in the above theorem is crucial.

Corollary 4.1. *Let X be a smooth contractible surface. Then $X \times \mathbf{C}$ is diffeomorphic to $\mathbf{C}^3 \simeq \mathbf{R}^6$, and so $X \times \mathbf{C}^k$ is diffeomorphic to $\mathbf{R}^{2k+4}$, $k \geq 1$.*

Proof. We indicate, following Ramanujam [Ram], an alternative direct proof of this corollary. According to Proposition 4.1, it suffices to show that $X \times \mathbf{C}$ is diffeomorphic to the interior of a smooth compact manifold D with a simply connected boundary ∂D. There are two natural ways to compactify $X \times \mathbf{C}$. First, consider any smooth affine variety $Z \hookrightarrow \mathbf{C}^N$. Then the restriction φ of the real polynomial $||z||^2$ to Z has only a finite number of critical values, and $\varphi^{-1}[R, \infty[$ for R large enough is diffeomorphic to $[R, \infty[\times T$, where $T := \varphi^{-1}(R)$. Thus, Z is diffeomorphic to $Z_0 := \varphi^{-1}[0, R[$, the interior of the manifold with boundary $\overline{Z}_0 := \varphi^{-1}[0, R]$, $\partial\overline{Z}_0 = T$. Represent in this way $X \hookrightarrow \mathbf{C}^n$ attaching the boundary ∂X, and $Y := X \times \mathbf{C} \hookrightarrow \mathbf{C}^{n+1}$ attaching the boundary $\partial_1 Y = \varphi^{-1}(R_1)$. Since Y is diffeomorphic to $X \times \Delta$, where $\Delta = \{|z| < 1\}$, Y can also be compactified by attaching the non-smooth boundary $\partial_2 Y := (\partial X \times \overline{\Delta}) \bigcup (\overline{X} \times \mathbf{S}^1)$. In fact, $\partial_2 Y = \psi^{-1}(R_2)$, where $\psi(\bar{x},\, z) := \max\,\{||\bar{x}||^2,\, |z|^2\}$, and $R_2 > 0$ is large enough. By the Van Kampen Theorem, $\partial_2 Y$ is simply connected.

We may assume that sufficiently large $R_1',\, R_1'',\, R_2',\, R_2''$ are chosen in such a way that $\partial_1 Y = \varphi^{-1}(R_1) \subset \psi^{-1}([R_2',\, R_2'']) \subset \varphi^{-1}([R_1',\, R_1''])$, and that $\varphi^{-1}([R_1',\, R_1'']) \approx \partial_1 Y \times [R_1',\, R_1'']$, $\psi^{-1}([R_2',\, R_2'']) \approx \partial_2 Y \times [R_2',\, R_2'']$. Thus, the composition of embeddings $\partial_1 Y \hookrightarrow \partial_2 Y \times [R_2',\, R_2''] \hookrightarrow \partial_1 Y \times [R_1',\, R_1'']$ provides a homotopical equivalence. Respectively, the induced isomorphism $\pi_1(\partial_1 Y) \xrightarrow{\simeq} \pi_1(\partial_1 Y \times [R_1',\, R_1''])$ factors through the trivial one $\pi_1(\partial_1 Y) \to \pi_1(\partial_2 Y \times [R_2',\, R_2'']) \simeq \pi_1(\partial_2 Y) = \mathbf{1}$. This proves simply connectedness of the boundary $\partial_1 Y$, and the assertion follows. $\square$

Theorem 4.3. *(a) Let S be a smooth contractible surface non-isomorphic to $\mathbf{C}^2$. Then $S \times \mathbf{C}^{n-2}$ $(n > 2)$ is a smooth affine variety diffeomorphic to $\mathbf{C}^n$, but non-isomorphic to $\mathbf{C}^n$* (in what follows such a variety is called an *exotic* $\mathbf{C}^n$).

(b) Furthermore, if two smooth contractible surfaces S_1, S_2 *are not isomorphic, then* $S_1 \times \mathbf{C}^{n-2}$, $S_2 \times \mathbf{C}^{n-2}$ $(n > 2)$ *are two non-isomorphic exotic* $\mathbf{C}^n$*− s.*

Proof. (a) By Lemma 4.1, $S \times \mathbf{C}^{n-2}$ is diffeomorphic to $\mathbf{R}^{2n}$ for $n \geq 3$. By the Miyanishi-Sugie-Fujita Theorem 2.3.(a), $\overline{k}(S) \neq -\infty$ (otherwise $S \simeq \mathbf{C}^2$), whence $\overline{k}(S) \geq 0$. But if $S \times \mathbf{C}^{n-2}$ were isomorphic to $\mathbf{C}^n$, then we would have $\overline{k}(S) = -\infty$, a contradiction.

(b) By virtue of the Strong Cancellation Theorem 4.1 of Iitaka-Fujita, the classification of exotic product structures on $\mathbf{C}^n$ of the type $S \times \mathbf{C}^{n-2}$, where S is a surface as above, is reduced to the classification of surfaces S themselves. Indeed, $S_1 \times \mathbf{C}^{n-2} \simeq S_2 \times \mathbf{C}^{n-2}$ and $\overline{k}(S_1) \geq 0$ would imply that $S_1 \simeq S_2$. Since $S_1 \not\simeq S_2$, and both surfaces are acyclic, by the Miyanishi-Sugie-Fujita Theorem 3.3(a), $\overline{k}(S_i) \geq 0$ for at least one value of i , say, for $i = 1$, and so, the assertion follows. □

Remark 1. For instance, pairwise non-isomorphic surfaces X_T of log-general type (see Example 2.3 above) yield sequences of exotic $\mathbf{C}^n$. Since contractible surfaces S with $\overline{k}(S) = 1$ admit deformations (see the Classification Theorem 3.3(d)), the corresponding exotic $\mathbf{C}^n$ -s of the type $S \times \mathbf{C}^{n-2}$ admit deformations, too [FlZa 1].

Remark 2. Let $X = \prod_{i=1}^n S_i$ be a product of $n \geq 2$ contractible surfaces. Then X is diffeomorphic to the interior of a compact contractible variety with boundary. By the Van Kampen Theorem, the boundary ∂X is simply connected. Therefore, X is diffeomorphic to $\mathbf{C}^{2n}$. Also, $\overline{k}(X) = \sum_{i=1}^n \overline{k}(S_i)$. Hence, X is of log-general type iff S_i are so for all $i = 1, \ldots, n$; $\overline{k}(X) = -\infty$ if $\overline{k}(S_i) = -\infty$ for at least one value of i.

Remark 3. If $\overline{k}(S) = 2$, then $X = S \times \mathbf{C}$ contains no copy of $\mathbf{C}^2$, i.e. there is no embedding $\mathbf{C}^2 \hookrightarrow S \times \mathbf{C}$ [Za 3]. (This is based on the fact that S contains no simply connected curve; see [Za 1] and Theorem 3.4 above.) In the next section we present examples of exotic $\mathbf{C}^3$ with many copies of $\mathbf{C}^2$ (see Example 5.1).

Remark 4. Due to the Ramanujam Theorem 2.2(b), there is no exotic $\mathbf{C}^2$.

Remark 5. Actually, the Zariski Cancellation Problem can be reformulated as follows: *Given an exotic* $\mathbf{C}^n$, *denote it* X, *should also the product* $X \times \mathbf{C}^k$ *be an exotic* $\mathbf{C}^m$ $(m = n + k)$ *?*

Another question is a *generalized Serre Problem*:

Is any vector bundle over an exotic $\mathbf{C}^n$ *trivial?*

Exercise (**4.1**) Verify that a smooth irreducible quadric hypersurface in $\mathbf{C}^{n+1}$ is contractible if and only if it is isomorphic to $\mathbf{C}^n$.

5 The Kaliman modification

Definition 5.1. Consider a triple (M, D, C), where $M \supset D \supset \operatorname{reg} D \supset C$, M and C are smooth affine varieties, D is an irreducible hypersurface in M, and C is proper in D, so that $\operatorname{codim}_M D = 1$ and $\operatorname{codim}_M C \geq 2$. Let $\sigma_C : \widehat{M} \longrightarrow M$ be the blow up of M along C with the exceptional divisor $E = \sigma^{-1}(C)$. Then $\sigma_C| \, E : E \longrightarrow C$ is a fiber bundle with the fiber $\mathbf{P}^k$, $k = \dim E - \dim C$; E and the proper transform D' of D meet transversally, and $\sigma_C : E \bigcap D' \longrightarrow C$ is a fiber bundle with the fiber $\mathbf{P}^{k-1}$. The variety $M' := \widehat{M} \setminus D'$ is called the *Kaliman transform* or the *Kaliman modification* of the triple (M, D, C) along D' with the center C. Let $E' = E \setminus D'$; clearly, the restriction $\sigma_C| \, E' : E' \longrightarrow C$ is a fiber bundle with the fiber $\mathbf{C}^k$.

Lemma 5.1. (Kaliman [Ka 2, Lemma 3.4]) $\pi_1(M') \simeq \pi_1(M)$.

Proof. The restriction $\sigma_C| \, (M' \setminus E') : M' \setminus E' \longrightarrow M \setminus D$ is an isomorphism. Thus, we may consider the following commutative diagram (left) and the induced commutative triangle (right):

$$\begin{array}{ccccc} & & M' & & \\ & {\scriptstyle i} \nearrow & & \searrow {\scriptstyle \sigma_C} & \\ M \setminus D & & \xrightarrow{\ j\ } & & M\ , \end{array} \qquad \begin{array}{ccccc} & & \pi_1(M') & & \\ & {\scriptstyle i_*} \nearrow & & \searrow {\scriptstyle (\sigma_C)_*} & \\ \pi_1(M \setminus D) & & \xrightarrow{\ j_*\ } & & \pi_1(M)\ . \end{array}$$

It is easily seen that both i_* and j_* are surjections (since a complex hypersurface has real codimension 2). Thus, $(\sigma_C)_*$ is also surjective. Denote by α_D a vanishing loop of D. By Lemma 2.3.(a), $\operatorname{Ker} j_* = << \alpha_D >>$, where $<< S >>$ denotes the subgroup of a group G generated by the conjugacy classes of the elements $s \in S \subset G$ ($<< S >>$ is said to be *normally generated* by S). We choose α_D in such a way that near D it is a boundary circle of a small transversal disc ω centered at a point $c_0 \in C$. Then the proper transform ω' of ω in M' is a disc centered at a point of $E' = E \setminus D'$. Thus, $i_*(\alpha_D) = 1 \in \pi_1(M')$, i.e. $\alpha_D \in \operatorname{Ker} i_*$. This implies that $\operatorname{Ker} j_* \subset \operatorname{Ker} i_*$. But since $j_* = (\sigma_C)_* \circ i_*$, $\operatorname{Ker} i_* = \operatorname{Ker} j_*$, and so $(\sigma_C)_* : \pi_1(M') \longrightarrow \pi_1(M)$ is an isomorphism. $\square$

Lemma 5.2. (cf. Kaliman [Ka 2, Proof of Thm. 3.5]) *Suppose that (i) D is a topological manifold, and (ii) D and C are acyclic. Then M' is acyclic iff M is.*

Proof. As follows from Lemma 5.1, $(\sigma_C)_* : H_1(M') \to H_1(M)$ is an isomorphism (hereafter all the homology groups are with coefficients in $\mathbf{Z}$). Note that

- $\sigma_C : E' \longrightarrow C$ is a smooth fibration with a contractible fiber, and so, it yields a homotopy equivalence between E' and C (*exercise*). Therefore, $(\sigma_C)_* : H_*(E') \to H_*(C)$ is an isomorphism. Hence, E' is also acyclic.
- Let $\dot{X}$ be the one-point compactification of a manifold X. Then we have

$$\tilde{H}^i(\dot{X}) \simeq H^i(\dot{X}, *) \simeq H^i_c(X) \cong H_{m-i}(X)\,,$$

where $\cong$ stands for the Lefschetz–Poincaré duality, and $m = \dim_{\mathbf{R}} X$. Thus, under our assumptions $\dot{D}$ and $\dot{E}'$ are homology spheres, and so is M resp. M' iff it is acyclic.

Assume first that M is acyclic. Then by the exact cohomology sequence of a pair

$$\begin{array}{ccc} & H^*(\dot{M}) & \\ {\scriptstyle i^*}\swarrow & & \nwarrow{\scriptstyle r^*} \\ H^*(\dot{D}) & \xrightarrow{\ \delta^*\ } & H^*(\dot{M}, \dot{D})\ , \end{array}$$

where $\deg\ r^* = \deg\ i^* = 0$, $\deg\ \delta^* = 1$, we have $H^{2n-1}(\dot{M}, \dot{D}) \simeq H^{2n}(\dot{M}, \dot{D}) \simeq \mathbf{Z}$, where $n = \dim_{\mathbf{C}} M$, and $\tilde{H}^j(\dot{M}, \dot{D}) = 0$ for $j \le 2n-2$. Since $\dot{M} \backslash \dot{D} = M \backslash D \approx M' \backslash E'$, we have the homeomorphisms

$$\dot{M}/\dot{D} \approx (M \setminus D)^{\cdot} \approx (M' \setminus E')^{\cdot} \approx \dot{M}'/\dot{E}'$$

(*exercise*).[13] Hence,

$$H^*(\dot{M}', \dot{E}') \simeq \tilde{H}^*(\dot{M}'/\dot{E}') \simeq \tilde{H}^*(\dot{M}/\dot{D}) \simeq H^*(\dot{M}, \dot{D}).$$

Thus, $H^{2n-1}(\dot{M}', \dot{E}') \simeq H^{2n}(\dot{M}', \dot{E}') \simeq \mathbf{Z}$, and the other groups are zero. From the exact cohomology sequence of the pair $(\dot{M}', \dot{E}')$ we obtain $H^j(\dot{M}') \simeq H^j(\dot{E}) = 0$, $1 \le j \le 2n-3$, and

$$0 = H^{2n-2}(\dot{M}', \dot{E}') \longrightarrow H^{2n-2}(\dot{M}') \longrightarrow H^{2n-2}(\dot{E}') \simeq \mathbf{Z} \xrightarrow{\partial^*}$$

[13] More generally, one may show that if D is a non-compact connected closed subspace of a smooth connected manifold M , then the identity mapping of the complement $M \setminus D$ extends to a homeomorphism of Hausdorff compact spaces $\dot{M}/\dot{D} \xrightarrow{\approx} (M \setminus D)^{\cdot}$.

$$\longrightarrow H^{2n-1}(\dot{M}', \dot{E}') \simeq \mathbf{Z} \longrightarrow H^{2n-1}(\dot{M}') \longrightarrow H^{2n-1}(\dot{E}') = 0. \qquad (*)$$

By the Poincaré duality, we have

$$H_{2n-j}(M') = \tilde{H}^j(\dot{M}').$$

Hence, $H_i(M') = 0$ for $i \geq 3$, and $H_1(M') \simeq H_1(M) = 0$. Thus, by the Poincaré duality, $H^{2n-1}(\dot{M}') = 0$ in $(*)$, whence $\partial^* : H^{2n-2}(\dot{E}') \simeq \mathbf{Z} \longrightarrow H^{2n-1}(\dot{M}', \dot{E}') \simeq \mathbf{Z}$ is onto, and so, it is an isomorphism. This implies that $H^{2n-2}(\dot{M}') = 0$, and also, by the Poincaré duality, $H_2(M') = 0$. Finally, we have that $\tilde{H}_*(M') = 0$, which means that M' is acyclic.

Vice versa, assuming that M' is acyclic, we can prove that so is M repeating word-in-word the above arguments, but exchanging the roles of the pairs (M, D) and (M', E'). This completes the proof. □

Theorem 5.1. (Kaliman [Ka 2, Thm. 3.5]) *Suppose that (i) D is a topological manifold, and (ii) D and C are acyclic. Then M' is contractible iff M is.*

Proof. By the Theorems of Hurewicz and Whitehead, M resp. M' is contractible iff it is acyclic and simply connected. Thus, the statement follows immediately from Lemmas 5.1 and 5.2. □

Remark. It can be shown that M' as in Theorem 5.1 is an affine variety once so is M; see [Ka 2, Lemma 3.3].

Lemma 5.3. (Kaliman [Ka 2]) $\overline{k}(M') \geq \overline{k}(M)$.

Proof. Indeed, $M' = \widehat{M} \setminus D'$ implies $\overline{k}(M') \geq \overline{k}(\widehat{M})$. Since $\sigma_C : \widehat{M} \longrightarrow M$ is a proper birational morphism, by Proposition 3.1.(f), we get $\overline{k}(M') \geq \overline{k}(M)$, as claimed. □

Example 5.1. (Kaliman [Ka 2]) Let $X = S \times \mathbf{C}$ be an exotic $\mathbf{C}^3$, where S is a contractible surface of log-general type. Chose a finite sequence of points $\{(s_i, z_i)\}_1^n \subset X$ as the centers C_i of the Kaliman modifications along the fibers $H_i = S \times \{z_i\}$ of the second projection $X \longrightarrow \mathbf{C}$, $i = 1, \ldots, n$. Then $E'_i \simeq \mathbf{C}^2$, and one can show that E'_i, $i = 1, \ldots, n$, are the only copies of $\mathbf{C}^2$ in X'. Thus, X' is an exotic $\mathbf{C}^3$, and the positions of E'_i, $i = 1, \ldots, n$, in X' or, what is the same, the positions of the points $\{(s_i, z_i)\}_1^n \subset X$, up to automorphisms of X, provide deformation parameters.

6 The hyperbolic modification

Here we follow, up to minor changes, tom Dieck [tD 1] (cf. another treatment in Petrie [Pe]); but we restrict the consideration to the simplest possible case.

Definition 6.1. Let $h \in \mathbf{C}[x_1, \ldots, x_n]$ be an irreducible polynomial such that $h(\overline{0}) = 0$. Suppose that $\operatorname{grad}_{\overline{0}} h \neq \overline{0}$, and so the hypersurface $X = \{h = 0\} \subset \mathbf{C}^n$ is smooth at the origin. Define the *hyperbolic modification* q of h as follows:

$$q(\overline{x}, u) = \frac{h(u\overline{x})}{u} \in \mathbf{C}[x_1, \ldots, x_n, u].$$

Since $h(u\overline{x}) = uq(\overline{x}, u)$, we have the equalities

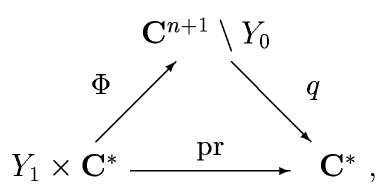
$$u\frac{\partial q}{\partial u}(\overline{x}, u) + q(\overline{x}, u) = \sum_1^n x_i \frac{\partial h(u\overline{x})}{\partial x_i},$$

$$\frac{\partial q}{\partial x_i}(\overline{x}, u) = \frac{\partial h(u\overline{x})}{\partial x_i}, \qquad i = 1, \ldots, n.$$

It follows that, once $(\overline{x}_0, u_0)$ is a critical point of q, i.e. $\operatorname{grad}_{(\overline{x}_0, u_0)} q = \overline{0}$, then also $\operatorname{grad}_{\overline{x}_0 u_0} h = \overline{0} = h(u_0\overline{x}_0)$, that is, $u_0\overline{x}_0 \in X$ is a singular point, and $(\overline{x}_0, u_0) \in Y_0 = \{q = 0\} \subset \mathbf{C}^{n+1}$. Thus, all the fibers $Y_c = \{q = c\}$, $c \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, of the polynomial q are smooth hypersurfaces, and the fiber $Y_0 = \{q = 0\}$ is smooth iff so is X, which will be assumed in the sequel. We denote $Y = Y_0$.

Lemma 6.1. *The restriction* $q \,|\, (\mathbf{C}^{n+1} \setminus Y) : \mathbf{C}^{n+1} \setminus Y \longrightarrow \mathbf{C}^*$ *is a trivial algebraic fiber bundle with the fiber* $Y_1 := \{q = 1\}$.

Proof. Consider the commutative triangle



where the map Φ is defined as follows:

$$(\overline{y}, \lambda) := ((\overline{x}, u), \lambda) \stackrel{\Phi}{\longmapsto} (\lambda\overline{x}, \lambda^{-1}u) := \overline{y}_\lambda \in Y_\lambda.$$

It is easy to check that Φ is a fibrewise (biregular) isomorphism, so we are done. $\square$

Define a $\mathbf{C}^*-$ action on $\mathbf{C}^{n+1} : (\lambda, (\overline{x}, u)) \stackrel{G_\lambda}{\longrightarrow} (\lambda\overline{x}, \lambda^{-1}u), \quad \lambda \in \mathbf{C}^*$. Then

$$q(G_\lambda(\overline{x}, u)) = \frac{h(u\overline{x})}{\lambda^{-1}u} = \lambda q(\overline{x}, u).$$

This means that q is a quasi-invariant of weight 1 of G. In particular, Y is invariant with respect to G, and $G_\lambda(Y_c) = Y_{\lambda c}$. In the above diagram this corresponds to the canonical $\mathbf{C}^*-$ action on the direct product, whence Φ is equivariant.

The monomials $ux_1, \dots, ux_n \in \mathbf{C}[x_1, \dots, x_n, u]$ are $G-$ invariants. It is easily seen that, in fact, $\mathbf{C}[x_1, \dots, x_n, u]^G = \mathbf{C}[ux_1, \dots, ux_n]$ (the algebra of $G-$ invariants). Hence, the *algebraic quotient* of $\mathbf{C}^{n+1}$ by this $\mathbf{C}^*-$ action is

$$\mathbf{C}^{n+1}//G \simeq \mathbf{C}^n = \text{spec } \mathbf{C}[x_1, \dots, x_n, u]^G .$$

The action of G on $\mathbf{C}^{n+1}$ is *hyperbolic*, that is, it has only one fixed point (the origin $\overline{0} \in \mathbf{C}^{n+1}$), and the weights at the origin $(1, \dots, 1, -1)$ are of different signs. The origin belongs to the closure of each orbit which is contained in the hyperplane $\{u = 0\}$, and of those in the axis $OU := \{\overline{0}\} \times \mathbf{C}$; all the other orbits are closed.

Denote by M the complement of the axis OU in $\mathbf{C}^{n+1}$. Then the $\mathbf{C}^*-$ action G restricts to M with closed orbits only. Let $\pi : M \longrightarrow T$ be the canonical morphism onto the orbit space (= the *geometric quotient*) $T = M/G$. Also, consider the morphism $\tau : M \longrightarrow \mathbf{C}^n$, $(\overline{x}, u) \longmapsto u\overline{x}$. Since τ is constant on any orbit, it factors as $\tau = \sigma \circ \pi$:

$$\begin{array}{ccc} & T & \\ {}^{\pi}\nearrow & & \searrow^{\sigma} \\ M & \xrightarrow{\quad\tau\quad} & \mathbf{C}^n . \end{array}$$

The restriction of π to the hypersurface $M \bigcap \{u = 0\} := \widehat{E} \simeq \mathbf{C}^n \setminus \{\overline{0}\}$ coincides with the standard projection $\mathbf{C}^n \setminus \{0\} \longrightarrow \mathbf{P}^{n-1}$, $\overline{x} \longmapsto \{\lambda \overline{x}\}_{\lambda \in \mathbf{C}^*}$. Set $\pi(\widehat{E}) = E \subset T$; thus, $\sigma(E) = \{\overline{0}\}$, i.e. E is the exceptional divisor of σ; it is straightforward that $\sigma | \, (T \setminus E) \, : \, T \setminus E \longrightarrow \mathbf{C}^n \setminus \{0\}$ is an isomorphism. Therefore, $\sigma : T \longrightarrow \mathbf{C}^n$ is the blow up of the origin.

Furthermore, $\pi(Y \bigcap M) := X'$ is the proper transform of X in T (indeed, Y is saturated by the orbits, whence $\pi(Y \bigcap M)$ is an irreducible closed hypersurface in T containing the proper transform $\sigma'(X)$). Therefore, $T \setminus X'$ is the Kaliman transform of $\mathbf{C}^n$ along X with center at the origin $\overline{0} \in \mathbf{C}^n$.

Lemma 6.2. *There is an isomorphism* $Y_1 \simeq T \setminus X'$.

Proof. Fix a point $y = (\overline{x}, u) \in Y_1$. Since q is a G-quasi-invariant of weight 1, $G_\lambda(y) \in Y_\lambda$, and hence, the orbit Gy of y meets Y_1 at y only. This means that the morphism $\pi | \, Y_1 : Y_1 \longrightarrow T \setminus X'$ is injective. On the other hand, any $G-$ orbit outside Y

meets Y_1; thus, this morphism is also surjective. Finally, a bijective morphism of smooth varieties is an isomorphism. $\square$

Corollary 6.1. *The hypersurface $Y_1 \subset \mathbf{C}^{n+1}$ is isomorphic to the Kaliman modification of $\mathbf{C}^n$ along the hypersurface $X \subset \mathbf{C}^n$ with center at the origin.*

Lemma 6.3. *The hypersurface $Y = Y_0 \subset \mathbf{C}^{n+1}$ is isomorphic to the Kaliman modification of $X \times \mathbf{C}$ along the hypersurface $X \times \{0\}$ with center at the point $(\overline{0}, 0) \in X \times \mathbf{C}$.*

Proof. The morphism

$$Z' := \mathbf{C}^{n+1} \xrightarrow{\sigma} \mathbf{C}^{n+1} =: Z, \quad (y_1, \dots, y_n, u) \longmapsto (uy_1, \dots, uy_n, u)\,,$$

is nothing but the Kaliman modification of Z along the hyperplane $H_0 := \{u = 0\}$ with center at the origin, and with the exceptional divisor $E' = \{u = 0\} \subset Z'$. Consider the natural embedding $i : X \times \mathbf{C} \hookrightarrow Z$. Set $\widehat{h}(\overline{x}, u) = h(\overline{x})$; then the image of i is the hypersurface $\widehat{h} = 0$ in $Z \simeq \mathbf{C}^{n+1}$.

We have

$$\widehat{h} \circ \sigma(\overline{y}, u) = h(u\overline{y}) = uq(\overline{y}, u).$$

Hence, for a point $(\overline{y}, u) \in Y$ (i.e. such that $q(\overline{y}, u) = 0$), we get $\widehat{h} \circ \sigma(\overline{y}, u) = 0$, i.e. $\sigma(\overline{y}, u) \in X \times \mathbf{C}$, and so $\sigma(Y) \subset X \times \mathbf{C}$. Furthermore, the total preimage of $X \times \mathbf{C}$ in Z' is the union of Y and of the exceptional divisor $E = \{u = 0\}$. Therefore, Y is the proper transform of $X \times \mathbf{C}$ in Z', and the assertion follows. $\square$

Remark. The $\mathbf{C}^*-$ action $\lambda(y_1, \dots, y_n, u) = (\lambda y_1, \dots, \lambda y_n, \lambda^{-1} u)$ on Z' provides the $\mathbf{C}^*-$ action $\lambda(x_1, \dots, x_n, u) = (x_1, \dots, x_n,\, \lambda^{-1} u)$ on Z and on $X \times \mathbf{C}$.

Exercise (6.1) Show that, under the embedding $\mathbf{C}^n \hookrightarrow \mathbf{C}^{n+1}$ given as $\overline{x} \longmapsto (\overline{x}, 1)$, X is naturally isomorphic to the hyperplane section $Y \bigcap H_1$, where $H_1 := \{u = 1\} \subset \mathbf{C}^{n+1}$. Furthermore, show that the exceptional divisor $E \subset Y$ of the Kaliman transform $\sigma : Y \longrightarrow X \times \mathbf{C}$ coincides with the linear subspace $Y \bigcap \{u = 0\}$. If $\sigma' : Y \longrightarrow X$ is composed of the contraction σ and the first projection, verify that

$$\sigma' : (\overline{x}, u) \longmapsto (\frac{\overline{x}}{u}, 1) \in Y \bigcap H_1 \simeq X$$

outside E , and $\sigma'(\overline{x}, 0) = (\overline{0}, 1)$ on E . Deduce that Y is the closure in $\mathbf{C}^{n+1}$ of the $\mathbf{C}^*-$ orbit of the subvariety $Y \bigcap H_1 \simeq X$.

Theorem 6.1. (tom Dieck [tD 1]) *Let $X \subset \mathbf{C}^n$ be a smooth contractible hypersurface given by an irreducible polynomial $h \in \mathbf{C}[x_1, \dots, x_n]$, $h(\overline{0}) = 0$. Then any fiber $Y_c = q^{-1}(c)$, $c \in \mathbf{C}$, of the hyperbolic modification $q(\overline{x}, u) = \frac{h(u\overline{x})}{u} \in \mathbf{C}[x_1, \dots, x_n, u]$ of h is a smooth contractible hypersurface in $\mathbf{C}^{n+1}$. Thus, $q : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ yields a foliation of $\mathbf{C}^{n+1}$ by smooth contractible hypersurfaces.*

Proof. Indeed, by Lemma 6.1, $Y_c \simeq Y_1$ for $c \neq 0$. By Corollary 6.1 and Lemma 6.3, $Y = Y_0$ and Y_1 are both Kaliman modifications of (triples of) smooth contractible varieties. By Kaliman's Theorem 5.1, Y and Y_1 are contractible. □

Remark 1. The inequality $\overline{k}(M') \geq \overline{k}(M)$ of Lemma 5.3 does not provide here a useful information; indeed, $\overline{k}(\mathbf{C}^n) = \overline{k}(X \times \mathbf{C}) = -\infty$. However, sometimes the *intermediate Eisenman-Kobayashi intrinsic measures* serve as appropriate analytic invariants (see Kaliman [Ka 2]).

Remark 2. The Kaliman Theorem 5.1 is still applied if X is only assumed being a contractible topological manifold smooth at the origin (and $C \subset \operatorname{reg} H$). In this case we still have that all the hypersurfaces Y_c, $c \neq 0$, are smooth and contractible, but the central fiber $Y = Y_0$ can be singular, as it is in the following example.

Example 6.1. (Petrie, tom Dieck [PtD 2]; see Example 3.1 above). Up to automorphisms of $\mathbf{C}^2$, $\Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbf{C}^2$, $(k, l) = 1$, $k > l \geq 2$, are the only contractible irreducible singular affine plane curves [LiZa] (see Theorem 3.4 above). Starting with $\Gamma_{k,l}$, perform the hyperbolic modification at the smooth point $(1, 1) \in \Gamma_{k,l}$. We obtain a foliation $p_{k,l} : \mathbf{C}^3 \longrightarrow \mathbf{C}$ of $\mathbf{C}^3$ by the fibers of the polynomial

$$p_{k,l} := \frac{(xz+1)^k - (yz+1)^l}{z} \in \mathbf{C}[x, y, z].$$

All of them are irreducible contractible surfaces; all but the central one $p_{k,l}^{-1}(0)$ are smooth. One can see that $\overline{k}(X_{k,l}) = 1$, where $X_{k,l} := p_{k,l}^{-1}(1)$ (see Exercise 6.2 below). Now, starting with $X_{k,l}$, by means of hyperbolic modifications one can construct non-trivial foliations of $\mathbf{C}^4$, $\mathbf{C}^5$, etc. by smooth contractible hypersurfaces. Moreover, the corresponding polynomials are quasi-invariants of hyperbolic $\mathbf{C}^*-$ actions on $\mathbf{C}^n$. In particular, for $n \geq 4$ the zero fiber of such a polynomial is a smooth contractible hypersurface in $\mathbf{C}^n$ endowed with a hyperbolic $\mathbf{C}^*-$ action. Furthermore, one can obtain new exotic $\mathbf{C}^{n-1}$ -s by passing to cyclic $\mathbf{C}^*-$ coverings over such a hypersurface (see the next section).

Exercise (6.2) Verify that $\overline{k}(X_{k,l}) = 1$, where $(k, l) = 1$, $k > l \geq 2$.

Indication. One can proceed, for instance, as follows. Lifting the meromorphic function x^k/y^l on $\mathbf{C}^2$ to the function $f := (xz+1)^k/(yz+1)^l \mid X_{k,l}$ on $X_{k,l}$, we obtain a $\mathbf{C}^*$– fibration $f : X_{k,l} \to \mathbf{P}^1$. Hence, by Iitaka's Easy Addition Theorem (Proposition 3.1(c)), $\overline{k}(X_{k,l}) \le 1$. Since $X_{k,l}$ is acyclic, by the Classification Theorem 3.3(b), $\overline{k}(X_{k,l}) = 1$ as soon as $X_{k,l} \not\simeq \mathbf{C}^2$. Recall that the surface $X_{k,l}$ is the Kaliman modification of $\mathbf{C}^2$ along the curve $\Gamma_{k,l} \subset \mathbf{C}^2$ with center at the point $(1,1) \in \Gamma_{k,l}$. Resolving singularities of the plane projective curve $\overline{\Gamma}_{k,l} \cup l_\infty \subset \mathbf{P}^2$ and blowing up at the point $(1,1) \in \overline{\Gamma}_{k,l}$, we obtain a completion $V_{k,l}$ of $X_{k,l}$. Contracting, if necessary, the (-1)– boundary components of valence at most two in the dual graph, we come to a minimal completion $V_{k,l}^{\min}$ of $X_{k,l}$. The dual graph of its boundary divisor $D_{k,l}^{\min}$ is non-linear (what is this graph?). Therefore, by the Ramanujam Theorem 2.2(a), $X_{k,l} \not\simeq \mathbf{C}^2$.

7 Cyclic $\mathbf{C}^*$– coverings

Definition 7.1. (cf. [KoRu 2, Prop. 2.11]) Let X be an affine variety, and let $q \in \mathbf{C}[X]$ be a regular function which defines an effective divisor $F_0 = q^*(0)$ on X. Fix an integer $s > 1$. The variety $Y_s = \{(x,u) \in X \times \mathbf{C} \mid q(x) = u^s\}$ together with the projection $\varphi_s : Y_s \longrightarrow X$, $(x,u) \overset{\varphi_s}{\longmapsto} x$, yields a *cyclic covering of* X *branched to order* s *along* F_0. We suppose that F_0 is smooth and reduced; then Y_s is also smooth (indeed, grad $_{(x,u)}(q(x) - u^s) = (\text{grad}\,_x q, -su^{s-1})$), as well as the hypersurface $F_{s,0} := \varphi_s^{-1}(F_0)$ in Y_s (*exercise*).

If X is endowed with a regular $\mathbf{C}^*$– action $t : \mathbf{C}^* \times X \longrightarrow X$, and q is a quasi-invariant of t of weight d, i.e.

$$q(t_\lambda x) = \lambda^d q(x),$$

where $d \in \mathbf{Z}$, then the $\mathbf{C}^*$– action $\lambda(x,u) = (\lambda^s(x), \lambda^d u)$ on $X \times \mathbf{C}$ restricts to Y_s making the following commutative diagram equivariant

$$\begin{array}{ccc} Y_s & \xrightarrow{(x,u) \mapsto x} & X \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{q}} \\ \mathbf{C} & \xrightarrow{u \mapsto u^s} & \mathbf{C}\,, \end{array}$$

where the original $\mathbf{C}^*$– action G on X is replaced by its 's-th power' $(\lambda, x) \longmapsto \lambda^s(x) := t(\lambda^s, x)$. Indeed, we have for $(x,u) \in Y_s$:

$$q(\lambda^s(x)) = \lambda^{sd} q(x) = \lambda^{sd} u^s = (\lambda^d u)^s,$$

whence $(\lambda^s(x),\ \lambda^d u) \in Y_s$, which shows that the above diagram is equivariant.

If, in addition, $(d,s)=1$, then the monodromy of the cyclic covering $\varphi_s : Y_s \to X$ is represented via the action on Y_s of the subgroup $\omega_s \subset \mathbf{C}^*$ of the $s-$ th roots of unity. Indeed, the ω_s- orbit of a point (x,u) in Y_s is

$$\omega_s(x,u) = \{(x, \lambda^d u) \mid \lambda^s = 1\} = \varphi_s^{-1}(x),$$

since $(s,d)=1$. The fixed point set $Y_s^{\omega_s} = \{(x,u) \in Y_s \mid u = 0\}$ of the monodromy action on Y_s can be identified with the hypersurface $F_0 \subset X$. Thus, we get $X = Y_s/\omega_s$ with the quotient action of $\mathbf{C}^*/\omega_s \simeq \mathbf{C}^*$ on X.

The equivariant covering $Y_s \longrightarrow X$ as above is called a *cyclic* $\mathbf{C}^*-$ *covering.*

Remarks. 1. The action of the monodromy group $\omega_s \simeq \mathbf{Z}/s\mathbf{Z}$ on Y_s is homologically trivial. Indeed, this is so for the continuous group action $\mathbf{C}^* \supset \omega_s$ on Y_s.

2. The above observations are equally applied in the more general setting when the regular $\mathbf{C}^*-$ action is only given on the Zariski open subset $X^* := X \setminus F_0$ of X. In particular, if $(d,s)=1$, then the monodromy group ω_s of the cyclic covering $\varphi_s : Y_s^* \to X^*$, where $Y_s^* := Y_s \setminus \varphi_s^{-1}(F_0)$, acts trivially in the homology $H_*(Y_s^*;\, \mathbf{Z})$.

The aim of this section is the following result due to Kaliman; it is a generalization of Theorem A in[14] [Ka 1].

Theorem 7.1. (Kaliman) *Let X be a smooth contractible affine variety, and let a smooth reduced irreducible divisor $F_0 = q^*(0) \subset X$ be the zero fiber of a regular function $q \in \mathbf{C}[X]$. Denote $G = \pi_1(X \setminus F_0)$, and fix a vanishing loop $\alpha \in G$ of F_0. Assume that*

($\sharp$) q is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbf{C}^*-$ action defined on $X \setminus F_0$.

($\sharp_1$) For an integer $s > 0$ such that $(s,c) = (s,d) = 1$, the hypersurface F_0 is $\mathbf{Z}_p-$ acyclic[15] *for each prime divisor p of s.*

($\sharp_2$) For some integer $c \neq 0$, α^c is an element of the center $Z(G)$ of the group G.

Consider the cyclic covering $\varphi_s : Y_s \to X$ branched to order s along F_0. Then Y_s is a smooth contractible affine variety.

Due to the Theorems of Hurewicz and Whitehead, it is enough to show that Y_s is acyclic and simply connected. This is done, respectively, in Theorems 7.2 and 7.3 below.

[14] Exposing this result in [Za 5, Thm. 6.9], the condition ($\sharp_2$) below has been missed. It should be used in the proof of this theorem instead of Lemma 6.8 in [Za 5], which is wrong; see the proof of Theorem 7.3 below.

[15] hereafter $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$.

Notice that the conditions $\sharp$ and $\sharp_1$ guarantee acyclicity of Y_s, whereas the condition $\sharp_2$ provides its simply connectedness.

7.1 Acyclicity of cyclic $\mathbf{C}^*-$ coverings: elements of Smith's Theory

Theorem 7.2. (Kaliman [Ka 1]; tom Dieck [tD 2]) *Let X be an acyclic smooth affine variety, and $F_0 = q^*(0)$, where $q \in \mathbf{C}[X]$, be a smooth reduced irreducible divisor in X. Consider a cyclic covering $\varphi_s : Y_s \to X$ branched to order s along F_0. Assume that*

($\sharp$) q is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbf{C}^*-$ action defined on $X \setminus F_0$.

($\sharp_1$) For an integer $s > 0$ such that $(s, d) = 1$, the hypersurface F_0 is $\mathbf{Z}_p-$ acyclic for each prime divisor p of s.

Then Y_s is acyclic, too.

Before proving Theorem 7.2, we recall the Smith theory (see [Bre, Ch.III]).

Elements of Smith's Theory. Consider a finite simplicial polyhedron Y endowed with a simplicial action of a finite group ω. Usually, passing to the second barycentric subdivision, one obtains some additional regularity properties of the action, which are always to be assumed (see [Bre, III.1]). Let k be a field, and let $\mathbf{Z}[\omega]$, $\mathbf{Z}_p[\omega]$, $k[\omega]$ be the group rings of ω (e.g. $\mathbf{Z}[\omega] = \{\sum_{g\in\omega} n_g g \mid n_g \in \mathbf{Z}\}$ with natural ring operations). The simplicial chain complexes $C(Y), C(Y) \otimes \mathbf{Z}_p, C(Y) \otimes k$ are, respectively, $\mathbf{Z}[\omega]-$, $\mathbf{Z}_p[\omega]-$, $k[\omega]-$ modules (indeed, given a simplex δ of Y, we set

$$(\sum n_g g)(\delta) = \sum n_g g(\delta) \in C(Y)).$$

In the sequel, ω is assumed to be a finite cyclic group $\mathbf{Z}_s = \mathbf{Z}/s\mathbf{Z}$ acting on Y in such a way that the fixed point set Y^ω of ω coincides with the individual fixed point set Y^g for every $g \in \omega$, $g \neq e$. In particular, the $\omega-$ action on the complement $Y \setminus Y^\omega$ is free. We denote $X = Y/\omega$ the orbit space, $\pi : Y \longrightarrow X$ the natural projection, and we identify Y^ω with its image in X. Consider the following three homomorphisms of chain complexes:

$$\pi_* : C(Y) \longrightarrow C(X),$$

$$\sigma : C(Y) \longrightarrow C(Y), \qquad \sigma = \sum_{g\in\omega} g \in \mathbf{Z}[\omega]\,,$$

$$\mu = \pi^* : C(X) \longrightarrow C(Y), \qquad \mu(\delta) = \pi^{-1}(\delta) \text{ if } \delta \cap Y^\omega = \emptyset; \qquad \mu(\delta) = \sigma(c) \text{ if } \pi_*(c) = \delta$$

(note that π_* is surjective). Then we have [Bre, III.2] Ker $\pi_* =$ Ker σ, and there is an isomorphism

$$\sigma C(Y) \simeq C(Y)/\text{Ker } \sigma \simeq C(Y)/\text{Ker } \pi_* = C(X),$$

whence $\mu\pi_* = \sigma$. But $\pi_*\mu(c) = |\omega|\pi_*(c)$. On the homology level, this leads to the following assertions.

Lemma 7.1. [Bre, III(2.2), (2.3)]

$$\pi_*\mu_* = |\omega| : H_*(X) \longrightarrow H_*(X),$$

$$\mu_*\pi_* = \sigma_* = \sum_{g\in\omega} g_* : H_*(Y) \longrightarrow H_*(Y).$$

Here μ_* *is called a* transfer. *On the invariant part of homology we have*

$$\mu_*\pi_*|H_*(Y)^\omega = |\omega| : H_*(Y)^\omega \longrightarrow H_*(Y)^\omega.$$

This implies

Corollary 7.1. [Bre, III(2.4)] *If* k *is a field of characteristic* char $k = q$ *with* $(q, |\omega|) = 1$, *then*

$$\pi_*|H_*(Y;k)^\omega : H_*(Y;k)^\omega \longrightarrow H_*(X;k)$$

is an isomorphism, and its inverse is the transfer μ_*. *Moreover,*

$$H_*(Y;k) = \mu_*H_*(X;k) \oplus \text{Ker } \pi_*,$$

where Ker $\pi_* =$ Ker σ_*.

Corollary 7.2. *Suppose that* ω *acts trivially in homology:*

$$\omega_*|H_*(Y) = \text{id}\,.$$

Then, for any field k *with* $(\text{char } k, |\omega|) = 1$, *we have the isomorphism of transfer*

$$\pi_* = \mu_*^{-1} : H_*(Y;k) \xrightarrow{\simeq} H_*(X;k).$$

In particular, if $\omega \simeq \mathbf{Z}_q$, *where* q *is a prime number, then the elements of the kernel and of the cokernel of the homomorphism* $\pi_* : H_*(Y;\mathbf{Z}) \longrightarrow H_*(X;\mathbf{Z})$ *are torsions of order* q.

The last assertion follows by the Universal Coefficient Formula:

$$\tilde{H}_j(Y;\mathbf{Z}_q) = \tilde{H}_j(Y;\mathbf{Z}) \otimes \mathbf{Z}_q \oplus \text{Tor } (H_{j-1}(Y;\mathbf{Z});\mathbf{Z}_q) \qquad \forall j.$$

Definition 7.2. [Bre, III.3] In what follows $\omega \simeq \mathbf{Z}_p$ is a multiplicative cyclic group of prime order p with a generator $t \in \omega$, so that $\sigma = 1 + t + \ldots + t^{p-1} \in \mathbf{Z}_p[\omega]$. We set $\tau = 1 - t \in \mathbf{Z}_p[\omega]$. We have $t^p = 1$, $\sigma\tau = \tau\sigma = 0$, and $\sigma = \tau^{p-1}$ (indeed, $(-1)^i \binom{p-1}{i} \equiv 1 \mod p$). For an element $\rho = \rho_i := \tau^i \in \mathbf{Z}_p$ set $\overline{\rho} = \tau^{p-i}$; then $\overline{\sigma} = \tau$ and $\overline{\tau} = \sigma$. Given $\rho = \rho_i = \tau^i$, consider the chain complex $\rho C(Y; \mathbf{Z}_p)$. Its graded homology group $H_*^\rho(Y; \mathbf{Z}_p) := H_*(\rho C(Y; \mathbf{Z}_p))$ is called the *special Smith's homology group.*

There are the exact sequences of chain complexes with coefficients in $\mathbf{Z}_p$ [Bre, III(3.1),(3.8)]:

$$0 \longrightarrow \overline{\rho} C(Y) \oplus C(Y^\omega) \xrightarrow{\ i\ } C(Y) \xrightarrow{\ \rho\ } \rho C(Y) \longrightarrow 0\,,$$

$$0 \longrightarrow \sigma C(Y) \xrightarrow{\ i\ } \tau^j C(Y) \xrightarrow{\ \tau\ } \tau^{j+1} C(Y) \longrightarrow 0, \ \ j = 1, \ldots, p-1\,.$$

Besides, the kernels of the homomorphism $\sigma : C(Y;\, \mathbf{Z}_p) \to C(Y;\, \mathbf{Z}_p)$ and of the composition $C(Y;\, \mathbf{Z}_p) \to C(Y,\, Y^\omega;\, \mathbf{Z}_p) \to C(X,\, Y^\omega;\, \mathbf{Z}_p)$, where Y^ω is indentified with its image in X, are the same [Bre, p. 124]. These observations lead to the following

Proposition 7.1. [Bre, III(3.3),(3.4),(3.8)] *For the homology groups with* $\mathbf{Z}_p$ *coefficients, one has*

(a) an isomorphism $H_*^\sigma(Y) \simeq H_*(X; Y^\omega)$ *,*

and the following two Smith's exact homology sequences*:*

(b)

$$\begin{array}{ccc} & H_*(Y) & \\ {\scriptstyle \rho_*}\swarrow & & \nwarrow {\scriptstyle i_*} \\ H_*^\rho(Y) & \xrightarrow{\quad \delta_* \quad} & H_*^{\overline{\rho}}(Y) \oplus H_*(Y^\omega) \end{array} \ ,$$

(c)

$$\begin{array}{ccc} & H_*^{\rho_j}(Y) & \\ {\scriptstyle \tau_*}\swarrow & & \nwarrow {\scriptstyle i_*} \\ H_*^{\rho_{j+1}}(Y) & \xrightarrow{\quad \delta_* \quad} & H_*^\sigma(Y) \end{array} \ ,$$

where $\deg\, \rho_* = \deg\, \tau_* = \deg\, i_* = 0, \deg\, \delta_* = -1.$

Proposition 7.2. *Suppose that*[16]

[16] The numeration of the conditions that we use here agrees with those in the next Corollary and Exercise.

(ii) the fixed point set Y^ω *is non-empty and* $\mathbf{Z}_p$ *–acyclic:* $\tilde{H}_*(Y^\omega;\, \mathbf{Z}_p) = 0$, *and*

(iii) $X = Y/\omega$ *is* $\mathbf{Z}_p$ *–acyclic:* $\tilde{H}_*(X;\, \mathbf{Z}_p) = 0$.

Then also Y *is* $\mathbf{Z}_p$ *–acyclic:* $\tilde{H}_*(Y;\, \mathbf{Z}_p) = 0$.

Proof. In view of the vanishing

$$\tilde{H}_*(X;\mathbf{Z}_p) = \tilde{H}_*(Y^\omega;\mathbf{Z}_p) = 0\,,$$

from the usual exact homology sequence of a pair

$$\dots \xrightarrow{i_*} H_j(X;\mathbf{Z}_p) \xrightarrow{r_*} H_j(X,\, Y^\omega;\mathbf{Z}_p) \xrightarrow{\delta_*} H_{j-1}(Y^\omega;\mathbf{Z}_p) \xrightarrow{i_*} \dots$$

it follows that $H_*(X,\, Y^\omega;\mathbf{Z}_p) = 0$, and thus, by Proposition 7.1(a), also $H^\sigma_*(Y;\mathbf{Z}_p) = 0$. Therefore, by the Smith's exact sequence (c), $H^\rho_*(Y;\mathbf{Z}_p) = 0\,\forall\rho = \rho_j,\, j = 1,\dots,p-1$. Now, by the Smith's exact sequence (b), $\tilde{H}_*(Y;\mathbf{Z}_p) \simeq H^\rho_*(Y;\mathbf{Z}_p) = 0$. □

Corollary 7.3. *Suppose that*

(i) $\omega \simeq \mathbf{Z}_p$ *acts trivially in homology:* $\omega_*|H_*(Y) = \mathrm{id}$,

(ii) the fixed point set Y^ω *is non-empty and* $\mathbf{Z}_p$ *–acyclic:* $\tilde{H}_*(Y^\omega;\, \mathbf{Z}_p) = 0$, *and*

(iii) $X = Y/\omega$ *is acyclic:* $\tilde{H}_*(X;\, \mathbf{Z}) = 0$.

Then also Y *is acyclic:* $\tilde{H}_*(Y;\, \mathbf{Z}) = 0$.

Proof. By Corollary 7.2, $H_*(Y;\, \mathbf{Z}_q) \simeq H_*(X;\, \mathbf{Z}_q) = 0$ for any prime $q \neq p$. By Proposition 7.2, also $H_*(Y;\, \mathbf{Z}_p) = 0$. Thus, by the Universal Coefficient Formula, $\tilde{H}_*(Y;\, \mathbf{Z}) \otimes \mathbf{Z}_q = 0$ for all prime q. Then $\tilde{H}_*(Y;\, \mathbf{Z}) = 0$. □

Exercise (7.1) Assume that $\omega \simeq \mathbf{Z}_s$ acts on Y in such a way that

(0) $Y^g = Y^\omega \neq \emptyset$ for every $g \in \omega$, $g \neq e$;

(i) the action is homologically trivial, i.e. $\omega_*|H_*(Y;\mathbf{Z}) = \mathrm{id}$;

(ii) the fixed point set Y^ω is $\mathbf{Z}_p-$ acyclic for any prime divisor p of s;

(iii) $X = Y/\omega$ is acyclic: $\tilde{H}_*(X;\, \mathbf{Z}) = 0$.

Show that Y is acyclic, too: $\tilde{H}_*(Y;\, \mathbf{Z}) = 0$.

Remark. Assume, for a moment, that the $\mathbf{C}^*-$ action in Theorem 7.2 is regular on the whole X. Then the monodromy action on Y_s is homologically trivial (see the remark preceding Theorem 7.1), that is, the above condition (i) is fulfilled. This provides a proof of Theorem 7.2 in that case. Notice that this proof does not use the assumption of smoothness of X and F_0. In general case, following tom Dieck [tD 2], we need to consider branched coverings over smooth varieties and to use the Thom classes.

Thom's classes and Thom's isomorphisms. Recall the following definitions and facts (see e.g. [Do, VIII.11], [MilSta, §9, §10]). Consider an oriented smooth connected manifold X and a codimension 2 closed oriented submanifold F_0 of X. Let $N \to F_0$ be the (oriented) normal bundle of F_0 in X with the zero section $Z_0 \simeq F_0$, and let $U \subset X$ be a tubular neighborhood of F_0 in X such that the pair $(U,\, F_0)$ is diffeomorphic to the pair $(N,\, Z_0)$. Denote $U^* := U \setminus F_0$ and $N^* := N \setminus Z_0$. By excision, we have the isomorphisms $\tilde{H}_*(X,\, X^*;\, \mathbf{Z}) \simeq \tilde{H}_*(U,\, U^*;\, Z) \simeq \tilde{H}_*(N,\, N^*;\, \mathbf{Z})$, and similarly for the cohomology groups. The *Thom class* $t(F_0) \in H^2(X,\, X^*;\, \mathbf{Z}) \simeq H^2(N,\, N^*;\, \mathbf{Z})$ is a unique class which takes the value 1 on any oriented relative two-cycle $(F,\, F^*) \in H_2(N,\, N^*;\, \mathbf{Z})$ defined by a fiber F of the normal bundle N.

The cup-product with the Thom class $t(F_0) \in H^2(X,\, X^*;\, \mathbf{Z}_q)$ yields the *Thom isomorphism*[17]

$$\tilde{H}_i(X,\, X^*;\, \mathbf{Z}_q) \simeq H_{i-2}(F_0;\, \mathbf{Z}_q),\ i = 0,\, 1, \dots .$$

Let $\varphi_s : Y_s \to X$ be a smooth cyclic ramified covering of X branched to order s along F_0, i.e. Y_s is an oriented manifold equipped with an action of a group $\omega \simeq \mathbf{Z}_s$ of orientation preserving diffeomorphisms; the fixed point set $Y_s^\omega \subset Y_s$ is a codimension 2 closed oriented submanifold; ω acts freely in the complement $Y_s \backslash Y_s^\omega$, and $\varphi_s : Y_s \to X$ is the orbit map, which provides a natural identification of Y_s^ω with $F_0 \subset X$.

Note that under the assumption ($\sharp$) of Theorem 7.2, the monodromy group $\omega \simeq \mathbf{Z}_s$ acts trivially in the homology $H_*(Y_s \backslash F_0;\, \mathbf{Z})$. Thus, the next proposition[18] yields Theorem 7.2.

Proposition 7.3. [tD 2, Thm. 2.9] *Let, in the notation as above, $\varphi_s : Y_s \to X$ be a smooth cyclic ramified covering of X branched to order s along F_0. Suppose that*

(i) the covering group $\omega \simeq \mathbf{Z}_s$ acts trivially in the homology of the complement $Y_s \backslash Y_s^\omega$: $\omega_* | H_*(Y_s \setminus Y_s^\omega; \mathbf{Z}) = \mathrm{id}$;

(ii) the fixed point set Y_s^ω is $\mathbf{Z}_p-$ acyclic for any prime divisor p of s;

(iii) $X = Y_s/\omega$ is acyclic: $\tilde{H}_*(X;\, \mathbf{Z}) = 0$.

Then Y_s is acyclic, too: $\tilde{H}_*(Y_s;\, \mathbf{Z}) = 0$.

Proof. Assume, for simplicity, that $s = p$ is a prime number (the general case can be reduced to this one; cf. Exercise 7.1 above). By Proposition 7.2, we have $\tilde{H}_*(Y_p;\, \mathbf{Z}_p) = 0$.

[17] the homology groups with negative indices are considered being zero.

[18] cf. Corollary 7.3.

By the Universal Coefficient Formula, it suffices to prove that $\tilde{H}_*(Y_p;\, \mathbf{Z}_q) = 0$ (i.e. Y_p is $\mathbf{Z}_q-$ acyclic) for any prime $q \neq p$.

Denote $Y_p^* = Y_p \setminus Y_p^\omega$ and $X^* = X \setminus F_0$. The restriction $\pi : Y_p^* \to X^*$ is a non-ramified cyclic covering of order p. By Corollary 7.1, $(\varphi_s)_* : H_*(Y_p^*;\, \mathbf{Z}_q) \to H_*(X^*;\, \mathbf{Z}_q)$ is an isomorphism for any prime $q \neq p$.

We have the Thom isomorphisms

$$\tilde{H}_i(X,\, X^*;\, \mathbf{Z}_q) \simeq H_{i-2}(F_0;\, \mathbf{Z}_q), \qquad \tilde{H}_i(Y_p,\, Y_p^*;\, \mathbf{Z}_q) \simeq H_{i-2}(Y_p^\omega;\, \mathbf{Z}_q),$$

given by cup-products with the Thom classes $t(F_0) \in H^2(X,\, X^*;\, \mathbf{Z}_q)$ resp. $t(Y_p^\omega) \in H^2(Y_p,\, Y_p^*;\, \mathbf{Z}_q)$. It is easily seen that $(\varphi_s)^*(t(F_0)) = p \cdot t(Y_p^\omega)$. Since the multiplication by p is an invertible operation in $\mathbf{Z}_q-$ (co)homology for $q \neq p$, it follows that $(\varphi_s)_* : H_*(Y_p,\, Y_p^*;\, \mathbf{Z}_q) \longrightarrow H_*(X,\, X^*;\, \mathbf{Z}_q)$ is an isomorphism.

Consider the following commutative diagram, where the horizontal lines are exact homology sequences of pairs with $\mathbf{Z}_q-$ coefficients:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & H_{j+1}(Y_p,\, Y_p^*) & \longrightarrow & H_j(Y_p^*) & \longrightarrow & H_j(Y_p) & \longrightarrow & H_j(Y_p,\, Y_p^*) & \longrightarrow & H_{j-1}(Y_p^*) & \longrightarrow \ldots \\
& \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq & \\
\ldots \longrightarrow & H_{j+1}(X,\, X^*) & \longrightarrow & H_j(X^*) & \longrightarrow & H_j(X) & \longrightarrow & H_j(X,\, X^*) & \longrightarrow & H_{j-1}(X^*) & \longrightarrow \ldots
\end{array}$$

By the above observations, we may conclude that the four indicated vertical arrows are isomorphisms induced by the projection φ_s. By the 5-lemma, the middle vertical arrow is an isomorphism, too. Hence, since X is acyclic, $\tilde{H}_*(Y_p;\, \mathbf{Z}_q) \simeq \tilde{H}_*(X;\, \mathbf{Z}_q) = 0$ for any prime q. This yields the assertion. □

Thus, the proof of Theorem 7.2 is completed.

Example 7.1. Let X be a smooth acyclic surface, $F_0 = q^*(0)$ be a smooth reduced irreducible contractible curve in[19] X, where $q \in \mathbf{C}[X]$ is a quasi-invariant of weight $d \neq 0$ of a regular $\mathbf{C}^*-$ action on $X \setminus F_0$. Then by Theorem 7.2, $Y_s := \{z^s = q(x)\} \subset X \times \mathbf{C}$, where $(d,\, s) = 1$, is a smooth acyclic surface, too.

For instance, for $k,\, l,\, s$ pairwise relatively prime, the surface $Y_{k,\,l,\,s} \subset \mathbf{C}^3$ given by the equation

$$\frac{(xz^s + 1)^k - (yz^s + 1)^l}{z^s} = 1$$

[19] see Theorems 3.3, 3.4 above for a description of such pairs $(X,\, F_0)$.

is a smooth acyclic one, and $\overline{k}(Y_{k,l,s}) = 1$ (cf. [tD 2]; see Examples 3.1, 6.1 and Exercise 6.2 above). Indeed, there is a cyclic $\mathbf{C}^*-$ covering $Y_{k,l,s} \to X_{k,l}$ branched to order s along the curve $L_{k,l} := X_{k,l} \cap \{z = 0\} \simeq \mathbf{C}$ in $X_{k,l}$, where the $\mathbf{C}^*-$ action in $X_{k,l} \backslash L_{k,l}$ is induced via the isomorphism $X_{k,l} \backslash L_{k,l} \simeq \mathbf{C}^2 \backslash \Gamma_{k,l}$ by the linear $\mathbf{C}^*-$ action $(\lambda, (x, y)) \longmapsto (\lambda^l x, \lambda^k y)$ on $\mathbf{C}^2$. Thus, we may apply Theorem 7.2 to show that the surface $Y_{k,l,s}$ is acyclic.

7.2 Simply connectedness of cyclic $\mathbf{C}^*-$ coverings

Theorem 7.3. (Kaliman) *Let X be a simply connected smooth affine variety, and $F_0 = q^*(0)$, where $q \in \mathbf{C}[X]$, be a smooth reduced irreducible divisor in X. Fix a vanishing loop $\alpha \in G := \pi_1(X \setminus F_0)$ of F_0. Consider a cyclic covering $\varphi_s : Y_s \to X$ branched to order s along F_0. Assume that*

($\sharp_2$) For an integer $c \neq 0$ such that $(s, c) = 1$, α^c is an element of the center $Z(G)$ of the group G.

Then Y_s is simply connected, too.

Remark. In [Ka 1, Lemmas 7 and 8] conditions on a polynomial $q \in \mathbf{C}[x_1, \dots, x_n]$ are given which ensure that $\pi_1(\mathbf{C}^n \setminus F_0) \simeq \mathbf{Z}$. In particular, repeating word-in-word the proof of Lemma 8 in [Ka 1] (based on the Seifert-van Kampen Theorem) one can easily see that $\pi_1(X \setminus F_0) \simeq \mathbf{Z}$ if F_0 is a generic fibre of a regular function q on a simply connected smooth affine variety X, that is, the restriction of q onto a preimage $q^{-1}(\Delta_\epsilon)$ of a small disc $\Delta_\epsilon \subset \mathbf{C}$ centered at the origin yields a smooth fibre bundle over Δ_ϵ. Thus, in this case also the assumption ($\sharp_2$) of Theorem 7.3 holds.

We need the following definition.

Definition 7.3. We say that a subgroup H of a group G is *normally generated by elements* $a_1, \dots, a_n \in H$ if it is generated by the set of all elements conjugate with $a_1, \dots, a_n$, i.e. if H is the minimal normal subgroup of G which contains $a_1, \dots, a_n$. We denote it by $<< a_1, \dots, a_n >>$. G is said to be *normally one–generated* if $G = << a >>$ for some element $a \in G$.

Lemma 7.2. *Let X be a smooth irreducible affine variety, and $F_0 = q^*(0)$ be a reduced irreducible divisor in X, where $q \in \mathbf{C}[X]$. Fix a vanishing loop $\alpha \in G := \pi_1(X \setminus F_0)$ of F_0. We have:*

(a) $\pi_1(X) = \mathbf{1}$ *iff* $G = << \alpha >>$.

(b) Let $\varphi_s : Y_s \to X$ *be a cyclic covering branched to order* s *along* F_0. *Set* $\widehat{G}_s =<< \alpha^s >>$. *Assume that* F_0 *is a smooth divisor. Then* $\pi_1(Y_s) = \mathbf{1}$ *iff* $G/\widehat{G}_s \simeq \mathbf{Z}/s\mathbf{Z}$.

Proof. (a) By Lemma 2.3(a), we have that Ker $(i_* : \pi_1(X \setminus F_0) \to \pi_1(X)) =<< \alpha >>$, and the assertion follows.

(b) Set $F_{s,0} = \varphi_s^{-1}(F_0) \subset Y_s$, $X^* = X \setminus F_0$ and $Y_s^* = Y_s \setminus F_{s,0}$. Then $\varphi_s : Y_s^* \to X^*$ is a non–ramified cyclic covering of order s . The induced homomorphism

$$(\varphi_s)_* : \pi_1(Y_s^*) \to \pi_1(X^*) =: G$$

is an injection onto a normal subgroup G_s of G of index s, and $G/G_s \simeq \mathbf{Z}/s\mathbf{Z}$. Observe that, by (a), $G =<< \alpha >>$, and that $\alpha^s \in G_s$ is the image of a vanishing loop $\beta \in \pi_1(Y_s^*)$ of the smooth irreducible divisor $F_{s,0} \subset Y_s$, i.e. $(\varphi_s)_*(\beta) = \alpha^s$. Therefore, $\widehat{G}_s :=<< \alpha^s >> \subset G_s$, and $\widehat{G}_s = G_s$ iff $G/\widehat{G}_s \simeq G/G_s \simeq \mathbf{Z}/s\mathbf{Z}$.

Denote also $\widehat{\widehat{G}}_s =<< \alpha^s >>_{G_s}$ the subgroup of G_s normally generated (in G_s) by the element $\alpha^s \in G_s$.

Claim. $\widehat{\widehat{G}}_s = \widehat{G}_s$.

Proof of the claim. Clearly, $\widehat{\widehat{G}}_s \subset \widehat{G}_s \subset G_s$. Since the quotient $G_s/\widehat{G}_s \simeq \mathbf{Z}/s\mathbf{Z}$ is abelian, we have that $K := [G, G] \subset G_s$. Since $G =<< \alpha >>$, the abelianization $G_{\text{ab}} := G/K$ is a cyclic group generated by the class $K\alpha$. Hence, any element $g \in G$ can be written as $g = g'\alpha^t$, where $g' \in K \subset G_s$ and $t \in \mathbf{Z}$. Thus, we have $g\alpha^s g^{-1} = g'\alpha^s g'^{-1} \in \widehat{\widehat{G}}_s$ for any $g \in G$. Therefore, $\widehat{G}_s \subset \widehat{\widehat{G}}_s$, and the claim follows. $\square$

By (a), $\pi_1(Y_s) = \mathbf{1}$ iff $\pi_1(Y_s^*) =<< \beta >>$, or, what is the same, iff $\widehat{\widehat{G}}_s = G_s$. Due to the above Claim, the latter holds iff $\widehat{G}_s = G_s$, or, equivalently, iff $G/\widehat{G}_s \simeq \mathbf{Z}/s\mathbf{Z}$. This proves (b). $\square$

Proof of Theorem 7.3. Since $G =<< \alpha >>$, any element $g \in G$ can be written as $g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1}$, where $g_i \in G$ and $r_i \in \mathbf{Z}$, $i = 1, \dots, n$. Let $\rho : G \to G/K \simeq H_1(X \setminus F_0; \mathbf{Z}) \simeq \mathbf{Z}$ be the canonical surjection. Then, clearly, $\rho(\alpha) = 1$, and so, $\rho(g) = \sum_{i=1}^n r_i \in \mathbf{Z}$.

Since $K \subset G_s := (\varphi_s)_*(\pi_1(Y_s^*))$ and $G/G_s \simeq \mathbf{Z}/s\mathbf{Z}$, we have that $\rho(G_s) = s\mathbf{Z}$. That is, $g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1} \in G_s$ iff $\rho(g) = \sum_{i=1}^n r_i \equiv 0 (\text{mod } s)$.

Using the assumption $(s, c) = 1$ write $r_i = k_i s + l_i c$, where $k_i, l_i \in \mathbf{Z}$, $i = 1, \dots, n$. By our assumption ($\sharp_2$), $\alpha^c \in Z(G)$, and hence $g_i \alpha^{r_i} g_i^{-1} = g_i \alpha^{sk_i} g_i^{-1} \alpha^{cl_i}$, $i = 1, \dots, n$, and furthermore,

$$g = \prod_{i=1}^n g_i \alpha^{r_i} g_i^{-1} = \left(\prod_{i=1}^n g_i \alpha^{sk_i} g_i^{-1} \right) \alpha^{mc} ,$$

where $m = \sum_{i=1}^n l_i$. For an element $g \in G_s$ it follows that $\rho(g) = s\sum_{i=1}^n k_i + mc \equiv 0 (\text{mod } s)$, or, equivalently, $m \equiv 0 (\text{mod } s)$. Set $m = ls, l \in \mathbf{Z}$. Whence, we have $g = \left(\prod_{i=1}^n g_i \alpha^{sk_i} g_i^{-1}\right) (\alpha^s)^{lc} \in \widehat{G}_s$. Therefore, $\widehat{G}_s \subset G_s \subset \widehat{G}_s$, and so, $\widehat{G}_s = G_s$, as required (see Lemma 7.2(b)). □

Now the proof of Kaliman's Theorem 7.1 is completed. In exercises 7.2–7.7 below we expose some additional properties of the fundamental group $G = \pi_1(X \setminus F_0)$ in the situation where the variety $X \setminus F_0$ is equipped with a $\mathbf{C}^*-$ action. In Example[20] 7.2 we show that, without the assumption ($\sharp_2$) (or, perhaps, a weaker one which has to be precised), the fundamental group of a cyclic $\mathbf{C}^*-$ covering Y_s of a contractible smooth affine variety (even surface) X can be quite big.

Exercises (7.2) [21] Let $G = << \alpha >>$ be a normally one-generated group. Denote $K = [G, G]$. Show that $\alpha^c \in Z(G)$ iff [22] $[a^c, K] = \mathbf{1}$, and that under this condition $K \subset \widehat{G}_s := << \alpha^s >>$ for any $s \in \mathbf{Z}$ prime to c.

(**7.3**) Let X and $q \in \mathbf{C}[X]$ be as in Theorem 7.3 (in particular, $\pi_1(X) = \mathbf{1}$). Assume that the restriction $q \,|\, (X \setminus F_0) \,:\, X \setminus F_0 \to \mathbf{C}^*$ is a smooth fiber bundle with a connected fiber $F_1 := q^{-1}(1)$. Show that $q_*(\alpha) = 1 \in \mathbf{Z}$, $i_*\pi_1(F_1) = K := [G, G]$, $G_{\text{ab}} := G/K \simeq H_1(X \setminus F_0; \mathbf{Z}) \simeq \mathbf{Z}$, and $q_* = \rho \,:\, G \to G/K = \mathbf{Z}$ is the canonical surjection. Deduce that $G = \pi_1(X \setminus F_0) \simeq \mathbf{Z}$ if and only if $\pi_1(F_1) = \mathbf{1}$, which, in turn, implies the condition ($\sharp_2$).

(**7.4**) Show, furthermore, that under the condition ($\sharp$) of Theorem 7.1, the above assumption is fulfilled, and, moreover, the group G contains a normal subgroup G_d of index d with the cyclic quotient $G/G_d \simeq \mathbf{Z}/d\mathbf{Z}$ such that $G_d \simeq K \times \mathbf{Z} \simeq \pi_1(F_1) \times \mathbf{Z}$. Let an element $\gamma \in G_d$ correspond to a generator of the second factor $\mathbf{Z}$ of this decomposition. Verify that[23] $\gamma\alpha^{-d} \in K$, and that the centralizer subgroup C_γ of γ in G contains G_d.

Hint. Put $G_d = (\varphi_d)_*(\pi_1(Y_d^*))$, where $\varphi_d \,:\, Y_d \to X$ is the $d-$ fold branched cyclic covering, $X^* = X \setminus F_0$ and $Y_d^* = Y_d \setminus \varphi_d^{-1}(F_0)$. The induced $\mathbf{C}^*-$ action on Y_d^* yields an equivariant isomorphism $F_1 \times \mathbf{C}^* \to Y_d^*$, $(x, \mu) \longmapsto (t_\mu(x), \mu)$, which provides, in

[20] These exercises and example were elaborated jointly with Sh. Kaliman.

[21] See [Za 5, Appendix]. A much shorter proof was proposed by H. Flenner during the lecture course.

[22] If $A, B \subset G$, then $[A, B]$ denotes the subgroup generated by all the commutators $[a, b] = aba^{-1}b^{-1}$, where $a \in A, b \in B$.

[23] In the proof of Lemma 6.8 in [Za 5] it was taken $\gamma = \alpha^d$. In general, this is not true; see the next exercise.

turn, the desired decomposition of the subgroup G_d of G. Thus, γ is the image of a generator of the group $\pi_1(\mathbf{C}^*) \simeq \mathbf{Z}$ under the homomorphism induced by the mapping $\mathbf{C}^* \to \mathcal{O}_{x_0} \subset X^*$ onto the $\mathbf{C}^*-$ orbit $\mathcal{O}_{x_0}$ of a base point $x_0 \in F_1$.

(**7.5**) Set $X = \mathbf{C}^2$ and $q(x, y) = x^2 - y^3 \in \mathbf{C}[x, y]$. Show that the group $G = \pi_1(X \setminus F_0)$ can be identified with the 3-braid group $B_3 :=< \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 >$, the generators $\sigma_1, \sigma_2 \in G$ being vanishing loops of the divisor $F_0 = \Gamma_{2,3} := q^*(0) \subset X$. Describe the subgroups G_d and $K = [G, G]$ in this example (see the preceding exercise). Verify that one can take for γ the element $(\sigma_1\sigma_2)^3$ which generates the center $Z(G) \simeq Z(B_3) \simeq \mathbf{Z}$. Putting $\alpha = \sigma_1$ check that $G =<< \alpha >>$, and $\alpha^c \notin Z(G)$ whatever $c \in \mathbf{Z} \setminus \{0\}$ is.

Hint. One can use the presentation of $-q$ as the discriminant of the cubic polynomial $t^3 - (y/\sqrt[3]{4})t + x/\sqrt{27} \in \mathbf{C}[t]$, and consider the Vieta covering $\mathbf{C}^2 \to \mathbf{C}^2$, which is a branched Galois covering ramified over $\Gamma_{2,3}$, with the Galois group S_3.

(**7.6**) Let X be an irreducible quasiprojective variety, $D \subset X$ be an irreducible hypersurface which contains the singularity locus sing X of X, and $C \subset D$ be a non-empty smooth subvariety such that $C \cap (\text{ sing } X \cup \text{sing } D) = \emptyset$. Consider the Kaliman transform $\sigma_C : X' \to X$ of X along D with center C. Show that $\pi_1(X') \simeq \pi_1(X \setminus \text{sing } X)$.

Hint. Replace the triple (X, D, C) by the triple $(X \setminus \text{ sing } X, D \setminus \text{ sing } X, C)$ and apply Lemma 5.1.

(**7.7**) For $q \in \mathbf{C}[x_1, \ldots, x_n]$, $q(\overline{0}) = 0$, denote $h_q = q \circ \sigma_n(x_1, \ldots, x_n)/x_n$, where

$$\sigma_n : \mathbf{C}^n \to \mathbf{C}^n, \;\; \sigma_n(x_1, \ldots, x_n) := (x_1x_n, \ldots, x_{n-1}x_n, x_n)\,.$$

Put $X = q^{-1}(0) \subset \mathbf{C}^n$ and $X' = h_q^{-1}(0) \subset \mathbf{C}^n$. Assume that $\overline{0} \in X$ is a smooth point. Verify that X' is the Kaliman transform of X along $D := X \cap \{x_n = 0\}$ with center at the origin.

Hint. Notice that $\sigma_n : \mathbf{C}^n \to \mathbf{C}^n$ is the Kaliman transform of $\mathbf{C}^n$ along the hyperplane $\{x_n = 0\}$ with center at the origin (cf. the proof of Lemma 6.3).

Example 7.2. [24] Consider the smooth surfaces $X_{k,l,s,m} := p_{k,l,s,m}^{-1}(0) \subset \mathbf{C}^3$ defined by the polynomials

$$p_{k,l,s,m} := \frac{(xz^m+1)^k - (yz^m+1)^l - z^s}{z^m} \in \mathbf{C}[x, y, z]\,,$$

[24] We are thankful to V. Sergiescu for useful discussions related to this example.

where $1 \le m \le s$. For $(k,\,l)=(k,\,s)=(l,\,s)=1$ and $m=s$ we have that $X_{k,l,s,s} = Y_{k,l,s}$ is an acyclic surface (see Example 7.1)[25]. Moreover, it can be represented as a cyclic $\mathbf{C}^*-$ covering of a contractible surface $X_{k,l} := X_{k,l,1,1} \subset \mathbf{C}^3$ branched to order s along the curve $L_{k,l} := X_{k,l} \cap \{z=0\} \simeq \mathbf{C}$ in $X_{k,l}$. However, in general, the acyclic surface $Y_{k,l,s}$ is not contractible and possesses quite a big fundamental group.

Indeed, Exercise 7.7 above shows that $\sigma_3 \,|\, X_{k,l,s,m} \,:\, X_{k,l,s,m} \to X_{k,l,s,m-1}$ is the Kaliman transform of $X_{k,l,s,m-1}$ along $D := X_{k,l,s,m-1} \cap \{z=0\}$ with center at the origin. The repeated application of Lemma 5.1 and Exercise 7.6 yield the isomorphisms

$$\pi_1(Y_{k,l,s}) = \pi_1(X_{k,l,s,s}) \simeq \pi_1(X_{k,l,s,s-1}) \simeq \ldots \simeq \pi_1(X_{k,l,s,1}) \simeq \pi_1(X_{k,l,s,0} \setminus \{p_0\})\,,$$

where $X_{k,l,s,0} \simeq X_{k,l,s} := \{x^k - y^l - z^s = 0\} \subset \mathbf{C}^3$, and $p_0 = (-1,\,-1,\,0) \in X_{k,l,s,0}$ is the only singular point of $X_{k,l,s,0}$. Whence, $X_{k,l,s,0} \simeq X_{k,l,s}$ is homotopically equivalent to the cone over the *Pham–Brieskorn 3-manifold* $M_{k,l,s} := X_{k,l,s} \cap S^5$ (the *link* of the surface singularity of $X_{k,l,s}$ in the sphere S^5). In turn, $X_{k,l,s} \setminus \{\bar{0}\}$ is homotopically equivalent to the link $M_{k,l,s}$, and thus $\pi_1(Y_{k,l,s}) \simeq \pi_1(M_{k,l,s}) := G'_{k,l,s}$.

The structure of these groups is well known (see [Mil 3]). The groups $G'_{k,l,s}$ are finite iff $1/k+1/l+1/s>1$, infinite nilpotent iff $1/k+1/l+1/s=1$. If $1/k+1/l+1/s<1$, then $G'_{k,l,s} = [G_{k,l,s},\, G_{k,l,s}]$, where

$$G_{k,l,s} :=< \gamma_1,\, \gamma_2,\, \gamma_3 \,|\, \gamma_1^k = \gamma_2^l = \gamma_3^s = \gamma_1\gamma_2\gamma_3 >$$

is a central extention of the *Schwarz triangular group*

$$T_{k,l,s} :=< b_1,\, b_2,\, b_3 \,|\, b_1^2 = b_2^2 = b_3^2 = 1,\;\, (b_1b_2)^k = (b_2b_3)^l = (b_3b_1)^s = 1 >,$$

which is a discrete group of isometries of the non-euclidean plane generated by reflections in the sides of an appropriate triangle.

Note that for $1/k+1/l+1/s<1$ the triangular group $T_{k,l,s}$ contains a free subgroup with two generators. Therefore, the group $G'_{k,l,s}$ also contains such a subgroup (*exercise*); in particular, it is not solvable. Observe that this group is *perfect*, i.e. it coinsides with its commutator subgroup; indeed, its abelianization $H_1(Y_{k,l,s};\,\mathbf{Z})$ is trivial. On the other hand, it is known that for $(k,\,l)=(k,\,s)=(l,\,s)=1$ the Pham–Brieskorn manifold $M_{k,l,s}$ is a homology 3-sphere; see [HNK, Appendix I.8].

Recall that $X_{k,l} \setminus L_{k,l} \simeq \mathbf{C}^2 \setminus \Gamma_{k,l}$, where $\Gamma_{k,l} := \{x^k - y^l = 0\} \subset \mathbf{C}^2$ (see Example 6.1). The group $B_{k,l} := \pi_1(X_{k,l} \setminus L_{k,l}) \simeq \pi_1(\mathbf{C}^2 \setminus \Gamma_{k,l})$ has the presentation $B_{k,l} =<$

[25]By Lemma 5.2, all the surfaces $X_{k,l,s,m}$, $m=1,\ldots,s$, are acyclic, too.

$a, b \,|\, a^k = b^l >$ (see e.g. [Di 2]). In turn, the group $\pi_1(Y_{k,l,s} \setminus \{z = 0\})$ is isomorphic to an index s subgroup of the group $B_{k,l}$ with a cyclic quotient. By Lemma 2.3(a), Ker $(i_* : \pi_1(Y_{k,l,s} \setminus \{z = 0\}) \to \pi_1(Y_{k,l,s})) =<< \alpha^s >>$, where $\alpha \in B_{k,l}$ represents a vanishing loop of the line $L_{k,l} \subset X_{k,l}$. Let $p, q \in \mathbf{Z}$ be such that $kp + lq = 1$. Then one may take $\alpha = a^q b^p \in B_{k,l}$ (*exercise*).

Therefore, for $1/k + 1/l + 1/s < 1$ and $(k, l) = (k, s) = (l, s) = 1$ the group $G'_{k,l,s} \simeq \pi_1(Y_{k,l,s})$ is isomorphic to an index s subgroup of the quotient

$$B_{k,l,s} := B_{k,l}/ << \alpha^s >>=< a, b \,|\, a^k = b^l, \ (a^q b^p)^s = 1 > \ .$$

In particular, for $k = 2, l = 3,$ and $s \geq 7$ we have that $B_{2,3} = B_3$ is the 3-braid group with generators $\sigma_1, \sigma_2 \in B_3$ being vanishing loops of $L_{2,3}$ in $X_{2,3}$ (see Exercise 7.5 above), $a = \sigma_1\sigma_2\sigma_1$, $b = \sigma_1\sigma_2$, and $G'_{2,3,s}$ is isomorphic to an index s subgroup of the group

$$B_{2,3,s} = B_3/ << \sigma_1^s >>=< \sigma_1, \sigma_2 \,|\, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \ \sigma_1^s = \sigma_2^s = 1 > \ .$$

8 Multicyclic $\mathbf{C}^*-$ coverings

8.1 Contractibility of multicyclic $\mathbf{C}^*-$ coverings. Examples

To clarify the very idea of the construction of contractible multicyclic $\mathbf{C}^*-$ coverings due to Koras and Russell [KoRu 1, KoRu 2], let us start with simple examples. We exhibit two different approaches (compare, for instance, Examples 8.1 and 8.3 below).

Example 8.1. (*The Russell cubic threefold*) The polynomial $q_0 = x(xy + 1) \in \mathbf{C}[x, y]$ is the hyperbolic modification of the polynomial $h = x + x^2 \in \mathbf{C}[x]$. Thus, it is a quasi-invariant of weight 1 of the $\mathbf{C}^*-$ action $(\lambda, (x, y)) \longmapsto (\lambda x, \lambda^{-1} y)$ on $\mathbf{C}^2$. The zero fiber $\Gamma_0 = q_0^{-1}(0)$ is a disjoint union of two affine curves isomorphic to $\mathbf{C}$ and to $\mathbf{C}^*$, respectively. Consider the two-fold $\mathbf{C}^*-$ covering $F_0 \to \mathbf{C}^2$ branched along Γ_0, given as the surface $F_0 = \{x + x^2 y + z^2 = 0\} \subset \mathbf{C}^3$ with the projection $\varphi_2 : F_0 \longrightarrow \mathbf{C}^2$, $(x, y, z) \longmapsto (x, y)$. Thus, φ_2 is $\mathbf{C}^*-$ equivariant with respect to the actions $(\lambda, (x, y, z)) \longmapsto (\lambda^2 x, \lambda^{-2} y, \lambda z)$ on F_0 and $(\lambda, (x, y)) \longmapsto (\lambda^2 x, \lambda^{-2} y)$ on $\mathbf{C}^2$. The restriction of the above C^*- action on F_0 to the subgroup $\omega_2 = \{\lambda^2 = 1\} \simeq \mathbf{Z}/2\mathbf{Z}$ of $\mathbf{C}^*$ yields the monodromy of the covering $F_0 \to \mathbf{C}^2$. Since this monodromy acts trivially in the homology of F_0, by Corollary 7.2, F_0 is $\mathbf{Z}_3-$ acyclic.

Furthermore, by the Sebastiani–Thom–Némethi Theorem[26] (cf. e.g. [Di 2, Ne, tD 2, Proof of Thm. B]), the generic fibre $F_1 = q^{-1}(1)$ of the polynomial $q = (x + x^2y) + z^2 \in \mathbf{C}[x, y, z]$ is homotopically equivalent to the *join*[27] $\Gamma_1 \star \mathbf{Z}/2\mathbf{Z}$, where $\Gamma_1 := q_0^{-1}(1) \subset \mathbf{C}^2$, i.e. to the *suspension* [28] over Γ_1. Since the curve $\Gamma_1 \simeq \mathbf{C}^*$ is connected, the fibre F_1 is simply connected, and hence, $G := \pi_1(\mathbf{C}^3 \setminus F_0) \simeq \mathbf{Z}$ (see Exercise 7.3).

Next we pass to the three-sheeted cyclic $\mathbf{C}^*-$ covering over $\mathbf{C}^3$ branched along F_0, i.e. to the hypersurface

$$X = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbf{C}^4$$

with the projection $\varphi_3 : (x, y, z, t) \longmapsto (x, y, z)$ onto $\mathbf{C}^3$ (we call X the *Russell cubic threefold*). Since the polynomial q is a quasi-invariant of weight 2 of the above $\mathbf{C}^*-$ action on $\mathbf{C}^3$, we are under the assumptions of Kaliman's Theorem 7.1. Due to this theorem, X is a contractible smooth affine variety.

Exercise (8.1) Verify that the smooth cubic threefold

$$X' = \{x + x^2y + z^2 + t^2 = 0\} \subset \mathbf{C}^4$$

is simply connected, but not acyclic (what are the homology groups of X'?).

Example 8.2. (tom Dieck [tD 2, Thm. B]) More generally, let X be a smooth contractible affine variety equipped with a regular $\mathbf{C}^*-$ action t, and let $q \in \mathbf{C}[X]$ be a quasi-invariant of t of weight $d \neq 0$ such that $F_0 := q^*(0)$ is a smooth reduced (not necessarily irreducible) divisor in X. Fix $s_1, s_2 \in \mathbf{N}$ such that d, s_1, s_2 are pairwise relatively prime. Consider the smooth affine hypersurface

$$Y_{s_1, s_2} := \{q(x) + z^{s_1} + t^{s_2} = 0\} \subset X \times \mathbf{C}^2 .$$

We assert that Y_{s_1, s_2} is contractible.

Indeed, consider first the cyclic $\mathbf{C}^*-$ covering $Y_{s_1} \to X$, $Y_{s_1} = \{q(x) + z^{s_1} = 0\} \subset X \times \mathbf{C}$, branched to order s_1 along F_0. Then $Y_{s_1} \subset X \times \mathbf{C}$ is a smooth reduced irreducible divisor defined by the quasi-invariant $q_1(x, z) := q(x) + z^{s_1} \in \mathbf{C}[X \times \mathbf{C}]$ of

[26] This theorem says that a generic fibre of a polynomial $p(x) + q(y)$, $p \in \mathbf{C}[x_1, \ldots, x_k]$, $q \in \mathbf{C}[y_1, \ldots, y_l]$, is homotopically equivalent to the join of generic fibres of the polynomials p and q.

[27] Recall that the join $X \star Y$ of two topological spaces X, Y is the cycinder $(X \times Y) \times [0, 1]$ with the base $(X \times Y) \times \{0\}$ resp. $(X \times Y) \times \{1\}$ being retracted to $X \times \{0\}$ resp. to $Y \times \{1\}$.

[28] that is, to the cycinder $\Gamma_1 \times [0, 1]$ with the bases $\Gamma_1 \times \{0\}, \Gamma_1 \times \{1\}$ being contracted each one to a point.

weight ds_1 of the $\mathbf{C}^*-$ action $(\lambda, (x, z)) \longmapsto (t(\lambda^{s_1}, x), \lambda^d z)$ on $X \times \mathbf{C}$. Since the monodromy group $G \simeq \mathbf{Z}_{s_1}$ of the covering acts trivially in the homology of Y_{s_1}, by Corollary 7.2, Y_{s_1} is $\mathbf{Z}_p-$ acyclic for any prime p which is prime to s_1, and hence, for any prime divisor p of s_2.

Besides, the fibre $F_1 = q^{-1}(1)$ of the regular function $q \in \mathbf{C}[X]$ is connected (see Exercise 7.3). As above, it follows from the Sebastiani–Thom–Némethi Theorem that the fibre $q_1^{-1}(1) = \{q(x) + z^{s_1} = 1\}$ of the function $q_1 \in \mathbf{C}[X \times \mathbf{C}]$ is simply connected, and hence, we have $\pi_1((X \times \mathbf{C}) \setminus Y_{s_1}) \simeq \mathbf{Z}$.

Therefore, by Kaliman's Theorem 7.1, the total space of the cyclic $\mathbf{C}^*-$ covering $Y_{s_1, s_2} \to X \times \mathbf{C}$ branched to order s_2 over Y_{s_1} is a smooth contractible affine variety.

Applying Kaliman's Theorem 7.1 successively in the same way as above, one can easily get the following result (cf. Koras and Russell [KoRu 2, (7.14)]).

Theorem 8.1. *Let X be a smooth contractible affine variety equipped with an effective $\mathbf{C}^*-$ action. Let $q_i \in \mathbf{C}[X]$, $i = 1, \ldots, k$, be a sequence of quasi-invariants of positive weights $d_1, \ldots, d_k$, respectively, and let $s_1, \ldots, s_k$ be a sequence of positive integers. Suppose that the following conditions are fulfilled:*

(i) For each $i = 1, \ldots, k$, $F_i := q_i^*(0)$ is a smooth reduced irreducible divisor, the union $\bigcup_1^k F_i$ is a divisor with normal crossings, and the group $\pi_1(X \setminus \bigcup_1^k F_i)$ is abelian;

(ii) $(d_i, s_i) = (s_i, s_j) = 1$ for all $i, j = 1, \ldots, k$;

(iii) F_i is $\mathbf{Z}_p-$ acyclic for each prime divisor $p|s_i$, $i = 1, \ldots, k$.

Let $Y \longrightarrow X$ be a multicyclic covering branched to order s_i along F_i, $i = 1, \ldots, k$, i.e. $Y = Y_{s_1 \ldots s_k}$ in the tower of cyclic $\mathbf{C}^*-$ coverings

$$Y_{s_1 \ldots s_k} \longrightarrow Y_{s_1 \ldots s_{k-1}} \longrightarrow \ldots \longrightarrow Y_{s_1 s_2} \longrightarrow Y_{s_1} \longrightarrow X,$$

where $Y_{s_1 \ldots s_i} \longrightarrow Y_{s_1 \ldots s_{i-1}}$ is a $\mathbf{C}^*-$ covering branched to order s_i over the preimage of the divisor[29] F_i in $Y_{s_1 \ldots s_{i-1}}$.

Then Y is a smooth contractible affine variety given in $X \times \mathbf{C}^k$ by the equations $z_i^{s_i} = q_i(x)$, $i = 1, \ldots, k$.

Remark 1. In the case when $X \subset \mathbf{C}^n$, and q_j is a variable, $q_j = x_j$ say, the equations of the cyclic covering $Y_{s_j} \longrightarrow X$ can be obtained from the equations $P_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, m$, which define X, by the substitution $x_j \longmapsto x_j^{s_j}$,

[29] In other words, if $A = \mathbf{C}[X]$, then $Y = \text{spec}\, A[\sqrt[s_1]{q_1}, \ldots, \sqrt[s_k]{q_k}]$.

i.e. $Y_{s_j} = \{P_i(x_1, \dots, x_j^{s_j}, \dots, x_n) = 0, \ i = 1, \dots, m\} \subset \mathbf{C}^n$. In particular, if X is a hypersurface in $\mathbf{C}^n$, so is Y_{s_j}.

Remark 2. To construct contractible $\mathbf{C}^*-$ invariant hypersurfaces, one can use the hyperbolic modification in the same way as in Example 8.1 above. Recall that, if $X = \{h = 0\} \subset \mathbf{C}^n$ is a smooth contractible hypersurface, then $Y = Y_0 = q^{-1}(0) \subset \mathbf{C}^{n+1}$, where $q(\overline{x}, y) = \frac{h(u\overline{x})}{u}$, is isomorphic to the Kaliman transform of $X \times \mathbf{C}$ along X with center at the origin (Lemma 6.3), and so, by Theorem 5.1, Y is a smooth contractible hypersurface, too. Moreover, q is a quasi-invariant of weight 1 of the $\mathbf{C}^*-$ action $(\lambda, (\overline{x}, u)) \longmapsto (\lambda\overline{x}, \lambda^{-1}u)$ on $\mathbf{C}^{n+1}$ with the only fixed point at the origin (of hyperbolic type with weights $(1, \dots, 1, -1)$).

If we take k smooth hypersurfaces $H_i = \{h_i(\overline{x}) = 0\} \bigcap X$ in X, $i = 1, \dots, k$, which satisfy the condition (i) above, and put $q_i(\overline{x}, u) = \frac{h_i(u\overline{x})}{u}$, then the q_i are $\mathbf{C}^*-$ invariants of weight 1 of the above $\mathbf{C}^*-$ action on Y. The hypersurfaces $F_i := q_i^{-1}(0) \bigcap Y$, $i = 1, \dots, k$, in Y also satisfy the condition (i) (note that F_i is the closure of the $\mathbf{C}^*-$ orbit of the subvariety $H_i \subset X \simeq Y \bigcap H \subset Y$, $i = 1, \dots, k$, where $H := \{u = 1\}$; see Exercise 6.1). This construction can be illustrated by the following simple example.

Example 8.3. (*The Russell cubic threefold once again*; see Koras-Russell [KoRu 2]; cf. Example 8.1.) Starting with $X = \mathbf{C}^2$, fix two smooth curves (f) and (g) , where $f, g \in \mathbf{C}[x, z]$, isomorphic to $\mathbf{C}$ and such that $(f), (g)$ meet transversally at the origin and in k other points, $k \geq 1$. For instance, take $f = z$, $g = z + x + x^2$. Then the Kaliman modification Y of $X \times \mathbf{C}$ along X with center at the origin is nothing but $\mathbf{C}^3$. The plane curves (f) and (g) give rise, respectively, to the surfaces (F) and (G) in $\mathbf{C}^3$, where $F = \frac{f(yx, yz)}{y}$, and $G = \frac{g(yx, yz)}{y}$. Observe that (F) and (G) are isomorphic to $\mathbf{C}^2$, meet transversally, and $\pi_1(\mathbf{C}^3 \setminus ((F) \cup (Q))) \simeq \mathbf{Z}$ (*exercise*).

In our particular example $F = z$ and $G = z + x + x^2y$. The polynomials F and G are $\mathbf{C}^*-$ quasi-invariants of weight 1 with respect to the action $(\lambda, (x, y, z)) \longmapsto (\lambda x, \lambda^{-1}y, \lambda z)$ on $\mathbf{C}^3$. We may also take $H_0 = \{y = 0\}$ for the third surface transversal to the first two (F) and (G) .

Fix two relatively prime positive integers s_1, s_2. Passing to the bicyclic $\mathbf{C}^*-$ covering of $Y \simeq \mathbf{C}^3$ branched to order s_1 along (F) and to order s_2 along (G), we obtain a hypersurface $Y_{s_1, s_2} \subset \mathbf{C}^4$ given by the equation

$$x + x^2y + z^{s_1} + t^{s_2} = 0,$$

which is a smooth contractible threefold. If $s_1 = 2, s_2 = 3$, we get the Russell cubic.

More generally, passing to the tricyclic covering of $\mathbf{C}^3$ branched to order s_0 resp. s_1, s_2 along the surface H_0 resp. (F), (G), where $(s_i, s_j) = 1$, $i \neq j$, yields the smooth contractible hypersurface $\{x + x^2y^{s_0} + z^{s_1} + t^{s_2} = 0\}$ in $\mathbf{C}^4$.

Remark. A theorem due to Koras and Russell [KoRu 2, Thm. 4.1] says that any smooth contractible affine threefold with a 'good' hyperbolic $\mathbf{C}^*-$ action appears in the same way as in the above example.

8.2 The logarithmic Kodaira dimension of multicyclic coverings

Lemma 8.1. *Let V be a smooth projective variety, and let L be a line bundle on V .*

(a) (Mori [Mo, Prop. 1.9]) *L is big (i.e. $k(V, L) = \dim_{\mathbf{C}} V$) iff for some $k \in \mathbf{N}$ the multiple kL can be written as $kL = A + E$, where A is an ample line bundle on V , and E is an effective one*[30].

(b) (Kleiman-Kodaira; see e.g. [Wil, (2.3)], [KMM, Lemma 0-3-3]) *If L is ample (resp. big), then for any line bundle L' on V and for any $k \in \mathbf{N}$ large enough, $kL - L'$ is ample (resp. big), too.*

Proof of (b). Recall the *Kleiman criterion of ampleness* [Kl]: L is ample iff it is positive on the cone $NE_1(V)$ (with the origin deleted) of numerically effective $1-$ cycles on V modulo numerical equivalence. This finite dimensional cone is closed, and hence, it has a compact support on the unit sphere. Thus, the openness of ampleness follows.

Let L be big, and let $k_0L = A + E$ be a decomposition as in (a). Then for $n_0 \in \mathbf{N}$ large enough, we have $n_0k_0L - L' = (n_0A - L') + n_0E$, where $n_0A - L'$ is ample. Therefore, by (a), $n_0k_0L - L'$ is big. It follows that for any $k \geq n_0k_0$ the divisor $kL - L' = (n_0k_0L - L') + (k - n_0k_0)L$ is big. $\square$

Proposition 8.1. (Kaliman [Ka 1, Lemma 11]) *Let X be a quasi-projective variety, (V, D) be an SNC-completion of X, and $Z = \sum_1^k Z_i$ be an SNC-divisor on V such that $D \bigcup Z$ is also an SNC-divisor, and D and Z have no irreducible component in common. Let $Y = Y_{\overline{s}} \longrightarrow X$, $\overline{s} := (s_1, \ldots, s_k)$, be a ramified covering branched to order s_i over $Z_i \bigcap X$, $i = 1, \ldots, k$. Then*

$$\overline{k}(Y) = k(V,\, K_V + D + \sum_{i=1}^{k}(1 - \frac{1}{s_i})Z_i).$$

[30]that is, E admits a non-zero holomorphic section.

Proof. One can compactify Y by an SNC-divisor D' (i.e. $Y = V' \setminus D'$) to obtain a commutative diagram of morphisms

$$\begin{array}{ccc} Y & \hookrightarrow & V' \\ \downarrow & & \downarrow \varphi \\ X & \hookrightarrow & V\,. \end{array}$$

Then $\varphi^*(Z_i) = s_i Z_i' + E_i$, where each E_i is $\varphi-$ exceptional, i.e. $\text{codim}\,_V \varphi(E_i) \geq 2$. The restriction $\varphi\,|\,(V' \setminus (D' \cup Z'))\,:\, V' \setminus (D' \cup Z') \to V \setminus (D \cup Z)$ is an étale covering. Therefore, by the Logarithmic Ramification Formula (R), we have

$$K_{V'} + D' + Z' = \varphi^*(K_V + D + Z) + R\,,$$

where R is an effective $\varphi-$ exceptional divisor in V' (see Proposition 3.1.(f),(g)). Hence,

$$K_{V'} + D' = \varphi^*(K_V + D + Z) + R - Z' = \varphi^*(K_V + D + Z) +$$
$$+\varphi^*(\sum_{i=1}^k (-\frac{1}{s_i}) Z_i) + R + \sum_{i=1}^k \frac{1}{s_i} E_i = \varphi^*(K_V + D + \sum_{i=1}^k (1 - \frac{1}{s_i}) Z_i) + E\,,$$

where $E := \sum_{i=1}^k \frac{1}{s_i} E_i + R$ is a $\varphi-$ exceptional $\mathbf{Q}-$ divisor. By [Ii 1, Lemma 1] or [Ii 3, Thm. 10.5], $\overline{k}(V',\,\varphi^*(D_1) + E) = \overline{k}(V,\,D_1)$ for any $\mathbf{Q}-$ Cartier divisor D_1 on V , where E is a $\varphi-$ exceptional divisor in V' (indeed, a meromorphic section of the associated line bundle $[\varphi^*(D_1)]$ with poles at most along E has no pole). Thus, the assertion follows. $\square$

Corollary 8.1. ([Ka 1], [KoRu 2, Cor.6.2]) *If* $\overline{s}' = (s_1', \dots, s_k')$ *and* $s_i' \geq s_i,\, i = 1, \dots, k,$ *then* $\overline{k}(Y_{\overline{s}'}) \geq \overline{k}(Y_{\overline{s}})$.

Corollary 8.2. *If* $X \setminus Z$ *is of log-general type (i.e.* $\overline{k}(X \setminus Z) = \dim\,_{\mathbf{C}} X$ *), then for* $s_i,\ i = 1, \dots, k,$ *large enough* $Y_{\overline{s}}$ *is of log-general type, too.*

Proof. Indeed, by Lemma 8.1(b), for $s_i >> 1,\ i = 1, \dots, k,$ we have

$$\overline{k}(Y_{\overline{s}}) = k(V,\,K_V + (D+Z) - \sum_{i=1}^k \frac{1}{s_i} Z_i) = k(V,\,K_V + D + Z) = \overline{k}(X \setminus Z) = \dim_{\mathbf{C}} X\,.$$

$\square$

Proposition 8.2. (see [Ka 1], [KoRu 2, Prop. 6.5]) *Let* $Y_{\overline{s}} \subset \mathbf{C}^4$ *, where* $\overline{s} = (s_1,\, s_2,\, s_3),$ *and* $(s_i,\, s_j) = 1,\ i \neq j,$ *be given as*[31]

$$Y_{\overline{s}} = \{x + x^2 y^{s_1} + z^{s_2} + t^{s_3} = 0\}\,.$$

[31] This is a particular kind of the Koras-Russell threefolds; see Example 8.3 above.

If $s_1,\, s_2,\, s_3 >> 1$, *then* $Y_{\overline{s}}$ *is an exotic* $\mathbf{C}^3$, *and* $\overline{k}(Y_{\overline{s}}) = 2$.

Proof. Set

$$X = \{x + x^2u_1 + u_2 + u_3 = 0\} \subset \mathbf{C}^4, \quad \text{and} \quad Z_i = \{u_i = 0\} \subset X,\; i = 1, 2, 3.$$

Evidently, $X \simeq \mathbf{C}^3$, $Z_i \simeq \mathbf{C}^2$, $i = 1, 2, 3,$ and $Z := Z_1 \cup Z_2 \cup Z_3$ is an SNC-divisor in X . The threefold $Y_{\overline{s}}$ is a tricyclic covering of X branched to order s_i along Z_i, $i = 1, 2, 3,$ with the covering morphism $\varphi_{\overline{s}} \,:\, (x,\, y,\, z,\, t) \longmapsto (x,\, u_1,\, u_2,\, u_3) := (x,\, y^{s_1},\, z^{s_2},\, t^{s_3})$. By Theorem 8.1, it follows that $Y_{\overline{s}} \subset \mathbf{C}^4$ is a smooth contractible affine hypersurface. Due to the Dimca-Ramanujam Theorem 4.2, $Y_{\overline{s}}$ is diffeomorphic to $\mathbf{R}^6$. It remains to show that $\overline{k}(Y_{\overline{s}}) = 2$ when $s_1,\, s_2,\, s_3$ are large enough.

Due to Corollary 8.1, $\overline{k}(Y_{\overline{s}}) \geq 2$ for sufficiently large $s_1,\, s_2,\, s_3$ if it is so for a particular choice of $\overline{s} = (s_1,\, s_2,\, s_3)$ (even without the assumption of relative primeness, which guarantees the contractibility).

Note that $Y_{\overline{s}} \subset \mathbf{C}^4$ is invariant under the hyperbolic linear $\mathbf{C}^*-$ action on $\mathbf{C}^4$

$$G \,:\, (\lambda,\, (x,\, y,\, z,\, t)) \longmapsto (\lambda^a x,\, \lambda^{-b} y,\, \lambda^c z,\, \lambda^d t)\,,$$

where

$$a = s_1 s_2 s_3,\; b = s_2 s_3,\; c = s_1 s_3,\; d = s_1 s_2\,.$$

The morphism $\varphi_{\overline{s}} \,:\, Y_{\overline{s}} \to X$ is a $\mathbf{C}^*-$ covering with respect to the $\mathbf{C}^*-$ action G on $Y_{\overline{s}}$ and the $\mathbf{C}^*-$ action

$$\overline{G} \,:\, (\lambda,\, (x,\, u_1,\, u_2,\, u_3)) \longmapsto (\lambda^a x,\, \lambda^{-a} u_1,\, \lambda^a u_2,\, \lambda^a u_3)$$

on X . We have spec $(\mathbf{C}[X])^{\overline{G}} := X//\overline{G} \simeq S,$ where $S := \{u + u^2 + v + w = 0\} \subset \mathbf{C}^3$ (clearly, $S \simeq \mathbf{C}^2$). Indeed, $(\mathbf{C}[X])^{\overline{G}} = \mathbf{C}[u,\, v,\, w]$, where $u := u_1 x$, $v := u_1 u_2$, $w := u_1 u_3 \in (\mathbf{C}[X])^{\overline{G}}$ are the basic $\overline{G}-$ invariants. This yields the following commutative diagram of morphisms:

$$\begin{array}{ccc} Y_{\overline{s}} & \xrightarrow{\rho_{\overline{s}}} & Y_{\overline{s}}//G = S_{\overline{s}} \\ {\scriptstyle \varphi_{\overline{s}}}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}_{\overline{s}}} \\ X & \xrightarrow{\rho} & X//\overline{G} = S\;, \end{array}$$

where $S_{\overline{s}} := Y_{\overline{s}}//G = \text{spec }(\mathbf{C}[Y_{\overline{s}}])^G$ is a normal surface. A generic fiber of the quotient morphism $\rho_{\overline{s}} \,:\, Y_{\overline{s}} \to S_{\overline{s}}$ (i.e. a generic orbit) is isomorphic to $\mathbf{C}^*$. Since $\overline{k}(\mathbf{C}^*) = 0$, from the Addition Theorems 3.1(c),(d) [32] we obtain

$$2 = \dim\, S_{\overline{s}} \geq \overline{k}(Y_{\overline{s}}) \geq \overline{k}(S_{\overline{s}})\,.$$

Thus, it remains to find a particular triple $\overline{s} = (s_1,\, s_2,\, s_3)$ such that $\overline{k}(S_{\overline{s}}) = 2$.

Note that the threefold $Y_{\overline{s}}$ is the closure of the $G-$ orbit of the surface $T_{\overline{s}} := Y_{\overline{s}} \cap H$, where $H := \{y = 1\} \subset \mathbf{C}^4$ (see Exercise 6.1). The surface $T_{\overline{s}}$ is invariant under the induced action of the cyclic subgroup $\omega_b \subset \mathbf{C}^*$ on $Y_{\overline{s}}$, and $S_{\overline{s}} = Y_{\overline{s}}//G \simeq T_{\overline{s}}/\omega_b$.

Take $s_1 = pq,\, s_2 = p,\, s_3 = q$, where $p,\, q \in \mathbf{N}$ are prime and distinct. Then we have $Y_{\overline{s}} = \{x + x^2y^{pq} + z^p + t^q = 0\}$, and

$$G(\lambda,\, (x,\, y,\, z,\, t)) = (\lambda^{p^2q^2}x,\, \lambda^{-pq}y,\, \lambda^{pq^2}z,\, \lambda^{p^2q}t)\,.$$

Therefore, $\omega_b = \omega_{pq} \subset \mathbf{C}^*$ ($=$ the kernel of non-effectiveness of the $\mathbf{C}^*-$ action G on $Y_{\overline{s}}$) acts trivially on $T_{\overline{s}} = Y_{\overline{s}} \cap H = \{x + x^2 + z^p + t^q = 0\} \subset \mathbf{C}^3$. Hence, $S_{\overline{s}} = Y_{\overline{s}}//G \simeq T_{\overline{s}} \subset Y_{\overline{s}}$. The projection

$$\rho \circ \varphi_{\overline{s}} \,|\, T_{\overline{s}} \,:\, T_{\overline{s}} \to S, \quad (x,\, z,\, t) \longmapsto (u,\, v,\, w) = (x,\, z^p,\, t^q)\,,$$

is a bicyclic covering branched to order p resp. q over the curve $C_1 := \{v = 0\} \subset S$ resp. $C_2 := \{w = 0\} \subset S$. By Corollary 8.2 above, we have $\overline{k}(S_{\overline{s}}) = \overline{k}(T_{\overline{s}}) = \overline{k}(S \setminus (C_1 \cup C_2))$, if p and q are sufficiently large. Thus, the proof is completed by the following simple exercises. □

Exercises Show that
(**8.2**) $(S,\, C_1 \cup C_2) \simeq (\mathbf{C}^2,\, D_1 \cup D_2)$, where $D_1 := \{y = 1\},\, D_2 := \{y = x^2\} \subset \mathbf{C}^2$;
(**8.3**) and that[33] $\overline{k}(\mathbf{C}^2 \setminus (D_1 \cup D_2)) = 2$.

Remark (see [KoRu 2, Prop. 7.8.]) However, for $s_1 = 1$ the threefold $Y_{\overline{s}} \subset \mathbf{C}^4$ is dominated by $\mathbf{C}^3$; in particular, it has the log-Kodaira dimension $\overline{k} = -\infty$. Indeed, if $s_1 = 1$, then for any $x \neq 0$, y is expressed in terms of z and t, whence the part $\{x \neq 0\}$ of the threefold $Y_{\overline{s}}$ is isomorphic to $\mathbf{C}^2 \times \mathbf{C}^*$. The 'book-surface' $B := \{x = 0\} \subset Y$ is the product $\mathbf{C} \times \Gamma_{s_2,s_3}$, where $\Gamma_{s_2,s_3} := \{z^{s_2} + t^{s_3} = 0\} \subset \mathbf{C}^2$. Fix a smooth point $\rho \in \Gamma_{s_2,s_3}$, and perform the Kaliman modification $\sigma : Y'_{\overline{s}} \longrightarrow Y_{\overline{s}}$ of $Y_{\overline{s}}$ along B with the center $C := \mathbf{C} \times \{\rho\}$. In this way, we replace B by a smooth surface $E \simeq \mathbf{C}^2$, and replace the function x by a function $f \,:\, Y'_{\overline{s}} \longrightarrow \mathbf{C}$ such that all the fibers of f are

[32] they are still available, although the quotient surface $S_{\overline{s}}$ might be singular.

[33] cf. [Ka 1, Lemma 16], [KoRu 2, Lemma 6.3].

smooth reduced surfaces isomorphic to $\mathbf{C}^2$. By a theorem of Miyanishi [Miy 1], $Y'_{\overline{s}} \simeq \mathbf{C}^3$, and so $\sigma : \mathbf{C}^3 \simeq Y'_{\overline{s}} \longrightarrow Y_{\overline{s}}$ is a birational (whence, dominant) morphism.

In the case of Russell's cubic $X_0 = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbf{C}^4$, a dominant morphism $\mathbf{C}^3 \longrightarrow X_0$ can be given explicitly as $(u, v, w) \longmapsto (x, y, z, t)$, where

$$(x, y, z, t) = \left(-u, \frac{u - (u^2v+1)^2 - (u^2w + u/3 - 1)^3}{u^2}, u^2v + 1, u^2w + u/3 - 1\right).$$

9 The Makar-Limanov invariant of the Russell cubic threefold

Let X be an affine variety. If X is irreducible, then the algebra $A = \mathbf{C}[X]$ of regular functions on X is an integral domain. In the sequel, this is supposed to be the case. Makar-Limanov [ML] (see also [KaML 3]) introduced a subring $\mathrm{ML}(A)$ of a ring A such that $\mathrm{ML}(A)$ is invariant under ring isomorphisms; that is, if $B \simeq A$, then $\mathrm{ML}(B) \simeq \mathrm{ML}(A)$. He proved the following

Theorem 9.1. (Makar-Limanov [ML]) *Set* $A_0 = \mathbf{C}[X_0]$, *where* $X_0 = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbf{C}^4$ *is the Russell cubic threefold. Then* $\mathrm{ML}(A_0)$ *is not isomorphic to* $\mathbf{C} = \mathrm{ML}(\mathbf{C}[x, y, z])$. *Thus,* X_0 *is not isomorphic to* $\mathbf{C}^3$, *and hence,* X_0 *is an exotic* $\mathbf{C}^3$.

Later on, Kaliman and Makar-Limanov [KaML 3] extended this result to all the Koras-Russell threefolds. This was one of the crucial steps in the recent proof of the Linearization Conjecture for $n = 3$ (see Koras and Russell [KoRu 2, KoRu 3], Kaliman, Koras, Makar-Limanov, Russell [KaKoMLRu]).

Theorem 9.2. [KoRu 2, KoRu 3, KaKoMLRu, KrPo, Po] *Any regular* $\mathbf{C}^*-$ *action on* $\mathbf{C}^3$ *is linearizable (i.e. it is conjugate with a linear* $\mathbf{C}^*-$ *action on* $\mathbf{C}^3$ *). Moreover, any regular action of a connected reductive group on* $\mathbf{C}^3$ *is linearizable.*

Here we give an exposition of Makar-Limanov's result following a simplified approach due to Derksen [De].

9.1 $\mathbf{C}_+-$ actions and locally nilpotent derivations

Definition 9.1. Let, as before, X be an affine variety, $A = \mathbf{C}[X]$ be the algebra of regular functions on X, and $\mathrm{LND}(A)$ be the set of all *locally nilpotent derivations* of

A, i.e. the set of all $\mathbf{C}-$ linear homomorphisms $A \xrightarrow{\partial} A$ satisfying the Leibnitz rule, and such that for any $a \in A$, $\partial^n a = 0$ for some $n = n(a) \in \mathbf{N}$. Any $\mathbf{C}_+-$ action[34] $\lambda : \mathbf{C}_+ \times X \longrightarrow X$ induces an algebra homomorphism

$$A \xrightarrow{\varphi} A[t] \simeq \mathbf{C}[\mathbf{C}_+ \times X], \quad p \in A \xmapsto{\varphi} p(\lambda(t,x)) \in A[t].$$

Set $\partial p = \frac{d}{dt}|_{t=0}\,(p \circ \lambda)$. Then $\partial \in \mathrm{LND}(A)$ (*exercise*). Vice versa, given $\partial \in \mathrm{LND}(A)$, consider the algebra homomorphism $\varphi_\partial : A \longrightarrow A[t]$ given by

$$\varphi_\partial(p) = \exp(t\partial)(p) = \sum_{i=0}^{\infty} \frac{t^i \partial^i p}{i!}, \quad p \in A.$$

Exercises (**9.1**) Show that φ_∂ corresponds to a $\mathbf{C}_+-$ action on X, and that this action is trivial iff $\partial = 0$.

(**9.2**) Prove the equality $A^{\varphi_\partial} = A^\partial$, where $A^\partial := \mathrm{Ker}\,\partial$, and A^{φ_∂} is the subalgebra of invariants of the $\mathbf{C}_+-$ action φ_∂ on X. Verify that this subalgebra is algebraically closed.

(**9.3**) Let $\partial \in \mathrm{LND}(A) \setminus \{0\}$. Verify that the transcendence degree of the algebra extension $[A : A^\partial]$ is 1. More precisely, let r_0 be any element of A such that $\partial r_0 \in A^\partial$ and $r_0 \notin A^\partial$. Show that the subalgebra $A^\partial[r_0] \subset A$ is a free $A^\partial-$ module, and for any $a \in A$ there exists $b \in A^\partial \setminus \{0\}$ such that $ba \in A^\partial[r_0]$.

(**9.4**) Given a linear representation $\varphi : \mathbf{C}_+ \longrightarrow \mathrm{GL}_n(\mathbf{C})$, $t \xmapsto{\varphi} e^{tB}$, where $B \in L_n(\mathbf{C})$, verify that it provides a regular $\mathbf{C}_+-$ action on $\mathbf{C}^n$ iff it is unipotent, i.e. iff B is a nilpotent matrix. Or, equivalently, iff the associated derivation $\partial_\varphi(p) =< Bx, \mathrm{grad}\, p >$ of the polynomial algebra $\mathbf{C}^{[n]} = \mathbf{C}[x_1, \ldots, x_n]$ is locally nilpotent.

(**9.5**) Let Γ be an irreducible affine algebraic curve. Show that, if $\mathbf{C}_+$ acts on Γ in a non-trivial way, then $\Gamma \simeq \mathbf{C}$.

Definition 9.2. Let A be an algebra over $\mathbf{C}$. The *Makar-Limanov invariant* $\mathrm{ML}(A)$ of A is the subalgebra $\mathrm{ML}(A) := \bigcap_{\partial \in \mathrm{LND}(A)} A^\partial \subset A$.

The *Derksen invariant* $\mathrm{Dk}(A)$ of A is the smallest subalgebra of A which contains A^∂ for all $\partial \in LND(A) \setminus \{0\}$.

Clearly, $\mathrm{ML}(\mathbf{C}^{[n]}) = \mathbf{C}$, and $\mathrm{Dk}(\mathbf{C}^{[n]}) = \mathbf{C}^{[n]}$.

[34] As before, $\mathbf{C}_+$ stands for the additive group of the complex number field; also, $\mathbf{C}^{[n]} = \mathbf{C}[x_1, \ldots, x_n]$ denotes the polynomial algebra in n variables.

Theorem 9.3. (Derksen [De]) $\mathrm{Dk}(A_0) \neq A_0$. *Hence,* A_0 *is not isomorphic to* $\mathbf{C}^{[3]}$, *and so, the Russell cubic* X_0 *is not isomorphic to* $\mathbf{C}^3$, *i.e.* X_0 *is an exotic* $\mathbf{C}^3$.

Before proceeding with the proof, we recall the following notions (see e.g. [Bou]).

9.2 Degree functions, filtrations and the associated graded algebras

Let A be an integral domain (usually, it will be also an algebra over $\mathbf{C}$).

Definition 9.3. A *degree function* $\deg : A \longrightarrow \mathbf{Z} \bigcup \{-\infty\}$ on[35] A is a map which satisfies the following axioms:

(d1) $\deg 0 = -\infty$, and $\deg a \in \mathbf{Z}$ for all $a \neq 0$; $\deg 1 = 0$.
(d2) $\deg fg = \deg f + \deg g$ for all $f, g \in A$.
(d3) $\deg (f+g) \leq \max\{\deg f, \deg g\}$ for all $f, g \in A$.

Definition 9.4. A degree function determines an *ascending filtration* $F = \{F^iA\}$ on A, where $F^iA := \{a \in A \mid \deg a \leq i\}$. This filtration satisfies the following conditions:

(f1) F^iA is a $\mathbf{C}-$ linear subspace of A, and $F^iA \subset F^{i+1}A$ (*ascending*).
(f2) $A = \bigcup_{i \in \mathbf{Z}} F^iA$ (*exhaustive*); $\bigcap_{i \in \mathbf{Z}} F^iA = \{0\}$ (*separated*); $1 \in F^0A \setminus F^{-1}A$.
(f3) $(F^iA \setminus F^{i-1}A)(F^jA \setminus F^{j-1}A) \subset (F^{i+j}A \setminus F^{i+j-1}A)$.

Clearly, $F^0A \subset A$ is a subring (resp. subalgebra), and A represents as an F^0A- module.

Vice versa, given a filtered domain (A, F) which satisfies the conditions (f1)-(f3), one can define a degree function d_F on A as follows: $d_F(0) = -\infty$ and $d_F(a) = i$ iff $a \in F^iA \setminus F^{i-1}A$ (*exercise*).

Definition 9.5. The *associated graded algebra* $\mathrm{Gr}\ A = \oplus_{i \in \mathbf{Z}} \mathrm{Gr}\ ^iA$ of a filtered algebra (A, F), where $\mathrm{Gr}\ ^iA := F^iA/F^{i-1}A$, can be identified with the algebra of the Laurent polynomials $\{\sum_{i=k}^{k+l} \widehat{f}_i u^i\}$, where $\widehat{f}_i$ is either zero or is equal to $\mathrm{gr}\ f_i := f_i + F^{i-1}A \in \mathrm{Gr}\ ^iA$ for some $f_i \in F^iA \setminus F^{i-1}A$. Due to the property (f3) of filtrations, the mapping $\mathrm{gr} : A \longrightarrow \mathrm{Gr}\ A$, $\mathrm{gr}\ f = \widehat{f}$, is a homomorphism of multiplicative semigroups.

Definition 9.6. A *weight degree function* on the polynomial algebra $\mathbf{C}^{[n]}$ is a degree function d such that $d(p) = \max_i\{d(m_i)\}$, where $p \in \mathbf{C}^{[n]}$ is a non-zero polynomial, and m_i runs over the set $M(p)$ of all the monomials of p. Clearly, d is uniquely

[35] In a similar way, one may define a degree function with values in arbitrary ordered semigroup.

determined by the *weights* $d_i := d(x_i)$, $i = 1, \ldots, n$. A weight degree function d defines a grading $\mathbf{C}^{[n]} = \oplus_{j \in \mathbf{Z}} \mathbf{C}^{[n]}_{d,j}$, where $\mathbf{C}^{[n]}_{d,j} \setminus \{0\}$ consists of all the $d-$ quasihomogeneous polynomials of $d-$ degree j. Accordingly, for any $p \in \mathbf{C}^{[n]} \setminus \{0\}$ we have a decomposition $p = \sum_{i=m(p)}^{d(p)} p_i$ into a sum of $d-$ quasihomogeneous components; here $p_d := p_{d(p)}$ is called the *principal* $d-$ *quasihomogeneous component* of p. It is clear that $(pq)_d = p_d q_d$.

Let $X = (I) \subset \mathbf{C}^n$ be a reduced irreducible affine variety defined by a prime ideal $I \subset \mathbf{C}^{[n]}$. Denote $A = \mathbf{C}[X] = \mathbf{C}^{[n]}/I$, and let $\widehat{I}$ be the (graded) ideal in $\mathbf{C}^{[n]}$ generated by the principal $d-$ quasihomogeneous components p_d, where p runs over I. We say that the weight degree function d is *appropriate for* I if the following conditions hold:

$(*)$ $\bar{0} \in X$, i.e. $I \subset \alpha := (x_1, \ldots, x_n)$;

$(**)$ the ideal $\widehat{I}$ is also prime, and $x_i \notin \widehat{I}$ $\forall\, i = 1, \ldots, n$.

For $f \in A \setminus \{0\}$ set

$$d_A(f) = \min_{p \in [f]} \{d(p)\}, \qquad \text{where} \qquad [f] := \{p \in \mathbf{C}^{[n]} \mid p \,|\, X = f\}.$$

Exercises (**9.6**) Show that $d_A(f) = d(p)$ for a polynomial $p \in [f]$ iff $p_d \notin \widehat{I}$.

(**9.7**) Assuming that d is appropriate for I, deduce that d_A is a degree function on A, and that $d_A(\tilde{x}_i) = d(x_i) = d_i$, where $\tilde{x}_i := x_i \,|\, X = x_i + I \in A$, $i = 1, \ldots, n$. Hence, due to the property (d2) of a degree function, $d_A(m \,|\, X) = d(m)$ for any monomial $m \in \mathbf{C}^{[n]}$.

Indication. Suppose that $f \in A$ and $d_A(f) = -\infty$, that is, there exists a sequence $p_j \in \mathbf{C}^{[n]}$, $j = 1, \ldots,$ such that $p_j \mid X = f$ and $\lim_{j \to \infty} d(p_j) = -\infty$. For $p \in \mathbf{C}^{[n]}$ set $\mu(p) = \min_{m \in M(p)} \{\deg m\}$, where deg is the usual degree. Then $p \in \alpha^{\mu(p)}$ (as above, $\alpha \subset \mathbf{C}^{[n]}$ denotes the maximal ideal which corresponds to the origin of $\mathbf{C}^n$). By the condition $(*)$, $\tilde{\alpha} := (\tilde{x}_1, \ldots, \tilde{x}_n) \subset A$ is a proper ideal, and we have $f = p_j \,|\, X \in \tilde{\alpha}^{\mu(p_j)}$, $j = 1, \ldots$. Thus, $f \in \bigcap_{n \in \mathbf{N}} \tilde{\alpha}^n = \{0\} \subset A$ (by the Krull Theorem), and so, $f = 0$. Hence, $d_A(f) > -\infty$ for any $f \in A \setminus \{0\}$.

The rest of the exercise, including checking of the other properties of a degree function, can be done without difficulty.

(**9.8**) let $F = \{F^i A\}$ be the filtration on A determined by the degree function d_A, and let $\widehat{A} = \operatorname{Gr} A$ be the associated graded algebra. Verify that the elements $\widehat{x}_1, \ldots, \widehat{x}_n \in \widehat{A}$, where $\widehat{x}_i := \operatorname{gr} \tilde{x}_i \in \widehat{A}$, generate the graded algebra $\widehat{A}$.

Lemma 9.1. (Kaliman, Makar-Limanov[36]) *Keeping the same notation and assumptions as in the above exercises, we have*

$$\widehat{A} \simeq \mathbf{C}^{[n]}/\widehat{I} = \mathbf{C}[\widehat{X}]\,,$$

where $\widehat{X} = (\widehat{I}) \subset \mathbf{C}^n$ *is the affine variety defined by the prime ideal*[37] $\widehat{I}$.

Proof. According to Exercise 9.8, the elements $\widehat{x}_1, \dots, \widehat{x}_n \in \widehat{A}$ generate the graded algebra $\widehat{A}$. Henceforth, $\widehat{A} = \mathbf{C}^{[n]}/J$, where $J \subset \mathbf{C}^{[n]} = \mathbf{C}[\widehat{x}_1, \dots, \widehat{x}_n]$ is the ideal of relations between the generators $\widehat{x}_1, \dots, \widehat{x}_n$ in $\widehat{A}$. Thus, we must show that $J = \widehat{I}$.

Fix $p = \sum\limits_{i=m(p)}^{d(p)} p_i \in I$. Then $p \equiv 0 \mod I$, i.e. $p_d \equiv -\sum\limits_{i=m(p)}^{d(p)-1} p_i \mod I$, and hence

$$d_A(p_d \,|\, X) \le \max_{m(p) \le i \le d(p)-1} \{d_A(p_i \,|\, X)\} \le \max_{m(p) \le i \le d(p)-1} \{d(p_i)\} < d(p) = d(p_d)\,.$$

Therefore, $p_d \,|\, X \in F^{d(p)-1}A$.

Since d is appropriate for I, by Exercise 9.7 we have $d_A(m_j \,|\, X) = d(m_j) = d(p)$ for any monomial $m_j \in M(p_d)$. Thus, $m_j \,|\, X \in F^{d(p)}A \setminus F^{d(p)-1}A$ for any $m_j \in M(p_d)$, and $p_d \,|\, X = \sum_{m_j \in M(p_d)} (m_j \,|\, X) \in F^{d(p)-1}A$. It follows that $(m_j \,|\, X)^\wedge := \mathrm{gr}\,(m_j \,|\, X) = m_j(\widehat{x}_1, \dots, \widehat{x}_n) \in \widehat{A}^{d(p)}$, and $\sum_{m_j \in M(p_d)} (m_j \,|\, X)^\wedge = 0$ in $\widehat{A}^{d(p)}$, i.e. $p_d(\widehat{x}_1, \dots, \widehat{x}_n) = 0$ in $\widehat{A}^{d(p)}$. Whence, $p_d \in J$, and so, $\widehat{I} \subset J$.

Vice versa, fix $f = \sum\limits_{i=m(f)}^{d(f)} f_i \in J$. It is clear that $f_i(\widehat{x}_1, \dots, \widehat{x}_n) \in \widehat{A}^i$ (indeed, as above, this is true for any monomial $m \in M(f_i)$). Since $\sum\limits_{i=m(f)}^{d(f)} f_i(\widehat{x}_1, \dots, \widehat{x}_n) = f(\widehat{x}_1, \dots, \widehat{x}_n) = 0$, we have $f_i(\widehat{x}_1, \dots, \widehat{x}_n) = 0$ for each $i = m(f), \dots, d(f)$. Thus, J is a homogeneous ideal of the $d-$ graded algebra $\mathbf{C}^{[n]}$ (see Definition 9.6). Hence, it is enough to show that $J_r \subset \widehat{I}$ for any $d-$ homogeneous component J_r of J.

Let $f \in J_r$ be a $d-$ quasihomogeneous polynomial of $d-$ degree $r = d(f)$. For any monomial $m \in M(f)$ we have, as above, that $m \,|\, X \in F^r A \setminus F^{r-1}A$, and so, $m(\widehat{x}_1, \dots, \widehat{x}_n) \in \widehat{A}^r$. Since $\sum_{m \in M(f)} m(\widehat{x}_1, \dots, \widehat{x}_n) = f(\widehat{x}_1, \dots, \widehat{x}_n) = 0$, it follows that $f \,|\, X \in F^{r-1}A$, i.e. $d_A(f \,|\, X) < r = d(f)$. By Exercise 9.6, this implies that $f_d \in \widehat{I}$. But $f = f_d$, and so, we are done. $\square$

Gradings and $\mathbf{C}^*-$ actions (see e.g. [KamRu, Ru 3]) Let $\widehat{X}$ be an affine variety endowed with a $\mathbf{C}^*-$ action t. Then t induces a grading $\widehat{A} = \oplus_{n \in \mathbf{Z}} \widehat{A}^n$ on the algebra

[36] A personal communication. We place here this lemma and the preceding definition and exercises with a kind permission of Sh. Kaliman.

[37] In fact, the same is true under the weaker assumption that $\widehat{I}$ is a proper prime ideal, instead of the conditions $(**)$ above.

$\widehat{A} = \mathbf{C}[\widehat{X}]$ of regular functions on $\widehat{X}$, where $\widehat{A}^n := \{f \in \widehat{A} \mid f \circ t_\lambda = \lambda^n f\}$ consists of the quasi-invariants of weight n of t.

Vice versa, given a grading $\widehat{A} = \oplus_{n\in\mathbf{Z}} \widehat{A}^n$ of $\widehat{A} = \mathbf{C}[\widehat{X}]$, one can define a $\mathbf{C}^*-$ action on $\widehat{A}$ by setting $t_\lambda(f_n) = \lambda^n f_n$ for $f_n \in \widehat{A}^n, \ \ n \in \mathbf{Z}$, and extending it to the whole $\widehat{A}$ in a natural way. If $\widehat{A}$ is finitely generated, then it also has a finite system of homogeneous generators $(f_{n_1}, \ldots, f_{n_k}), \quad f_{n_i} \in \widehat{A}^{n_i}$. The morphism $F = (f_{n_1}, \ldots, f_{n_k}) : \widehat{X} \hookrightarrow \mathbf{C}^k$ is an embedding equivariant with respect to the linear $\mathbf{C}^*-$ action $t_\lambda(x_1, \ldots, x_k) = (\lambda^{n_1} x_{n_1}, \ldots, \lambda^{n_k} x_{n_k})$ on $\mathbf{C}^k$ and the induced $\mathbf{C}^*-$ action on $\widehat{X}$.

Gradings and locally nilpotent derivations (see e.g. [ML, KaML 3, De])

Definition 9.7. Let $\partial \in \mathrm{LND}(A)\setminus\{0\}$, where $(A,\, F)$ is a filtered domain. Suppose that

$(*)$ *there exists* $k \in \mathbf{Z}$ *such that* $\partial F^i A \subset F^{i+k} A$ *for all* $i \in \mathbf{Z}$.

Denote by $\deg \partial = k_0$ the minimal such k. Define $\widehat{\partial} = \mathrm{gr}\, \partial : \mathrm{Gr}\, A \longrightarrow \mathrm{Gr}\, A$ as follows: for $f \in F^i A \setminus F^{i-1} A$, set $\widehat{\partial} \widehat{f} = \partial f + F^{i+k_0-1} A$, and then naturally extend $\widehat{\partial}$ to the whole algebra $\mathrm{Gr}\, A$.

Exercises (**9.9**) Given $\partial \in LND(A) \setminus \{0\}$, verify that $\widehat{\partial} \in \mathrm{LND}_{\mathrm{gr}}(\mathrm{Gr}\, A) \setminus \{0\}$, where $\mathrm{LND}_{\mathrm{gr}}(\widehat{A})$ denotes the set of all homogeneous locally nilpotent derivations of a graded algebra $\widehat{A} = \oplus_{n\in\mathbf{Z}} \widehat{A}^n$.

(**9.10**) Suppose that a filtered domain $(A,\, F)$ is finitely generated. Show that, given $\partial \in \mathrm{LND}(A) \setminus \{0\}$, the condition $(*)$ is fulfilled.

(**9.11**) Let $\widehat{A} = \oplus_{n\in\mathbf{Z}} \widehat{A}^n$ be a graded algebra. Show that, given any $\widehat{\partial} \in \mathrm{LND}_{\mathrm{gr}}(\widehat{A})$, there exists $k_0 = k_0(\widehat{\partial}) \in \mathbf{Z}$ called the *degree* of $\widehat{\partial}$ such that $\widehat{\partial}(\widehat{A}^n) \subset \widehat{A}^{n+k_0}$. Furthermore, show that, if $f = \sum_{i=k}^{k+l} f_i \in \mathrm{Ker}\, \widehat{\partial} = \widehat{A}^{\widehat{\partial}}$, where $f_i \in \widehat{A}^i$, then $f_i \in \widehat{A}^{\widehat{\partial}}$, $i = k, \ldots, k+l$. Therefore, $\widehat{A}^{\widehat{\partial}}$ is a graded subalgebra of $\widehat{A}$.

(**9.12**) Let A be an integral domain. Given $\partial \in \mathrm{LND}(A)$, set $\deg_\partial f = n$ if $\partial^{n+1} f = 0$ and $\partial^n f \neq 0$; $\deg_\partial 0 = -\infty$. Verify that $\deg_\partial$ is a degree function on A (over $\mathbf{N}$). Given $f, g \in A \setminus \{0\}$, show that $\partial(fg) = 0$ implies $\partial f = \partial g = 0$. Check that, if A is a $\mathbf{C}-$ algebra, then $\partial \lambda = 0$ for any $\lambda \in \mathbf{C}$.

(**9.13**) Let $A = \mathbf{C}[x,\, y]$. Consider the $\mathbf{C}_+-$ action $\varphi_\lambda : (x, y) \longmapsto (x, y + \lambda x^2)$ on $\mathbf{C}^2$. Let ∂_φ be the locally nilpotent derivation which corresponds to φ. Show that $\partial_\varphi(x) = 0, \quad \partial_\varphi(y) = x^2$ and $\partial_\varphi^2(y) = 0$. Deduce that $\deg_\varphi x = 0, \quad \deg_\varphi y = 1$ for the associated degree function $\deg_\varphi$ on A, and so, that $\deg_\varphi f = \deg_y f$ for any polynomial $f = f(x, y) \in A$.

9.3 Gradings and LND's on Russell's cubic

From now on $A = A_0 = \mathbf{C}[x,y,z,t]/(p_0)$, where $p_0 = x + x^2y + z^3 + t^2$, will be the algebra of regular functions on the Russel cubic threefold $X_0 \subset \mathbf{C}^4$. Consider the weight degree function $\deg x = -1, \deg y = 2, \deg z = \deg t = 0$ on $\mathbf{C}^{[4]}$. It is easily seen that deg is appropriate for the principal ideal $I := (p_0)$ (see Definition 9.6). Hence, by Exercise 9.7, it induces a degree function d_A on A, which, in turn, defines a filtration on A. Let $\widehat{A} := \mathrm{Gr}\, A$ be the associated graded algebra. From Lemma 9.1 we obtain such a corollary.

Corollary 9.1. *$\widehat{A} \simeq \mathbf{C}[\widehat{x},\widehat{y},\widehat{z},\widehat{t}]/(q_0)$, where $q_0(\widehat{x},\widehat{y},\widehat{z},\widehat{t}) = \widehat{x}^2\widehat{y} + \widehat{z}^3 + \widehat{t}^2$, i.e. $\widehat{A} = \mathbf{C}[\widehat{X_0}]$, where $\widehat{X_0} = \{\widehat{x}^2\widehat{y} + \widehat{z}^3 + \widehat{t}^2 = 0\}$.*

Lemma 9.2. (a) (exercise) *Any $f \in A$ has a unique representation*

$$f = a(x,z,t) + yb(y,z,t) + xyc(y,z,t)\,,$$

where a, b, c are polynomials.
(b) *We have:*
(i) $\widehat{f} = \widehat{x}^r h(\widehat{z},\widehat{t})$ iff $\deg f = r \le 0$;
(ii) $\widehat{f} = \widehat{y}^r h(\widehat{z},\widehat{t})$ iff $\deg f = 2r > 0$;
(iii) $\widehat{f} = \widehat{x}\widehat{y}^r h(\widehat{z},\widehat{t})$ iff $\deg f = 2r - 1 > 0$,
where $h(\widehat{z},\widehat{t}) \in \mathbf{C}[\widehat{z},\widehat{t}]$.

Proof of (b). Since the degree $\deg\,[yb(y,z,t)]$ is even and positive when $b \neq 0$, $\deg\,[xyc(y,z,t)]$ is odd and positive when $c \neq 0$, and $\deg a(x,z,t) \le 0$, we have that, in the case (i), $f = a(x,z,t)$, and hence $\widehat{f} = \widehat{x}^r h(\widehat{z},\widehat{t})$; in the case (ii), $\widehat{f} = \mathrm{gr}\,[yb(y,z,t)] = \widehat{y}^r h(\widehat{z},\widehat{t})$; finally, in the case (iii), $\widehat{f} = \mathrm{gr}\,[xyc(y,z,t)] = \widehat{x}\widehat{y}^r h(\widehat{z},\widehat{t})$ for some $h(\widehat{z},\widehat{t}) \in \mathbf{C}[\widehat{z},\widehat{t}]$. □

It follows that $\widehat{A}^0 = \mathbf{C}[\widehat{z},\widehat{t}]$ (and thus $\widehat{A}$ is a $\mathbf{C}[\widehat{z},\widehat{t}]-$ module); $\widehat{A}^i = \widehat{x}^{-i}\mathbf{C}[\widehat{z},\widehat{t}]$ for $i \le 0$; $\widehat{A}^{2r} = \widehat{y}^r\mathbf{C}[\widehat{z},\widehat{t}]$, and $\widehat{A}^{2r-1} = \widehat{x}\widehat{y}^r\mathbf{C}[\widehat{z},\widehat{t}]$ for $r > 0$.

Lemma 9.3. *$A^\partial \subset F^0A$ for any $\partial \in \mathrm{LND}(A) \setminus \{0\}$. Hence, $\mathrm{Dk}(A) \subset F^0A \neq A$, which proves Theorem 9.3.*

Proof. Assume the contrary, i.e. that for some $f \in A$, where $\deg f > 0$, and for some $\partial \in \mathrm{LND} \setminus \{0\}$ we have $\partial f = 0$. Then $\deg \widehat{f} > 0$ as well, and $\widehat{\partial}\widehat{f} = 0$ (see Exercise 9.9). Thus, it suffices to prove the following

Claim 1. *For any* $\widehat{\partial} \in \mathrm{LND}_{\mathrm{gr}}(\widehat{A}) \setminus \{0\}$, *and for any homogeneous* $\widehat{f} \in \widehat{A}^n$, $\widehat{\partial}\widehat{f} = 0$ *implies that* $n := \deg \widehat{f} \le 0$.

Proof of Claim 1. Assume, on the contrary, that $n > 0$. Suppose first that n is odd, i.e. $n = 2r - 1$ for some $r \in \mathbf{N}$. Then, by Lemma 9.2(b), $\widehat{f} = \widehat{x}\widehat{y}^r h(\widehat{z}, \widehat{t})$ for some non-zero polynomial $h(\widehat{z}, \widehat{t}) \in \mathbf{C}[\widehat{z}, \widehat{t}]$, and $\widehat{\partial}\widehat{f} = 0$ implies that $\widehat{\partial}\widehat{x} = \widehat{\partial}\widehat{y} = \widehat{\partial}h = 0$ (see Exercise 9.12 above). Thus, $\widehat{x}, \widehat{y} \in \widehat{A}^{\widehat{\partial}}$ are invariants of the associated $\mathbf{C}_+-$ action $\varphi = \varphi_{\widehat{\partial}}$ on $\widehat{X}_0 = \{\widehat{x}^2\widehat{y} + \widehat{z}^3 + \widehat{t}^2 = 0\}$ (see Exercise 9.2 above). Therefore, each orbit of φ is contained in a curve $\Gamma_{c_1,c_2} = \{\widehat{x} = c_1, \widehat{y} = c_2\} \subset \widehat{X}_0$. Conversely, any such curve in $\widehat{X}_0$ consists of $\mathbf{C}_+-$ orbits. Since a generic curve $\Gamma_{c_1,c_2} = \{\widehat{z}^3 + \widehat{t}^2 = -c_1^2 c_2\}$ is elliptic, it admits no embedding of $\mathbf{C}$ (cf. Exercise 9.5 above), and hence all the points of Γ_{c_1,c_2} are fixed by φ. It follows that the $\mathbf{C}_+-$ action on $\widehat{X}_0$ is trivial, i.e. $\widehat{\partial} = 0$, a contradiction.

Now consider the case when $n > 0$ is even, i.e. $n = 2r$ for some $r \in \mathbf{N}$. In this case, $\widehat{f} = \widehat{y}^r h(\widehat{z}, \widehat{t})$, where $h(\widehat{z}, \widehat{t}) \in \mathbf{C}[\widehat{z}, \widehat{t}]$, and $h \ne 0$. Hence, $\widehat{\partial}\widehat{y} = 0$, and so $\widehat{y} \in \widehat{A}^{\widehat{\partial}}$ is an invariant of the $\mathbf{C}_+-$ action φ on $\widehat{X}_0$. Thus, $\widehat{X}_0$ is foliated by the $\varphi-$ invariant surfaces $S_c = \{\widehat{y} = c\}$, $c \in \mathbf{C}$ (see Exercise 9.2). Denote by $\partial_c \in \mathrm{LND}(A_c)$ the corresponding locally nilpotent derivation on $A_c = \mathbf{C}[S_c]$, that is, the infinitesimal generator of the $\mathbf{C}_+-$ action $\varphi_c := \varphi \mid S_c$ on S_c. For a generic $c \in \mathbf{C}$, φ_c is a non-trivial $\mathbf{C}_+-$ action on S_c, whence $\partial_c \ne 0$.

Next we show that there exists a non-constant $\varphi-$ invariant function $h_1(\widehat{z}, \widehat{t}) \in \widehat{A}^{\widehat{\partial}}$. Indeed, since $\mathrm{tr.deg}\, \widehat{A}^{\widehat{\partial}} = \mathrm{tr.deg}\, \widehat{A} - 1 = 2$ (see Exercise 9.3 above), $\widehat{A}^{\widehat{\partial}}$ contains a function g such that $\widehat{y}$ and $\widehat{g}$ are algebraically independent; in particular, $\widehat{g} \notin \mathbf{C}[\widehat{y}]$. Furthermore, $\widehat{g}$ and $\widehat{y}$ are both $\varphi-$ invariants, and so, for $s \in \mathbf{N}$ sufficiently large $\widehat{g}\widehat{y}^s$ is a $\varphi-$ invariant of a positive degree. We have proven above that the equality $\widehat{g}\widehat{y}^s = \widehat{x}\widehat{y}^r h_1(\widehat{z}, \widehat{t})$ is impossible. Hence, we get $\widehat{A}^{\widehat{\partial}} \ni \widehat{g}\widehat{y}^s = \widehat{y}^r h_1(\widehat{z}, \widehat{t})$ for some $r > 0$, where $h_1 \in \widehat{A}^{\widehat{\partial}}$ is non-constant.

To get a final contradiction, we prove

Claim 2. *For* $c \ne 0$, *the surface* $S_c = \{c\widehat{x}^2 + \widehat{z}^2 + \widehat{t}^3 = 0\} \subset \mathbf{C}^3$ *does not admit any non-trivial* $\mathbf{C}_+-$ *action with an invariant function* $h_1(\widehat{z}, \widehat{t}) \ne \mathrm{const}$.

Proof of Claim 2.[38] Assume the contrary, i.e. that for some $c \ne 0$, the surface S_c does admit a non-trivial $\mathbf{C}_+-$ action with a non-constant invariant function $h_1(\widehat{z}, \widehat{t})$. We may suppose that the general fibers $C_\lambda = \{h_1(\widehat{z}, \widehat{t}) = \lambda\}$ of the polynomial h_1 in $\mathbf{C}^2$ are smooth irreducible affine plane curves. The generic fiber $F_\lambda = (h_1 \mid S_c)^{-1}(\lambda) \subset S_c$

[38] We give a simplified proof suggested by Sh. Kaliman; cf. [De].

of the regular function h_1 on S_c is represented as a two-sheeted ramified covering of C_λ under the projection $\pi : S_c \to \mathbf{C}^2,\ (\hat{x}, \hat{z}, \hat{t}) \mapsto (\hat{z}, \hat{t})$. Since F_λ coincides with a one-dimensional orbit of the $\mathbf{C}_+-$ action, the curve C_λ admits a dominant morphism $\mathbf{C} \simeq F_\lambda \to C_\lambda$, and hence, it is isomorphic to $\mathbf{C}$ (*exercise*; cf. Exercise 9.5 above). Moreover, it meets the ramification locus $\Gamma_{2,3} = \{\hat{z}^2 + \hat{t}^3 = 0\} \subset \mathbf{C}^2$ of the projection π at most at one point (indeed, a quadratic polynomial of one variable has only one critical value). If a generic curve C_λ does not meet $\Gamma_{2,3}$, then it should be contained in an elliptic curve $\hat{z}^2 + \hat{t}^3 = \text{const} \neq 0$, which is impossible. Therefore, the restriction $h_1 \mid \Gamma_{2,3} : \Gamma_{2,3} \to \mathbf{C}$ is generically one-to-one, which is impossible, too. This completes the proof. □

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