Non-Hyperbolic Complex Spaces with Hyperbolic Normalizations

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A complex space is called *hyperbolic* if the Kobayashi pseudodistance on it is a distance [2]. Recall the following

Theorem ([2], [3]). *If $\tilde{E}$ is a normalization of a hyperbolic complex space E, then $\tilde{E}$ is also hyperbolic.*

One may ask whether the converse is true. More precisely, if E is a complex space with hyperbolic normalization $\tilde{E}$, is E also hyperbolic? The answer is negative, and we present two very simple examples below. In the first of them E and $\tilde{E}$ are affine algebraic surfaces, while in the second one they are projective surfaces.

Example 1.

Let (x, y, u, v) be a coordinate system in $\mathbf{C}^4$, and let E be the algebraic surface in $\mathbf{C}^4$ given by the equations

$$\begin{cases} y^4 &= x^4 - 1 \\ u^4 &= y^4(v^4 - 1). \end{cases}$$

The surface E is not hyperbolic, since the hyperplane section $E \cap \{y = 0\}$ consists of complex affine lines $\simeq \mathbf{C}$, which are not hyperbolic.

Let $\tilde{E}$ be the algebraic surface in $\mathbf{C}^4$ given by the equations

$$\begin{cases} y^4 &= x^4 - 1 \\ u^4 &= v^4 - 1. \end{cases}$$

Then $\tilde{E}$ is smooth and, in particular, normal. One can consider $\tilde{E}$ as the direct product $R \times R$, where R is the smooth affine algebraic curve defined by the equation $x_1^4 = x_2^4 - 1$ in $\mathbf{C}^2 = \{(x_1, x_2)\}$. Since R is hyperbolic, $\tilde{E}$ is also hyperbolic ([2, Proposition 4.1]). Put $p(x, y, u, v) = (x, y, yu, v)$ and $\pi = p\,|_{\tilde{E}}$. Clearly, $\pi : \tilde{E} \to E$ is a normalization of E. This is a desired example.

Example 2.

Next we give an example in the compact case. Choose a homogeneous coordinate system $(X : Y : Z)$ in $\mathbf{P}^2 = \mathbf{P}^2_{\mathbf{C}}$. Put $\overline{R} = \{(X : Y : Z) \in \mathbf{P}^2 \mid X^4 - Y^4 - Z^4 = 0\}$.

Then $\overline{R}$ is a smooth Riemann surface of genus 3, and therefore, $\overline{R}$ is hyperbolic. Set $R_1 = \overline{R} - \{Z = 0\}$ and $R_2 = \overline{R} - \{X = 0\}$. Let $q : B \to \overline{R}$ be a holomorphic fiber bundle over $\overline{R}$ with the fiber $\mathbf{P}^2$ such that $q^{-1}(R_k) \cong R_k \times F_k$, where $F_k \cong \mathbf{P}^2$ with a homogeneous coordinate system $(U_k : V_k : W_k)$, $k = 1, 2$, and the transition mapping is given in $q^{-1}(R_1 \cap R_2)$ as follows: $(U_2 : V_2 : W_2) = (ZU_1 : XV_1 : XW_1)$. Let $E \subset B$ be the surface defined by the equations

$$E \cap q^{-1}(R_1) = \begin{cases} Y^4 &= X^4 - Z^4 \\ Z^4 U_1^4 &= Y^4(V_1^4 - W_1^4) \end{cases}$$

and

$$E \cap q^{-1}(R_2) = \begin{cases} Y^4 &= X^4 - Z^4 \\ X^4 U_2^4 &= Y^4(V_2^4 - W_2^4) \end{cases}$$

Then E is not hyperbolic, since the intersection $E \cap \{Y = 0\}$ consists of Riemann spheres $\simeq \mathbf{P}^1$, which are not hyperbolic.

Let $\tilde{E} \subset B$ be the surface defined by the equations

$$\tilde{E} \cap q^{-1}(R_1) = \begin{cases} Y^4 &= X^4 - Z^4 \\ Z^4 U_1^4 &= V_1^4 - W_1^4 \end{cases}$$

and

$$\tilde{E} \cap q^{-1}(R_2) = \begin{cases} Y^4 &= X^4 - Z^4 \\ X^4 U_2^4 &= V_2^4 - W_2^4 \end{cases}$$

It is easily seen that $\tilde{E}$ is smooth and that $q\mid_{\tilde{E}} : \tilde{E} \to \overline{R}$ is a holomorphic fiber bundle over $\overline{R}$ with the fiber $\overline{R}$. Thus, $\tilde{E}$ is hyperbolic [1]. The mapping

$$Q : ((X : Y : Z), (U_k : V_k : W_k)) \longmapsto ((X : Y : Z), (YU_k : V_k : W_k)), \quad k = 1, 2,$$

makes $\tilde{E}$ a normalization of E.

References

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