NON-ARCHIMEDEAN MELLIN TRANSFORM AND p-ADIC L-FUNCTIONS

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The purpose of this paper is to describe some new general constructions of p-adic L-functions attached to certain arithmetically defined complex L-functions. These constructions are based on the use of the p-adic Mellin transform. We explain that these constructions are equivalent to proving some generalized Kummer congruences for critical special values of these complex L-functions. The paper is based on a talk of the author in the French-Vietnamese Colloquium on Mathematics held in Ho Chi Minh City from March 3 to March 8, 1997. A part of the work was done in MSRI in 1995 [PaMSRI].

1. Kummer congruences and p-adic integration

The starting point in the theory of complex and p-adic L-functions is the expansion of the Riemann zeta-function $\zeta(s)$ into the Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \quad (\mathrm{Re}(s) > 1)\,.$$

The set of arguments s for which $\zeta(s)$ is defined can be extended to all $s \in \mathbf{C}$, $s \neq 1$, and we may regard $\mathbf{C}$ as the group of all continuous quasicharacters

$$\mathbf{C} = \mathrm{Hom}(\mathbf{R}_+^\times, \mathbf{C}^\times), \quad y \mapsto y^s$$

of $\mathbf{R}_+^\times$. The special values $\zeta(1-k)$ at negative integers are rational numbers:

$$\zeta(1-k) = -\frac{B_k}{k} \quad (k \geq 1)$$

where B_k are Bernoulli numbers.

The proof of these facts which is due to Riemann is based on the *Mellin transform*: the construction which associates to a function $h(y)$ on $\mathbf{R}_+^\times$ (with certain growth conditions for $y \to \infty$ and $y \to 0$) the integral

$$L_h(s) = \int_{\mathbf{R}_+^\times} h(y)\, y^s \,\frac{dy}{y}$$

(which probably converges not for all values of s). For example, if $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(y) = 1/(1-e^{-y})$:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{1-e^{-y}} y^s \frac{dy}{y},$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. This identity is immediately deduced from the well known integral representation for the gamma-function

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}, \quad n^{-s}\Gamma(s) = \int_0^\infty e^{-ny} y^s \frac{dy}{y} \quad (\mathrm{Re}(s) > 0)$$

where $\frac{dy}{y}$ is a measure on the group $\mathbf{R}_+^\times$ which is invariant under the group translations (Haar measure). The idea of Riemann was to replace the integral $\int_0^\infty \frac{1}{1-e^{-y}} y^s \frac{dy}{y}$ by a certain contour integral giving an analytic continuation to all $s \in \mathbf{C}$. The above formulas for $\zeta(1-k)$ are obtained using the Taylor expansion of the function

$$\frac{y\, e^y}{e^y - 1} = \sum_{k=0}^{\infty} \frac{B_k y^k}{k!}$$

and an evaluation of such a contour integral in terms of residues.

For an arbitrary function of the type

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

with $z = x + iy \in \mathfrak{H}$ in the upper half plane $\mathfrak{H}$ and with the growth condition $a(n) = \mathcal{O}(n^c)$ $(c > 0)$ on its Fourier coefficients, we see that the zeta function

$$L(s, f) = \sum_{n=1}^{\infty} a(n) n^{-s},$$

essentially coincides with the Mellin transform of $f(z)$, that is

$$\frac{\Gamma(s)}{(2\pi)^s} L(s, f) = \int_0^\infty f(iy) y^s \frac{dy}{y}.$$

Both sides of this equality converge absolutely for $\mathrm{Re}(s) > 1 + c$.

The numbers $\zeta(1-k)$ have remarkable integrality properties: by the classical Sylvester-Lipschitz theorem we know that

$$c \in \mathbf{Z} \text{ implies } c^k(c^k - 1)\frac{B_k}{k} \in \mathbf{Z}.$$

The proof [Mi-Sta] uses the function

$$f(y) = \frac{e^{cy} - 1}{e^y - 1} = e^{(c-1)y} + \ldots + e^y + 1, \;\; c \geq 1,$$

and the Taylor expansion of the function

$$\frac{d}{dy}\log\left(\frac{e^{cy}-1}{e^y-1}\right)=\frac{f'(y)}{f(y)}=c-1+\sum_{k=1}^{\infty}(c^k-1)\frac{B_k y^{k-1}}{k!}.$$

One sees on one hand that $f(0)=c$ and $f^{(k)}(0)\in\mathbf{Z}$, and on the other hand

$$(c^k-1)\frac{B_k}{k}=\frac{d^{k-1}}{dy^{k-1}}\frac{d}{dy}\log f(y)|_{y=0}.$$

The result follows from the identity:

$$\frac{d^{k-1}}{dy^{k-1}}\frac{d}{dy}\log f(y)=\frac{d^{k-1}}{dy^{k-1}}\left(\frac{f'(y)}{f(y)}\right)=\frac{P(f,f',\dots,f^{(k-1)})}{f(y)^k}\ (k\geq 1)$$

where $P(X_1,\dots,X_k)\in\mathbf{Z}[X_1,\dots,X_k]$ a universal polynomial with integral coefficients (this is easily proved by induction).

The theory of non-Archimedean zeta-functions originates in the work of Kubota and Leopoldt containing p-adic interpolation of these special values. Their construction turns out to be equivalent to classical Kummer congruences for the Bernoulli numbers, which we recall here in the following form.

Theorem (Kummer). — *Let p be a fixed prime number, $c>1$ an integer prime to p. Put*

$$\zeta_{(p)}^{(c)}(-k)=(1-p^k)(1-c^{k+1})\zeta(-k)$$

and let $h(x)=\sum_{i=0}^{n}\alpha_i x^i\in\mathbf{Z}[x]$ be a polynomial over $\mathbf{Z}$ such that

$$(n,p)=1\Longrightarrow h(n)\equiv 0 \mod p^N.$$

Then we have that

$$\sum_{i=0}^{n}\alpha_i\,\zeta_{(p)}^{(c)}(-i)\in p^N\mathbf{Z}_p\,.$$

Note that $\zeta_{(p)}^{(c)}(-k)\in\mathbf{Z}_p\cap\mathbf{Q}$ is p-integral due to the theorem of Sylvester-Lipschitz and the theorem of Kummer implies in particular that the numbers $\zeta_{(p)}^{(c)}(-k)$ depend continuously on k in the p-adic sense: if we take $h(x)=x^{k'}-x^k$ with $k'\equiv k \mod (p-1)p^N$ we have by Euler's theorem that $h(n)\equiv 0 \mod (p-1)p^N$, and the theorem implies that

$$\zeta_{(p)}^{(c)}(-k')\equiv\zeta_{(p)}^{(c)}(-k) \mod (p-1)p^N.$$

Proof. — The proof of the theorem is deduced from the known formula for the sum of k-th powers:

$$S_k(N)=\sum_{n=1}^{N-1}n^k \;=\; \frac{1}{k+1}[B_{k+1}(N)-B_{k+1}]$$

in which $B_k(x) = (x+B)^k = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$ denotes the Bernoulli polynomial. Indeed, all summands in $S_k(N)$ depend p-adic analytically on k, if we restrict ourselves to numbers n, prime to p, so that the desired congruence follows if we express the numbers $\zeta_{(p)}^{(c)}(-k)$ in terms of Bernoulli numbers. More precisely we express the Bernoulli numbers in terms of $S_k(N)$:

$$B_k = \lim_{m\to\infty} \frac{1}{p^m} S_k(p^m),$$

(the p-adic limit) which follows directly from the above formula for $S_k(N)$. Consider now the sum $S_k(p^m) = \sum_{n=1}^{p^m-1} n^k$. For each n with $(p,n) = 1$ we have the congruence $h(n) \equiv 0 \pmod{p^N}$. Let

$$S_k^*(p^m) = \sum_{\substack{n=1\\(n,p)=1}}^{p^m-1} n^k = S_k(p^m) - p^k S_k(p^{m-1}).$$

Then

$$\lim_{m\to\infty} \frac{1}{p^m} S_k^*(p^m) = \lim_{m\to\infty} \frac{1}{p^m}[S_k(p^m) - p^k S_k(p^{m-1})] =$$
$$\lim_{m\to\infty} \frac{1}{p^m} S_k(p^m) - p^{k-1} \lim_{m\to\infty} \frac{1}{p^m} S_k(p^{m-1}) = (1-p^{k-1})B_k\,.$$

In order to prove the congruence

$$\sum_i \alpha_i\, \zeta_{(p)}^{(c)}(-i) \equiv 0 \pmod{p^N}$$

where $\zeta_{(p)}^{(c)}(-k) = (1-p^k)(1-c^{k+1})\zeta(-k)$, we rewrite it as

$$\sum_i \alpha_i\,(1-p^i)(1-c^{i+1})\frac{B_{i+1}}{i+1} \equiv 0 \pmod{p^N},$$

and we choose $m > N$ such that

$$(1-p^{k-1})B_k \equiv \frac{1}{p^m} S_k^*(p^{m-1}) \mod p^N.$$

then the left hand side transforms to

$$\sum_i \alpha_i\,(1-c^{i+1})\frac{1}{i+1}\frac{S_k^*(p^m)}{p^m} \equiv \sum_i \alpha_i\,\frac{1}{i+1} \sum_{\substack{n=1\\(n,p)=1}}^{p^m-1} \frac{n_c^{i+1} - (nc)^{i+1}}{p^m} \pmod{p^N}$$

where $n_c \in \{1,2,\cdots,p^m-1\}$ with $n_c \equiv nc \mod p^m$. Write $t = \frac{n_c-(nc)}{p^m} \in \mathbf{Z}$,

$$n_c^{i+1} - (nc)^{i+1} = (nc + p^m t)^{i+1} - (nc)^{i+1} \equiv (i+1)p^m t(nc)^i \pmod{p^{2m}}.$$

We see that the left hand side becomes

$$\sum_{\substack{n=1\\(n,p)=1}}^{p^m-1} \frac{n_c-(nc)}{p^m} \sum_i \alpha_i (nc)^i \pmod{p^m}.$$

and it remains to notice that $\sum_i \alpha_i (nc)^i \equiv 0 \,(\bmod p^N)$, $m > N$ Q.E.D.

The domain of definition of p-adic zeta functions is the p-adic analytic Lie group

$$X_p = \mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$$

of all continuous p-adic characters of the profinite group $\mathbf{Z}_p^\times$, where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$), so that all integers k can be regarded as the characters $x_p^k : \; y \mapsto y^k$. The construction of Kubota and Leopoldt is equivalent to existence a p-adic analytic function $\zeta_p : \; X_p \to \mathbf{C}_p$ with a single pole at the point $x = x_p^{-1}$, which becomes a bounded holomorphic function on X_p after multiplication by the elementary factor $(x_p \, x - 1)$ $(x \in X_p)$, and this function is *uniquely determined* by the condition

$$\zeta_p({x_p}^k) = (1 - p^k)\zeta(-k) \quad (k \geq 1).$$

This result has a very natural interpretation in framework of the theory of non-Archimedean integration (due to Mazur). We recall that a p-adic measure μ on a profinite group $G = \varprojlim_i G_i$ $(i \in I)$ is a bounded $\mathbf{C}_p$-linear form on $\mathcal{C}(G, \mathbf{C}_p)$, notation: $\mu \in$ *Meas*$(G, \mathbf{C}_p)$. Then the theorem of Kummer is equivalent to the fact that there exists a p-adic measure $\mu^{(c)}$ on $\mathbf{Z}_p^\times$ with values in $\mathbf{Z}_p$ such that $\int_{\mathbf{Z}_p^\times} x_p^k \, \mu^{(c)} = \zeta_{(p)}^{(c)}(-k)$. Indeed, if we integrate $h(x)$ over $\mathbf{Z}_p^\times$ we exactly get the above congruence. On the other hand, in order to define a measure $\mu^{(c)}$ satisfying the above condition it suffices for any continuous function $\phi : \mathbf{Z}_p^\times \to \mathbf{Z}_p$ to define its integral $\int_{\mathbf{Z}_p^\times} \phi(x) \, \mu^{(c)}$. For this purpose we approximate $\phi(x)$ by a polynomial (for which the integral is already defined), and then pass to the limit. The Kummer congruences guarantee that the limit is well defined.

2. The non-Archimedean Mellin transform

Let μ be a (bounded) $\mathbf{C}_p$-valued measure on $\mathbf{Z}_p^\times$. Then the *non-Archimedean Mellin transform* of a measure μ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbf{Z}_p^\times} x \, d\mu \quad (x \in X_p),$$

which represents a bounded $\mathbf{C}_p$-analytic function

$$L_\mu : X_p \to \mathbf{C}_p.$$

Indeed, the boundedness of the function L_μ is obvious since all characters $x \in X_p$ take values in $\mathcal{O}_p$ and μ is also bounded. The analyticity of this function expresses a general property of the integral, namely, that it depends analytically on the parameter $x \in X_p$.

However, there is a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description also implies that every bounded $\mathbf{C}_p$-analytic function on X_p is the Mellin transform of a certain measure μ.

The Iwasawa algebra. Let $\mathcal{O}$ be a closed subring in $\mathcal{O}_p = \left\{z \in \mathbf{C}_p \mid |z|_p \le 1\right\}$, and let $G = \varprojlim_i G_i \ \ (i \in I)$ be a profinite group. Then the canonical homomorphism $G_i \overset{\pi_{ij}}{\longleftarrow} G_j$ induces a homomorphism of the corresponding group rings

$$\mathcal{O}[G_i] \longleftarrow \mathcal{O}[G_j].$$

Then *the completed group ring* $\mathcal{O}[[G]]$ is defined as the projective limit

$$\mathcal{O}[[G]] = \varprojlim_i \mathcal{O}[[G_i]] \ \ (i \in I).$$

Let us consider also the set *Distr* $(G, \mathcal{O})$ of all $\mathcal{O}$-valued distributions on G (finite-additive functions on open-compact subsets of G with values in $\mathcal{O}$, which itself is an $\mathcal{O}$-module and a ring with respect to multiplication given by *the convolution of distributions*, which is defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \to \mathcal{O}$$

(see the previous section) as follows:

$$(\mu_1 * \mu_2)^{(i)}(y) \ = \ \sum_{y = y_1 y_2} \mu_1^{(i)}(y_1)\mu_2^{(i)}(y_2) \ \ (y_1,\ y_2 \in G_i)$$

Then $Meas(G, \mathbf{C}_p) = Distr\,(G, \mathcal{O}_p) \otimes_{\mathcal{O}_p} \mathbf{C}_p$ and

$$\int_G \phi(y)(\mu_1 * \mu_2)(y) = \int_G \phi(y_1 y_2)\mu_1(y_1)\mu_2(y_2).$$

Now we describe an isomorphism of $\mathcal{O}$-algebras $\mathcal{O}[[G]]$ and *Distr* $(G, \mathcal{O})$. In the case when $G = \mathbf{Z}_p$ the algebra $\mathcal{O}[[G]]$ is called *the Iwasawa algebra*.

Theorem (see [PaLNM], Ch.1).

(a) There is the canonical isomorphism of $\mathcal{O}$-algebras

$$Distr\,(G, \mathcal{O}) \overset{\sim}{\to} \mathcal{O}[[G]];$$

(b) If $G = \mathbf{Z}_p$ then there is an isomorphism

$$\mathcal{O}[[G]] \overset{\sim}{\to} \mathcal{O}[[X]],$$

where $\mathcal{O}[[X]]$ is the ring of formal power series in X over $\mathcal{O}$. The isomorphism depends on a choice of the topological generator of the group $G = \mathbf{Z}_p$.

In order to prove this result one needs to construct a measure (an $\mathcal{O}$-valued distribution) attached to a power series in $\mathcal{O}[[X]]$. A convenient tool to construct p-adic measures is given by the following

THEOREM (abstract Kummer congruences, see [KaCM], p.258). — *Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(G, \mathcal{O}_p)$ in the ring $\mathcal{C}(G, \mathcal{O}_p)$ of all continuous functions on a profinite group G with values in the ring of integers $\mathcal{O}_p$ of $\mathbf{C}_p$ such that $\mathbf{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(G, \mathbf{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on G with the property*

$$\int_G f_i \, d\mu = a_i$$

is equivalent to the following congruences: for an arbitrary choice of elements $b_i \in \mathbf{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \text{ for all } y \in G \text{ implies } \sum_i b_i a_i \in p^n \mathcal{O}_p.$$

Remark. — Since $\mathbf{C}_p$-measures are characterized as bounded $\mathbf{C}_p$-valued distributions, every $\mathbf{C}_p$-measure on G becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.

Proof. — The necessity is obvious since

$$\begin{aligned}\sum_i b_i a_i &= \int_G (p^n \mathcal{O}_p\text{−valued function })\, d\mu \\ &= p^n \int_G (\mathcal{O}_p\text{−valued function })\, d\mu \in p^n \mathcal{O}_p.\end{aligned}$$

In order to prove the sufficiency we need to construct a measure μ from the numbers a_i. For a function $f \in \mathcal{C}(G, \mathcal{O}_p)$ and a positive integer n there exist elements $b_i \in \mathbf{C}_p$ such that only a finite number of b_i does not vanish, and

$$f = \sum_i b_i f_i \in p^n \mathcal{C}(G, \mathcal{O}_p)$$

according to the density of the $\mathbf{C}_p$-span of $\{f_i\}$ in $\mathcal{C}(G, \mathbf{C}_p)$. By the assumption the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo p^n (i.e. does not depend on the choice of b_i). We denote this value by "$\int_Y f \, d\mu \mod p^n$". Then we have that the limit procedure

$$\int_G f \, d\mu = \lim_{n\to\infty} \text{“} \int_G f \, d\mu \mod p^n \text{”} \in \varprojlim_n \mathcal{O}_p / p^n \mathcal{O}_p = \mathcal{O}_p,$$

gives the measure μ.

Formulas for coefficients of power series. We have noticed above that $\mathbf{C}_p$-analytic bounded functions on X_p can be described in terms of measures. Indeed, these functions are defined on analyticity components of the decomposition X_p as certain power series with p-adically bounded coefficients, that is, power series, whose coefficients belong to

$\mathcal{O}_p$ after multiplication by some non-zero constant from $\mathbf{C}_p^\times$. We give a direct computation of these coefficients in terms of the corresponding measures. Let us consider the decomposition $\mathbf{Z}_p^\times \cong \Delta \times \Gamma$ where $\Delta = (\mathbf{Z}/p^\nu\mathbf{Z})^\times$, $\Gamma = (1 + p^\nu\mathbf{Z}_p)^\times$, where $\nu = 1$ for $p > 2$ and $\nu = 2$ for $p = 2$. Then the group $\Gamma \cong \mathbf{Z}_p$ is topologically cyclic with a generator $\gamma = 1 + p^\nu$. Consider $a \in \Delta$, and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on Γ defined by restriction of μ to the subset $a\Gamma \subset \mathbf{Z}_p^\times$. Consider the isomorphism $a\Gamma \cong \mathbf{Z}_p$ given by

$$y = a\gamma^x \ (x \in \mathbf{Z}_p,\ y \in \Gamma).$$

Let μ'_a be the corresponding measure on $\mathbf{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbf{Z}_p} \binom{x}{i}\, d\mu'_a(x) = a_i,$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbf{C}_p$-span of the family

$$\left\{ \binom{x}{i} \right\} (i \in \mathbf{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbf{Z}_p, \mathcal{O}_p)$ according to the Mahler's interpolation theorem which says that any continuous function $f : \mathbf{Z}_p \to \mathbf{Q}_p$ can be written in the form:

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

with $a_n \to 0$ (p-adically) for $n \to \infty$. For a function $f(x)$ defined for $x \in \mathbf{Z},\ x \geq 0$ one can write formally

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n},$$

where the coefficients can be found from the following system of linear equations

$$f(n) = \sum_{m=0}^{n} a_m \binom{n}{m},$$

that is

$$a_m = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(i).$$

This property of the interpolation polynomials implies that

$$\sum_i b_i \binom{x}{i} \equiv 0 \ (\operatorname{mod} p^n) \text{ (for all } x \in \mathbf{Z}_p) \Longrightarrow b_i \equiv 0 \ (\operatorname{mod} p^n).$$

We can now apply the abstract Kummer congruences, which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exists a measure with the desired property.

On the other hand we state that the Mellin transform L_{μ_a} of the measure μ_a is given by the power series $F_a(t)$, that is

$$F_a(t) = \int_\Gamma \chi_{(t)}(y)\, d\mu(ay) = \sum_{i=0}^{\infty} \Big(\int_{\mathbf{Z}_p} \binom{x}{i} d\mu'_a(x) \Big) (t-1)^i$$

for all characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that this identity is valid for all characters of the type $y \mapsto y^m$, where m is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

which implies that

$$\int_\Gamma y^m\, d\mu(ay) = \int_{\mathbf{Z}_p} \gamma^{mx} d\mu'_a(x) = \sum_{i=0}^{\infty} \Big(\int_{\mathbf{Z}_p} \binom{x}{i} d\mu'_a(x) \Big) (\gamma^m - 1)^i,$$

establishing the formulas for the coefficients of $F_a(t)$.

Example. — The p-adic Mazur measure and the non-Archimedean Kubota-Leopoldt zeta function. Consider again a positive integer $c \in \mathbf{Z}_p^\times \cap \mathbf{Z}$, $c > 1$ coprime to p. Then for each complex number $s \in \mathbf{C}$ there exists a complex distribution μ_p^c on $G_p = \mathbf{Z}_p^\times$ which is uniquely determined by the following condition

$$\mu_s^c(\chi) = (1 - \chi^{-1}(c) c^{-1-s})(1 - \chi(p) p^s) L(-s, \chi).$$

The right hand side of is holomorphic for all $s \in \mathbf{C}$ including $s = -1$. If $s = k \geq 0$ is a natural number then the right hand side belongs to the field

$$\mathbf{Q}(\chi) \subset \mathbf{Q}^{\mathrm{ab}} \subset \overline{\mathbf{Q}}$$

generated by values of the character χ, and we get a distribution with values in $\mathbf{Q}^{\mathrm{ab}}$. If we now apply the fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ we get a $\mathbf{C}_p$-valued distribution $\mu^{(c)} = i_p(\mu_0^c)$ which turns out to be an $\mathcal{O}_p$-measure, and the following equality holds

$$\mu^{(c)}(\chi x_p^r) = i_p(\mu_r^c(\chi)).$$

This identity is verified as above using the abstract Kummer congruences for characters $\chi(y) y^k$; it relates the special values of the Dirichlet L-functions at different non-positive points. The function

$$\zeta_p(x) = (1 - c^{-1} x(c)^{-1})^{-1} L_{\mu^{(c)}}(x) \quad (x \in X_p)$$

is well defined and it is holomorphic on X_p with the exception of a simple pole at the point $x = x_p \in X_p$, and we have that

$$\zeta_p(\chi(y) y^k) = (1 - \chi(p) p^k) L(-k, \chi) \quad (k \geq 0, \chi \in X_p^{\mathrm{tors}}).$$

The function ζ_p is called the *non-Archimedean zeta function of Kubota-Leopoldt*. The corresponding measure $\mu^{(c)}$ is be called the *p-adic Mazur measure*.

The original construction of Kubota and Leopoldt was successfully used by Iwasawa for the description of the class groups of cyclotomic fields. According to a conjecture of Iwasawa proved by Mazur and Wiles in 1984 [Maz-Wi1], the power series representing ζ_p describes the structure of p-ideal class groups in certain cyclotomic extensions of $\mathbf{Q}$ as Galois modules over $\mathbf{Z}_p^\times = \mathrm{Gal}(\mathbf{Q}^{\mathrm{ab}}(p,\infty)/\mathbf{Q})$ where $\mathbf{Q}^{\mathrm{ab}}(p,\infty) = \mathbf{Q}(\sqrt[p^\infty]{1})$ is the maximal abelian extension of $\mathbf{Q}$ unramified outside of p and ∞. Since then the class of functions admitting p-adic analogues has gradually extended.

3. Admissible measures

Now we recall the notion of the h-admissible measures on G_p and properties of their Mellin transform. These Mellin transform are certain p-adic analytic functions on the $\mathbf{C}_p$-analytic Lie group X_p. Recall that a p-adic measure on G_p may be regarded as a bounded $\mathbf{C}_p$-linear form μ on the space $\mathcal{C}(G_p)$ of all continuous $\mathbf{C}_p$-valued functions

$$\varphi \to \mu(\varphi) = \int_{G_p} d\mu \in \mathbf{C}_p\,,\ \varphi \in \mathcal{C}(G_p),$$

which is uniquely determined by its restriction to the subspace $\mathcal{C}^1(G_p)$ of locally constant functions. We denote by $\mu(a+(Q))$ the value of μ on the characteristic function of the set

$$a + (Q) = \left\{x \in G_p \mid x \equiv a \mod Q\right\} \subset G_p.$$

The Mellin transform L_μ of μ is a bounded analytic function

$$L_\mu : X_p \to \mathbf{C}_p, \quad L_\mu(\chi) = \int_{G_p} \chi d\mu \in \mathbf{C}_p\,,\ \chi \in X_p,$$

on X_p, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi \in \mathcal{X}_S^{\mathrm{tors}}$.

A more delicate notion of an h-admissible measure was introduced by Amice-Vélu and Višik (see [Am-V]). Let $\mathcal{C}^h(G_p)$ denote the space of $\mathbf{C}_p$-valued functions which can be locally represented by polynomials of degree less than a natural number h of the variable $x_p \in X_p$ introduced above.

DEFINITION. — *A* $\mathbf{C}_p$*-linear form*

$$\mu : \mathcal{C}^h(G_p) \to \mathbf{C}_p$$

is called h-admissible measure if for all $a \in G_p$ and for all $r = 0, 1, \dots, h-1$ the following growth condition is satisfied

$$\left| \sup_{a \in G_p} \int_{a+(Q)} (x_p - a_p)^r d\mu \right| = o(|Q|_p^{r-h})$$

It is known due to Amice-Vélu and Višik that each h-admissible measure can be uniquely extended to a linear form on the $\mathbf{C}_p$-space of all locally analytic functions so that one can associate to its Mellin transform

$$L_\mu : X_p \to \mathbf{C}_p, \ \ L_\mu(\chi) = \int_{G_p} \chi d\mu \in \mathbf{C}_p, \quad \chi \in X_p,$$

which is a $\mathbf{C}_p$-analytic function on X_p of the type $o(\log x_p^h)$. Moreover, the measure μ is uniquely determined by the special values of the type

$$L_\mu(\chi x_p^r) \quad (\chi \in X_p, r = 0, 1, \dots, h-1).$$

4. Further generalizations

L-functions (of complex variable) can be attached as certain Euler products to various objects such as diophantine equations, representations of Galois groups, modular forms etc., and they play a crucial role in modern number theory. Deep interrelations between these objects discovered in last decades are based on identities for the corresponding L-functions which presumably all fit into a general concept of the Langlands of L-functions associated with automorphic representations of a reductive group G over a number field K.

From this point of view the study of arithmetic properties of these zeta function is becoming especially important. The major sources of such L-functions are:

1) *Galois representations* of $G_K = \mathrm{Gal}(\overline{K}/K)$ for algebraic number fields K, $r : G_K \to \mathrm{GL}(V)$, V a finite dimensional vector space, and one can attach to r an Euler product due to Artin.

2) *Algebraic varieties* X defined over an algebraic number field K. In this case one can attach to X/K its Hasse-Weil zeta function.

3) *Automorphic forms and automorphic representations.* In the classical case one associates to a modular form $f(z) = \sum_{n=0}^{\infty} a_n \exp(2\pi i n z)$ its Mellin transform $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$. In general an automorphic form generates an automorphic representations in

the space of smooth functions over an adelic reductive group, and one can attach an Euler product to it using a decomposition of such a representation into a tensor product indexed by prime numbers p and ∞.

Conjecturally, all the three type of L-functions can be related to each other using a general theory of motives over $\mathbf{Q}$ with coefficients in a number field T, $[T:\mathbf{Q}]<\infty$ (this field coincides with the field $\mathbf{Q}(\{a_n\}_{n\geq 1})$ generated by the coefficients of the corresponding L-function $L(M,s)=\sum_{n=1}^{\infty} a_n n^{-s}$.) For a fixed prime number p one can also attach in many cases to the above complex L-function a p-adic L-function. These p-adic L-functions are certain analytic functions in a p-adic domain obtained by an interpolation procedure of certain special values of the corresponding complex analytic L-functions. Their existence is equivalent to certain generalized Kummer congruences for the special values.

5. Non-archimedean L-functions of Jacquet-Langlands

These L-functions correspond to certain automorphic representations on the group $G=\mathrm{GL}_2$ and $G=\mathrm{GL}_2\times\mathrm{GL}_2$ over a totally real field F, and they reduce to zeta functions of the form

$$L(s,\mathfrak{f})=\sum_{\mathfrak{n}} C(\mathfrak{n},\mathfrak{f})\,\mathcal{N}(\mathfrak{n})^{-s},\quad L(s,\mathfrak{f},\mathbf{g})=\sum_{\mathfrak{n}} C(\mathfrak{n},\mathfrak{f})\,C(\mathfrak{n},\mathbf{g})\mathcal{N}(\mathfrak{n})^{-s},$$

where $\mathfrak{f}$, $\mathbf{g}$ are *Hilbert automorphic forms of "holomorphic type"* over F, where $C(\mathfrak{n},\mathfrak{f})$, $C(\mathfrak{n},\mathbf{g})$ are their normalized Fourier coefficients (indexed by integral ideals $\mathfrak{n}$ of the maximal order $\mathcal{O}_F\subset F$) which also coincide with the eigenvalues of Hecke operators $T(\mathfrak{m})$. On can regard $\mathfrak{f}$, $\mathbf{g}$ as functions on the adelic group $G_{\mathbf{A}}=\mathrm{GL}_2(\mathbf{A}_F)$, where $\mathbf{A}_F$ is the ring of adeles of F and we suppose that $\mathfrak{f}$ is a *primitive cusp form* of scalar weight $k\geq 2$, of conductor $\mathfrak{c}(\mathfrak{f})\subset\mathcal{O}_F$, and the character ψ and $\mathbf{g}$ a primitive cusp form of weight $l<k$, the conductor $\mathfrak{c}(\mathbf{g})$, and the character ω, $(\psi,\omega:\mathbf{A}_F^\times\to\mathbf{C}^\times$ are Hecke characters of finite order). The non-Archimedean construction is based on the algebraic properties of the special values of the function $L(s,\mathfrak{f},\mathbf{g})$ at the points $s=l,\cdots,k-1$ up to some constant, which is expressed in terms of the Petersson inner product $\langle\mathfrak{f},\ \mathfrak{f}\rangle$ of the automorphic form $\mathfrak{f}$. Our theorem on non-Archimedean interpolation is equivalent to certain generalized Kummer congruences for these special values.

We need some more notation for the precise formulation of the result (in a simplified form). Let ψ^*,ω^* be the characters of the ideal group of F associated with ψ,ω and let

$$L_{\mathfrak{c}}(s, \psi\omega) = \sum_{\mathfrak{n}+\mathfrak{c}=\mathcal{O}_F} \psi^*(\mathfrak{n})\omega^*(\mathfrak{n})\mathcal{N}(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}+\mathfrak{c}=\mathcal{O}_F} (1 - \psi^*(\mathfrak{p})\omega^*(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{-s})^{-1}$$

be the corresponding Hecke L-function with $\mathfrak{c} = \mathfrak{c}(\mathfrak{f})\mathfrak{c}(\mathbf{g})$. We now define the normalized Rankin zeta function by setting

$$\Psi(s, \mathfrak{f}, \mathbf{g}) = \gamma_n(s)L_{\mathfrak{c}}(2s + 2 - k - l, \psi\omega)L(s, \mathfrak{f}, \mathbf{g}),$$

where $n = [F : \mathbf{Q}]$ is the degree of F,

$$\gamma_n(s) = (2\pi)^{-2ns}\Gamma(s)^n\Gamma(s + 1 - l)^n$$

is the gamma-factor. Then the function $\Psi(s, \mathfrak{f}, \mathbf{g})$ admits a holomorphic analytic continuation onto the entire complex plane and it satisfies a certain functional equation. Put $\Omega(\mathfrak{f}) = \langle \mathfrak{f}, \mathfrak{f}\rangle_{\mathfrak{c}(\mathfrak{f})}$, then we know due to Shimura [Shi] that the number

$$\frac{\Psi(l + r, \mathfrak{f}, \mathbf{g})}{(2\pi i)^{n(1-l)}\Omega(\mathfrak{f})} \text{ is algebraic for all integers } r \text{ with } 0 \leq r \leq k - l - 1\,.$$

For the non-Archimedean construction we introduce the p-adic completion

$$\mathcal{O}_{F,p} = (\mathcal{O}_F \otimes \mathbf{Z}_p) = \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p}}$$

of the ring $\mathcal{O}_F$. Put

$$S_F = \{\mathfrak{p}|\mathfrak{p} \text{ divides } p\}$$

and let $G_{F,p} = \mathrm{Gal}(F^{\mathrm{ab}}(p, \infty)/F)$ be the Galois group of the maximal abelian extension $F^{\mathrm{ab}}(p, \infty)$ of F unramified outside places over p and ∞.

The domain of definition of our non-Archimedean L-functions is the p-adic analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{\mathrm{contin}}(G_{F,p}, \mathbf{C}_p^{\times})$$

of all continuous p-adic characters of the Galois group $G_{F,p}$ with $\mathbf{C}_p$ being the Tate field. Elements of finite order $\chi \in \mathcal{X}_p$ can be identified with those Hecke characters of finite order whose conductors are divisible only by prime divisors belonging to S_F, via the decomposition

$$\chi : \mathbf{A}_F^{\times} \stackrel{\text{class field theory}}{\longrightarrow} G_{F,p} \to \overline{\mathbf{Q}}^{\times} \stackrel{i_p}{\to} \mathbf{C}_p^{\times}.$$

We shall use the same symbol χ to denote both Hecke character and the corresponding element of $\mathcal{X}_p$. Since $\mathbf{Q}^{\mathrm{ab}}(p, \infty) \subset F^{\mathrm{ab}}(p, \infty)$, the restriction of Galois automorphisms to $\mathbf{Q}^{\mathrm{ab}}(p, \infty)$ determines a natural homomorphism

$$\mathcal{N} : G_{F,p} \to G_p \stackrel{\sim}{\to} \mathbf{Z}_p^{\times}.$$

We let $\mathcal{N}x_p$ denote the composition of this homomorphism with the inclusion $\mathbf{Z}_p^{\times} \subset \mathbf{C}_p^{\times}$.

We note first that an analogue of the Kubota-Leopoldt zeta function in this case was constructed by Deligne and Ribet [De-Ri]. In 1976 Yu.I. Manin [Man] has constructed a p-adic analogue of the series $L(s,\mathfrak{f}) = \sum_{\mathfrak{n}} C(\mathfrak{n},\mathfrak{f})\,\mathcal{N}(\mathfrak{n})^{-s}$ under the assumption that the cusp form $\mathfrak{f}$ is *p-ordinary*, i.e. that for the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ and for all $\mathfrak{p}|p$ there exists a root $\alpha(\mathfrak{p})$ of Hecke $\mathfrak{p}$-polynomial of $\mathfrak{f}$ such that $|i_p(\alpha(\mathfrak{p}))|_p = 1$. This p-adic L functions satisfy a certain functional equation , similar to the complex analytic one. In the case $F = \mathbf{Q}$ Ha Huy Khoai [HHKh] proved that the p-adic functional equation characterize $\mathfrak{f}$.

In order to construct p-adic L functions of two automorphic forms $\mathfrak{f}$ and $\mathbf{g}$ we make again the assumption that the cusp form $\mathfrak{f}$ is *p-ordinary*. We then fix roots $\alpha(\mathfrak{p})$ with $|i_p(\alpha(\mathfrak{p}))|_p = 1$ and extend the definition of $\alpha(\mathfrak{m})$ to all integral ideals $\mathfrak{m} \subset \mathcal{O}_F$ by multiplicativity.

Theorem. — *(On the non-Archimedean convolutions of Hilbert modular forms)*

Under the above notations and assumptions there exists a bounded $\mathbf{C}_p$-analytic function $\Psi_p : \mathcal{X}_p \to \mathbf{C}_p$ which is uniquely determined by the condition: for each Hecke character of finite order $\chi \in \mathcal{X}_p^{\mathrm{tors}}$ the following equality holds

$$\Psi_p(\chi) = i_p\left(D_F^{2l}\omega^*(\mathfrak{m})\frac{\tau(\chi)^2\mathcal{N}\mathfrak{m}^{l-1}}{\alpha(\mathfrak{m})^2}\frac{\Psi(l,\mathfrak{f},\mathbf{g}^\rho(\bar{\chi}))}{(-2\pi i)^{n(1-l)}\langle\mathfrak{f},\ \mathfrak{f}\rangle}\right),$$

where D_F is the discriminant of F, $\tau(\chi)$ the Gauss sum of χ, and $\mathbf{g}^\rho(\chi)$ the cusp form obtained from $\mathbf{g}$ by complex conjugation of its Fourier coefficients and by twisting it then with the character χ.

This result is also valid for the special values $\Psi(l+r,\mathfrak{f},\mathbf{g}(\chi))$ with $r = 1,\dots,k-l$, if we replace $\chi \in \mathcal{X}_p$ by $\chi\mathcal{N}x_p^r \in \mathcal{X}_p$ (see [PaLNM], Ch. 4). Note that this result was established by H.Hida [Hi] in a much more general, but p-ordinary, situation. On the other hand, this construction was extended by My Vinh Quang [MyVQ] to the *non-p-ordinary*, i.e. *supersingular* case, when $|i_p(\alpha(\mathfrak{p})|_p < 1$ for both roots $\alpha(\mathfrak{p})$ and $\mathfrak{p}|p$. In this situation the functions Ψ_p are also uniquely determined by the above condition provided that they have a prescribed logarithmic growth.

6. p-adic families of motives and their L-functions

We consider in the rest of the paper a motive M over $\mathbf{Q}$ with coefficients in a number field T (see Section 7 for definitions), and its L-functions $L(M,s)$. We describe a general conjecture on the existence of a p-adic family M_P of motives coming from M. The most important known example of such a family is given by Hida's families of p-ordinary cusp forms

$$\left\{ f_k = \sum_{n=1}^{\infty} a_n(k) \exp(2\pi i n z) \right\}$$

of weight k. In this case $P = P_k$, $k \geq 2$, and motives of the family M_{P_k} are characterized by the condition $L(M_{P_k}, s) = L(s, f_k)$ (the Mellin transform of f_k). The functions $k \mapsto a_n(k) \in \overline{\mathbf{Q}}$ admit an interpolation to certain Iwasawa functions of k. A famous application of Hida's families was given in the proof of Wiles of the Fermat Last theorem and the Shimura-Taniyama conjecture [Wi]: in order to associate to an elliptic curve E over $\mathbf{Q}$ a modular form of weight $k = 2$ one finds first a modular form $g \mod p$ attached to E, with $p = 3$ or 5. Then one lifts this form to characteristic 0, and includes it to a family of Hida f_k. By putting $k = 2$ one gets the answer. Other interesting examples of p-adic families of automorphic forms were given recently by J.Tilouine and E.Urban, [Til-U]

In a more general situation this family is parametrised by some dense subset of algebraic characters P of a p-adic commutative algebraic group (which we call the group of Hida). This group can be regarded as a maximal torus of the p-adic part of the motivic Galois group G_M of M (the Tannakian group for the tensor category generated by M). The important condition of motives M_P of the above family is that they have the same fixed p-invariant $h = h_P$, which is defined as the difference between the Newton polygon and the Hodge polygon of a motive at certain point d^+ (the dimension of the subspace M^+ of the Betti realization M_B of M). The corresponding p-adic L-functions of this family can be unbounded (of Amice-Vélu type [Am-Ve]) but they form a family which is conjecturally bounded in the "weight direction", that is for P parametrized by algebraic characters of $G_{M,p}$.

More precisely, the values of the function $P \mapsto L(M_P, 0)$ satisfy generalised Kummer congruences in the following sense: for any finite linear combination $\sum_P b_P \cdot P$ with $b_P \in \mathbf{C}_p$ which has the property $\sum_P b_P \cdot P \equiv 0 \ (\mathrm{mod}\, p^N)$ we have that for some constant $C \neq 0$ the corresponding linear combination of the normalized L-values

$$C \sum_P b_P c_p(M_P) \cdot \frac{L_{(p,\infty)}(M_P, 0)}{c_\infty(M_P)} \equiv 0 \ (\mathrm{mod}\, p^N).$$

Here $c_p(M_P)$ and $c_\infty(M_P)$ denote a p-adic and a complex period of M_P so that the ratio "$\dfrac{c_p(M_P)}{c_\infty(M_P)}$" is uniquely defined, and $L_{(p,\infty)}(M_P, s)$ denotes the above L-function $L(M_P, s)$ normalized by multiplying by a certain canonically defined Deligne's p-factor corresponding to a choice of inverse roots $\alpha^{(1)}(p), \ldots, \alpha^{(d^+)}(p) \in \mathbf{C}_p$ of p-local polynomial of M such that

$$\mathrm{ord}_p(\alpha^{(1)}(p)) \leq \mathrm{ord}_p(\alpha^{(2)}(p)) \leq \ldots \leq \mathrm{ord}_p(\alpha^{(d)}(p)),$$

d being the common rank of the family M_P, d^+ the T-dimension of the Deligne'a subspace M^+ of M_B (the fixed subspaces of the canonical involution ρ of M over T).

Recent examples of such families related to modular forms were constructed by R. Coleman [Col] who proved the following

THEOREM (R. Coleman). — *Suppose $\alpha \in \mathbf{Q}$ and $\varepsilon : (\mathbf{Z}/pZ)^\times \to \mathbf{C}_p^\times$ is a character. Then there exists a number n_0 which depends on p, N and ε, and α with the following property: If $k \in \mathbf{Z}$, $k > \alpha + 1$ and there is a unique normalized cusp form F on $X_1(Np)$ of weight k, character $\varepsilon\omega^{-k}$ and slope α and if $k' > \alpha+1$ is an integer congruent tok modulo p^{n+n_0}, for any positive integer n, then there exists a unique normalized cusp form F' on $X_1(Np)$ of weight k', character $\varepsilon\omega^{-k'}$ and slope α (ω denotes the Teichmüller character). Moreover his form satisfies the congruence*

$$F'(q) \equiv F(q) \ (\operatorname{mod} p^{n+1}).$$

This result can be regarded as a generalization of the work of Hida [HiGal] who considered the case $\alpha = 0$ and constructed interesting families of Galois representations of the type

$$\rho_p : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]), \ \ G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}),$$

which are non ramified outside p. These representations have the following property: if we consider the homomorphisms

$$\mathbf{Z}_p[[T]] \xrightarrow{s_k} \mathbf{Z}_p, \ \ 1 + T \ \mapsto (1+p)^{k-1},$$

then we obtain a family of Galois representations

$$\rho_p^{(k)} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p),$$

which is parametrized by $k \in \mathbf{Z}$, and for $k = 2, 3, \cdots$, these representations are equivalent over $\mathbf{Q}_p$ to the p-adic representations of Deligne, attached to modular forms of weight k. This means that the representations of Hida are obtained by the p-adic interpolation of Deligne's representations. A geometric interpretation of Hida's representations was given by Mazur and Wiles [Maz-W2], cf. [Maz]. For example, for the modular form Δ of weight 12 Hida has constructed a representation

$$\rho_{p,\Delta} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{Z}_p[[T]]),$$

as an example of his general theory, where the prime number p have the property $\tau(p) \not\equiv 0 \ (\operatorname{mod} p)$ (e.g. $p < 2041$, $p \neq 2, 3, 5$ and 7). The boundedness property is the subject of a recent research by G.Stevens, B.Mazur and F.Q.Gouvêa. Note that other examples may include Rankin products, Garrett triple products of elliptic and Hilbert modular forms and standard L-functions of Siegel modular forms.

To describe this conjecture more precisely, let M is a motive over $\mathbf{Q}$ of with coefficients in T i.e.

$$M_B,\ M_{DR},\ M_\lambda,\ I_\infty,\ I_\lambda,$$

where M_B is the Betti realization of M which is a vector space over T of dimension d endowed with a T-rational involution ρ ; $M_B = M^+ \oplus M^-$ denotes the corresponding decomposition into the sum of $+1$ and -1-eigenspaces of ρ.

M_{DR} is the de Rham realization of M, a free T-module of rank d, endowed with a decreasing filtration $\{F^i_{DR}(M) \subset M_{DR} | i \in \mathbf{Z}\}$ of T-modules;

M_λ is the λ-adic realization of M at a finite place λ of the coefficient field T (a T_λ-vector space of degree d over T_λ, a completion of T at λ) which is a Galois module over $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ so that we a have a compatible system of λ-adic representations denoted by

$$r_{M,\lambda} = r_\lambda : G_{\mathbf{Q}} \to GL(M_\lambda).$$

Also,

$$I_\infty : M_B \otimes_T \mathbf{C} \to M_{DR} \otimes_T \mathbf{C}$$

is the complex comparison isomorphism of complex vector spaces

$$I_\lambda : M_B \otimes_T T_\lambda \to M_\lambda$$

is the λ-adic comparison isomorphism of T_λ-vector spaces. It is assumed in the notation that the complex vector space $M_B \otimes_{\mathbf{Q}} \mathbf{C}$ is decomposed in the Hodge bigraduation

$$M_B \otimes_T \mathbf{C} = \oplus_{i,j} M^{i,j}$$

in which $\rho(M^{i,j}) \subset M^{j,i}$ and

$$h(i,j) = h(i,j,M) = \dim_{\mathbf{C}} M^{i,j}$$

are the Hodge numbers. Moreover,

$$I_\infty(\oplus_{i' \geq i} M^{i',j}) = F^i_{DR}(M) \otimes \mathbf{C}.$$

Also, I_λ takes ρ to the r_λ-image of the Galois automorphism which corresponds to the complex conjugation of $\mathbf{C}$. We assume that M is pure of weight w (i.e. $i + j = w$).

The L-function $L(M,s)$ of M is defined as the following Euler product:

$$L(M,s) = \prod_p L_p(M, p^{-s}),$$

extended over all primes p and where

$$\begin{aligned} L_p(M,X)^{-1} &= \det(1 - X \cdot r_\lambda(Fr_p^{-1}) \mid M_\lambda^{I_p}) \\ &= (1 - \alpha^{(1)}(p)X) \cdot (1 - \alpha^{(2)}(p)X) \cdot \ldots \cdot (1 - \alpha^{(d)}(p)X) \\ &= 1 + A_1(p)X + \ldots + A_d(p)X^d; \end{aligned}$$

here $Fr_p \in G_{\mathbf{Q}}$ is the Frobenius element at p, defined modulo conjugation and modulo the inertia subgroup $I_p \subset G_p \subset G_{\mathbf{Q}}$ of the decomposition group G_p (of any extension of p to $\overline{\mathbf{Q}}$). We make the standard hypothesis that the coefficients of $L_p(M, X)^{-1}$ belong to T, and that they are independent of λ coprime to p. Therefore we can and we shall regard this polynomial both over $\mathbf{C}$ and over $\mathbf{C}_p$. We shall need the following twist operation: for an arbitrary motive M over $\mathbf{Q}$ with coefficients in T an integer m and a Hecke character χ of finite order one can define the twist $N = M(m)(\chi)$ which is again a motive over $\mathbf{Q}$ with the coefficient field $T(\chi)$ of the same rank d and weight w so that we have

$$L(N, s) = \prod_p L_p(M, \chi(p)p^{-s-n}).$$

7. The group of Hida and the algebra of Iwasawa-Hida

Now let us fix a motive M with coefficients in $T = \mathbf{Q}(\langle a(n)_n \rangle)$ of rang d and of weight w, and let $\mathrm{End}_T\, M$ denote the endomorphism algebra of M (i.e. the algebra of T-linear endomorphisms of any M_B, which commute with the Galois action under the comparison isomorphisms). Let

$$G_p = \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}_p^\times$$

denotes the Galois group of the maximal abelian extension $\mathbf{Q}_{p,\infty}^{ab}$ of $\mathbf{Q}$ unramified outside p and ∞. Define $\mathcal{O}_{T,p} = \mathcal{O}_T \otimes \mathbf{Z}_p$.

Definition. — *The* group of Hida $GH_M = GH_{M,p}$ *is the following product*

$$GH_M = G_{M,p} \times G_p,$$

where $G_{M,p} = (\mathrm{End}_T\, M)^\times(\mathcal{O}_{T,p})$ denotes the p-adic group of $\mathcal{O}_{T,p}$-points of a maximal torus of the algebraic T-group $(\mathrm{End}_T\, M)^\times$ of invertible elements of $\mathrm{End}_T\, M$ (it is implicitly supposed that the group $\mathrm{End}_T\, M^\times$ possesses an $\mathcal{O}_T$-integral structure given by an appropriate choice of an $\mathcal{O}_T$-lattice).

Consider next the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Hida group GH_M, which contains the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(G_p,\ \mathbf{C}_p^\times)$$

consisting of all continuous characters of the Galois group G_p (via the projection of GH_M onto G_p.

The group $\mathcal{X}_{M,p}$ contains the discrete subgroup $\mathcal{A}$ of arithmetical characters of the type

$$\chi \cdot \eta \cdot x_p^m = (\chi, \eta, m),$$

where

$$\chi \in \mathcal{X}_{M,p}^{tors}$$

is a character of finite order of GH_M, η is a T-algebraic character of $G_{M,p}$, $m \in \mathbf{Z}$, and x_p denotes the following natural homomorphism

$$x_p : G_p = \mathrm{Gal}(\mathbf{Q}_{p,\infty}^{ab}/\mathbf{Q}) \cong \mathbf{Z}_p^\times \to \mathbf{C}_p^\times,\ x_p \in \mathcal{X}_p.$$

Definition. — *The* algebra of Iwasawa-Hida $I_M = I_{M,p}$ *of* M *at* p *is the completed group ring* $\mathcal{O}_p[[GH_M]]$, *where* $\mathcal{O}_p$ *denotes the ring of integers of the Tate field* $\mathbf{C}_p$.

Note that this definition is completely analogous to the usual definition of the Iwasawa algebra Λ as the completed group ring $\mathbf{Z}_p[[\mathbf{Z}_p]]$ if we take into account that $\mathbf{Z}_p$ coincides with the factor group of $\mathbf{Z}_p^\times$ modulo its torsion subgroup.

Now for each arithmetic point $P = (\chi, \eta, m) \in \mathcal{A}$ we have a homomorphism

$$\nu_P : I_{M,p} \to \mathcal{O}_p$$

which is defined by the corresponding group homomorphism

$$P : GH_M \to \mathcal{O}_p^\times \subset \mathbf{C}_p^\times.$$

For a I_M-module N and $P \in \mathcal{A}$ we put

$$N_P = N \otimes_{I_M, \nu_P} \mathcal{O}_p$$

("reduction of N modulo P", or a fiber of N at P). Therefore, for a Galois representation

$$r_N : G_\mathbf{Q} \to \mathrm{GL}(N)$$

of the above type its reduction $r_{N_P} = r \mod P$ is defined as the natural composition:

$$G_\mathbf{Q} \to \mathrm{GL}(N) \to \mathrm{GL}(N_P).$$

Remark. — In his very recent work [HiGen] Hida gives another version of the above definition, but he starts from a Galois representation $\varphi : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_n(\mathbf{I})$, where $\mathbf{I} = \mathcal{O}_K[[T_n(\mathbf{Z}_p)]]$ and T_n the maximal split torus of $\mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\, GL(n)$ for the integer ring $\mathcal{O}_F$ of F, and for the integer ring $\mathcal{O}_K$ of a sufficiently large finite extension K of $\mathbf{Q}_p$. He is interested in representations φ satisfying the following condition:

> *There are arithmetic points P "densely populated" in* $\mathrm{Spec}(\mathbf{I}(K))$ *such that the Galois representation $\varphi_P = P \circ \varphi$ is the p-adic étale realization of a rank n pure motive M_P of weight w defined over F with coefficients in a number field E_P in $\overline{\mathbf{Q}}$.*

We are trying to resolve an inverse problem and to include a given motive M in a maximal possible p-adic family M_P parametrized by arithmetic characters of a certain group which we suppose to consist of an "algebraic part" $G_{M,p}$ and of a "Galois part" G_p.

8. A conjecture on the existence of p-adic families of Galois representations attached to motives

Note first that the fixed embeddings $T \hookrightarrow \mathbf{C}$,

$$i_\infty : \overline{\mathbf{Q}} \to \mathbf{C}, \;\; i_p : \overline{\mathbf{Q}} \to \mathbf{C}_p$$

define a place $\lambda(p)$ of T attached to the corresponding composition

$$T \hookrightarrow \overline{\mathbf{Q}} \stackrel{i_p}{\to} \mathbf{C}_p.$$

Conjecture I. — *For every M of rang d with coefficients in T there exists a free I_M-module M_I of the same rang d, a Galois representation*

$$r_I : G_F \to \mathrm{GL}(M_I),$$

a dense subset $\mathcal{A}' \subset \mathcal{A}$ of characters, and a distinguished point $P_0 \in \mathcal{A}$ such that

(a) the reduced Galois representation

$$r_{I,P_0} : G_F \to \mathrm{GL}(M_{I,P_0})$$

is equivalent over $\mathbf{C}_p$ to the $\lambda(p)$-adic representation $r_{M,\lambda(p)}$ of M at the distinguished place $\lambda(p)$;

(b) for every $P \in \mathcal{A}'$ there exists a motive M_P over $\mathbf{Q}$ of the same rang d such that its $\lambda(p)$-adic Galois representation is equivalent over $\mathbf{C}_p$ to the reduction

$$r_{I,P} : G_{\mathbf{Q}} \to \mathrm{GL}(M_{I,P}).$$

We call the module M_I the *realization of Iwasawa* of M.

A generalization of the Hasse invariant for a motive. — We define the generalized *Hasse invariant* of a motive in terms of the *Newton polygons* and the *Hodge polygons* of a motive. Properties of these polygons are closely related to the notions of a p-ordinary and a p-admissible motive.

Now we are going to define the Newton polygon $P_{Newton}(u) = P_{Newton}(u, M)$ and the Hodge polygon $P_{Hodge}(u) = P_{Hodge}(u, M)$ attached to M. First we consider (using i_∞) the local p-polynomial

$$\begin{aligned} L_p(M, X)^{-1} &= 1 + A_1(p)X + \cdots + A_d(p)X^d \\ &= (1 - \alpha^{(1)}(p)X) \cdot (1 - \alpha^{(2)}(p)X) \cdot \ldots \cdot (1 - \alpha^{(d)}(p)X), \end{aligned}$$

and we assume that its inverse roots are indexed in such a way that

$$\mathrm{ord}_p \alpha^{(1)}(p) \leq \; \mathrm{ord}_p \alpha^{(2)}(p) \leq \cdots \leq \; \mathrm{ord}_p \alpha^{(d)}(p)$$

DEFINITION. — *The* Newton polygon $P_{Newton}(u)(0 \leq u \leq d)$ *of* M *at* p *is the convex hull of the points* $(i, \mathrm{ord}_p A_i(p))$ $(i = 0, 1, \cdots, d)$.

The important property of the Newton polygon is that the length the horizontal segment of slope $i \in \mathbf{Q}$ is equal to the number of the inverse roots $\alpha^{(j)}(p)$ such that $\mathrm{ord}_p \alpha^{(j)}(p) = i$ (note that i may not necessarily be integer but this will be the case for the p-ordinary motives below).

The *Hodge polygon* $P_{Hodge}(u)$ $(0 \leq u \leq d)$ of M is defined using the Hodge decomposition of the d-dimensional $\mathbf{C}$-vector space

$$M_B = M_B \otimes_T \mathbf{C} = \oplus_{i,j} M^{i,j}$$

where $M^{i,j}$ as a $\mathbf{C}$-subspace.

DEFINITION. — *The* Hodge polygon $P_{Hodge}(u)$ *is a function* $[0, d] \to \mathbf{R}$ *whose graph consists of segments passing through the points*

$$(0, 0), \ldots, (\sum_{i' \leq i} h(i', j), \sum_{i' \leq i} i' h(i', j)),$$

so that the length of the horizontal segment of the slope $i \in \mathbf{Z}$ *is equal to the dimension* $h(i, j)$.

Now we recall the definition of a p-ordinary motive (see [Co], [Co-PeRi]). We assume that M is pure of weight w and of rank d. Then M is called p ordinary at p if the the Hodge polygon and the Newton polygon of M coincide:

$$P_{Newton}(u) = P_{Hodge}(u).$$

If furthermore M is critical at $s = 0$ then it is easy to verify that the number d_p of the inverse roots $\alpha^{(j)}(p)$ with

$$\mathrm{ord}_p \alpha^{(j)}(p) < 0 \text{ is equal to } d^+ = d^+(M) \text{ of } M_B^+.$$

However, it turns out that the notion of a p-ordinary motive is too restrictive, and we have introduced the following weaker version of it.

DEFINITION. — *The motive* M *over* F *with coefficients in* T *is called* admissible *at* p *if*

$$P_{Newton}(d^+) = P_{Hodge}(d^+)$$

here $d^+ = d^+(M)$ *is the dimension of the subspace* $M^+ \subset M_B$.

In the general case we use the following quantity (a “generalized slope") $h = h_p$ which is defined as the difference between the Newton polygon and the Hodge polygon of M:

$$h_p = P_{Newton}(d^+) - P_{Hodge}(d^+).$$

of M at p. Note the following important properties of h:

(i) $h = h(M)$ does not change if we replace M by its Tate twist.

(ii) $h = h(M)$ does not change if we replace M by its twist $M = M(\chi)$ with a Dirichlet character χ of finite order whose conductor is prime to p.

(iii) $h = h(M)$ does not change if we replace M by its dual $M^\vee$.

In the next section we state in terms of this quantity a general conjecture on p-adic L-functions.

A conjecture on the existence of certain families of p-adic L-functions. — We are going to describe families of p-adic L-functions as certain analytic functions on the total analytic space, the $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_{M,p} = \mathrm{Hom}_{contin}(GH_M,\ \mathbf{C}_p^\times),$$

which contain the $\mathbf{C}_p$-analytic Lie subgroup (the cyclotomic line) $\mathcal{X}_p \subset \mathcal{X}_{M,p}$:

$$\mathcal{X}_p = \mathrm{Hom}_{contin}(G_p,\ \mathbf{C}_p^\times).$$

In order to do this we need a modified L-function of a motive. Following J.Coates this modified L-function has a form appropriate for further use in the p-adic construction. First we multiply $L(M, s)$ by an appropriate factor at infinity and define

$$\Lambda_{(\infty)}(M,\ s) = E_\infty(M,\ s)L(M, s)$$

where $E_\infty(M, s) = E_\infty(\tau, R_{F/\mathbf{Q}}M, \rho, s)$ is the modified Γ-factor at infinity which actually does not depend on the fixed embedding τ of T into $\mathbf{C}$. Also we put

$$c^\nu(M) = (c^\nu(M)^{(\tau)}) = c^\nu(RM)(2\pi i)^{r(RM)} \in (T \otimes \mathbf{C})^\times$$

where

$$\nu = (-1)^m, r(M) = \sum_{j<0} jh(i, j) = \sum_{j<0} jh(i, j),$$

$c^\nu(M)$ is the period of M. Note that the quantity $r(M)$ has a natural geometric interpretation as the minimum of the Hodge polygon $P_{Hodge}(M)$.

We define

$$\Lambda_{(p,\infty)}(M(m)(\chi), s) = G(\chi)^{-d^{\varepsilon_0}(M(m)(\chi))} \prod_{p|p} A_p(M(m)(\chi), s) \cdot \Lambda_{(\infty)}(M(m)(\chi), s),$$

where

$$A_p(M(\chi), s) = \begin{cases} \prod_{i=d^++1}^{d} (1 - \chi(p)\alpha^{(i)}(p)p^{-s}) \prod_{i=1}^{d^+} (1 - \chi^{-1}(p)\alpha^{(i)}(p)^{-1}p^{s-1}) & \text{for } p \nmid \mathfrak{c}(\chi) \\ \prod_{i=1}^{d^+} \left(\dfrac{p^s}{\alpha^{(i)}(p)}\right)^{\mathrm{ord}_p\, \mathfrak{c}(\chi)}, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}$ be the discrete subgroup $\mathcal{A}$ of arithmetical characters,

$$\chi \cdot \eta \cdot x_p^m = (\chi, \eta, m) \in \mathcal{A},$$

$\mathcal{A}' \subset \mathcal{A}$ a certain "dense" subset of characters, $P_0 \in \mathcal{A}$ a distinguished point of conjecture I. Let $\mathcal{A}'' \subset \mathcal{A}'$ be the subset of critical elements, which consists of those P, for which the corresponding motives M_P are critical (at $s = 0$). Now we are ready to formulate the following

Conjecture II. — *There exists a certain choice of complex periods $\Omega_\infty(P) \in \mathbf{C}^\times$ and p-adic periods $\Omega_p(P) \in \mathbf{C}_p^\times$ for all $P \in \mathcal{A}''$ such that "the ratio" $\Omega_p(P)/\Omega_\infty(P)$ is canonically defined, and there exists a $\mathbf{C}_p$-meromorphic function*

$$Ł_M : \mathcal{X}_{M,p} \to \mathbf{C}_p$$

with the properties:

(i)

$$Ł_M(P) = \Omega_p(P) \frac{\Lambda_{p,(\infty)}(M(m)(\chi), 0)}{\Omega_\infty(P)}$$

for almost all $P \in \mathcal{A}''$;

(ii) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with η fixed there exists a finite set $\Xi \subset \mathcal{X}_{M,p}$ of p-adic characters and positive integers $n(\xi)$ (for $\xi \in \Xi$) such that for any $g_0 \in G_p$ we have that the function

$$\prod_{\xi \in \Xi} (x(g_0) - \xi(g_0))^{n(\xi)} Ł_M(x \cdot P)$$

is holomorphic on $\mathcal{X}_p$;

(iii) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with η fixed the function in (ii) is bounded if and only if the invariant $h(P) = h(M_P)$ vanishes;

(iv) In the general case the function $Ł_M(P \cdot x)$ of $x \in \mathcal{X}_p$ is of logarithmic growth type $o(\log(\cdot)^{h}{}_0)$ with

$$h_0 = [h] + 1.$$

(v) For arithmetic points of type

$$P = (\chi, \eta, m) \in \mathcal{A}''$$

with χ and m fixed the function in (ii) is always bounded if the Hasse invariant $h(P) = h(M_P)$ does not depend on η.

Note that the assertion (v) means in particularly that the values of the function

$$P \mapsto \Omega_p(P) \frac{\Lambda_{(p,\infty)}(M_P, 0)}{\Omega_\infty(P)}$$

satisfy generalised Kummer congruences in the following sense: for any finite linear combination $\sum_P b_P \cdot P$ with $b_P \in \mathbf{C}_p$ which has the property $\sum_P b_P \cdot P \equiv 0 \,(\bmod p^N)$ we have that for some constant $C \neq 0$ the corresponding linear combination of the normalized L-values

$$C \sum_P b_P \Omega_p(P) \cdot \frac{\Lambda_{(p,\infty)}(M_P, 0)}{\Omega_\infty(P)} \equiv 0 \,(\bmod p^N).$$

In the case of families of supersingular modular forms studied by R. Coleman [Col] the invariant $h(P)$ reduces to the slope of a modular form in such a family.

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