Geometrically finite Kleinian groups:
the completeness of Nayatani's metric

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Abstract. We give a criteria for the completeness of Nayatani's metric on the domain of discontinuity of a geometrically finite kleinian group.

Complétude de la métrique de Nayatani

pour les groupes kleiniens géométriquement finis

Résumé. Nous donnons un critère de complétude pour la métrique de Nayatani sur le domaine de discontinuité d'un groupe kleinien géométriquement fini.

Version Française Abrégée

Soit Γ un groupe kleinien non élémentaire. Γ est par définition un sous-groupe du groupe de Möbius $M(n)$: ses éléments agissent par isométries sur le modèle en boule $\mathbb{B}^{n+1}$ de l'espace hyperbolique de dimension $n+1$ et par transformations conformes sur son bord géométrique, la sphère unité $\mathbb{S}^n$. En utilisant les mesures de Patterson-Sullivan, S. Nayatani [3] a construit une métrique g sur l'ensemble de discontinuité $\Omega(\Gamma) \subset \mathbb{S}^n$ de Γ, conforme à la métrique standard g_0 de la sphère et invariante par Γ. En notant $\Lambda(\Gamma)$ l'ensemble limite de Γ, $\delta = \delta(\Gamma)$ son exposant critique, μ une mesure de Patterson-Sullivan sur $\Lambda(\Gamma)$ et $|.|$ la norme euclidienne de $\mathbb{R}^{n+1}$, g est donnée par:

$$g = \left(\int_{\Lambda(\Gamma)} \frac{2^\delta}{|x-y|^{2\delta}} \, d\mu(y) \right)^{2/\delta} g_0.$$

Si Γ agit librement sur $\Omega(\Gamma)$, on obtient une métrique compatible sur la variété kleinienne

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$M = \Omega(\Gamma)/\Gamma$. S. Nayatani a montré que la courbure scalaire de g s'exprime simplement en fonction de l'exposant critique $\delta(\Gamma)$ de Γ, ce qui permet de faire le lien avec les travaux de R. Schoen et S.T. Yau (cf [5]) sur les variétés conformément plates.

Si Γ est convexe cocompact, la variété quotient M est compacte et donc la métrique g est complète sur $\Omega(\Gamma)$. Nous nous intéressons dans cette note à la complétude de g dans le cas où Γ est seulement supposé géométriquement fini. En comparant le facteur conforme de g avec les masses totales de mesures de Patterson-Sullivan bien choisies, et grâce à l'article [7] de D. Sullivan, on voit apparaître une condition portant sur le rang des cusps. Plus précisément, on montre que g est complète si et seulement si l'ensemble limite $\Lambda(\Gamma)$ de Γ ne contient pas de point fixe parabolique de rang $k < \delta(\Gamma)$ (théorème 1).

Can one realize conformally flat Riemannian manifolds as Kleinian manifolds, that is, quotients of (connected) domains of the unit sphere $\mathbb{S}^n$ by Kleinian groups? Schoen and Yau [5] answered positively this question for a large class of conformally flat manifolds, which in particular contains those which admit a compatible complete metric of non-negative scalar curvature.

Conversely, given a non-elementary Kleinian group Γ, Nayatani [3] constructed a metric g on the domain of discontinuity $\Omega(\Gamma)$ of Γ, which is conformal to the standard metric g_0 of $\mathbb{S}^n$:

$$g = \left(\int_{\Lambda(\Gamma)} \frac{2^\delta}{|x-y|^{2\delta}} \, d\mu(y) \right)^{2/\delta} g_0,$$

where $\Lambda(\Gamma) \subset \mathbb{S}^n$ is the limit set of Γ, $\delta = \delta(\Gamma)$ its critical exponent, $|.|$ is the euclidian norm of $\mathbb{R}^{n+1}$ and μ is a borel measure supported on $\Lambda(\Gamma)$ of unit total mass such that for every borel subset E of $\mathbb{S}^n$ and for every γ in Γ,

$$\mu(\gamma(E)) = \int_E |\gamma'(y)|^\delta d\mu(y)$$

(see Patterson [4] and Sullivan [6]). Following Sullivan, we call μ a geometric measure of unit total mass.

This metric is Γ-invariant and hence if Γ acts freely on $\Omega(\Gamma)$, it gives rise to a compatible metric on the Kleinian manifold $M = \Omega(\Gamma)/\Gamma$. Nayatani also showed that the scalar curvature of g is closely related to the critical exponent of Γ.

If M is compact, for example if Γ is convex cocompact (cf [8]), the metric g is automatically complete on $\Omega(\Gamma)$. When Γ is geometrically finite with parabolic elements, the situation becomes somewhat more complicated. The purpose of this note is to prove the following theorem for a geometrically finite Kleinian group:

THEOREM 1. — *Let Γ be a non-elementary geometrically finite Kleinian group, $\Omega(\Gamma) \subset \mathbb{S}^n$ its domain of discontinuity and $\delta(\Gamma)$ its critical exponent. The Nayatani metric g on $\Omega(\Gamma)$ is complete if and only if the limit set $\Lambda(\Gamma)$ of Γ does not contain any parabolic fixed point of rank $k < \delta(\Gamma)$.*

Let $\mathbb{B}^{n+1}$ be the unit ball model of the hyperbolic $(n+1)$-space, and let $\mathbb{S}^n = \partial_\infty \mathbb{B}^{n+1}$ be its boundary at infinity. Sullivan showed that, for Γ non-elementary and geometrically finite (what we now assume), there is, up to a constant multiple, only one geometric measure. Hence we may assume that $\mu = \mu_0$, the Patterson-Sullivan measure at $0 \in \mathbb{B}^{n+1}$, and that $g = f\,g_0$, whith

$$\begin{array}{rcl} f : \Omega(\Gamma) & \longrightarrow & \mathbb{R}^+ \\ x & \longmapsto & f(x) = \left(\int_{\Lambda(\Gamma)} \frac{2^\delta}{|x-y|^{2\delta}}\, d\mu_0(y) \right)^{2/\delta}. \end{array}$$

For $x \in \mathbb{B}^{n+1}$, let μ_x denote the Patterson-Sullivan measure at x and let φ_μ be the map "total mass" on $\mathbb{B}^{n+1}$ defined by $\varphi_\mu(x) = \mu_x(\mathbb{S}^n)$. φ_μ is continuous and Γ-invariant. In fact, we have that

$$\varphi_\mu(x) = \int_{\Lambda(\Gamma)} \left(\frac{1-|x|^2}{|x-y|^2} \right)^\delta d\mu_0(y).$$

Thanks to the work of Sullivan, φ_μ is a much more known object than f. Thus we would like to compare $f(x)$ for $x \in \Omega(\Gamma)$ with $\varphi_\mu(x')$ for $x' \in \mathbb{B}^{n+1}$ well chosen.

LEMMA 1. — *Let $x \in \Omega(\Gamma)$, $r(x) = \inf_{y\in\Lambda(\Gamma)} |x-y|$ and $\xi \in \Lambda(\Gamma)$ such that $|x-\xi| = r(x)$. Let $\nu(\xi, r(x))$ be the point on the geodesic $[0,\xi)$ in $\mathbb{B}^{n+1}$ at hyperbolic distance $\ln(1/r(x))$ from 0. Then we have:*

$$f(x) \asymp \frac{1}{r(x)^2}\, \varphi_\mu(\nu(\xi, r(x)))^{2/\delta},$$

where $\asymp$ means that there exist a constant $C > 0$, which depends only on Γ (and maybe on the equivalence class of ξ under Γ), such that

$$\frac{1}{C}\frac{1}{r(x)^2}\, \varphi_\mu(\nu(\xi, r(x)))^{2/\delta} \le f(x) \le C\frac{1}{r(x)^2}\, \varphi_\mu(\nu(\xi, r(x)))^{2/\delta}.$$

Proof. — Let $r = r(x)$. From the hyperbolic formulas (cf [1]), we have

$$|\nu(\xi, r)| = \frac{1-r}{1+r} \quad \text{and} \quad |\nu(\xi, r) - \xi| = \frac{2r}{1+r}.$$

The following is just an obvious computation: let $y \in \Lambda(\Gamma)$,

$$\begin{array}{rcl} |\nu(\xi,r) - y| & \le & |\nu(\xi,r) - \xi| + |\xi - x| + |x-y| \\ & \le & \frac{2r}{1+r} + 2|x-y| \\ & \le & 4|x-y| \end{array}$$

In the same way, we obtain the reverse inequality and thus

$$|x-y| \asymp |\nu(\xi, r) - y|.$$

Hence

$$f(x) \asymp f(\nu(\xi, r)),$$

and

$$f(x) \asymp \frac{(1+r^2)^2}{r^2}\, \varphi_\mu(\nu(\xi,r))^{2/\delta},$$

which yields the result. □

From now on, we will assume that $0 \in \mathbb{B}^{n+1}$ belongs to the hyperbolic convex hull $C(\Lambda(\Gamma))$ of $\Lambda(\Gamma)$ in $\mathbb{B}^{n+1}$ (if not, just take an L-neighbourhood $V_L(C(\Lambda(\Gamma)))$ of $C(\Lambda(\Gamma))$ containing it and replace $C(\Lambda(\Gamma))$ by $V_L(C(\Lambda(\Gamma)))$ in the proof).

If Γ is geometrically finite without parabolic element, that is, convex cocompact, $C(\Lambda(\Gamma))/\Gamma$ is by definition compact. Hence by continuity and Γ-invariance, φ_μ is bounded from below on $C(\Lambda(\Gamma))$ by a strictly positive constant. We denote the g_0-distance from x to $\Lambda(\Gamma)$ by $\rho(x)$. Since $\nu(\xi, r(x)) \in C(\Lambda(\Gamma))$ and $\rho(x) \asymp r(x)$, we have

$$\exists C > 0 \text{ such that } \forall x \in \Omega(\Gamma), f(x) \geq \frac{C}{\rho(x)^2},$$

which gives another proof of the completeness of g on $\Omega(\Gamma)$.

But when Γ is geometrically finite with parabolic elements, $C(\Lambda(\Gamma))/\Gamma$ consists of a compact piece with boundary and a finite number of exponentially skinny ends attached, called cuspidal ends, one for each class under Γ of parabolic fixed points ([7]) . φ_μ is then bounded from below only in the compact part of $C(\Lambda(\Gamma))$ mod Γ and we need a control of φ_μ inside the cuspidal ends, which is precisely given by the following result of Sullivan ([7]).

THEOREM 2. — *Let $\xi \in \Lambda(\Gamma)$ be a rank k parabolic fixed point. There is a constant r_0, which depends only on Γ, k, and the class of ξ under Γ, such that:*

$$\forall r \leq r_0, \quad \varphi_\mu(\nu(\xi,r)) \asymp \left(\frac{1}{r}\right)^{k-\delta}.$$

Proof. — It is more convenient to work in the upper half-space model $\mathbb{H}^{n+1}$ of the hyperbolic $(n+1)$-space. We write $\overline{\mathbb{H}^{n+1}} = \mathbb{R}^k \oplus \mathbb{R}^{n-k} \oplus \mathbb{R}^+$ and we denote by $[x]_k$, $[x]_{n-k}$, $[x]_{n+1}$, the images of $x \in \overline{\mathbb{H}^{n+1}}$ under the projections on $\mathbb{R}^k$, $\mathbb{R}^{n-k}$, $\mathbb{R}^+$.

We come back to the construction of the Patterson-Sullivan measures. Let d denote the hyperbolic distance in $\mathbb{H}^{n+1}$. Fix a point y in $\mathbb{H}^{n+1}$ and define the absolute Poincaré series by:

$$g_s(x,y) = \sum_{\gamma\in\Gamma} \exp(-s\, d(x,\gamma y)), \text{ for } x \in \mathbb{H}^{n+1}.$$

By definition, $\delta = \delta(\Gamma) = \inf\{s \in \mathbb{R}^+ / g_s(x,y) < \infty\}$. For $s > \delta$, form the measures

$$\mu_{s,x} = \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} \exp(-s\, d(x,\gamma y))\, \delta(\gamma y),$$

where $\delta(\gamma y)$ is the Dirac atomic mass at γy. Since Γ is geometrically finite, $g_\delta(y,y) = \infty$ ([6]) and the Patterson-Sullivan measure at x is given by:

$$\mu_x = \lim_{s\to\delta} \mu_{s,x}.$$

Hence

$$\varphi_\mu(x) = \mu_x(\mathbb{S}^n) = \lim_{s\to\delta} \mu_{s,x}(\mathbb{H}^{n+1}) = \lim_{s\to\delta} \frac{g_s(x,y)}{g_s(y,y)}.$$

Thus we need to estimate $g_s(x,y)$, for $s > \delta$, when $x \in \mathbb{H}^{n+1}$ goes to the rank k parabolic fixed point ξ. Up to conjugation, we may assume that $\xi = \infty$ and so we compute $g_s(x_z, y)$ for $x_z = (0, \cdots, 0, z)$. Let $\Gamma_\infty \subset \Gamma$ be the stabilizer of ∞.

$$\begin{array}{rcl} g_s(x_z,y) & = & \sum_{\gamma\in\Gamma} \exp(-s\, d(x_z, \gamma y)) \\ & = & \sum_{h|h(y)\in P_\infty} \sum_{\gamma\in\Gamma_\infty} \exp(-s\, d(x_z, \gamma^{-1} h(y))) \\ & = & \sum_{h|h(y)\in P_\infty} \sum_{\gamma\in\Gamma_\infty} \exp(-s\, d(\gamma x_z, hy)) \end{array}$$

where P_∞ is a convex fundamental polyhedra for the action of Γ_∞ on $\mathbb{H}^{n+1}$.

We have, see Beardon [1], that

$$\exp(d(\gamma x_z, hy)) \asymp \cosh(d(\gamma x_z, hy)) = 1 + \frac{|\gamma x_z - hy|^2}{2z[hy]_{n+1}}.$$

But, up to conjugation, one may assume that Γ_∞ leaves $\mathbb{R}^k$ globally invariant and acts cocompactly on it: $\mathbb{R}^k/\Gamma_\infty$ is compact. Then, the action of Γ_∞ on $\overline{\mathbb{H}^{n+1}}$ is given by

$$\gamma(x) = ([x]_k + a_\gamma\, ,\, \alpha_\gamma([x]_{n-k})\, ,\, [x]_{n+1}), \text{ where } \left\{ \begin{array}{l} a_\gamma \in \mathbb{R}^k \\ \alpha_\gamma \in O(n-k), \end{array}\right.$$

so that

$$|\gamma x_z - hy|^2 = |a_\gamma - [hy]_k|^2 + |[hy]_{n-k}|^2 + |z - [hy]_{n+1}|^2.$$

The next theorem (see Tukia [8]) describes the structure of the limit set near a rank k parabolic fixed point:

Theorem 3. — *Let Γ be a geometrically finite group, and ∞ a rank k parabolic fixed point. Then there is a neighbourhood U_∞ of ∞ in $\overline{\mathbb{H}^{n+1}}$ of the form*

$$\overline{\mathbb{H}^{n+1}} \cap \left(\mathbb{R}^{n+1} \backslash (\mathbb{R}^k \times \overline{B^{n-k}(0,R)})\right)$$

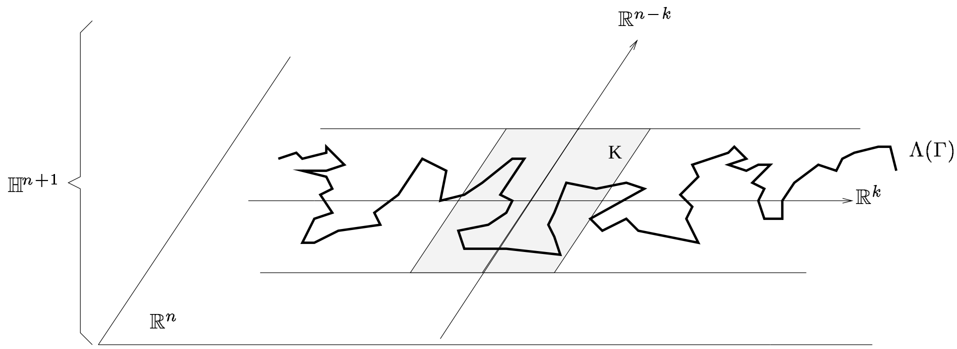
where $B^{n-k}(0,R)$ is the euclidian ball in $\mathbb{R}^{n-k}$ of center 0 and radius R, such that $\Lambda(\Gamma) \cap U_\infty = \{\infty\}$, U_∞ is Γ_∞-invariant and $h(U_\infty) \cap U_\infty = \emptyset$ for every $h \in \Gamma\backslash\Gamma_\infty$. Such a neighbourhood is called cuspidal.

and allows us to show the

Lemma 2. — *Let Γy be the orbit of y under Γ and P_∞ a convex fundamental polyhedra for the action of Γ_∞ on $\mathbb{H}^{n+1}$. $\Gamma y \cap P_\infty$ is bounded for the euclidian norm $|.|$ on $\overline{\mathbb{H}^{n+1}} \subset \mathbb{R}^{n+1}$.*

Proof of the lemma. — From theorem 3, there exists a constant R such that for all ξ in $\Lambda(\Gamma)\backslash\{\infty\}$, $|[\xi]_{n-k}| \leq R$. Hence there exists a compact $K \subset \mathbb{R}^n$ such that

$$\forall \xi \in \Lambda(\Gamma)\backslash\{\infty\}, \exists \gamma \in \Gamma_\infty \mid \gamma\xi \in K.$$



Now, assume that $\Gamma y \cap P_\infty$ is unbounded: there exists a sequence $(h_m)_{m\in\mathbb{N}}$ of distinct elements of Γ such that $h_m(y) \in P_\infty$ and $|h_m(y)| \longrightarrow \infty$. Then,

$$\begin{cases} |[h_m(y)]_k| \text{ is bounded, for } \mathbb{R}^k/\Gamma_\infty \text{ compact implies } P_\infty \cap \mathbb{R}^k \text{ bounded,} \\ |[h_m(y)]_{n-k}| + |[h_m(y)]_{n+1}| \longrightarrow \infty. \end{cases}$$

We may assume that $y \in P_\infty$ and $h_m \in \Gamma\backslash\Gamma_\infty$. Let P be a convex fundamental polyhedra for the action of Γ on $\mathbb{H}^{n+1}$. We can choose P so that $\infty \in \overline{P}$. Indeed, consider the Dirichlet polyhedra P based at $p \in U_\infty$ with $|[p]_{n+1}| > R$. Since $g(U_\infty) \cap U_\infty = \emptyset$ for all $h \in \Gamma\backslash\Gamma_\infty$, $|[\gamma p]_{n+1}| \leq |[p]_{n+1}|$ for every $\gamma \in \Gamma$. Hence $\overline{P}$ must contain ∞.

Let x be a point in P and σ be the geodesic $[x, \infty)$: $\sigma \subset P$ because P is convex. $d(h_m(x), h_m(y)) = d(x, y)$ and so $|[h_m(x)]_{n-k}| + |[h_m(x)]_{n+1}| \longrightarrow \infty$. Since h_m doesn't belong to Γ_∞, $h_m(\infty) \in \Lambda(\Gamma)\backslash\{\infty\}$ and there exists $\gamma_m \in \Gamma_\infty$ such that $\gamma_m \circ h_m(\infty) \in K$. Remark that we still have $|[\gamma_m \circ h_m(x)]_{n-k}| + |[\gamma_m \circ h_m(x)]_{n+1}| \longrightarrow \infty$.

Hence we can find a compact $K' \subset \mathbb{H}^{n+1}$, for example $V_1(K) \times \{1\}$ (here $V_1(K)$ denotes the set of points of $\mathbb{R}^n$ at euclidian distance from K less than one), so that $\gamma_m \circ h_m(\sigma) \cap K' \neq \emptyset$ for m large enough. But the covering of $\mathbb{H}^{n+1}$ by ΓP must be locally finite. This contradiction ends the proof. $\square$

Thus there exists a constant M such that $|hy - x_1| \leq M$, for every $h \in \Gamma$ with $hy \in P_\infty$. From now on, we assume that $z \geq z_0 \gg M$.

Then we have

$$|\gamma x_z - hy|^2 \asymp |a_\gamma - [hy]_k|^2 + |z|^2,$$

and

$$\exp(d(\gamma x_z, hy)) \asymp 1 + \frac{|a_\gamma - [hy]_k|^2 + |z|^2}{2z[hy]_{n+1}} \asymp \frac{z}{[hy]_{n+1}} \left(1 + \frac{|a_\gamma - [hy]_k|^2}{z^2}\right).$$

Hence

$$g_s(x_z, y) \asymp z^{-s} \sum_{h|h(y)\in P_\infty} ([hy]_{n+1})^s \sum_{\gamma\in\Gamma_\infty} \left(1 + \frac{|a_\gamma - [hy]_k|^2}{z^2}\right)^{-s}$$

$$g_s(x_z, y) \asymp z^{-s} \sum_{h|h(y)\in P_\infty} ([hy]_{n+1})^s \sum_{m\in\mathbb{N}} \sum_{\substack{\gamma \in \Gamma_\infty \\ m \le |a_\gamma - [hy]_k| < m+1}} \left(1 + \frac{|a_\gamma - [hy]_k|^2}{z^2}\right)^{-s}.$$

For every $\gamma \in \Gamma_\infty$ such that $m \le |a_\gamma - [hy]_k| < m+1$, $1 + \frac{|a_\gamma - [hy]_k|^2}{z^2} \asymp 1 + \frac{m^2}{z^2}$. Moreover, $card\{\gamma \in \Gamma_\infty \,|\, m \le |a_\gamma - [hy]_k| < m+1\} \asymp m^{k-1}$, with the convention that $0^{k-1} = 1$. Thus,

$$g_s(x_z, y) \asymp z^{-s} \sum_{h|h(y)\in P_\infty} ([hy]_{n+1})^s \sum_{m\in\mathbb{N}} m^{k-1} \left(1 + \frac{m^2}{z^2}\right)^{-s}.$$

We remark that

$$\sum_{m\in\mathbb{N}} m^{k-1} \left(1 + \frac{m^2}{z^2}\right)^{-s} \asymp \int_0^\infty \frac{t^{k-1}dt}{\left(1 + \frac{t^2}{z^2}\right)^s},$$

and after the change of variables $t \to u = t/z$ in the integral, we finally obtain

$$g_s(x_z, y) \asymp z^{k-s} \int_0^\infty \frac{u^{k-1}dt}{(1+u^2)^s} \sum_{h|h(y)\in P_\infty} ([hy]_{n+1})^s.$$

The integral $\int_0^\infty \frac{u^{k-1}dt}{(1+u^2)^s}$ is finite since a rank k parabolic fixed point implies $\delta > k/2$ (cf [2]).

The same computation is valid for $z = z_0$ and thus

$$g_s(x_z, y) \asymp \left(\frac{z}{z_0}\right)^{k-s} g_s(x_{z_0}, y).$$

But z_0 is fixed and we have

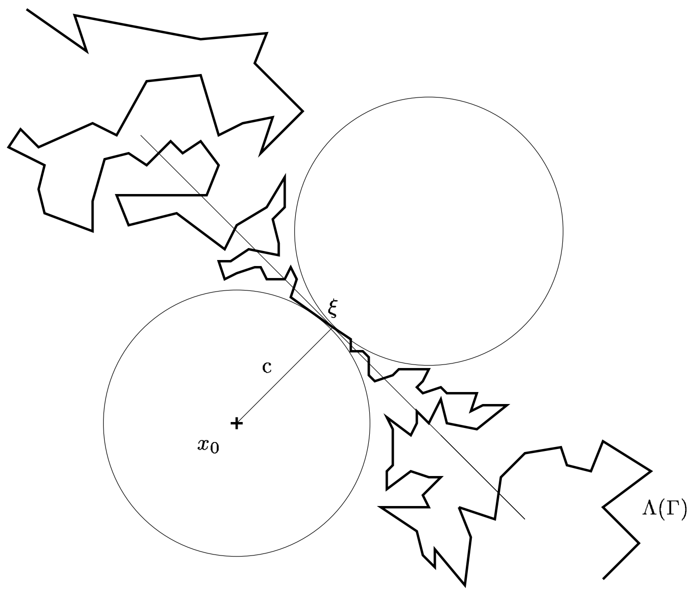
$$\mu_{s,x_z} \asymp z^{k-s} \mu_{s,x_{z_0}},$$

which, making $s \to \delta$, yields $\varphi_\mu(x_z) \asymp z^{k-\delta}$. (The constant, implicit in the notation $\asymp$, depends only on the geometry of Γ, on the class of the parabolic fixed point ∞, and on the point $y \in \mathbb{H}^{n+1}$ we chose to define the Patterson-Sullivan measures.) □

We are now in position to prove theorem 1. Assume that the rank of any parabolic fixed point of $\Lambda(\Gamma)$ is not less than δ. Then, for every parabolic fixed point ξ, $\varphi_\mu(\nu(\xi, r))$ is bounded from below when $r \to 0$. This lower bound depends only on the class of ξ and there is a finite number of these: $\varphi_\mu(\nu(\xi, r(x)))$ is bounded from below on $C(\Lambda(\Gamma))$ by a stricly positive constant, independently of $\xi \in \Lambda(\Gamma)$. Hence there is a constant $A > 0$ such that $f(x) \ge \frac{A}{\rho(x)^2}$ for all $x \in \Omega(\Gamma)$: g is complete on $\Omega(\Gamma)$.

Remark. — It is known that if Γ is geometrically finite with only rank n parabolic fixed points then M is compact ([8]) and hence g is complete on $\Omega(\Gamma)$. Here we obtain this last completness result from the fact that $\delta(\Gamma) < n$ for Γ geometrically finite ([7]).

On the contrary, assume that there exists a parabolic fixed point ξ whose rank k is (stricly) less than δ. Then $1 \leq k \leq n-1$ since for a geometrically finite group, $\delta < n$. From the existence of a cuspidal neighbourhood U_ξ of ξ, we see that there exists x_0 in $\Omega(\Gamma)$ such that $|x_0 - \xi| = r(x_0)$ (Indeed, this cuspidal neighbourhood is the image of a set of the form U_∞ by the conformal transformation mapping $\mathbb{H}^{n+1}$ into $\mathbb{B}^{n+1}$ and sending ∞ on ξ).



Let $(c_t)_{t\in[0,l_0]}$ be the unit speed geodesic from x_0 to ξ for the g_0 metric of $\mathbb{S}^n$ (l_0 is the g_0-distance between x_0 and ξ). c is a divergent curve of $\Omega(\Gamma)$: it escapes from every compact of $\Omega(\Gamma)$. Let's compute the g-length of c:

$$l_g(c) = \int_0^{l_0} f(c_t)^{1/2} dt \leq C \int_0^{l_0} \frac{1}{r(c_t)} \left(\frac{1}{r(c_t)^{k-\delta}} \right)^{1/\delta} dt = C \int_0^{l_0} \frac{dt}{r(c_t)^{k/\delta}}.$$

For $r(c_t) \asymp \rho(c_t) = l_0 - t$ and $k/\delta < 1$, the last integral is finite. c is a divergent curve in $\Omega(\Gamma)$ of finite g-length: $(\Omega(\Gamma), g)$ is not complete and theorem 1 is proved.

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