Computation of a universal deformation ring in the Borel case

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1. Introduction

Let p be an odd prime number. Let $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(\mathbf{F}_p)$ be an odd continuous representation unramified outside a finite set $S_{\mathbf{Q}}$ of rational primes. In that situation $\bar{\rho}$ factors through $G_S = \mathrm{Gal}(K_{S,p}/\mathbf{Q})$ where K denotes the field fixed by $\mathrm{Ker}\bar{\rho}$ in $\bar{\mathbf{Q}}$, S a finite set of places of K containing the places over $S_{\mathbf{Q}}$ in K and $K_{S,p}$ the maximal pro-p-extension of K, unramified outside S.

Mazur's deformation theory of Galois representations shows the existence of a (uni)versal deformation ring $R_{G_S}(\bar{\rho})$ and the associated (uni)versal representation ρ_{G_S} which allow us to parametrize all deformations $\rho : G_S \to \mathrm{Gl}_2(R)$ of $\bar{\rho} : G_S \to \mathrm{Gl}_2(\mathbf{F}_p)$ where R stands for any complete nœtherian local ring with residual field $\mathbf{F}_p$.

To our knowledge, there is not yet any general method for computing the (uni)versal deformation ring $R_{G_S}(\bar{\rho})$ although partial results suggest that the structure of this ring is closely related to the adjoint representation $Ad(\bar{\rho})$. By application of Schlessinger's criterion [Sc] Mazur ([Ma] subsection 1.2) shows that $R_{G_S}(\bar{\rho})$ is a quotient ring of a formal series ring whose minimal number of variables is $d' = \dim_{\mathbf{F}_p} H^1(G_S, Ad(\bar{\rho}))$. To be more specific, $R_{G_S}(\bar{\rho}) = \mathbf{Z}_p[[Y_1, \cdots, Y_{d'}]]/I$ where I will be called the *ideal of relations* of $R_{G_S}(\bar{\rho})$, so that the determination of $R_{G_S}(\bar{\rho})$ amounts to that of I.

The knowledge of I would allow to discuss Mazur's question on the Krull dimension of $R_{G_S}(\bar\rho)/pR_{G_S}(\bar\rho)$ (in the case $\dim\bar\rho = 1$, this is the celebrated Leopoldt conjecture, [Ma] subsections 1.6 and 1.10). The general theory of obstructions shows that $I = 0$ (i.e the (uni)versal deformation ring is free) if $H^2(G_S, Ad(\bar\rho)) = 0$. Almost all explicit examples that we know of rely on the latter assumption ([Bo1],[Ra]), which occurs for instance if the maximal pro-p-quotient P_{G_S} of G_S is a free pro-p-group or if some irreducible components of the representation $P_{G_S}^{ab}\otimes\mathbf{F}_p$ are prime to the irreducible components of the representation $Ad(\bar\rho)$. This suggests a precise connection (via $\bar\rho$) between the relations of the pro-p-group P_{G_S} and the relations of the (uni)versal deformation ring R_{G_S}. It should be stressed though, that even the precise knowledge of the relations of P_{G_S} does not imply straightforwardly the knowledge of the ideal I.

In this paper, we present an approach for determining I, via Iwasawa theory, built on an example studied by Boston. In [Bo1] subsection 9.3 Boston considers representations $\bar\rho$ which factor through $\bar\rho_G : G \to \mathrm{Gl}_2(\mathbf{F}_p)$, where G is a natural quotient of G_S which occurs in Iwasawa theory. By construction there exists a surjective morphism of local rings $R_{G_S}(\bar\rho) \to R_G(\bar\rho_G)$ making $R_G(\bar\rho_G)$ an approximation of $R_{G_S}(\bar\rho)$. We make more precise this morphism in the sequel. Moreover if the order of $\mathrm{Im}\bar\rho$ is prime to p (this is the *tame case*), then $R_{G_S}(\bar\rho)$ and $R_G(\bar\rho_G)$ have the same minimal number of variables.

In the sequel we shall develop Boston's example along two directions: First we present a systematic way of deriving, from a minimal system of relations of the group P_{G_S}, a minimal set of relations for the ring $R_G(\bar\rho_G)$, using the method of Fox derivatives described in [Ng1]. Second we extend the framework of Iwasawa theory to study not only the $\mathbf{Z}_p$-cyclotomic extension, but also pro-p-free extensions, which allows us to enlarge the group G which approximates G_S and to replace the classical Iwasawa algebra by the (non commutative) Magnus algebra as in [Ng2].

More precisely it will be shown that the computation of $R_G(\bar\rho_G)$ is possible provided that $\mathrm{Im}\bar\rho \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ with an additional hypothesis on the diagonal characters (i.e *Borel case*).
If the relations of P_{G_S} are known, [Bo1] proposition 6.1 allows us in principle to determine $R_{G_S}(\bar\rho)$ in the tame case, though in most cases the computations

involved cannot be carried out. Our method here works fine even if the relations of P_{G_S} are not explicitly known.
The method involves two steps: first (sections 4-5) we find the images of the generators of G using the action of G/P_G (where P_G is the p-Sylow subgroup of G). We proceed exactly as if P_G were free. Second (sections 6-7) we express the relations between selected generators (the images of which are simple).
In the special case considered by Boston, we obtain a presentation of the universal ring $R_G(\bar{\rho}_G) = \mathbf{Z}_p[[Y_1, \cdots, Y_{d'_G}]]/I$ where $d'_G = \dim_{\mathbf{F}_p} H^1(G, Ad\bar{\rho}_G)$ and I is the ideal of relations generated by

$$\Big(\prod_{i=1}^{u_{X_\infty}} (1+Y_i)^{a_i^j} - 1\Big)_{1\leq j\leq \ell}$$

and

$$\Big(a_n^j + \sum_{k=1}^{\infty} b_{n,k}^j \Big(\frac{1+Y_{d'_G-1}}{1+Y_{d'_G}} - 1\Big)^k + \sum_{i=1+u_{X_\infty}}^{v_{X_\infty}} \Big(a_i^j + \sum_{k=1}^{\infty} b_{i,k}^j \Big(\frac{1+Y_{d'_G-1}}{1+Y_{d'_G}} - 1\Big)^k\Big) Y_i\Big)_{1\leq j\leq \ell}$$

where the coefficients $a_i^j, b_{i,k}^j$ are derived from the relations of P_{G_S} (see theorem 7.2.1 and compare [Bo1] subsection 9.3).
The comparison between the universal deformation rings $R_G(\bar{\rho}_G)$ and $R_{G_S}(\bar{\rho})$ is done in theorem 7.4.2, where we make precise the natural surjective morphism $R_{G_S}(\bar{\rho}) \to R_G(\bar{\rho}_G)$.
Lastly (sections 7-8) we give some applications of our results to representations associated to elliptic curves or to modular forms. For the latter representations, cyclotomic fields (we correct an imprecise result of [Bo1] proposition 9.2 and [Bo2] section 6) or fields having a so-called Wingberg presentation appear naturally.

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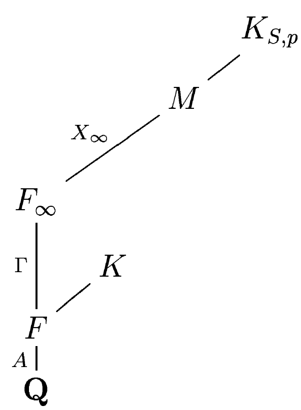
2. Notations

Let $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(\mathbf{F}_p)$ be an odd continuous representation unramified outside a finite set of primes $S_{\mathbf{Q}}$ of $\mathbf{Q}$.
Assume that $\mathrm{Im}\bar{\rho}$ is upper triangular: $\mathrm{Im}\bar{\rho} \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.
Let K denote the subfield of $\bar{\mathbf{Q}}$ fixed by $\ker\bar{\rho}$. Hence, one has $\mathrm{Gal}(K/\mathbf{Q}) \cong \mathrm{Im}\bar{\rho}$ (finite). Let F be a subextension of K such that $\mathrm{Gal}(K/F)$ (possibly trivial) is the Sylow p-subgroup of $\mathrm{Gal}(K/\mathbf{Q})$.
We assume the extension K/F is ramified at p.

Let S_F be a finite set of places of F containing the places over $S_{\mathbf{Q}}$ in F. Suppose S_F contains the primes over p in F and the archimedean primes. Let S_K be the places in K over S_F. We write S for S_F or S_K when no confusion can arise.

The maximal algebraic pro-p-extensions of F or K unramified outside S_F or S_K are the same: $K_{S,p} = F_{S,p}$. We work with the following setup



where F_∞ is a free pro-p-extension (not necessary abelian) of rank $k \geq 1$ of F: this means that $\Gamma = \mathrm{Gal}(F_\infty/F)$ is pro-p-free on k-generators. We assume that A acts on Γ by conjugation.

The field M is the maximal abelian pro-p-extension of F_∞ unramified outside S,
and $H = \mathrm{Gal}(K_{S,p}/F_\infty)$,

and $X_\infty = \mathrm{Gal}(M/F_\infty) = H^{ab}$.

Remark 2.2 We assume that K and F_∞ are linearly disjoint.

Define furthermore $G_S = \mathrm{Gal}(K_{S,p}/\mathbf{Q})$,
$P_{G_S} = \mathrm{Gal}(K_{S,p}/F)$,
$G = \mathrm{Gal}(M/\mathbf{Q})$,
$P_G = \mathrm{Gal}(M/F)$;
The group $\bar{P}_{G_S}$ (resp. $\bar{P}_G$, resp. $\bar{\Gamma}$) is the quotient of P_{G_S} (resp. P_G, resp. Γ) by its Frattini subgroup.

Remark 2.3 Even if G and P_G have no index S, they depend on S_F and on the choice of Γ.

In the sequel we will work in particular with
-the $\mathbf{Z}_p$-cyclotomic extension of F, (then we denote $\Gamma = \Gamma_{cyc}$) in order to maximize information on the module X_∞,
-a maximal free pro-p-extension of F, (then we denote $\Gamma = \Gamma_{max}$) in order to maximize the size of G. Note that such a maximal free pro-p-extension is not necessary unique (see [Ya2] remark).
In these cases it is clear that A acts on Γ.

After Schur-Zassenhaus theorem (see proposition 2.1 [Bo1]) the group G contains a subgroup A mapping isomorphically to G/P_G.
We assume

$$\bar{\rho}_{|A} = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \text{ such that } \chi_1 \neq \pm\chi_2,$$

where $\chi_1, \chi_2 : A \to \mathbf{F}_p^*$ are the diagonal characters defined by $\mathrm{Im}\bar{\rho}$. The case where $\mathrm{Im}\bar{\rho}$ is upper triangular and $\chi_1 \neq \chi_2$ is termed *the Borel case.*

We denote by $Ad(\bar{\rho})$ the vector space $\mathrm{M}_2(\mathbf{F}_p)$ equipped with the action of G_S defined by conjugation by $\bar{\rho}$ (or the action of G). Depending upon the context we shall instead of $Ad(\bar{\rho})$ write $Ad(\bar{\rho}_G)$ when necessary, to avoid confusion.
At last we define $\Gamma_2(R) = \ker(\mathrm{Gl}_2(R) \to \mathrm{Gl}_2(\mathbf{F}_p))$ for all $R \in \mathcal{C}$ where $\mathcal{C}$ is the category of the complete noetherian local rings of residual field $\mathbf{F}_p$. We denote by $\mathcal{M}_R$ the maximal ideal of R.

Two liftings ρ and ρ' to R of $\bar{\rho}$ are *strictly equivalent* if there exists $M' \in \Gamma_2(R)$ such that $\rho' = M'\rho M'^{-1}$.
Mazur's functor of deformations of $\bar{\rho}$, $\mathcal{F} : \mathcal{C} \to$ Sets defined by $\mathcal{F}(R)$ is the set of the strict equivalent classes of liftings of $\bar{\rho}$ onto R. One takes $\rho \in \mathcal{F}(R)$.

There exists a canonical surjective morphism $G_S \to G$ and, by construction, the representation $\bar{\rho}$ factors through G; after [Ma] subsection 1.3 there exists a surjective morphism of local rings $R_{G_S}(\bar{\rho}) \to R_G(\bar{\rho}_G)$.

Remark 2.4 By definition we have the exact sequence:

$$0 \to P_{G_S} \to G_S \to \mathrm{Gal}(F/\mathbf{Q}) \to 0$$

In the tame case the order of $\mathrm{Gal}(F/\mathbf{Q})$ is prime to p then $H^1(G_S, Ad(\bar{\rho})) \cong H^1(P_{G_S}, Ad(\bar{\rho}))^{G_S/P_{G_S}}$. Moreover in the tame case, $P_{G_S} \subset \mathrm{Ker}\bar{\rho}$, i.e the action of P_{G_S} on $Ad(\bar{\rho})$ is trivial; hence

$$H^1(P_{G_S}, Ad(\bar{\rho}))^{G_S/P_{G_S}} \cong \mathrm{Hom}(\bar{P}_{G_S}, Ad(\bar{\rho}))^{G_S/P_{G_S}} \cong H^1(G_S, Ad(\bar{\rho})).$$

This reasoning is also valuable if we replace G_S and $Ad(\bar{\rho})$ by G resp. $Ad(\bar{\rho}_G)$, and since $\bar{P}_{G_S} \cong \bar{P}_G$ we obtain

$$H^1(G_S, Ad(\bar{\rho})) \cong H^1(G, Ad(\bar{\rho}_G)).$$

Hence $R_{G_S}(\bar{\rho})$ and $R_G(\bar{\rho}_G)$ have the same minimal number d' of variables in the tame case (for example if $\mathrm{Im}\bar{\rho}$ is diagonal).

Recall at last that, following [Ra] theorem 1.1, if $Ad(\bar{\rho})^{G_S} = \mathbf{F}_p\mathrm{Id}$ there exists a universal deformation ring; otherwise $R_{G_S}(\bar{\rho})$ will merely stand for a versal deformation ring.

3. Strategy

We use the notation ρ for ρ_G whenever no confusion shall arise.
The profinite group G has a normal p-Sylow subgroup P_G of finite index and finite type. After lemma 2.4 [Bo1] there exists exactly one semi-direct product $G = A \ltimes P_G$ with given action of A on $\bar{P}_G$, where $\bar{P}_G = P_G/P_G^p[P_G, P_G]$. One then writes $\rho(G) = \rho(A) \ltimes \rho(P_G)$. To be more precise (proposition 2.3

[Bo1])

Lemma 3.1 *If V is an $\mathbf{F}_p[A]$-submodule of $\bar{P}_G$, then there exists an A-invariant subgroup B of P_G with $dim_{\mathbf{F}_p}V$ generators mapping onto V.*

Since $(\sharp A, p) = 1$ and $\bar{\rho} : A \to \mathrm{Gl}_2(\mathbf{F}_p)$ is injective, $\rho(A)$ is known by proposition 3.1 [Bo3]; the universal deformation ring of $\bar{\rho}_{|A}$ is $\mathbf{Z}_p$ and $\rho(A)$ is given by the canonical homomorphism $\mathbf{Z}_p \to R$.

Boston replaces the functor $\mathcal{F}$ by the functor $\mathcal{E} : \mathcal{C} \to$ Sets defined by $\mathcal{E}(R) = \mathrm{Hom}_A(P_G, \Gamma_2(R))$ which is always representable. To be more specific:

- If $K = F$, $\mathcal{F} \cong \mathrm{Hom}_A(P_G, \Gamma_2(R))$.
- Otherwise $\rho(P_G) \not\subset \Gamma_2(R)$ but to compute the universal representation ρ it suffices to know $\rho(A)$, $\mathrm{Hom}_A(\mathrm{Gal}(M/K), \Gamma_2(R))$, the images $\rho(\mathrm{Gal}(K/F))$ and $\rho(\mathrm{Gal}(M/K))$, as well as the action of $\rho(\mathrm{Gal}(K/F))$ on $\rho(\mathrm{Gal}(M/K))$.

Remark 3.2 These considerations are also valid if we replace G and M by G_S and $K_{S,p}$.

Remark 3.3 The above considerations justify the method announced in the introduction: on the one hand we find the image of the generators of P_G using the action of A; for $x \in \mathrm{Ker}\bar{\rho}$, $\rho(x) = \begin{pmatrix} 1+Y_1 & Y_2 \\ Y_3 & 1+Y_4 \end{pmatrix}$, $Y_i \in \mathcal{M}_R$, $1 \leq i \leq 4$; this image introduces four variables. Expressing the action of A allows to reduce this number of variables. We begin exactly as if P_G were free. We choose a system of generators the images of which are particularly simple.
On the other hand we express the relations between selected generators. We need the relations of P_G. For this, we use that $\Gamma \cong P_G/X_\infty$ is free so we merely have to determine the relations of X_∞ as a $\mathbf{Z}_p[[\Gamma]]$-module. These relations will yield relations between the variables in a straightforward fashion.

4. Preliminary computations

Following Boston's approach, we begin by expressing the action of $\rho(A)$ on $\rho(G)$. In particular, subsection 4.2 shows that A possesses a uniquely determined complex conjugation c. This complex conjugation plays an important rôle because it allows to reduce the number of variables. We then express the action of the other elements of A, if there are any. When $\rho(A)$ acts on a commutative group, its action is expressed in terms of basic linear algebra. We therefore find it convenient to work with $\bar{P}_G$.

4.1. Image of the residual representation

To express the action of A, informations on the image of the residual representation $\bar{\rho}$ are needed. In the following lemma we show that $\mathrm{Gal}(K/F) \cong \bar{\rho}(P_G)$ and A are abelian.

Lemma 4.1.1 *One has* $\bar{\rho}(P_G) \subset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ *and A is isomorphic to a subgroup of invertible diagonal matrices.*

Proof : The group P_G is a pro-p-group, thus $\bar{\rho}(P_G)$ is a p-group which acts on the p-group $\mathrm{M}_2(\mathbf{F}_p)$, hence $\bar{\rho}(P_G) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ (for a valuable basis). If $\begin{pmatrix} 1 & b \\ 0 & b' \end{pmatrix} \in \bar{\rho}(P_G)$, then there exists $s \in \mathbf{N}$ such that $b'^{p^s} = 1$ since $\mathrm{Gal}(K/F)$ is a p-group; yet $b' \in \mathbf{F}_p$, hence $b' = 1$ and

$$\bar{\rho}(P_G) \subset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

One has $A \cong \mathrm{Im}\bar{\rho}/\bar{\rho}(P_G)$, hence A is isomorphic to a subgroup of the group of invertible diagonal matrices. □

Remark 4.1.2 As $\bar{\rho}$ is odd and A is abelian, A possesses a uniquely determined complex conjugation.

4.2. Action of complex conjugation on P_G

We can assume

$$\rho(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

because we want ρ up to strict equivalence.

Lemma 4.2.1 *Let $x \in P_G$, $\rho(x) = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \in Gl_2(R)$, then*

- *If x is invariant under complex conjugation then*

$$\rho(x) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix} \text{ with } Y, Y' \in \mathcal{M}_R,$$

- *If x is inversed by complex conjugation then*

$$\rho(x) = \begin{pmatrix} (1+UU')^{1/2} & U \\ U' & (1+UU')^{1/2} \end{pmatrix}, \text{ with } U \in R, U' \in \mathcal{M}_R.$$

Proof : It suffices to compare $\rho(x)$ or $\rho(x)^{-1}$ to

$$\rho(c \cdot x) = \rho(c)\rho(x)\rho(c)^{-1} = \begin{pmatrix} U_1 & -U_2 \\ -U_3 & U_4 \end{pmatrix}$$

and to use lemma 4.1.1 to fing the values of the introduced coefficients. □

4.3. Action of A

If $x \in P_G$, $\bar{x}$ denotes the image of x in $\bar{P}_G$.
Since A is abelian, the group $\bar{P}_G$ decomposes into A-invariant subgroups $< \bar{x} >$ of dimension 1. After lemma 3.1 there exists $x \in P_G$ that maps onto $\bar{x}$ and χ, a character of A such that

$$x \in P_{G,\chi} := \{u \in P_G : a \cdot u = u^{\chi(a)}, \forall a \in A\}.$$

In the Borel case the action of $\rho(A)$ on $\rho(X)$ is known explicitly.
Recall that $\rho(A)$ is abelian; since it contains $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, one has $\rho_{|A} = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ and:

Proposition 4.3.1 *Let $x \in P_{G,\chi}$ with χ an odd character; then the action of A on x imposes:*

$$\rho(x) = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix} \text{ if } \chi = \chi_1\chi_2^{-1},$$

$$\rho(x) = \begin{pmatrix} 1 & 0 \\ U' & 1 \end{pmatrix} \text{ if } \chi(a) = \chi_2\chi_1^{-1},$$

$$\rho(x) = Id \text{ otherwise.}$$

PROOF : Let $a \in A$; we denote by

$$\rho(a) = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}, \; a_i \in R^*.$$

Recall that $\bar{\rho}(x) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in \mathbf{F}_q$. One writes $\rho(a \cdot x) = \rho(x)^{\chi(a)}$ and one uses the invariance of the trace by conjugation in order to obtain that if $\chi(a) \neq \pm 1$ then $UU' = 0$. Then the action of a on x imposes one condition out of the four following ones:

- Prime-to-adjoint condition $U = U' = 0$
  if $\chi(a) = \pm 1$ and $a_1 \neq \pm a_4$,
  if $\chi(a) \neq \pm 1$ and $a_1 \neq \chi(a)^{\pm 1} a_4$.

- Condition-free
  if $\chi(a) = \pm 1$ and $a_1 = \pm a_4$.

- $U' = 0$
  if $\chi(a) \neq \pm 1$, and $a_1 = \chi(a) a_4$.

- $U = 0$
  if $\chi(a) \neq \pm 1$ and $a_4 = \chi(a) a_1$.

Recalling that $\bar{\rho}_{|A} \neq \begin{pmatrix} \chi_1 & 0 \\ 0 & \pm\chi_1 \end{pmatrix}$, there must exist $a \in A$ such that $\rho(a) = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}$ with $a_1 \neq \pm a_4$, which shows the lemma in all cases. □

One notices that the conditions imposed by the action of A reduce the number of deformation variables, but introduce no relations between these variables.

Remark 4.3.2 In subsection 3.B [Böl] Böckle solved the Borel case for an even representation ($\bar{\rho}(c) = \pm\text{Id}$). If $\bar{\rho}$ is odd, the deformations can introduce such images as

$$\begin{pmatrix} (1+UU')^{1/2} & U \\ U' & (1+UU')^{1/2} \end{pmatrix}$$

which render the computation inextricable. To avoid this ugly image, we have imposed

$$\bar{\rho}_{|A} \neq \begin{pmatrix} \chi_1 & 0 \\ 0 & \pm\chi_1 \end{pmatrix}.$$

If $x \in P_{G,\chi}$ with χ even, we obtain in the same fashion

Proposition 4.3.3 *Let $x \in P_{G,\chi}$ with χ even; then*

$$\rho(x) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix} \; Y, Y' \in \mathcal{M}_R.$$

Moreover if $\chi(A) \neq 1$, $\rho(x) = Id$.

If $\Gamma = \Gamma_{cyc} =< \gamma >$ then $\bar{\gamma}$ (image of γ in $\bar{P}_G$) is invariant by the action of A. Hence one can lift it to an element s_d in $P_{G,triv}$.

Lemma 4.3.4 *One has*

$$\rho(s_d) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix} \; \textit{with } Y, Y' \in \mathcal{M}_R.$$

PROOF : The element $s_d \in \text{Ker}\, \bar{\rho}$ is invariant by the action of A; the expression of the action of the complex conjugation allows us to conclude. □

4.4. Choice of representatives of the strict equivalence classes

The chosen form above for the image of complex conjugation under ρ does not completely fix a unique representative of a strict equivalence class. The representation ρ can be replaced by its conjugate $\rho' = M\rho M^{-1}$ for some $M \equiv \mathrm{Id} \bmod \mathcal{M}_R$ without changing the strict equivalence class of ρ provided M commutes with $\rho(c)$. The matrix M can be chosen in the form (see lemma 4.2.1)

$$M = \begin{pmatrix} 1+Z_1 & 0 \\ 0 & 1+Z_2 \end{pmatrix} \; Z_1, Z_2 \in \mathcal{M}_R.$$

This remark allows us to establish

Lemma 4.4.1 *If Im$\bar{\rho}$ is not diagonal, there exists $x_n \in P_G$ such that we can impose*

$$\rho(x_n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Moreover if $x \in P_{G,\chi}$ commutes with x_n and
-if $\chi = \chi_1\chi_2^{-1}$ then $\rho(x) = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$,
-if $\chi = 1$ then $\rho(x) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y \end{pmatrix}$,
-$\rho(x)$ =Id otherwise.

PROOF : If Im$\bar{\rho}$ is not diagonal, there exists x_n such that $\bar{\rho}(x_n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
Hence $x_n \in P_G$ is inversed by c and using proposition 4.3.1,

$$\rho(x_n) = \begin{pmatrix} 1 & U_n \\ 0 & 1 \end{pmatrix} \text{ where } U_n \equiv 1 \bmod \mathcal{M}_R.$$

Take M as before. One has

$$M\rho(x_n)M^{-1} = \begin{pmatrix} 1 & \frac{1+Z_1}{1+Z_2}U_n \\ 0 & 1 \end{pmatrix}.$$

As U_n is invertible one can impose $U_n = 1$ by fixing the representative of the strict equivalence class of ρ.
The conditions of commutativity with x_n give the results announced in the lemma. □

5. Images of the generators

5.1. Minimal system of generators

Let $d = \dim_{\mathbf{F}_p} H^1(G, \mathbf{F}_p) = \dim_{\mathbf{F}_p} H^1(P_G, \mathbf{F}_p)$ and $k = \dim_{\mathbf{F}_p} H^1(\Gamma, \mathbf{F}_p)$.

Lemma 5.1.1 *A minimal system of ($\Lambda = \mathbf{Z}_p[[\Gamma]]$)-generators of the Λ-module X_∞ contains $n = d - k$ elements.*

PROOF : We write the inflation-restriction sequence of

$$0 \to X_\infty \to P_G \to \Gamma \to 0$$

using Γ is free. The obtained exact sequence allows to conclude

$$0 \to H^1(\Gamma, \mathbf{F}_p) \to H^1(P_G, \mathbf{F}_p) \to H^1(X_\infty, \mathbf{F}_p)^\Gamma \to 0.$$

□

In the next paragraph we fix some notations.
If X is a p-group and $x \in X$, $\bar{X}$ denotes the quotient of X by its Frattini subgroup and $\bar{x}$ the image of x in $\bar{X}$.
As $\bar{\Gamma}$ and $\overline{(X_\infty)_\Gamma}$ are invariant by A, we can decompose $\bar{P}_G$ as an $\mathbf{F}_p[A]$-module into:

$$\bar{P}_G = \overline{(X_\infty)_\Gamma} \oplus \bar{\Gamma}.$$

To be more specific $\bar{\Gamma}$ is a $\mathbf{F}_p$-vector subspace of $\bar{P}_G$ with a $\mathbf{F}_p$-basis of eigenvectors for the action of A $(\bar{s}_{n+1}, \dots, \bar{s}_d)$ which admits as a complementary vector space $\overline{(X_\infty)_\Gamma}$. Let $(\bar{s}_1, \dots \bar{s}_n)$ be a basis of $\overline{(X_\infty)_\Gamma}$ composed of eigenvectors for the action of A. Using lemma 3.1 we can lift these vectors to a minimal system $(s_1, \dots, s_d)$ of generators of P_G.
We denote by $\tilde{\Gamma}$ the closed subgroup of P_G generated by $(s_{n+1}, \dots, s_d)$.
With these notations lemma 5.1.1 reads: since the quotient $P_G/X_\infty = \Gamma$ is pro-p-free, there exists a section $\Gamma \to P_G$ with image $\tilde{\Gamma}$ and we have

$$P_G = \tilde{\Gamma} \ltimes X_\infty.$$

5.2. Definition of the special generators

By the previous subsection, there exists some minimal systems generators of $\tilde{\Gamma}$, P_G and a minimal system of Λ-generators of X_∞, for which we control the action of A. Thus we know the image under ρ of such generators. In this subsection, we rearrange these sytems following the images by ρ.

Definition 5.2.1 If $\mathrm{Im}\bar{\rho}$ is not diagonal, for the subgroup X_∞ of P_G, there exists a minimal system of generators $(s_1, \ldots, s_n)$ and some integers $u_{X_\infty}, v_{X_\infty} \in \mathbf{N}$ such that:
- $\rho(s_i) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y \end{pmatrix}$, $Y \in \mathcal{M}_R$, $1 \leq i \leq u_{X_\infty}$
- $\rho(s_i) = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$, $Y \in \mathcal{M}_R$, $u_{X_\infty} + 1 \leq i \leq v_{X_\infty}$
- $\rho(s_i) = \mathrm{Id}$, $v_{X_\infty} + 1 \leq i < n$,
- $\rho(s_n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

These generators are termed *special generators.*

Remark 5.2.2 If $(s_1 \ldots, s_n)$ is a minimal system of special generators then $s_i \in P_{G,\chi'_i}$ where $\chi'_i = \mathrm{Id}$ if $1 \leq i \leq u_{X_\infty}$, $\chi'_i = \chi_1\chi_2^{-1}$ if $u_{X_\infty} + 1 \leq i \leq v_{X_\infty}$ or $i = n$ (see propositions 4.3.1 and 4.3.3).

Definition 5.2.3 If $\mathrm{Im}\bar{\rho}$ is not diagonal, let $(s_1, \ldots, s_d)$ be a minimal system of generators of P_G such that:
-the system $(s_1, \ldots, s_n)$ is a minimal system of special Λ-generators of X_∞ and
-the system $(s_{n+1}, \ldots, s_d)$ is a minimal system of $\tilde{\Gamma}$ and $u_\Gamma, v_\Gamma, w_\gamma \in \mathbf{N}$ such that
- $\rho(s_{n+i}) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix}$, $Y \neq Y' \in \mathcal{M}_R$, $1 \leq i \leq u_\Gamma$
- $\rho(s_{n+i}) = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}$, $Y \in \mathcal{M}_R$, $u_\Gamma + 1 \leq i \leq v_\Gamma$
- $\rho(s_{n+i}) = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}$, $Y \in \mathcal{M}_R$, $v_\Gamma + 1 \leq i \leq w_\Gamma$
- $\rho(s_{n+i}) = \mathrm{Id}$, $w_\Gamma + 1 \leq i < k$.

The generators $(s_{n+1}, \ldots, s_d)$ of $\tilde{\Gamma}$ are termed *special generators of* $\tilde{\Gamma}$.

The generators $(s_1, \ldots, s_d)$ of P_G are termed *special generators of* P_G.
At last, a minimal system of generators of P_{G_S}, the image of which in the quotient P_G is a minimal system of special generators of P_G is termed minimal system of *special generators of* P_{G_S}.

Remark 5.2.4 Since $\mathrm{Gal}(F/\mathbf{Q})$ is abelian, Leopoldt's conjecture holds for F. It implies that $u_\Gamma \leq 1$.

We denote by Y_i the variable introduced by $\rho(s_i)$, $1 \leq i \leq v_{X_\infty}$; we denote by $Y_{1+v_{X_\infty}}$ and $Y_{2+v_{X_\infty}}$ the variables introduced by

$$\rho(s_{n+i}) = \begin{pmatrix} 1+Y_{1+v_{X_\infty}} & 0 \\ 0 & 1+Y_{2+v_{X_\infty}} \end{pmatrix},\ 1 \leq i \leq u_\Gamma$$

and by $Y_{i+u_\Gamma+v_{X_\infty}}$ the variable introduced by $\rho(s_{i+n})$, $u_\Gamma + 1 \leq i \leq w_\Gamma$.

Remark 5.2.5 If $\mathrm{Im}\bar{\rho}$ is diagonal, we can define the special generators of X_∞ as for $\tilde{\Gamma}$ and assume that $w_{X_\infty} = 0$ in order to recover an situation analogous to the previous one.

6. Determination of the universal deformation ring

The study of sections 3-4-5 allows us to choose a minimal system of special generators of P_G, the images of which are simple. Since P_G is not always free, we have to express the image by ρ of the relations between selected generators in order to obtain the ideal of relations I of $R_G(\bar{\rho})$. For this, we use the decomposition $P_G \cong \tilde{\Gamma} \ltimes X_\infty$.

6.1. Action of Γ on X_∞. Choice of Γ.

Let Γ be pro-p-free on k generators. Let π denote the projection $P_G \to \Gamma$ and $(\pi(s_{n+1}), \ldots, \pi(s_{n+k}))$ the image in Γ of a minimal system of special generators of $\tilde{\Gamma}$. It is known that the correspondance that maps $\pi(s_{n+i})$ onto $1+T_i$ induces an isomorphism of $\mathbf{Z}_p$-algebra $\Lambda = \mathbf{Z}_p[[\Gamma]] \cong \mathbf{Z}_p[[T_1, \ldots, T_k]]_{nc}$: this is the so called *Magnus algebra* which is non commutative except for

$k = 1$, in which case it coincides with the usual Iwasawa algebra.
We can express the image under ρ of the action of Λ on X_∞:

$$\rho(T_i x) = \rho((s_i - 1)\cdot x) = \rho(s_i)\rho(x)\rho(s_i)^{-1}\rho(x)^{-1}, \;\; n+1 \le i \le d, \; x \in X_\infty.$$

To be more specific:

Lemma 6.1.1 *Let s (resp. x) be a special generator of $\tilde{\Gamma}$ (resp. X_∞);*

(i) if $\rho(s) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix}$ *and if* $\rho(x) = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$ *then*

$$\rho((s-1)\cdot x) = \begin{pmatrix} 1 & (\frac{1+Y}{1+Y'} - 1)U \\ 0 & 1 \end{pmatrix}$$

(ii) if $\rho(s) = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}$ *and if* $\rho(x) = \begin{pmatrix} 1 & U \\ 0 & 1 \end{pmatrix}$ *then*

$$\rho((s-1)\cdot x) = \begin{pmatrix} 1 - YU & YU^2 \\ -Y^2U & Y^2U^2 + YU + 1 \end{pmatrix}$$

(iii) $\rho((s-1)\cdot x) = Id$ *otherwise.*

This lemma shows that the action of Λ on X_∞ is very simple provided that $w_\Gamma = 0$ i.e for all special generators s of $\tilde{\Gamma}$

$$\rho(s) \neq \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}.$$

If $\Gamma = \Gamma_{cyc}$ then $w_{\Gamma_{cyc}} = 0$.

We first assume that $\Gamma = \Gamma_{max}$ and $(s_{n+1}, \dots, s_d)$ is a system of special generators of $\tilde{\Gamma}_{max}$. We define $v_{\Gamma_{max}}, w_{\Gamma_{max}}$ as in subsection 5.2. Let $\tilde{\Gamma}'$ be a subgroup of the subgroup of $\tilde{\Gamma}_{max}$ isomorphic to the quotient of Γ_{max} by the normal subgroup generated by $(s_{n+v_{\Gamma_{max}}+1}, \dots, s_{w_{\Gamma_{max}}})$.
It suffices also to replace G by G' defined through $\pi(\tilde{\Gamma}')$, where G' is defined by $\pi(\tilde{\Gamma}')$ in the same way as G is by $\pi(\tilde{\Gamma})$.

6.2. Expression of the ideal of relations

We first need to express the relations of $R_G(\bar{\rho})$ in terms of the relations of the ($\Lambda = \mathbf{Z}_p[[\Gamma]]$)-module X_∞. We shall propose here a generalization of Boston's method ([Bo2] subsection 9.3).

Let $(f_j^i)_{1\leq i\leq n, 1\leq j\leq \ell}$ be the matrix of relations for a chosen system of special generators of the Λ-module X_∞, where ℓ denotes the number of relations of X_∞ for the above chosen system of special generators. To be more precise, the matrix of relations defined the endomorphism ϕ such that

$$\Lambda^\ell \xrightarrow{\phi} \Lambda^n \to X_\infty \to 0$$

Proposition 6.2.1 *Let* $(s_1, \ldots, s_d)$ *a minimal system of special generators of* P_G *(see subsection 5.2 for notations* $w_\Gamma, u_\Gamma, u_{X_\infty}, v_{X_\infty}$*). If* $w_\Gamma = 0$ *and* $u_\Gamma = 1$, *(i.e* Γ *contains the cyclotomic* $\mathbf{Z}_p$*-extension, then* $R_G(\bar{\rho}) = \mathbf{Z}_p[[Y_1, \cdots, Y_{d'}]]/I$, I *being the ideal of relations generated by*

$$\prod_{i=1}^{u_{X_\infty}} (1+Y_i)^{f_i^j(0)} - 1,\ 1 \leq j \leq \ell$$

and for $1 \leq j \leq \ell$,

$$f_n^j\Big(\frac{1+Y_{v_{X_\infty}+1}}{1+Y_{v_{X_\infty}+2}} - 1, 0, \ldots, 0\Big) +$$

$$\sum_{i=1+u_{X_\infty}}^{v_{X_\infty}} f_i^j\Big(\frac{1+Y_{v_{X_\infty}+1}}{1+Y_{v_{X_\infty}+2}} - 1, 0, \ldots, , 0\Big) Y_i$$

where $(\sum_{i=1}^n f_i^j(T_1, \ldots, T_k) s_i)_{1\leq j\leq \ell}$ *is a system of relations of the* Λ*-module* X_∞.

PROOF : As in [Bo2] subsection 9.3 apply ρ to the relations of the Λ-module X_∞, using lemma 6.1.1 (iii) and (i). □

6.3. Computation of the relations of the Λ-module X_∞

The relations of the Λ-module X_∞ are in general not known. We shall give a way to derive them from the relations of P_{G_S}. To this end we shall proceed

as follows: first the Fox derivative allows one to describe an intermediate Λ-module Y_∞ with the help of generators and relations; second the diagram of lemma 6.3.1. yields a description of X_∞ from that of Y_∞.
We want to emphasize that the ideal of relations of $R_G(\bar{\rho})$ results directly from the relations of P_{G_S} along a systematic line.

In [Ng1] (proposition 1.7) Nguyen Quang Do proposes a description of the Λ-module $Y_\infty = H_0(\mathrm{Gal}(K_{S,p}/F_\infty), I(K_{S,p}/F))$ based upon the exact sequence:

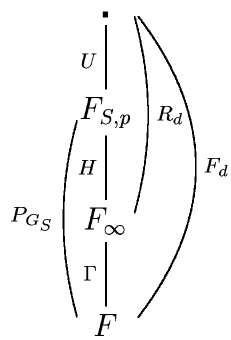
$$0 \to \Delta(F_\infty) \to \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/F)]]^r \xrightarrow{\phi} \mathbf{Z}_p[[\mathrm{Gal}(F_\infty/F)]]^d \to Y_\infty \to 0$$

where ϕ denotes the composition of the projection of Γ by the Fox derivative of the relations of $\mathrm{Gal}(M/F)$, and where $d = \dim_{\mathbf{F}_p} H^1(G_S, \mathbf{F}_p)$, and $r = \dim_{\mathbf{F}_p} H^2(G_S, \mathbf{F}_p)$. One also denotes by $I(K_{S,p}/F)$ the augmentation ideal of $\mathrm{Gal}(K_{S,p}/F)$.
Moreover one has $\Delta(F_\infty) = H^2(\mathrm{Gal}(K_{S,p}/F_\infty), \mathbf{Q}_p/\mathbf{Z}_p)^* = 0$ since, (see theorem 2.2 [Ng1]), Leopoldt's conjecture holds in view of $\mathrm{Gal}(F/\mathbf{Q})$ being abelian. The matrix of ϕ has rank $r - d$ and describes the relations of Y_∞. In our case these relations give the relations of X_∞ using the following lemma (see [Ja] lemma 4.3):

Lemma 6.3.1 *In the commutative diagram below the two horizontal sequences and the two vertical sequences are exact:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Lambda^{d-n} & & \Lambda^{d-n} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda^r & \xrightarrow{\text{Fox}} & \Lambda^d & \longrightarrow & Y_\infty & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Lambda^r & \longrightarrow & \Lambda^n & \longrightarrow & X_\infty & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

PROOF : In the diagram



F_d denotes a free presentation of rank d of P_{G_S}; we assume F_d is too a free presentation of Γ and R_d is defined by

$$0 \to R_d \to F_d \to \Gamma \to 0$$

The augmentation ideal of F_d is defined by the exact sequence:

$$0 \to I(F_d) \to \mathbf{Z}_p[[F_d]] \to \mathbf{Z}_p \to 0$$

One applies to this exact sequence the functor $H_i(R_d, \cdot)$. The freeness of $\mathbf{Z}_p[[F_d]]$ implies the Lyndon exact sequence ([Ng1] proposition 1.1):

$$0 \to H_1(R_d, \mathbf{Z}_p) \to H_0(R_d, I(F_d)) \to H_0(R_d, \mathbf{Z}_p[[F_d]]) \to H_0(R_d, \mathbf{Z}_p)$$

We deduce:

$$0 \longrightarrow R_d^{ab} \longrightarrow \Lambda^d \longrightarrow I(\Lambda) \longrightarrow 0$$

where $R_d^{ab} = R_d/[R_d, R_d]$. Similarly one applies the functor $H_i(H, \cdot)$ to the exact sequence which defines the augmentation ideal of P_{G_S}, so that ([Ng1] proposition 1.7)

$$0 \to H_1(H, \mathbf{Z}_p) \to H_0(H, I(P_{G_S}) \to H_0(H, \mathbf{Z}_p[[P_{G_S}]]) \to H_0(H, \mathbf{Z}_p) \to 0$$

Hence,

$$0 \longrightarrow X_\infty \longrightarrow Y_\infty \longrightarrow I(\Lambda) \longrightarrow 0$$

The following commutative diagram

$$\begin{array}{ccc} R_d & \longrightarrow & H \\ \curvearrowright & & \curvearrowright \\ F_d & \longrightarrow & P_{G_S} \end{array}$$

yields functorial morphisms

$H_0(R_d, I(F_d)) \to H_0 = (H, I(P_{G_S}))$ and $H_0(R_d, \mathbf{Z}_p[[F_d]]) \to H_0(H, \mathbf{Z}_p[[P_{G_S}]])$;

this proves the commutativity of the following diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
I(\Lambda) & \longrightarrow & I(\Lambda) & & \\
\uparrow & & \uparrow & & \\
\Lambda^d & \longrightarrow & Y_\infty & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
R_d^{ab} & \longrightarrow & X_\infty & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
0 & & 0 & &
\end{array}$$

We shall complete the horizontal sequences by identifying them with known exact sequences.
By the equalities

$$\Lambda^d = \mathbf{Z}_p[[\Gamma]]^d = H_0(R_d, I(F_d)) = H_0(\mathrm{Gal}(F_{S,p}/F_\infty), H_0(U, I(F_d)))$$

the arrows between these modules and Y_∞ identify by functoriality. The corresponding line is completed by the Fox derivative: recall that the morphism 'Fox' is obtained thanks to the exact sequences:

$$0 \to H_0(H, U^{ab}) \to H_0(H, H_0(U, I(F_d)) \cong \Lambda^d \to Y_\infty \to 0$$

The same reasoning applies to the horizontal line which then identifies to

$$0 \longrightarrow H_1(U, \mathbf{Z}_p) \longrightarrow R_d^{ab} \longrightarrow X_\infty \longrightarrow 0$$

where $R_d^{ab} = R_d/[R_d, R_d]$.

Recall $k = \dim_{\mathbf{F}_p} H^1(\Gamma, \mathbf{F}_p)$ we can use the corollary 1.2 of [Ng1], to show that $R_d^{ab} \cong R_k^{ab} \times \Lambda^{d-k}$, where $R_k^{ab} = R_k/[R_k, R_k]$ and R_k is defined by a presentation of Γ:

$$0 \to R_k \to F_k \to \Gamma \to 0$$

Since Γ is free, one knows ([Br] lemma 5.1) that $I(\Lambda)$ is a free $\mathbf{Z}_p[[\Lambda]]$-module of rank k. Hence $R_k^{ab} = 0$, which yields the bottom horizontal exact sequence:

$$H_0(H, U^{ab}) \cong \Lambda^r \text{ and } H_1(U, \mathbf{Z}_p) \cong \Lambda^r$$

and $I(\Lambda) \cong \Lambda^k = \Lambda^{d-n}$. □

6.4. General presentation by generators and relations

We now apply our results using a presentation of P_{G_S} by generators and relations. The pro-p-group P_{G_S} has a minimal presentation by the exact sequence:

$$0 \to R \to F_d \to P_{G_S} \to 0$$

where F_d is a free pro-p-group of rank d and R is the pro-p-group of the relations of P_{G_S}.
Let $\{s_i,\ 1 \le i \le d\}$ be a minimal system of generators of P_{G_S} and $\{r_j, 1 \le j \le r\}$ a minimal system of relations of P_{G_S} .
In order to render the injective morphism $X_\infty \to Y_\infty$ explicit, it is necessary to choose judiciously the generators of P_{G_S}. To this end we notice that the elements of X_∞ are characterized by their image being trivialin the quotient Γ.

As in subsection 6.1, π denotes the projection $P_G \to \Gamma$ and if $x \in P_{G_S}$, we denote $\tilde{x}$ its projection on P_G.

Let $(s_1, \ldots, s_d)$ be a minimal system of special generators of P_{G_S} such that: $\pi(\tilde{s}_i) = 1,\ 1 \le i \le n$ (recall that $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is a minimal system of generators of X_∞).
We can now compute the Fox matrix $M_\phi = \left(\pi\left(\frac{\partial r_j}{\partial s_i}\right)\right)_{1 \le j \le l,\ 1 \le i \le d}$, hence Y_∞ as well. To obtain X_∞ it suffices to omit the last k lines of M_ϕ, which gives the matrix $M_{\phi'}$ of ϕ' describing the exact sequence

$$0 \to \Lambda^r \xrightarrow{\phi'} \Lambda^n \to X_\infty \to 0$$

thus describing X_∞ as a Λ-module.

Proposition 6.4.1 *Let*

$$0 \to R \to F_d \to P_{G_S} \to 0$$

be a minimal presentation of P_{G_S}, with $\{r_j,\ 1 \le j \le r\}$ a minimal system of relations and $\{s_i,\ 1 \le i \le d\}$ a minimal system of special generators such that $\pi(\tilde{s}_i) = 0$, $1 \le i \le n$. Then the Λ-module X_∞ is described by

$$0 \to \Lambda^r \xrightarrow{\phi'} \Lambda^n \to X_\infty \to 0$$

the matrix $M_{\phi'}$ of ϕ' being

$$M_{\phi'} = \Big(\pi\Big(\frac{\partial r_j}{\partial s_i}\Big)\Big)_{1 \le i \le n,\ 1 \le j \le r}.$$

Remark 6.4.2 Recall that the injectivity of ϕ' is obtained thanks to the weak Leopoldt conjecture. In order to discuss Mazur's conjecture (see [Ma] subsections 1.6 and 1.10) we remark that this injectivity means that the relations of X_∞ obtained in this way are independent.

7. Explicit universal deformation ring

7.1. General computation of the Fox matrix

We define by induction

$$P^{(0)} = P_{G_S} \text{ and } P^{(n+1)} = [P^{(n)}, P_{G_S}].$$

There exists a decomposition with coefficients in $\mathbf{Z}_p$, using the 'Hall collecting process':

$$r_j \bmod P^{(n+1)} \equiv$$

$$\prod_{i=1}^{d} s_i^{a_i^j} \prod_{1 \le i < k \le d} [s_i, s_k]^{a_{i,k}^j} \cdots \prod_{1 \le i_1 < i_2 \le d}^{i_3, \dots, i_n \in [1,d]} [\cdots[[s_{i_1}, s_{i_2}], s_{i_3}], \dots, s_{i_n}]^{a_{i_1,\dots,i_n}^j}.$$

Remark 7.1.1 Let p^l be the lcm of the orders p^{l_i} or ∞ of the elements $s_i[P_{G_S}, P_{G_S}]$, $1 \le i \le d$, with the wild convention that $p^\infty = 0$.
The coefficients $a_i^j \bmod (p^l)^2$ and $a_{i,k}^j \bmod p^l$ are known ([Ko1] proposition 7.23). They are related to the transgression map $H^2(R, \mathbf{Z}/p^l\mathbf{Z}) \to H^1(P_{G_S}, \mathbf{Z}/p^l\mathbf{Z})$ and to the cup product of elements of $H^1(P_{G_S}, \mathbf{Z}/p^l\mathbf{Z})$.

The following two lemmas (which are straightforward) allow to compute the Fox derivative of r_j.

Lemma 7.1.2 *Let $u, v \in F_d$, $a \in \mathbf{N}^*$ then*

$$\frac{\partial(uv)}{\partial s} = \frac{\partial u}{\partial s} + u\frac{\partial v}{\partial s}, \quad hence \quad \frac{\partial u^a}{\partial s} = \sum_{k=0}^{a-1} u^k \frac{\partial u}{\partial s}.$$

Lemma 7.1.3 *Let $u \in F_d$, $i, j \in [1, d]$, then*

$$\frac{\partial}{\partial s_i}[u, s_j] = \frac{\partial u}{\partial s_i} + us_j \frac{\partial u^{-1}}{\partial s_i} \ \textit{if } i \neq j$$

$$\frac{\partial}{\partial s_i}[u, s_i] = \frac{\partial u}{\partial s_i} + u + us_i \frac{\partial u^{-1}}{\partial s_i} - us_i u^{-1} s_i^{-1}.$$

7.2. The $\mathbf{Z}_p$-cyclotomic extension of F

We make precise the computation of $R_G(\bar{\rho}_G)$ in the case $\Gamma = \Gamma_{cyc} =< \gamma >$ (recall $w_{\Gamma_{cyc}} = 0$). We may assume that $\pi(s_d) = \gamma$, $\pi(s_i) = 1$, $1 \leq i \leq d-1$. After lemma 7.1.3 the Fox matrix reads

$$M_\phi = \Big(\pi\Big(\frac{\partial r_j}{\partial s_i}\Big)\Big) = \Big(a_i^j + \sum_{k=1}^{\infty} b_{i,k}^j (\gamma - 1)^k\Big)_{1 \leq i \leq d,\ 1 \leq j \leq r}$$

where $b_{i,k}^j$ denotes the coefficient of $[\cdots[[s_i, \gamma], \gamma], \cdots, \gamma] \in P^{(k)} - P^{(k+1)}$ in the relation r_j.

Recall lemma 4.3.4 and the notations of subsection 5.2; for $\Gamma = \Gamma_{cyc}$ the variables introduced by $\tilde{s_d}$ are $Y_{d'-1}, Y_{d'}$ in $R_G(\bar{\rho}_G)$ (see introduction: the number of variables of $R_G(\bar{\rho})$ is $d' = \dim_{\mathbf{F}_p} H^2(G, Ad\bar{\rho}_G)$), hence:

$$\rho_G(\tilde{s}_d) = \begin{pmatrix} 1 + Y_{d'-1} & 0 \\ 0 & 1 + Y_{d'} \end{pmatrix} \text{ with } Y_{d'-1}, Y_{d'} \in \mathcal{M}_R$$

We can also apply proposition 6.2.1 so as to obtain $R_G(\bar{\rho}_G)$:

Theorem 7.2.1 *In the Borel case, if $Im\bar{\rho}$ is not diagonal and if $\Gamma = \Gamma_{cyc}$, we obtain*

$$R_G(\bar{\rho}_G) = \mathbf{Z}_p[[Y_1, \ldots, Y_{d'}]]/I$$

where I is the ideal of relations generated by

$$\prod_{i=1}^{u_{X_\infty}} (1+Y_i)^{a_i^j} - 1,\ 1 \le j \le \ell$$

and

$$a_n^j + \sum_{k=1}^{\infty} b_{n,k}^j \Big(\frac{1+Y_{d'-1}}{1+Y_{d'}} - 1\Big)^k + \sum_{i=1+u_{X_\infty}}^{v_{X_\infty}} \Big(a_i^j + \sum_{k=1}^{\infty} b_{i,k}^j \Big(\frac{1+Y_{d'-1}}{1+Y_{d'}} - 1\Big)^k\Big) Y_i,\ 1 \le j \le \ell.$$

7.3. Free pro-p-extension of rank k of F

Let us assume Γ has rank k.
If $\Gamma_{cyc} =< \gamma > \subset \Gamma$ and $w_\Gamma = 0$, the computation of $R_G(\bar\rho_G)$ is analogous to that of the previous case $\Gamma = \Gamma_{cyc}$.
After lemma 6.1.1 the image by ρ_G of the action of $\Lambda \cong \mathbf{Z}_p[[T_1, \cdots, T_k]]_{nc}$ is commutative, and even trivial if $i > 1$; i.e, if $f \in \mathbf{Z}_p[[T_1, \cdots, T_k]]_{nc}$ is monomonial and $i > t_\Gamma$

$$\rho_G(T_i f x) = \text{ Id.}$$

hence only the commutators of the form $r = [\cdots[[s_i, \gamma], \ldots, \gamma]$ have a non trivial image by $r \mapsto \rho_G(\pi(\frac{\partial r}{\partial s_i})x)$.
The theorem 7.3.1 also holds in the latter case (note the coefficients u_{X_∞}, d' have changed) and gives a better approximation $R_G(\bar\rho_G)$ of $R_{G_S}(\bar\rho)$.

Remark 7.3.1 Our method can also give an approximation of the ideal I: if the relations of P_{G_S} are known modulo P^{n+1}, theorem 7.2.1 allows us to control the approximation of I. Moreover, even if the computations are inextricable if $w_\Gamma \neq 0$, we can also work modulo P^{n+1} (for a given n depending on our patience).

7.4. Comparison between $R_{G_S}(\bar\rho)$ and $R_G(\bar\rho_G)$

The study carried out in section 3-4 also applies when G, P_G are replaced by G_S, P_{G_S}. Since $\bar P_G = \bar P_{G_S}$, if $< \bar x >$ is an A-invariant subgroup, we can not only lift $\bar x$ to $\tilde x \in P_G$ but also to $x \in P_{G_S}$. The computations of 4.2 and 4.3 give the image of $\rho(x)$ as if P_{G_S} were free.
Let us recall how to compute the universal ring in order to make precise the

surjection $R_{G_S}(\bar{\rho}) \to R_G(\bar{\rho}_G)$.
After remark 3.1, if $\tilde{x} \in \mathrm{Ker}\bar{\rho}$ then

$$\rho_G(\tilde{x}) = \begin{pmatrix} 1+Y_1 & Y_2 \\ Y_3 & 1+Y_4 \end{pmatrix}, \; Y_i \in \mathcal{M}_R, 1 \leq i \leq 4.$$

This image a priori introduces four variables (depend upon $\tilde{x}$); the action of A allows to reduce the number of variables to
• 1 or 0 if $\tilde{x}$ is inverted by complex conjugation,
• 2 or 0 if $\tilde{x}$ is invariant by complex conjugation.
These computations apply to ρ_G or ρ.
Taking G instead of G_S introduces additional conditions of commutativity which allow us to cancel the lower triangular images and to render scalar the diagonal images.
Then we express the image by ρ_G (resp. ρ) of the relations of P_G (resp. P_{G_S}). The universal deformation ring $R_G(\bar{\rho}_G)$ (resp. $R_{G_S}(\bar{\rho})$) is the quotient of the ring of formal series in the introduced variables by the images of the special generators of P_G (resp. P_{G_S}), by the image of the relations of P_G (resp. P_{G_S}) by ρ_G (resp. ρ).
The relations of P_G are obtained from the relations of P_{G_S} using the Fox derivative.
The pro-p-group P_{G_S} has a minimal presentation by the exact sequence:

$$0 \to R \to F_d \to P_{G_S} \to 0$$

where F_d is a free pro-p-group of rank d and R is the pro-p-group of the relations of P_{G_S}.
Let $\{s_i, \; 1 \leq i \leq d\}$ a minimal system of special generators of P_{G_S}.
Let r be a relation of P_{G_S}; we can think of r as a word in F_d. Let $\tilde{r} \in P_G$ the element obtained from r when we remplace the letters s_i by $\tilde{s}_i$, $1 \leq i \leq d$.
We observe that

$$\rho_G(\tilde{r}) = \rho_G\Big(\sum_{i=1}^{n} \pi\Big(\frac{\partial r}{\partial s_i}\Big)\tilde{s}_i\Big).$$

Hence we obtain

Theorem 7.4.1 *There exists a minimal system $\{s_i, 1 \leq i \leq d\}$ of (special) generators of P_{G_S} and choice of the representatives of the strict equivalence classes ρ, ρ_G such that the surjective morphism*

$$R_{G_S}(\bar{\rho}) \to R_G(\bar{\rho}_G)$$

maps the variables introduced by $\rho(s_i)$ *onto the variables introduced by* $\rho_G(\tilde{s}_i)$.

Remark 7.4.2 The surjection $R_{G_S}(\bar{\rho}) \to R_G(\bar{\rho}_G)$ yields

$$\dim_{Krull} R_{G_S}(\bar{\rho})/pR_{G_S}(\bar{\rho}) \geq \dim_{Krull} R_G(\bar{\rho}_G)/pR_G(\bar{\rho}_G).$$

This could allow to discuss Mazur's question ([Ma] subsection 1.10).

7.5. Examples

Cyclotomic fields

The first examples of representations $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(\mathbf{F}_p)$ in the Borel case appear in the study of elliptic curves. These representations introduce the cyclotomic fields $F = \mathbf{Q}(\zeta_p)$, where ζ_p denotes a primitive p-root of unit. We can find some examples of such representations associated to elliptic curves for $p = 5, 7$ in [Se] subsection 5.5. In these cases $R_{G_S}(\bar{\rho})$ is known because p is regular hence P_{G_S} is free.
In [Bo3] section 6, Boston gives an example of such a representation with $F = \mathbf{Q}(\zeta_p)$ for the irregular prime $p = 691$; this representation is associated to the unique normalized cusp form of weight 12.
We would like to correct an imprecision in [Bo2] proposition 9.2 and [Bo3] section 6, where a confusion seems to arise because 'Spiegelung' is overlooked (see below).
Let $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(F_p)$ be a continuous odd representation unramified outside $S_{\mathbf{Q}} = \{p, \infty\}$ with $\mathrm{Im}\bar{\rho} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ and $F = \mathbf{Q}(\zeta_p)$.
We denote by a a generator of $A \cong \mathrm{Gal}(F/\mathbf{Q})$ and $\omega : A \to \mathbf{Z}_p^*$ the Teichmüller lift of the cyclotomic character. There exists $k' \in \mathbf{N}$ even depending upon the representation such that

$$\bar{\rho}(a) = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{k'-1}(a) \bmod p \end{pmatrix}.$$

Assume that Vandiver's conjecture holds for p. Let S_F be the set of places in F over $S_{\mathbf{Q}}$, $\Gamma = \Gamma_{cyc}$ and $\tilde{\gamma}$ denotes a lift to P_G of a generator of Γ.
Since we assume Vandiver's conjecture to hold, we know the structure of X_∞ as a Λ-module. For any $n \geq 1$, let A_n be the p-class group of $\mathbf{Q}(\zeta_{p^n})$,

$$A_\infty = \lim_{\rightarrow} A_n, \ Z_\infty = \lim_{\leftarrow} A_n.$$

Then:
-It is well known that Vandiver's conjecture implies $A_n^+ = 0$ for any $n \geq 1$, so that the natural maps $A_n \to A_\infty$ are injective. It follows that the Λ-torsion free part of X_∞ is actually Λ-free, and hence we have an isomorphism of Λ-modules (and not only a pseudo-isomorphism)

$$X_\infty \cong \Lambda^{r_2} \oplus \mathrm{tor}_\Lambda X_\infty$$

where $r_2 = (p-1)/2$ is the number of complex places in F and $\mathrm{tor}_\Lambda X_\infty$ denotes the Λ-torsion of X_∞.
-The structure of $\mathrm{tor}_\Lambda X_\infty$ can be given by a 'mirror' ('Spiegelung') version of Z_∞. More precisely, using the fact that each $\mathbf{Q}(\zeta_{p^n})$ admits only one place above p and verifies Leopoldt's conjecture, we get (for details, see [Ng3] section 3):

$$\mathrm{tor}_\Lambda X_\infty \cong \lim_{\leftarrow} \mathrm{Hom}_{\Gamma_n}(Z_\infty, \mu_{p^\infty})$$

where $\Gamma_n = \mathrm{Gal}(\mathbf{Q}(\zeta_{p^\infty})/\mathbf{Q}(\zeta_{p^n}))$. Because $A_n^+ = 0$ for any $n \geq 1$, it follows that $X_\infty^+ = \mathrm{tor}_\Lambda X_\infty^+ = \mathrm{tor}_\Lambda X_\infty$, and that X_∞^- (which admits Z_∞ as a quotient) is a Λ-free on $r_2 = (p-1)/2$ generators $(\tilde{x}_1, \ldots, \tilde{x}_{p-2})$ where $\tilde{x}_i \in P_{G,\omega^i}$, $1 \leq i \leq p-2$ (see [Wa] corollary 10.15).

Recall $A \cong \mathrm{Gal}(F/\mathbf{Q})$ is cyclic of order $p-1$ prime to p. Hence the subgroup of invariants by A of X_∞ is isomorphic to image of the norm ν of X_∞:

$$X_\infty^A \cong \nu X_\infty.$$

By class field theory, νX_∞ is isomorphic to the analogue of X_∞ for the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_\infty/\mathbf{Q}$, hence is trivial. Recall that if $x \in X_{\infty,\omega^i}$ and i even then $\rho(x) = \mathrm{Id}$ if $i \not\equiv 0 \bmod p$. Thus the image of X_∞^+ by ρ is trivial by using subsection 4.3 we get

Proposition 7.5.1 *In the Borel case, if $\Gamma = \Gamma_{cyc}$, if $Im\bar{\rho}$ not diagonal such that $F = \mathbf{Q}(\zeta_p)$ and $p < 1 + 4.10^6$ (so that Vandiver's conjecture holds), then*

$$R_G(\bar{\rho}_G) = \mathbf{Z}_p[[Y_1, Y_2]]$$

and the universal representation is given by

$$\rho_G(\tilde{a}) = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{k'-1}(a) \end{pmatrix}, \; \rho_G(\tilde{\gamma}) = \begin{pmatrix} 1+Y_1 & 0 \\ 0 & 1+Y_2 \end{pmatrix},$$

$$\rho_G(\tilde{x}_{p-k'}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and all other given generators of P_G have a trivial image by ρ_G.

Using theorem 7.4.1 we obtain

Proposition 7.5.2 *In the Borel case, if Im$\bar{\rho}$ not diagonal such that $F = \mathbf{Q}(\zeta_p)$ and $p < 1 + 4.10^6$ (so that Vandiver's conjecture holds), then*

$$R_{G_S}(\bar{\rho}) = \mathbf{Z}_p[[Y_1, Y_2, Y_3]]/I;$$

the universal representation (with obvious notations) is given by

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{k'-1}(a) \end{pmatrix},\ \rho(\gamma) = \begin{pmatrix} 1+Y_1 & 0 \\ 0 & 1+Y_2 \end{pmatrix},$$

$$\rho(x_{p-k'}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \rho(x_{k'-1}) = \begin{pmatrix} 1 & 0 \\ Y_3 & 1 \end{pmatrix},$$

and all other given generators of P_{G_S} have a trivial image by ρ. Moreover if $r \in I$ we have $r \equiv 0$ mod Y_3, and $I = (0)$ if p is regular.

Remark 7.5.3 For these representations, we have

$$\dim_{Krull} R_{G_S}(\bar{\rho})/pR_{G_S}(\bar{\rho}) \leq 3.$$

A reasoning analogous to that of [Ma] subsection 1.10 proves that the latter Krull dimension is greater than 3 ([Bo1] remark 5.1). Thus this dimension is 3 and Mazur's question ([Ma] subsection 1.6) still holds in these cases.

Increasing the ramification

The problem of increasing the ramification has been studied by Boston [Bo4] theorem 1 and Böckle [Bö1] subsection 3.C (for even representations).
Let p be a regular odd prime, $F = \mathbf{Q}(\zeta_p)$ and S_p be the set of places over p and archimedean places.
In this case $G_{S_p} = \Gamma_{max}$ is uniquely determined by [Ya2] proposition 2.2 and is free with $(p+1)/2$ generators (see previous description). Remark that the quotient Γ_{max} of G_S does not depend on S with $S_p \subset S$.
If we increase the ramification (i.e we consider S instead of S_p), Neumann [Ne] corollary 5.3 gives a presentation of G_S:

Lemma 7.5.4 *Let $(x_i,\ i \in I)$ be a minimal system of topological generators of G_{S_p} and $(s_i,\ i \in I)$ be a system of elements of G_S with*

$$s_i \text{ mod } Gal(K_S/K_{S_p}) = x_i,\ i \in I.$$

Further let $t_q \in Gal(K_S/K_{S_p})$ for $q \in S - S_p$ with $N(q) \equiv 1$ mod p, such that t_q generates the inertia group of some prolongation of q.
Then the set

$$\{s_i, t_q,\ i \in I,\ q \in S - S_p\ N(q) \equiv 1 \text{ mod } p\}$$

forms a set of generators of G_S where the subset $\{t_q\}$ is free.

Hence $\tilde{\Gamma}_{max}$ is generated by $(s_i, i \in \{0\} \cup \{1, 3, \cdots (p-1)/2\})$ with $s_i \in P_{G_{S,\omega_i}}$.
Let $\bar{\rho} : G_S \to \mathrm{Gl}_2(F_p)$ in the Borel case, such that $F = \mathbf{Q}(\zeta_p)$ and there exists $q_0 \in S - S_p$ such that

$$\bar{\rho}(t_{q_0}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(see remark 2.2).
After the previous description, in order to have $w_\Gamma = 0$ (see subsection 6.1), we choose $\tilde{\Gamma} = \tilde{\Gamma}_{max}/ < s_{k'-1} >_{normal}$ (with k' as in the previous paragraph).
Hence we have the universal representation:

$$\rho_G(\tilde{s}_0) = \begin{pmatrix} 1 + Y_1 & 0 \\ 0 & 1 + Y_2 \end{pmatrix},\ \rho_G(\tilde{s}_{p-k'}) = \begin{pmatrix} 1 & Y_3 \\ 0 & 1 \end{pmatrix},$$

and $\rho_G(\tilde{s}_i) = \mathrm{Id}$ for the other $i \in I$;

$$\rho_G(\tilde{t}_{q_0}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\rho_G(\tilde{t}_q) = \begin{pmatrix} 1 & Y_q \\ 0 & 1 \end{pmatrix} \text{ or } \rho_G(\tilde{t}_q) = \begin{pmatrix} 1 + Y_q & 0 \\ 0 & 1 + Y_q \end{pmatrix}$$

where $q \in S - (S_p \cup \{q_0\})$ and some variables Y_q can vanish depending of the action of A.
In this particular situation we can easily apply theorem 7.3.2.

8. Wingberg's presentation

Recall the notations of the section 2: $\bar{\rho} : \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Gl}_2(\mathbf{F}_p)$ is an odd continuous representation in the Borel case unramified outside a finite set of primes $S_{\mathbf{Q}}$ of $\mathbf{Q}$, K denotes the subfield of $\bar{\mathbf{Q}}$ fixed by $\ker\bar{\rho}$, and F is a subextension of K such that $\mathrm{Gal}(K/F)$ is a Sylow p-subgroup of $\mathrm{Gal}(K/\mathbf{Q})$. The finite set S_F of places of F containing the places over $S_{\mathbf{Q}}$ in F, allows us to define G_S and P_{G_S}; from here on S denotes S_F.

8.1. Using Wingberg's explicit presentation

Our method is most effective whenever the relations of P_{G_S} are known. Explicit results are known (see [Bo1]) in the case where P_{G_S} is free. We shall now solve in a simple way the case where the relations of P_{G_S} do not contain any threefold commutator. Then we can compute $R_{G_S}(\bar{\rho})$. A particular case of this corresponds to what we call a *Wingberg presentation*

Let F_v denote the v-completion of F for $v \in S$.
Let μ_p denote the set of p-roots of unity.
Let $\delta_v = \begin{cases} 1 & \text{if } \mu_p \subset F_v \\ 0 & \text{otherwise} \end{cases}$ and $\delta = \begin{cases} 1 & \text{if } \mu_p \subset F \\ 0 & \text{otherwise.} \end{cases}$
Let $S_0 \subset S$ be a maximal subset of finite primes such that:

$$\sum_{v \in S_0} \delta_v = \delta.$$

Let

$$V_{S_0}^S = \{\alpha \in F^* \mid \alpha \in F_v^{*p} \text{ for } v \in S_0,\ \alpha \in U_v F_v^{*p},\ v \notin S\}/F^{*p}$$

where U_v denotes the group of units in the ring of integers of F_v.
In [Wi] Wingberg proves the following proposition:

Proposition 8.1.1 *The condition $V_{S_0}^S = 0$ is equivalent to P_{G_S} being the free product of the decomposition groups $\mathcal{P}_v$, $v \in S - S_0$ and a free pro-p-group of rank $r_f = 1 + \sum_{v \in S_p \cap S_0} [F_v : \mathbf{Q}_p] - \sharp(S - S_0)$.*

Remark 8.1.2 Leopoldt's conjecture for F implies that the rank k of Γ_{max} verifies $1 \leq k \leq r_2 + 1$ where r_2 is the number of complex places in F. In the situation of proposition 8.1.1, Yamagishi [Ya1] has determined k explicitely in terms of local datas; in particular $k \leq r_2 + 1$ (see also subsection 8.3).

Remark 8.1.3 If $\mu_p \not\subset F$ then the decomposition groups $\mathcal{P}_v$, $v \in S - S_0$ are free; hence P_{G_S} is free. The latter case is well known, so we now turn to $\mu_p \subset F$; in this case S_0 contains only one element.

If $p \neq 2$ then places that ramify in a pro-p-extension of a number field are primes q of the form $N(q) \equiv 1 \bmod p$ or q divides p. So we assume that $S_{\mathbf{Q}}$ and S_F contain only places of this type.

Let S' denote a set of r_f places different from the elements of S. Hence P_{G_S} is described

- by *free generators* s_v, $v \in S'$,
- by *tame generators* s_v, t_v $v \in S - S_0$ such that $v \equiv 1 \bmod p$,
- by *tame relations* $r_v = (t_v)^{q_v}[t_v, s_v]$ with $q_v = |\mu(F_v)|$ $v \in S - S_0$ such that $v \equiv 1 \bmod p$,
- by *wild generators* $s_v, t_v, s_v^2, t_v^2 \ldots, s_v^{n_v}, t_v^{n_v}$ with $v \in S - S_0$ such that v divides p,
- and by *wild relations* $r_v = (t_v)^{q_v}[t_v, s_v][t_v^2, s_v^2] \cdots [t_v^{n_v}, s_v^{n_v}]$ with $q_v = |\mu(F_v)| = p$, $v \in S - S_0$ such that v divides p, $2n_v = [F_v : \mathbf{Q}_p] + 2$ (see [Se2] corollary 4.3).

The archimedean places do not appear because p is odd.

Let $\mu_p(F)$ and $\mu_p(F_v)$ be the set of p^{th}-roots of unity in F and F_v respectively. Let A' be the subgroup of A of order two generated by c and let $\tilde{\mathbf{F}}_p$ be the non trivial irreducible $\mathbf{F}_p[A']$-module.
Let $V^S = V^S_{\emptyset}$. Then we have $V^S \subset V^S_{S_0}$ and after [BoU] proposition 3.2, if $V^S = 0$

$$\bar{P}_{G_S} \cong \mathrm{Ind}_{A'}^{A} \tilde{\mathbf{F}}_p \oplus \mathbf{F}_p \oplus \mathrm{Coker}(\mu_p(F) \to \oplus_{v \in S} \mu_p(F_v))$$

as an $\mathbf{F}_p[A]$-module. It allows to prove

Lemma 8.1.4 *Assume that* Γ *contains the cyclotomic* $\mathbf{Z}_p$*-extension and that* $V^S_{S_0} = 0$*, then*

$$X_\infty = X_\infty^-.$$

PROOF : Recall that a generator of the cyclotomic $\mathbf{Z}_p$-extension is invariant by complex conjugation.
Using [BoU] proposition 3.2, in order to prove the lemma it suffices to show that the action of c on $\mathrm{Ind}_{A'}^{A}\tilde{\mathbf{F}}_p$ and on $\mathrm{Coker}(\mu_p(F) \to \oplus_{v\in S}\mu_p(F_v))$ is not trivial. It is clear for the induced representation and for $\mu_p(F_v)$ if v divides p.
Let $v \in S$ be a place over l a prime number distinct of p. Since we have assumed $N(v) \equiv 1 \bmod p$, p divides $l-1$, hence $\mu_p \subset \mathbf{Q}_l$.
All decomposition subgroups of A at all $v|l$ are conjugate. Since A is abelian, these subgroups are the same, we denote them by A_l. The local action of A_l on $\mathcal{P}_v$ is known and A acts on $\oplus_{v|l}\mathcal{P}_v$ by the induced representation $\mathrm{Ind}_{A}^{A_l}\mathcal{P}_v$; then $c \in A$ permutes $\oplus_{v\in S, v|l}\mu_p(F_v)$.
The lemma follows. □

Remark 8.1.5 Wingberg's presentation yields a system of generators of P_{G_S} but this system is not special. We have to work in order to obtain a 'good' system of generators, i.e for which we know enough informations about the A-action.

8.2. The $\mathbf{Z}_p$-cyclotomic extension of F

Let us assume $\Gamma = \Gamma_{cyc} =< \hat{\gamma} >$. Recall that if $x \in P_{G_S}$ we denote $\tilde{x}$ its projection in P_G.

Choice of the generators of P_G

We have to choose a lift of $\hat{\gamma}$ in P_G. Recall that a tame generator $\tilde{s}_v$ comes from a generator of the local $\mathbf{Z}_p$-cyclotomic extension. We assume there exists $w \in S - S_0$ such that $N(w) \equiv 1 \bmod p$ and $N(w) \not\equiv 1 \bmod p^2$ hence $q_w = p$, so that we can assume that $\pi(\tilde{s}_w) = \hat{\gamma}$ generates the $\mathbf{Z}_p$-cyclotomic extension of F; if such a w does not exists we can choose a wild generator s_w such that $\pi(\tilde{s}_w) = \hat{\gamma}$. We denote $\gamma = s_w$ and $\tilde{\gamma} = \tilde{s}_w$.
We fix $\bar{\rho}(t_w^1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; this image depends upon the representation, but this choice does not change the method of computation.

In order to choose a 'good' minimal system of generators of P_G let us express the image by the projection onto Γ of the system of generators of P_G:
• The images by π of $\tilde{t}_v$ or $\tilde{t}^i_v, \tilde{s}^i_v$, $2 \leq i \leq n_v$ are trivial by definition.
• To find the projection of the tame generator $\tilde{s}_v$ in Γ, it suffices to compute the number of p^{th}-roots of unity in the localization of F in v; by Hensel's lemma it suffices to compute the number q_v of p^{th}-roots of unity in the residue field. Hence the image of $\tilde{s}_v$ in Γ is $\hat{\gamma}^{q'_v}$ and q'_v has the same p-adic valuation as q_v/p..
• The projection of the wild generator $\tilde{s}_v$, in Γ is $\hat{\gamma}^{q'_v}$.

Then P_{G_S} admits the following 'good' system of generators (only the element γ has a non trivial image by π):

$$\{s_v,\ v \in S'\} \cup \{\gamma, t_w\} \cup$$
$$\{s'_v = \gamma^{-q'_v} s_v, t_v,\ v \in S - (S_0 \cup \{w\})\} \cup$$
$$\{s^i_v, t^i_v,\ v \in S - S_0,\ v|p,\ 2 \leq i \leq n_v\}$$

with the relations

$$r_w = (t_w)^p [t_w, \gamma]$$
$$r'_v = (t_v)^{q_v} [t_v, \gamma^{q'_v} s'_v],\ v \in S - S_0,\ v \not| p,$$
$$r'_v = (t_v)^{q_v} [t_v, \gamma^{q'_v} s'_v][t^2_v, s^2_v] \cdots [t^{n_v}_v, s^{n_v}_v],\ v \in S - S_0, v|p.$$

Fox derivatives

For $v \in S - \{w\}$ the Fox derivatives of a tame relation are

$$\frac{\partial r'_v}{\partial t_v} = \sum_{i=0}^{q_v} (t_v)^i - (t_v)^{q_v+1} \gamma^{q'_v} s'_v (t_v)^{-1}$$
$$\frac{\partial r'_v}{\partial \gamma} = \sum_{i=0}^{q'_v-1} (t_v)^{q_v+1} \gamma^i - \sum_{i=1}^{q'_v} (t_v)^{q_v+1} \gamma^{q'_v} s'_v (t_v)^{-1} (s'_v)^{-1} \gamma^{-i}$$
$$\frac{\partial r'_v}{\partial s'_v} = (t_v)^{q_v} t_v - (t_v)^{q_v} [t_v, s'_v \gamma^{q'_v}]$$

Since $\pi(\tilde{\gamma}) = \hat{\gamma}$ and all the other elements of the good system of generators of P_G have a trivial image by π, the columns of the Fox matrix are of the form (for wild or tame relations):

$$(0, \cdots, 0, q_v - (\hat{\gamma}^{q'_v} - 1), 0, \cdots, 0),\ v \in S - S_0.$$

Image of the generators

By lemma 8.1.6 we have $X_\infty = X_\infty^-$. Then all the images by ρ_G of the elements of X_∞ are upper triangular. We denote:

$$\rho_G(\tilde{\gamma}) = \begin{pmatrix} 1+Y & 0 \\ 0 & 1+Y' \end{pmatrix},\ \rho_G(\tilde{t}_w) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\rho_G(\tilde{s}'_v) = \begin{pmatrix} 1 & Y'_v \\ 0 & 1 \end{pmatrix}\ v \in S \cup S' - \{w\},\ \rho_G(\tilde{t}_v) = \begin{pmatrix} 1 & Y_v \\ 0 & 1 \end{pmatrix},\ v \in S - \{w\}$$

and for wild generators with $v \in S - S_0$, $2 \leq i \leq n_v$

$$\rho_G(\tilde{s}^i_v) = \begin{pmatrix} 1 & {Y'}^i_v \\ 0 & 1 \end{pmatrix},\ \rho_G(\tilde{t}^i_v) = \begin{pmatrix} 1 & Y^i_v \\ 0 & 1 \end{pmatrix}.$$

possibly with some relations between the $Y_v, Y'_v, Y^i_v, {Y'}^i_v$ depending upon the action of A. Let us discuss these relations.

Action by A

Let l be a prime. To each $v|l$, $v \in S - S_0$, there corresponds a Demŭskin group $\mathcal{P}_v$ one which we would like to determine the action of the decomposition group A_l.
Let q be the highest power of p such that F_v contains a primitive q^{th}-root of unity. Then $V_q = F_v^*/F_v^{*q}$ is a symplectic space relatively to the q-Hilbert symbol $< \cdot, \cdot >$, and A_l acts on this space as a group of similarities, i.e

$$< a(x), a(y) >= \omega(a) < x, y >,\ \forall a \in A_l$$

where ω denotes the Teichmüller character.
Since the order of A_l is prime to p, we have both tame ramification and semi-simplicity. Results of Borevič, Jakovlev and Koch ([Ko2] proposition

6, [Ko2] proposition 9) on simplectic spaces with operators then allow us to decompose V_q as a direct sum of 'hyperbolic planes' (in an obvious sense), $V_q = \oplus_\chi H_\chi$, where H_χ is generated by a hyperbolic pair $s_\chi, t_{\omega\chi^{-1}} \in V_{q,\chi}$.
Taking this into account and replacing the Demŭskin refining process, is not enough to obtain a system of generators of $\mathcal{P}_v$ which verify the Demŭskin relation (see [Bö2] proposition 2.5). We can also obtain such a nice relation in terms of generators with the desired action of A_l via characters modulo $F_d^{(3,q)}$ where $F_d^{(i,q)}$ is defined by induction

$$F_d^{(1,q)} = F_d \text{ and } F_d^{(i+1,q)} = (F_d^{(i,q)})^q[F_d^{(i,q)}, F_d],$$

(see [Bö2] remark 2.4). For the wild generators, the action of A is more ambiguous. We can easily be convinced of the difficulty of describing the action of A upon recalling that in the Wingberg presentation, we forget a place S_0 over S.

To find the form of the ideal of relations I we do not need to know precisely the action of A:
Using the images of the elements of X_∞ we have linear relations between the Y_v, Y'_v of the form

$$Y_v = f(\text{variables})$$

where f(variables) denotes a linear combinaison of the variables introduced by the images of the elements of X_∞ (write $\rho_G(a{\cdot}x) = \rho_G(a)\rho_G(x)\rho_G(a)^{-1}$ $a \in A, x \in X_\infty$). Then the action of A allows us to eliminate some variable. To summarize:

Lemma 8.2.1 *In the Borel case, if $Im\bar{\rho}$ is not diagonal, if $V_{S_0}^S = 0$ and if $\Gamma = \Gamma_{cyc}$, then a presentation by generators and relations of P_{G_S} is known and the universal deformation ring of $\bar{\rho}_G$ reads*

$$R_G(\bar{\rho}_G) = \mathbf{Z}_p[[Y, Y', Y_v, Y'_{v'}, Y^i_{v''}, Y'^i_{v''}]]/I$$

with $v' \in S \cup S' - (S_0 \cup \{w\})$ and $v \in S - (S_0 \cup \{w\})$ and $v'' \in S - S_0$, $v''|p$, $2 \leq i \leq n_{v''}$; where I is the ideal of relations generated by

$$\Big(q_v - \Big(\Big(\frac{1+Y}{1+Y'}\Big)^{q'_v} - 1\Big)\Big)Y_v,\ v \in S - (S_0 \cup \{w\})$$

$$and\ p - \Big(\frac{1+Y}{1+Y'} - 1\Big)$$

and some linear combinaisons between the variables $Y_v, Y_{v,}, Y^i_{v''}, {Y'}^i_{v''}$ depending upon the action of A.

Let replace Y, Y' by Y_1, Y_1' with $Y_1' = \frac{1+Y'}{1+Y} - 1 = p$ which can be remove; then we have

$$R_G(\bar{\rho}_G) \cong \mathbf{Z}_p[[Y_1, \cdots, Y_{d''}]]/I'$$

where $d'' = \dim_{\mathbf{F}_p} H^1(P_G, Ad\bar{\rho}_G)$ and I' is an ideal generated by relations of the form p_i^Y with $r \geq 1$.

Remark 8.2.2 Let $\bar{R}_G(\bar{\rho}_G) = R_G(\bar{\rho}_G) \otimes \mathbf{F}_p$. Then $\bar{R}_G(\bar{\rho}_G)$ is free. It shows that some deformations of the residual representation $\bar{\rho}$ cannot lift to characteristic zero rings, while others can (see [Ti] subsection 5.2).

Example: Im$\bar{\rho}$ diagonal
In [Sa], Sauzet gives some examples of fields with a Wingberg presentation. Let $F = \mathbf{Q}(\sqrt{6}, \zeta_5)$, $p = 5$, $S = \{\infty, \mathfrak{p}, \mathfrak{p}'\}$ with $\mathfrak{p}, \mathfrak{p}'$ the two places in F which divide 5, $S_0 = \{\mathfrak{p}\}$. In this case $V^S_{S_0} = 0$, we can apply the previous results to the representation $\bar{\rho} : \mathrm{Gal}(F_{S,p}/\mathbf{Q}) \to \mathrm{Gl}_2(\mathbf{F}_5)$ given by

$$\mathrm{Gal}(F/\mathbf{Q}) \to \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Then this representation satisfies the Borel case.
Since $\mathrm{Gal}(F_{S,p}/F)$ is a pro-p-group, if we suppose $\mathrm{Gal}(F_{S,p}/F) \subset \mathrm{Ker}\bar{\rho}$, then $\bar{\rho}$ is well defined.
In this case P_{G_S} has six generators $(s_1, t_1, s_2, t_2, s_3, t_3)$ and a wild relation: $r = t_1^5[t_1, s_1][t_2, s_2][t_3, s_3]$.
• If $\Gamma = \Gamma_{cyc}$ then

$$R_G(\bar{\rho}_G) = \mathbf{Z}_p[[Y_1, \ldots, Y_{d''}]]/\Big(\Big(5 - \Big(\frac{1+Y_1}{1+Y_2} - 1\Big)\Big)Y_3\Big)$$

where $d'' = \dim_{\mathbf{F}_p} H^1(G, Ad\bar{\rho}_G)$.

Remark 8.2.3 This example does not correspond exactly to the situation explained in the introduction: the set S does not contain all the places where $\bar{\rho}$ is ramified: $2, 3 \notin S$.

8.3. The maximal free pro-p-extension of F

If P_{G_S} admits a Wingberg presentation, is not needed $w_\Gamma = 0$ because then the relations are simple, though the action by A on the generators introduced by Wingberg's presentation is not easy to write.
Let $\Gamma = \Gamma_{max}$.
We use the presentation of P_{G_S} and the notations of paragraph 8.1. Then we can choose Γ_{max} such that

$$(\pi(\tilde{s}_v)\ v \in S' \cup S - S_0,\ \pi(\tilde{s}^i_{v'})\ v' \in S - S_0\ v'|p\ 2 \leq i \leq n_v)$$

is a minimal system of generators of Γ (see lemma 4.7 [Ya1]).

Fox derivatives

The computation of the Fox derivatives as above gives the columns of the Fox matrix for a wild relation:

$$(0, \cdots, 0, q_v - (\pi(\tilde{s}_v) - 1), -(\pi(\tilde{s}^2_v) - 1), \ldots, -(\pi(\tilde{s}^{n_v}_v) - 1), 0, \cdots, 0), v \in S - S_0.$$

Action by A

Using the previous discussion on the action by A we have a good description modulo $F_d^{(p,3)}$ of P_{G_S} by generators and relations with action by A:

• Tame generators
Let l be a prime $l \neq p$. We assume that S contains all places over l in F and that $A/A_l \cong \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$. Then $A \cong A_l \times \mathbf{F}_p^*$.
Let v be a place over l in F. The local action by A_l on $\mathcal{P}_v \cong < \hat{t}_v, \hat{s}_v | \hat{r}_v >$ is known and A acts on $\oplus_{v'|l} \mathcal{P}_{v'}$ by the induced representation $\mathrm{Ind}_A^{A_l} \mathcal{P}_v^{ab}$. We can lift the local generators to s_v, t_v keeping the action by A_l modulo $F_d^{(2,p)}$; hence modulo $F_d^{(2,p)}$, A permutes the t_v, $v|l$ (resp. s_v).

• Wild generators
We assume there are at most two places over p in S_F, i.e we have at most one wild relation. In the same way as for the tame generators, we can lift modulo $F_d^{(2,p)}$ the wild generators with the action by A described in the previous subsection: wild generators $t_v, s_v, t_v^2, s_v^2, \ldots, t^{n_v}, s_v^{n_v}$ where $t_v \in P_{G,\omega}$,

$s_v \in P_{G,triv}$, $t_v^i \in P_{G,\omega\chi_i^{-1}}$, $s_v^i \in P_{G,\chi_i}$ modulo $F^{(2,p)}$ (you get image by ρ_G modulo $\mathrm{Gl}_2(R)^{(2,p)}$) with a relation $t_v^p[t_v, s_v][t_v^2, s_v^2]\cdots[t_v^{n_v}, s_v^{n_v}]$ modulo $F_d^{(3,p)}$.

Knowing the Fox derivatives and the action of A we can now compute an approximation of the universal deformation ring $R_G(\bar{\rho})$ (for a discussion of this approximation see [Bö2]).

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