ON THE EQUATIONS DEFINING MINIMAL VARIETIES

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Introduction

The classification of all projective varieties of minimal degree is due to the successive contributions of various authors, and spreads over a century. More than hundred years ago, in 1886, Del Pezzo [3] solved the surface case, then, in 1907, Bertini [1] extended the characterization to irreducible varieties of any dimension. He found a class of rationally ruled varieties, for which Harris [6] in 1976 gave a nice algebraic description: he proved that they are a particular class of determinantal varieties, which he called *scrolls*. Finally, in 1981, Xambó [9] completed the classification for varieties with more than one irreducible component. De Concini-Eisenbud-Procesi [2] state that "the precise equations satisfied by reducible subvarieties of minimal degree remain mysterious", hence they "stop short of giving a normal form for the equations of each type" of them. This problem is solved in the present paper.

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1 Preliminaries

Let K be an algebraically closed field, and let $\underline{T} = \{T_0, \ldots, T_n\}$ be a finite set of variables over K. Let $R = K[\underline{T}]$ be the corresponding polynomial ring.
For a subset S of R, by $< S >$ we shall denote the linear subspace of R generated by S. If A is a matrix with entries in R, we shall use the notation $< A >$ for the linear subspace generated by the set of all entries of A.
We recall some basic definitions.
A simple scroll matrix will be a matrix of the form

$$\begin{pmatrix} l_0 & l_1 & \ldots & l_{m-1} \\ l_1 & l_2 & \ldots & l_m \end{pmatrix},$$

where $l_0, \ldots, l_m$ are linearly independent linear elements of R.
A *scroll matrix* will be a matrix of the form

$$\left(\begin{array}{c|c|c|c} \beta_1 & \beta_2 & \ldots & \beta_s \end{array} \right),$$

where for all $i = 1, \ldots, s$, the submatrix β_i is a simple scroll matrix and

$$< \beta_i > \cap (\sum_{j \neq i} < \beta_j >) = 0.$$

A projective variety defined by the vanishing of the 2-minors of a scroll matrix will be called a *scroll*.
Let J be a reduced ideal of R having an irredundant prime decomposition

$$J = \bigcap_{i=1}^{r} J_i.$$

We shall throughout suppose that J is of pure dimension. Under this hypothesis we say that J is *connected in codimension 1* if – up to rearranging the indices – for all $k = 1, \ldots, r$ it holds:

$$\operatorname{codim}(J_k + \bigcap_{i=1}^{k-1} J_i) = \operatorname{codim} J_k + 1.$$

This definition coincides with the one given by Hartshorne [8].
The following result is quoted from Eisenbud-Goto ([4], Th. 4.2 and 4.3):

Theorem 1.1. (Del Pezzo-Bertini-Xambó) *Let J be a reduced ideal of R. Suppose it is connected in codimension 1, and it has pure dimension d. Let X be the variety of $\mathbf{P}^n$ defined by J. Let $\deg J$ denote the degree of J. If $\deg J \leq \operatorname{codim} J + 1$, then J is Cohen-Macaulay, and either:*

(1) X is a quadratic hypersurface; $R/J = K[T_0, \ldots, T_n]/(Q)$, for some quadratic polynomial Q;

(2) *X is a cone over the Veronese surface in $\mathbf{P}^5$; R/J is isomorphic to a polynomial ring over $K[T_{00}, T_{01}, T_{02}, T_{11}, T_{12}, T_{22}]$ modulo the ideal of 2-minors of the generic symmetric matrix:*

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} \\ T_{01} & T_{11} & T_{12} \\ T_{02} & T_{12} & T_{22} \end{pmatrix},$$

or

(3) *$\mathbf{P}^n$ contains linear subspaces $L_1, \ldots, L_r$ and there are d-dimensional scrolls $X_i \subseteq L_i$ such that $X = \bigcup_{i=1}^{r} X_i$, and for each $k = 1, \ldots, r$ we have*

$$X_k \cap (X_1 \cup \cdots \cup X_{k-1}) = L_k \cap (\overline{L_1 \cup \cdots \cup L_{k-1}}), \qquad (*)$$

which is a linear subspace of dimension $d - 1$.

We refer to the paper of Xambó [9] for a proof.
Our first aim is to provide an explicit description of the defining ideal $J \subseteq R$ of the variety X given in (3). This answers a question posed by De Concini-Eisenbud-Procesi (cf. [2], p. 54).

Let J_i be the defining ideal of X_i, for all $i = 1, \ldots, r$. Then

$$J_i = (M_i,\ Q_i),$$

where

- Q_i is a set of linear forms defining L_i, and
- M_i is the set of all 2-minors of a scroll matrix B_i consisting of c_i columns: (M_i) is the defining ideal of the scroll X_i in its space of immersion L_i ($M_i = \emptyset$ if $c_i = 1$).

Note that the entries of B_i can be considered as a system of coordinates of L_i. In particular

$$< B_i > \cap < Q_i >= 0.$$

Up to replacing R with a polynomial ring $S \subseteq R$ we may assume that

$$\bigcap_{i=1}^{r} < Q_i >= 0.$$

Of course the entries of B_i are defined up to linear combination with the elements of Q_i. Any such modification - which of course leaves the ideal J_i untouched - will be called an *admissible change.*

Condition $(*)$ in 1.1 can be re-formulated as follows:

$$J_k + \bigcap_{i=1}^{k-1} J_i = (Q_k) + (\bigcap_{i=1}^{k-1} < Q_i >) \qquad (**)$$

for all $k = 1, \ldots, r-1$. In the sequel we shall stick to the notation just introduced. We are now ready to state our main theorem.

2 The Main Theorem

Theorem 2.1. *The following two conditions are equivalent:*

(I) The ideal J is reduced of pure dimension d, is connected in codimension 1 and has minimal degree $n-d+1$.

(II) There exist, for all $i = 1, \ldots, r$, two subsets D_i, P_i of $< Q_i >$ such that

$$< P_i > \oplus < D_i > = < Q_i >,$$

and the following axioms are satisfied:

(a) $D_1 \supseteq D_2 \supseteq \ldots \supseteq D_r = \emptyset$, and $< D_1 > = < Q_1 >$;

(b) $|D_{i-1}| = c_i + |D_i|$ for all $i = 2, \ldots, r$;

(c) $< P_i > \cap < D_{i-1} > = 0$, and $|P_i| + |D_{i-1}| = n - d + 1$ for all $i = 2, \ldots, r$;

(d)

$$\bigcap_{i=1}^{k-1} (Q_i) \subseteq (P_k, D_{k-1})$$

for all $k = 2, \ldots, r$,

and, up to admissible changes for $B_1, \ldots, B_r$, one has that

(e) $M_i \subseteq (D_{i-1})$ for all $i = 2, \ldots, r$, and

(f) $M_i \subseteq (P_j)$ for all $i = 1 \ldots, r-1$ and all $j = i+1, \ldots, r$.

We prove this Theorem in several steps. The first auxiliary result generalizes Prop. 5.1 in [5]. The technique used in the proof is similar.

Lemma 2.2. *Assume that the axioms (a), (e) and (f) of* 2.1 *are satisfied. Then*

$$\bigcap_{i=1}^{k} J_i = (M_1, \ldots, M_k, \bigcap_{i=1}^{k} (Q_i))$$

for all $k = 1, \ldots r$.

Proof. For all $k = 1, \dots, r$ set

$$H_k = (M_1, \dots, M_k, \bigcap_{i=1}^{k} (Q_i)).$$

It follows from (a), (e), (f) and the definition of J_i that $M_i \subseteq J_j$ for all indices $i, j \in \{1, \dots, r\}$: this yields "$\supseteq$". We prove "$\subseteq$". Let

$$f \in \bigcap_{i=1}^{k} J_i.$$

Using induction on $j = 1, \dots, k$ we show that for all j there is a decomposition

$$f = f_j + q_j \qquad \text{for some } f_j \in H_k \text{ and some } q_j \in \bigcap_{i=1}^{j} (Q_i).$$

For $j = k$ this will yield the claim. Since $f \in J_1 = (M_1, Q_1)$ we can write

$$f = f_1 + q_1 \qquad \text{for some } f_1 \in (M_1) \subseteq H_k \text{ and some } q_1 \in (Q_1),$$

which proves our claim for $j = 1$. Now let $j > 1$, and suppose that

$$f = f_{j-1} + q_{j-1} \qquad \text{for some } f_{j-1} \in H_k \text{ and some } q_{j-1} \in \bigcap_{i=1}^{j-1} (Q_i).$$

Now $f \in J_j$, and $f_{j-1} \in J_j$ by the first part of the proof, so that $q_{j-1} \in J_j$. Hence

$$q_{j-1} = m_j + p_j + d_j \quad \text{for some } m_j \in (M_j),\ p_j \in (P_j) \text{ and } d_j \in (D_j).$$

But by virtue of (a) and (e)

$$m_j, d_j \in D_{j-1} \subseteq \bigcap_{i=1}^{j-1} (Q_i).$$

Hence we also have that

$$p_j \subseteq \bigcap_{i=1}^{j-1} (Q_i),$$

and consequently

$$p_j \in \bigcap_{i=1}^{j-1} (Q_i) \cap (P_j) \subseteq \bigcap_{i=1}^{j} (Q_i).$$

Furthermore

$$d_j \in \bigcap_{i=1}^{j-1} (Q_i) \cap (D_j) \subseteq \bigcap_{i=1}^{j} (Q_i).$$

Thus

$$f_j = f_{j-1} + m_j \in H_k \qquad \text{and} \qquad q_j = p_j + d_j \in \bigcap_{i=1}^{j} (Q_i)$$

give the required decomposition of f. □

PROOF OF 2.1 "(II) $\Rightarrow$ (I)". For all $k = 1, \ldots, r$ one has

$$\begin{aligned} \operatorname{codim} J_k &= \operatorname{codim}(M_k, P_k, D_k) = c_k + |P_k| + |D_k| - 1 \\ &= |P_k| + |D_{k-1}| - 1 = n - d, \end{aligned} \tag{1}$$

where the last two equalities are due to (b) and (c) respectively. This proves equidimensionality. In view of 2.2 for all $k = 1, \ldots, r-1$ we have

$$\begin{aligned} J_k + \bigcap_{i=1}^{k-1} J_i &= (M_1, \ldots, M_{k-1}, M_k, \bigcap_{i=1}^{k-1} (Q_i), P_k, D_k) \subseteq (P_k, D_k, D_{k-1}) \\ &= (P_k, D_{k-1}), \end{aligned}$$

where the inclusion follows easily from (d), (e) and (f), and the last equality is a consequence of (a). Since by (a)

$$D_{k-1} \subseteq \bigcap_{i=1}^{k-1} (Q_i) \subseteq \bigcap_{i=1}^{k-1} J_i,$$

and $P_k \subseteq J_k$, the above inclusion can be reversed, so that

$$J_k + \bigcap_{i=1}^{k-1} J_i = (P_k, D_{k-1}).$$

Therefore

$$\operatorname{codim}(J_k + \bigcap_{i=1}^{k-1} J_i) = \operatorname{codim}(P_k, D_{k-1}) = n - d + 1$$

because of (c). This together with (1) proves connectivity in codimension 1. It is well known that the degree of the cylinder over a scroll is equal to the number of its columns (see [7]). Hence for all $i = 1, \ldots, r$ one has that

$$\deg J_i = c_i, \tag{2}$$

and

$$\deg J = \sum_{i=1}^{r} \deg J_i = \sum_{i=1}^{r} c_i.$$

But in (1) we have seen that

$$n - d + 1 = c_i + |P_i| + |D_i| \qquad \text{for all } i = 1, \ldots, r.$$

Summing up both sides of this equality over all $i = 1, \ldots, r$ we get

$$\begin{aligned} r(n - d + 1) &= \sum_{i=1}^{r} c_i + \sum_{i=1}^{r} (|P_i| + |D_i|) = \sum_{i=1}^{r} c_i + \sum_{i=2}^{r} (|P_i| + |D_{i-1}|) \\ &= \sum_{i=1}^{r} c_i + (r-1)(n - d + 1), \end{aligned}$$

where we used (c) and the fact that $P_1 = D_r = \emptyset$. Thus

$$n - d + 1 = \sum_{i=1}^{r} c_i,$$

which shows the minimality of degree.
Now we turn to the proof of the other implication.

Definition 2.3. Let $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_r$ be a chain of sets such that for all $i = 1, \ldots, r$ the set D_i is a basis of $\cap_{j=1}^{i} < Q_i >$. We rewrite condition $(**)$:

$$J_k + \bigcap_{i=1}^{k-1} J_i = (Q_k, D_{k-1}). \qquad (**)$$

Note that axiom (a) is fulfilled. Now for all $i = 2, \ldots, r$ there is a decomposition

$$< Q_i, D_{i-1} >=< D_i > \oplus < \varDelta_i > \oplus < C_i >,$$

where $D_i \cup \varDelta_i$ is a basis of $< D_{i-1} >$ and $D_i \cup C_i$ is a basis of $< Q_i >$. In the sequel we shall throughout stick to the notation just introduced. Note that as an immediate consequence of the definition of $\varDelta_i$ we have that

$$(3) \qquad < D_i >= \bigoplus_{j=i+1}^{r} < \varDelta_j > \qquad \text{for all } i = 1, \ldots, r-1.$$

Lemma 2.4. *Assume condition $(**)$ is fulfilled. Then there are admissible changes for $B_1, \ldots, B_r$ such that $M_i \subseteq (\varDelta_i)$ for all $i = 2, \ldots, r$. In particular axiom (e) is satisfied.*

Proof. Fix an index $i \in \{2, \ldots, r\}$. Condition $(**)$ implies that

$$(4) \qquad M_i \subseteq J_i \subseteq (< D_i > \oplus < \varDelta_i > \oplus < C_i >).$$

Choose a system of variables $\underline{T}$ such that $D_i \cup \varDelta_i \cup C_i \subseteq \underline{T}$. For all entries x of B_i write $x = u + v$, where

$$u = \sum_{T \in D_i \cup C_i} \alpha_T T \qquad \text{and} \qquad v = \sum_{T \in \underline{T} \setminus (D_i \cup C_i)} \alpha_T T, \quad (\alpha_T \in K).$$

Replace x by v. This is an admissible change for B_i, because $u \in< Q_i >$. After this operation, condition (4) for the modified set M_i implies that $M_i \subseteq (\varDelta_i)$, because no variable of $D_i \cup C_i$ appears in the polynomials belonging to M_i. $\square$

Now the sets D_i are completely determined. Next we construct the sets P_i. To this end we shall again recur to admissible changes of $B_1, \ldots, B_r$. Simultaneously we shall have to modify the sets Δ_i (and, consequently, the sets D_i) in order to preserve the validity of the claim of 2.4. These modifications will also be called *admissible changes.*
Let $i \in \{2, \ldots, r\}$. For our construction we shall consider subsets P_i of $< Q_i >$ satisfying the following
Condition 2.5:

$$< P_i > \oplus < D_{i-1} > = < Q_i > + < D_{i-1} > .$$

Lemma 2.5. *Assume (I) is true. Set* $P_1 = \emptyset$ *and for all* $i = 2, \ldots, r$ *let* P_i *be a minimal subset of* Q_i *verifying Condition 2.5. Then axiom (d) is true. Moreover, for all* $i = 1, \ldots, r$ *it holds*

$$(5) \qquad < P_i > \oplus < D_i > = < Q_i >,$$

and the axioms (b) and (c) are verified, too.

Proof. With the notation given above, and in view of 2.3, condition $(**)$ can be written in the form:

$$J_k + \bigcap_{i=1}^{k-1} J_i = (P_k, D_k, D_{k-1}) = (P_k, D_{k-1}).$$

Hence for all $k = 1, \ldots, r$ it holds:

$$\bigcap_{j=1}^{k-1} (Q_j) \subseteq (P_k, D_{k-1}),$$

which proves (d). For $i = 1$ equality (5) follows from 2.3. Fix $i \in \{2, \ldots, r\}$. Since $D_i \subseteq D_{i-1}$ by 2.3 and $< P_i > \cap < D_{i-1} > = 0$ by Condition 2.5, one has that

$$(6) \qquad < P_i > \cap < D_i > = 0.$$

Moreover by $(**)$ and connectivity in codimension 1

$$(7) \qquad n - d + 1 = \operatorname{codim}(J_i + \bigcap_{j=1}^{i-1} J_j) = \operatorname{codim}(P_i, D_{i-1}) = |P_i| + |D_{i-1}|.$$

This proves (b). From (2) and minimality of degree it follows that

$$(8) \qquad n - d + 1 = \deg J = \sum_{i=1}^{r} \deg J_i = \sum_{i=1}^{r} c_i.$$

On the other hand, for all $i = 1, \dots, r$,

$$n - d = \operatorname{codim} J_i = c_i + |Q_i| - 1.$$

Finally

$$|Q_i| = n - d + 1 - c_i = \sum_{i=1}^{r} c_j - c_i = \sum_{\substack{j=1 \\ j \neq i}}^{r} c_j. \tag{9}$$

We show that for all $i = 1, \dots, r$

$$|D_i| = \sum_{j=i+1}^{r} c_j. \tag{10}$$

This will imply (c). We prove the two inequalities. For "$\leq$" we proceed by induction on $i = 1, \dots, r$. By definition $< D_1 >=< Q_1 >$, so that the claim follows from (9) for $i = 1$. Now let $i > 1$. By (7), (8) and induction

$$|P_i| = n - d + 1 - |D_{i-1}| \geq \sum_{i=1}^{r} c_j - \sum_{j=i}^{r} c_j = \sum_{j=1}^{i-1} c_j.$$

Hence, by (6) and (9)

$$|D_i| \leq |Q_i| - |P_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^{r} c_j - \sum_{j=1}^{i-1} c_j = \sum_{j=i+1}^{r} c_j.$$

Next we show "$\geq$". In view of (7) and (8) it suffices to show that

$$|P_i| \leq \sum_{j=1}^{i-1} c_j \quad \text{for all } i = 2, \dots, r. \tag{11}$$

We prove (11) using descending induction on $i = r, r-1, \dots, 1$. By definition $< P_r > \subseteq < Q_r >$, so that by (9)

$$|P_r| \leq |Q_r| = \sum_{i=1}^{r-1} c_i,$$

which yields the claim for $i = r$. Let $i < r$. Then by (7), (8) and induction

$$|D_i| = n - d + 1 - |P_{i+1}| \geq \sum_{j=1}^{r} c_j - \sum_{j=1}^{i} c_j = \sum_{j=i+1}^{r} c_j.$$

This, together with (6) and (9) implies that

$$|P_i| \leq |Q_i| - |D_i| \leq \sum_{j=1}^{i-1} c_j.$$

This completes the proof of (10). In particular it follows that

$$|P_i| = n - d + 1 - |D_{i-1}| = \sum_{j=1}^{i-1} c_j, \tag{12}$$

so that $|P_i| + |D_i| = |Q_i|$ because of (9). In view of (6) this yields

$$< P_i > \oplus < D_i > = < Q_i > . \qquad \square$$

Corollary 2.6. *For all $i = 2, \ldots, r$ it holds $|\Delta_i| = c_i$.*

Proof. The claim follows from (3) and (10).

Lemma 2.7. *Assume (I) is true. For all $i = 1, \ldots, r$ there is a subset P_i of $< Q_i >$ verifying Condition 2.5 such that after suitable admissible changes for $B_1, \ldots, B_r$ and $\Delta_2, \ldots, \Delta_r$ the following properties are satisfied.*

(i) For all $j = 2, \ldots, r$ let

$$P_j \times \Delta_j = \{p\delta \mid p \in P_j, \delta \in \Delta_j\}.$$

Set $G_1 = \emptyset$, and

$$G_i = \bigcup_{j=1}^{i} P_j \times \Delta_j$$

for all $j = 2, \ldots, r$. Then $G_i \subseteq (P_{i+1})$ for all $i = 1, \ldots, r-1$

(ii) $M_i \subseteq (P_j)$ for all $i = 1, \ldots, r-1$ and all $j = i+1, \ldots, r$.

(iii) For all $i = 1, \ldots, r-1$ there is an index $l(i+1)$, $1 \le l(i+1) \le i$, such that

$$< P_{i+1} > \supseteq < \Pi_{l(i+1),i+1} > + < P_{l(i+1)} > \oplus < \Delta_{l(i+1)+1} > \oplus \cdots \oplus < \Delta_i >,$$

where $\Pi_{l(i+1),i+1}$ is a subset of $< Q_{i+1} >$ for which $M_{l(i+1)} \subseteq (\Pi_{l(i+1),i+1})$.

Proof. Set $P_1 = \emptyset$. We define the remaining P_i and the subsets Δ_i by recursion. We proceed by induction on $k = 1, \ldots, r-1$ showing that there are subsets $P_1, \ldots, P_k$ and admissible changes for $B_1, \ldots, B_k$ and $\Delta_2, \ldots, \Delta_k$ such that the statements (i), (ii) and (iii) are fulfilled for $i \le k-1$ and $i < j \le k$. (Set $G_0 = M_0 = \emptyset$). At the k-th step we assume that this is true, and construct P_{k+1} in such a way that (i) and (ii) are true for $i = k$ and $j = k+1$. To this end we shall perform admissible changes on $B_1, \ldots, B_k$ and $\Delta_2, \ldots, \Delta_k$, and also suitable modifications on $P_1, \ldots, P_k$, so that the claim is finally true for $i \le k$ and $i < j \le k+1$.
Let P_{k+1} be a (minimal) subset of $< Q_{k+1} >$ satisfying Condition 2.5 for

$i = k$. Since $< P_{k+1} > \cap < D_k >= 0$, there is a system of variables $\underline{T}$ such that $P_{k+1} \cup D_k \subseteq \underline{T}$. Let $i \in \{2, \ldots, k\}$ and $x \in P_i$. Write $x = u + v$, where

$$u = \sum_{T \in D_k} \alpha_T T \qquad \text{and} \qquad v = \sum_{T \in \underline{T} \setminus D_k} \alpha_T T, \quad (\alpha_T \in K).$$

Replace x by v in P_i. Perform this substitution for all $x \in P_i$ and all entries x of B_i, for $i = 1, \ldots, k$. Since $D_k \subseteq D_i \subseteq (Q_i)$, these substitutions are admissible changes and respect the condition $P_i \subseteq < Q_i >$. Furthermore (ii) is preserved, and so is Condition 2.5 for $i = 1, \ldots, k-1$ and $j = i+1, \ldots, k$, because $D_k \subseteq D_{i-1}$. Then perform the above substitution for all $x \in \Delta_i$, $i = 2, \ldots, k$. Since $D_k \subseteq D_i \subseteq D_{i-1}$ for all $i = 2, \ldots, k$, this respects the definition of Δ_i. Moreover also the inclusion $M_i \subseteq (\Delta_i)$ is still true for $i = 2, \ldots, k$. Furthermore (i) is fulfilled for $i = 1, \ldots, k-1$ with respect to the modified G_i. Finally (iii) is true for $i = 1, \ldots, k-1$ with respect to the modified subsets. We prove that

$$G_k \subseteq \bigcap_{i=1}^{k} (Q_i).$$

By finite induction we may assume

$$G_{k-1} \subseteq \bigcap_{i=1}^{k-1} (Q_i) \cap (P_k) \subseteq \bigcap_{i=1}^{k} (Q_i).$$

On the other hand $\Delta_k \subseteq (D_i) \subseteq (Q_i)$ for $i = 1, \ldots, k-1$ and $P_k \subseteq (Q_k)$. It follows that

$$G_k = G_{k-1} \cup (P_k \times \Delta_k) \subseteq \bigcap_{i=1}^{k} (Q_i).$$

In view of $(**)$ and Condition 2.5 this implies

$$G_k \subseteq (P_{k+1}, D_k). \tag{13}$$

But by construction none of the elements of G_k contains a variable $T \in D_k$, and the same is true for the elements of P_{k+1}. Hence (13) implies

$$G_k \subseteq (P_{k+1}) \tag{14}$$

which proves (i) for $i = k$. Let $i \in \{1, \ldots, k-1\}$. By (e) $M_i \subseteq (D_{i-1}) \subseteq (D_j) \subseteq J_j$ for all $j = 1, \ldots, i-1$. On the other hand by induction $M_i \subseteq (P_j) \subseteq J_j$, for all $j = i+1, \ldots, k$, so that, in view of $(**)$:

$$M_i \subseteq \bigcap_{j=1}^{k} J_j \subseteq (P_{k+1}, D_k).$$

But by construction the elements of M_i do not contain any variable from D_k. Hence $M_i \subseteq (P_{k+1})$. Furthermore by 2.4 for all $i = 1, \ldots, k-1$ it holds that $M_k \subseteq (D_{k-1}) \subseteq (D_i)$. Moreover $M_k \subseteq J_k$. By virtue of $(**)$ it follows that

$$M_k \subseteq \bigcap_{j=1}^{k} J_j \subseteq (P_{k+1}, D_k),$$

which implies that $M_k \subseteq (P_{k+1})$, since the elements of M_k do not contain any variable of the subset D_k. This proves (ii) for $i \leq k$ and $j = k+1$. Finally we show (iii) for $i = k$. From (14) and the definition of G_k it follows that for all $j = 1, \ldots, k$ either $P_j \subseteq < P_{k+1} >$ or $\Delta_j \subseteq < P_{k+1} >$. Let

$$l = \min\{h \mid \Delta_j \subseteq < P_{k+1} >, \text{ for all } j \text{ such that } \ h \leq j \leq k\} - 1$$

If the set is empty set $l = k$. Since $\Delta_l \not\subseteq < P_{k+1} >$, necessarily $P_l \subseteq < P_{k+1} >$. Now, in view of (3) and 2.5

$$< P_l > \cap (< \Delta_{l+1} > \oplus \cdots \oplus < \Delta_k >) \subseteq < P_l > \cap < D_l > = 0.$$

Hence $< P_{k+1} > \supseteq < P_l > \oplus < \Delta_{l+1} > \oplus \cdots \oplus < \Delta_k >$. On the other hand by (ii) there must be a subset $\Pi_{l(k+1),k+1}$ of $< P_{k+1} >$ such that $M_l \subseteq (\Pi_{l(k+1),k+1})$. ☐

Corollary 2.8. *With respect to the data of* 2.7*, axiom (f) is fulfilled.*

Proof. By induction on $j = 2, \ldots, r$ we show that $M_i \subseteq (P_j)$ for all $i = 1, \ldots, j-1$. By 2.7 (iii) we have that

$$< P_2 > \supseteq < \Pi_{1,1} >,$$

where $M_1 \subseteq (\Pi_{1,1})$, so that the claim is true for $j = 2$. Let $j > 2$ and $i < j$. With the notation of 2.7 one has that $M_i \subseteq (\Delta_i)$ for $i = l(j), \ldots, j-1$, and by induction $M_i \subseteq (P_{l(j)})$ for $i = 1, \ldots, l(j) - 1$. This implies the claim. ☐

The Theorem is completely proven now: the required subsets D_i and P_i are those fulfilling 2.7.

3 A constructive method

In this last section we want to give an explicit description of the elements of the subsets P_i and D_i. This will permit us to develop an algorithm for the construction of the defining ideals of the irreducible components of any projective variety which is connected in codimension 1 and has minimal degree.

Lemma 3.1. *Let B be a scroll matrix with $c > 1$ columns. Let M be the set of its 2-minors. Let Q be a set of independent linear forms such that $M \subseteq (Q)$. Let L_1, L_2 be the row vectors of B. Then one of the following cases occurs:*

(1) There is $(\lambda,\mu)\in K^2\setminus\{(0,0)\}$ *such that* $<\lambda L_1+\mu L_2>\in<Q>$,

(2) B *contains an isolated column* $\hat\beta$ *such that* $<\hat\beta>\cap<Q>=0$, *and for every other small block* β *of* B *either*

(i) $<\beta>\subseteq<Q>$, *or*

(ii) β *is an isolated column and* $<\beta+\alpha\hat\beta>\in<Q>$ *for some* $\alpha\in K$.

Proof. We first prove the claim in the case where B is a simple scroll matrix, say

$$B=\begin{pmatrix} l_0 & l_1 & \dots & l_{c-1}\\ l_1 & l_2 & \dots & l_c\end{pmatrix}.$$

Let $\underline{T}$ be a set of variables such that $\underline{T}\supseteq Q$. For all $i=0,\dots,c$ write $l_i=l'_i+l''_i$, where

$$l'_i=\sum_{T\in Q}\alpha_T T \qquad\text{and}\qquad l''_i=\sum_{T\in\underline{T}\setminus Q}\alpha_T T \qquad (\alpha_T\in K).$$

Let

$$\bar B=\begin{pmatrix} l''_0 & l''_1 & \dots & l''_{c-1}\\ l''_1 & l''_2 & \dots & l''_c\end{pmatrix}$$

be the image of B in the polynomial ring $\bar R=K[\underline{T}]/(Q)$. All the 2-minors of $\bar B$ are zero in $\bar R$.
First assume that $l''_1=0$. Then

$$0=\begin{vmatrix} l''_1 & l''_2\\ l''_2 & l''_3\end{vmatrix}=l''_1l''_3-l''^2_2=l''^2_2,$$

so that $l''^2_2=0$. Suppose that $l''_0\neq 0$. Let $i\in\{3,\dots,m\}$. One has that

$$0=\begin{vmatrix} l''_0 & l''_{i-1}\\ l''_1 & l''_i\end{vmatrix}=l''_0l''_i-l''_1l''_{i-1}=l''_0l''_i,$$

so that $l''_i=0$. It follows that $L_2=(l'_1,l'_2,\dots,l'_m)$, whence $<L_2>\subseteq<Q>$.
Now suppose that $l''_0=0$. We prove that $l''_i=0$ for all $i=3,\dots,m-1$. Fix such an index i and assume that $l''_{i-1}=0$. We have that

$$0=\begin{vmatrix} l''_{i-1} & l''_i\\ l''_i & l''_{i+1}\end{vmatrix}=l''_{i-1}l''_{i+1}-l''^2_i=-l''^2_i,$$

so that $l''_i=0$. Hence $L_1=(l'_0,l'_1,\dots,l'_{m-1})$, so that $<L_1>\subseteq<Q>$.
Now assume that $l''_1\neq 0$. We prove that for all $i=0,\dots,m$ there is $\alpha_i\in K\setminus\{0\}$ such that $l''_i=\alpha_il''_1$. We proceed by induction on $i=0,\dots,m$. First note that

$$0=\begin{vmatrix} l''_0 & l''_1\\ l''_1 & l''_2\end{vmatrix}=l''_0l''_2-l''^2_1.$$

Since $\bar{R}$ is a UFD, it follows that $l''_0 = \alpha_0 l''_1$ and $l''_2 = \alpha_2 l''_1$ for some $\alpha_0, \alpha_2 \in K \setminus \{0\}$. Since $\alpha_1 = 1$, this shows the claim for $i = 0, 1, 2$. Now let $i \in \{3, \ldots, m\}$ and suppose that $l''_{i-2} = \alpha_{i-2} l''_1$, $l''_{i-1} = \alpha_{i-1} l''_1$ for some $\alpha_{i-2}, \alpha_{i-1} \in K \setminus \{0\}$. It holds

$$0 = \begin{vmatrix} l''_{i-2} & l''_{i-1} \\ l''_{i-1} & l''_i \end{vmatrix} = l''_{i-2} l''_i - l''^2_{i-1} = \alpha_{i-2} l''_1 l''_i - \alpha^2_{i-1} l''_1.$$

Hence

$$l''_i = \frac{\alpha^2_{i-1}}{\alpha_{i-2}} l''_1.$$

This completes the induction.
We have just proven that

$$\bar{B} = \begin{pmatrix} \alpha_0 l''_1 & \alpha_1 l''_1 & \ldots & \alpha_{m-1} l''_1 \\ \alpha_1 l''_1 & \alpha_2 l''_1 & \ldots & \alpha_m l''_1 \end{pmatrix}.$$

Since all minors of $\bar{B}$ are zero in $\bar{R}$, its rows are proportional: there is $\lambda \in K \setminus \{0\}$ such that $\alpha_i = \lambda \alpha_{i+1}$ for all $i = 0, \ldots, m-1$. Therefore $(L_1 - \lambda L_2) = (l'_0 - \lambda l'_1, \ldots, l'_m - \lambda l'_m)$, whose entries all belong to $< Q >$. This completes the proof of the claim in the case where B is simple.

Now suppose that B consists of more than one small block. Let

$$\widetilde{\beta} = \begin{pmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{pmatrix}$$

be a small block of B. If $\widetilde{\beta}$ is not an isolated column and $\widetilde{M}$ is the set of its minors, then

$$\widetilde{M} \subseteq M \subseteq (Q),$$

so that by the first part of the proof there is $(\widetilde{\lambda}, \widetilde{\mu}) \in K^2 \setminus \{(0,0)\}$ for which

$$< \widetilde{\lambda} \widetilde{L}_1, \widetilde{\mu} \widetilde{L}_2 > \subseteq < Q > .$$

Let $B' = (\beta_1 | \ldots | \beta_s)$ be the submatrix of B' formed by all small blocks with the above property. Then $B' = B$ if B has no isolated column. We show that the claim is true for B'. For all $i = 1, \ldots, s$ let $L_1^{(i)}$, $L_2^{(i)}$ be the row vectors of β_i. By assumption for all $i = 1 \ldots, s$ there is $(\lambda^{(i)}, \mu^{(i)}) \in K^2 \setminus \{(0,0)\}$ such that

$$< \lambda^{(i)} L_1^{(i)} + \mu^{(i)} L_2^{(i)} > \subseteq < Q > .$$

Suppose for a contradiction that the claim is not true for B'. Then there are two indices i, j, $1 \leq i < j \leq s$ such that there is an entry x of $\lambda^{(i)} L_1^{(j)} + \mu^{(i)} L_2^{(j)}$ and an entry y of $\lambda^{(j)} L_1^{(i)} + \mu^{(j)} L_2^{(i)}$ such that $x, y \notin < Q >$. This implies that

$$\begin{vmatrix} \lambda^{(i)} & \mu^{(i)} \\ \lambda^{(j)} & \mu^{(j)} \end{vmatrix} \neq 0.$$

Therefore there is an invertible row transformation mapping B into the matrix

$$\widetilde{B} = \begin{pmatrix} \lambda^{(i)} L_1 + \mu^{(i)} L_2 \\ \lambda^{(j)} L_1 + \mu^{(j)} L_2 \end{pmatrix}.$$

In particular the ideals of the 2-minors of B and $\widetilde{B}$ coincide. Consider the following minor of $\widetilde{B}$:

$$m = \begin{vmatrix} x & y' \\ x' & y \end{vmatrix} = xy - x'y'.$$

Then $m \in (M) \subseteq (Q)$. Note that x' is an entry of $\lambda^{(j)} L_1^{(j)} + \mu^{(j)} L_2^{(j)}$. Hence $x' \in < Q >$. But then $xy \in (Q)$. Since (Q) is a prime ideal and $x,\ y \notin (Q)$, this provides the required contradiction.

Now suppose that there is an isolated column

$$\hat{\beta} = \begin{pmatrix} l_0 \\ l_1 \end{pmatrix},$$

not belonging to $\widetilde{B}$. We fix another small block of B:

$$\beta = \begin{pmatrix} l_2 & l_3 & \dots & l_{m-1} \\ l_3 & l_4 & \dots & l_m \end{pmatrix}.$$

By the first part of the proof up to inverting the rows of B we may assume that either β is an isolated column or $l_i + \nu l_{i+1} \in < Q >$ for all $i = 2, \dots, m-1$. for some $\nu \in K \setminus \{0\}$. For all $i = 0, \dots, m$ let l_i' and l_i'' have the same meaning as above.
In $\bar{R}$ it holds

$$\begin{vmatrix} l_0'' & l_{m-1}'' \\ l_1'' & l_m'' \end{vmatrix} = l_0'' l_m'' - l_1'' l_{m-1}'' = 0.$$

First we show that l_0'' and l_1'' cannot be proportional. Suppose that $\lambda l_0'' + \mu l_1'' = 0$ for some $(\lambda, \mu) \in K^2 \setminus \{(0,0)\}$. Then

$$\lambda l_0 + \mu l_1 = \lambda(l_0' + l_0'') + \mu(l_1' + l_1'') = \lambda l_0' + \mu l_1' \in < Q >,$$

against our assumption on $\hat{\beta}$. Since $\bar{R}$ is a UFD, it follows that one of the following cases occurs:
(1) $l_m'' = l_{m-1}'' = 0$. We show that in this case $< \beta > \in < Q >$. We have that $l_m, l_{m-1} \in < Q >$. If β is an isolated column, there is nothing left to prove. Otherwise $l_i + \nu l_{i+1} \in < Q >$ for all $i = 2, \dots, m-1$ By finite descending induction one concludes that $l_i \in < Q >$ for $i = 2, \dots, m-1$. In particular β belongs to $\widetilde{B}$.
Now let γ be another small block of B. We show that $< \gamma > \in < Q >$, too. Up

to an elementary row transformation we may assume that the entries of the first row of γ all belong to $< Q >$. Let $(x,y)^t$ be a column of γ. Then

$$\begin{vmatrix} l_0 & x \\ l_1 & y \end{vmatrix} = l_0 y - x l_1 \in < Q >,$$

since this is a minor of B. Now $x \in < Q >$, and $l_0 \notin < Q >$. It follows that $y \in < Q >$.
(b) There is $\alpha \in K \setminus \{0\}$ such that $l''_0 = \alpha l''_{m-1}$ and $l''_1 = \alpha l''_m$. Then

$$l_0 - \alpha l_{m-1} = l'_0 + l''_0 - \alpha(l'_{m-1} + l''_{m-1}) = l'_0 - \alpha l'_{m-1} \in < Q >,$$

and

$$l_1 - \alpha l_m = l'_1 - \alpha l'_m \in < Q > .$$

Hence $< \beta - \alpha\hat{\beta} > \in < Q >$. We prove that in this case β is an isolated column. Suppose this were not the case. Then $l_{m-1} + \mu l_m \in < Q >$. But then

$$l_0 + \mu l_1 = l'_0 + \mu l'_1 + l''_0 + \mu l''_1 = l'_0 + \mu l'_1 + \alpha(l''_{m-1} + \mu l''_m) \in < Q >,$$

against our assumption on $\hat{\beta}$. □

As an immediate consequence we have:

Corollary 3.2. *Under the assumption of* 3.1 *let Π be a subset of $< Q >$ which is minimal with respect to the condition $M \subseteq (\Pi)$. Then $< \Pi > \subseteq < B >$. If B has no isolated columns, then Π is the set of entries of a non trivial linear combination of the row vectors of B.*

Remark 3.3. In [7] it is proven that a d-dimensional scroll $X \in \mathbf{P}^n$ is a *ruled variety*: if B is the associated scroll matrix, then X is the union of all $(n-d)$-planes (*rulings*) defined by the annulation of a non trivial linear combination of the row vectors of B. Note that the number of isolated columns of B is equal to the number of linear components of X. Our corollary shows that if X does not contain any d-plane, then it is "ruled" in a stronger sense: for every linear subspace V contained in X there is a ruling $W \subseteq X$ containing V.
In the next result we determine all $(n-d)$-planes contained in a scroll X. It follows easily from 3.1 and 3.2, and will be useful for the constructive algorithm later on.

Corollary 3.4. *Let B be a scroll matrix with $c > 1$ columns. Let M be the set of its 2-minors. Let Q be a set of linear forms such that $M \subseteq (Q)$. If $|Q| = c$, then either*

(i) $< Q >$ is generated by the entries of a non trivial linear combination of the row vectors of B, or

(ii) $B = (\beta_1|\beta_2)$, where β_1 is an isolated column, and $< Q > = < \beta_2 >$, or

(iii) $B = (\beta_1|\beta_2)$, *where* β_1 *and* β_2 *are isolated columns and* $< Q >$ *is generated by the entries of a non trivial linear combination of* β_1 *and* β_2*.*

Now we can precise the structure of the subsets P_i introduced in Section 2.

Proposition 3.5. *For all* $i = 2, \dots, r$ *there is an index* $l(i)$, $1 \le l(i) \le i - 1$ *such that we have a decomposition*

$$< P_i >=< P_{l(i)} > \oplus < \Pi_{l(i),i} > \oplus < \Delta_{l(i)+1} > \oplus \cdots \oplus < \Delta_{i-1} >,$$

where $\Pi_{l(i),i}$ *is a subset of* $< B_{l(i)} >$ *for which* $|\Pi_{l(i),i}| = c_{l(i)}$ *and* $M_{l(i)} \subseteq (\Pi_{l(i),i})$.

Proof. Fix an index $i \in \{2, \dots, r\}$. By 2.7 there is an index $l(i)$, $1 \le l(i) \le i - 1$ and there is a subset $\Pi_{l(i),i}$ such that

$$< P_i > \supseteq < P_{l(i)} > \oplus < \Pi_{l(i),i} > \oplus < \Delta_{l(i)+1} > \oplus \cdots \oplus < \Delta_{i-1} >,$$

and $M_{l(i)} \subseteq (\Pi_{l(i)})$. By 3.2 this subset can be chosen in such a way that it is contained in $< B_{l(i)} >$. Then

$$< \Pi_{l(i)} > \cap (< P_{l(i)} > \oplus < \Delta_{l(i)+1} > \oplus \cdots \oplus < \Delta_{i-1} >) \subseteq < B_{l(i)} > \cap < Q_{l(i)} >= 0.$$

On the other hand we have that

$$|\Pi_{l(i),i}| > codim(M_{l(i)}) = c_{l(i)} - 1.$$

Finally by (12) and 2.6

$$|P_i| - |P_{l(i)}| - |\Delta_{l(i)+1}| - \cdots |\Delta_{i-1}| = c_{l(i)}.$$

This suffices to conclude. □

Next we want to present an explicit construction of the defining equations of any projective variety satisfying the assumptions of 1.1 (3).

Theorem 3.6. *Let* $R = K[T_0, \dots, T_n]$ *be a polynomial ring over the field* K*. Let* $r, c_1, \dots, c_r$ *be positive integers such that* $c_1 + \cdots + c_r + r \le n + 1$*. Let* $\Delta_1, \Delta_2, \dots, \Delta_r$ *be independent sets of linear forms of* R*, such that* $|\Delta_i| = c_i$ *for all* $i = 1, \dots, r$*. Set*

$$D_i = \Delta_{i+1} \cup \cdots \cup \Delta_r,$$

for all $i = 1, \dots, r - 1$*, and let* $D_r = \emptyset$*. Then set* $P_1 = \emptyset$*, and* $P_2 = \Delta_1$*. Apply the following recursive construction.*
1. Let $i = 3$, $\Pi_1 = \Delta_1$*.*
2. Choose an index $l = l(i)$, $1 \le l \le i - 1$*. If* $l = l(j)$ *and* $< \Pi_{l,j} > \ne < \Delta_l >$,

for some $j < i$, then goto 5.
3. Choose a set $\Pi_{l,i}$ of c_l independent linear forms such that

$$< \Pi_{l,i} > \cap (< P_{l,i} > \oplus < \Delta_{l+1} > \oplus \cdots \oplus < \Delta_r >) = 0. \qquad (\star)$$

If $< \Pi_{l,i} > = < \Delta_l >$ or $c_l = 1$, then goto 6.
4. Choose a scroll matrix B_l with c_l columns such that the set M_l of its 2-minors is contained in (Δ_l) and $(\Pi_{l,i})$, and $\Delta_l, \Pi_{l,i} \subseteq < B_l >$. Goto 6.
5. Choose a set $\Pi_{l,i}$ of c_l independent linear forms such that $M_l \subseteq (\Pi_{l,i})$, and $(\star)$ is true.
6. Set

$$P_i = P_l \cup \Pi_{l,i} \cup \Delta_{l+1} \cup \cdots \cup \Delta_{i-1}.$$

If $i < r$, replace i with $i+1$ and goto 2.
7. For all $i = 1, \ldots, r$, if $i \neq l(j)$ for all indices j, choose B_i to be a scroll matrix with c_i columns such that the set M_i of its minors is contained in (Δ_i), $\Delta_i \subseteq < B_i >$, and $< B_i > \cap < P_i, D_i > = 0$. End.
Then the axioms (a)–(f) of 2.1 are satisfied.

Remark 3.7. Note that the above recursive construction is always possible. The only step that really has to be justified is 4. But one easily sees that it possible to find a scroll matrix B_l such that $< \Delta_l >$ and $< \Pi_l >$ (which are supposed to be distinct) are generated by the set of entries of the first and the second row of B_l respectively. And in step 5 it suffices to choose $\Pi_l = \Delta_l$.

For the rest of the section we shall refer to the sets and matrices constructed in the claim of 3.6. It is immediate to verify that the axioms (a)–(c) and (e), (f) are satisfied. The validity of axiom (d) follows from the next two results.

Lemma 3.8. *Let $i \in \{1, \ldots, r-1\}$. Then for all $j = 1, \ldots, i$ one has that $P_j \subseteq P_i$ or $\Delta_j \subseteq P_i$.*

Proof. We proceed by induction on i. For $i = 1$ the claim is trivial, since $P_1 = \emptyset$. Let $1 < i \leq r-1$, and $1 \leq j \leq i$. We refer to the decomposition of P_i given in 3.5. We have one of the following cases.

If $l(i) < j$, then $\Delta_j \subseteq P_i$,
if $l(i) = j$, then $P_j \subseteq P_i$,
if $l(i) > j$, then replace P_i by $P_{l(i)}$ and apply induction. □

In the claim of the next Proposition the sets G_k are those introduced in 2.7. As in Section 2 we set $Q_i = P_i \cup D_i$ for all $i = 1, \ldots, r$.

Proposition 3.9. *For all $k = 1, \ldots r$ one has that $G_k \subseteq (P_{k+1})$. Moreover it holds*

$$\bigcap_{i=1}^{k} (Q_i) = (G_k, D_k).$$

PROOF. The first part of the claim is an immediate consequence of 3.8. We prove the rest of the claim by induction on k. Since $G_1 = \emptyset$, the claim is trivial for $k = 1$. Let $1 < k \leq r$. By induction

$$\begin{aligned}\bigcap_{i=1}^{k}(Q_i) &= (\bigcap_{i=1}^{k-1}(Q_i)) \cap (Q_k) = (G_{k-1}, D_{k-1}) \cap (P_k, D_k) \\ &= (G_{k-1}, \Delta_k, D_k) \cap (P_k, D_k) = (G_{k-1}, P_k \times \Delta_k, D_k) = (G_k, D_k),\end{aligned}$$

where the last but one equality is due to the first part of the claim.

Now we are able to determine a minimal set of generators of J explicitely.

Corollary 3.10. *For all $i = 1, \dots, r$ let $J_i = (M_i, Q_i)$. Let $J = J_1 \cap \dots \cap J_r$. Then*

$$J = (M_1, \dots, M_r, G_r).$$

In particular the ideal J is generated by elements of degree 2.

Note that the last assertion has already been proven in [4].

Example 3.11. In $K[a, \dots, m]$ we consider the ideal

$$J = J_1 \cap \ldots \cap J_5,$$

where $J_i = (M_i, Q_i)$, and

- $M_2 = M_5 = \emptyset$ and M_1, M_3, M_4 are the set of 2-minors of the scroll matrices B_1, B_3, B_4 given below;
- for all $i = 1, \dots, 5$ the set Q_i is constructed as follows.

	B_1			B_2	B_3		B_4		B_5
	a	c	d		h	i	k	m	
	b	d	e		i	j	l	$a+j$	
Q_1				f	h	i	m	$a+j$	$l+b$
				Δ_2	Δ_3		Δ_4		Δ_5
Q_2	$a+b$	$c+d$	$d+e$		h	i	m	$a+j$	$l+b$
	P_2				Δ_3		Δ_4		Δ_5
Q_3	$a+b$	$c+d$	$d+e$	g			m	$a+j$	$l+b$
	P_2			Π_2	Δ_3		Δ_4		Δ_5
Q_4	c	d	e	f	h	i			$l+b$
	Π_1			Δ_2	Δ_3				Δ_5
Q_5	$a+b$	$c+d$	$d+e$	g	i	j	m	$a+j$	
	P_3				Π_3		Δ_4		

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