On Certain Algebras of Reduction Number One

Margherita Barile[1]
Dipartimento di Matematica, Università degli Studi di Bari, Via Orabona 4
70125 Bari (ITALY)

Marcel Morales
Université de Grenoble I, Institut Fourier, Laboratoire de Mathématiques associé au CNRS, URA 188, B.P.74, 38402 Saint-Martin D'Hères Cedex,
and IUFM de Lyon, 5 rue Anselme, 69317 Lyon Cedex (FRANCE)

Introduction

In this paper we consider a large class of coordinate rings of certain unions of projective scrolls. This class appears in several recent works studying properties of the special fibre $F(I)$ of an ideal I in a local ring ([5], [6], and [7]). We prove that the reduction number of these algebras is always equal to one ([14]). We also prove that these reduced algebras are Cohen-Macaulay of minimal degree, so that the above assertion also follows from a theorem by [4]. Our methods, however, are constructive, and we can explicitely describe a Noether subalgebra. This can be used to find explicit reductions for the ideal I (see 2.7). As an application we describe the special fibre $F(I)$ when I is the defining ideal of a projective monomial variety of codimension 2, and prove a conjecture contained in [5] and [6]: we show that the Rees algebra of I is defined by relations of degree two at most.

1 The ideal associated to a barred matrix

Let K be an algebraically closed field. Let $\underline{T} = \{T_a \mid 1 \leq a \leq n\}$ be a set of variables over K. We consider the following barred matrix:

$$\mathcal{N} = \left(\; B_1 \mid B_2 \mid \ldots \mid B_{s_1} \parallel B_{s_1+1} \mid \ldots \mid B_{s_2} \parallel \ldots \parallel B_{s_{r-1}+1} \mid \ldots \mid B_{s_r} \; \right),$$

where for all $\nu = 1, \ldots, s_r$,

$$B_\nu = \begin{pmatrix} T_{i_\nu+1} & T_{i_\nu+2} & \ldots & \ldots & T_{i_{\nu+1}} \\ T_{i_\nu+2} & T_{i_\nu+3} & \ldots & T_{i_{\nu+1}} & T_{j_\nu} \end{pmatrix}.$$

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We call B_ν the ν-th *small block* of $\mathcal{N}$. For all $i = 1, \ldots, r$, the submatrix

$$\mathcal{B}_i = \left(\; B_{s_{i-1}+1} \; \middle| \; B_{s_{i-1}+2} \; \middle| \; \ldots \; \middle| \; B_{s_i} \; \right)$$

will be called the i-th *big block* of $\mathcal{N}$.

We suppose that different indices correspond to different variables, and that the entries of each big block are pairwise distinct. Moreover we assume that the indices j_ν are pairwise distinct and for all $\nu = 1, \ldots, s_r$, the index j_ν verifies one of the following two conditions:

(i) T_{j_ν} does not appear anywhere else in the matrix $\mathcal{N}$, or

(ii) there exists a unique μ, $\nu < \mu \leq s_r$, such that B_ν and B_μ belong to different big blocks and $T_{j_\nu} = T_{i_\mu+1}$.

In other words, all the entries of $\mathcal{N}$ appear one time only, except for the last entry of the second row of each small block, which can appear a second time as the first entry of the first row of a small block belonging to one of the following big blocks.

Let J be the ideal of $K[\underline{T}]$ generated by the 2×2-minors of every big block of $\mathcal{N}$ and by every product T_aT_b, where T_a is in the first row of $\mathcal{B}_i$, and T_b is in the second row of $\mathcal{B}_j$ for some indices i and j, $i < j$. The latter will be called *transversal products.*

The following result generalizes Proposition 5.1 in [7]. We refer to this paper for the proof.

Proposition 1.1. *For all i, $1 \leq i \leq r$, let J_i be the ideal generated by the 2×2-minors of the i-th big block, by the entries of the first rows of the big blocks with the indices $1, \ldots, i-1$, and by the entries of the second rows of the big blocks with the indices $i+1, \ldots, r$. Then*

$$J = \bigcap_{i=1}^{r} J_i$$

is a primary decomposition of the ideal J.

We shall say that J is the *ideal associated to the barred matrix* $\mathcal{N}$. We introduce the following notation. For all i let M_i be the set of 2×2-minors of the big block $\mathcal{B}_i$. Let P_i be the set of entries of the first rows of the big blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{i-1}$ and D_i the set of entries of the second rows of the big blocks $\mathcal{B}_{i+1}, \ldots, \mathcal{B}_r$. Moreover, for all i let c_i be the number of columns of $\mathcal{B}_i$. One has that

$$\operatorname{codim}(M_i) = c_i - 1.$$

For the generalities on rational normal scrolls see [4] or [8].

Remark 1.2. Note that the ideal (M_i) defines a scroll not in $\mathbf{P}^n$, but in its space of immersion. The latter is the linear subspace of $\mathbf{P}^n$ defined by the annulation of the variables in $P_i \cup D_i$.

Next we give some immediate properties of sets P_i and D_i, which will be useful in the proofs of the next results.

(a) One has that $\emptyset = P_1 \subseteq P_2 \subseteq \cdots \subseteq P_r$, and $D_1 \supseteq D_2 \supseteq \cdots \supseteq D_r = \emptyset$.

(b) $P_i \cap D_i = \emptyset$ for all $i = 1, \ldots, r$.

(c) $|P_{i+1}| = \sum_{j=1}^{i} c_j$, and $|D_i| = \sum_{j=i+1}^{r} c_i$, for all $i = 1, \ldots, r-1$.

(d) $M_i \subseteq (D_{i-1})$, and $M_{i-1} \subseteq (P_i)$ for all $i = 2, \ldots, r$.

Corollary 1.3. *Let c be the number of columns of $\mathcal{N}$. Then*

$$\operatorname{codim} J_i = c - 1$$

for all $i = 1, \ldots, c$. In particular J is of pure codimension $c-1$.

Proof. By properties (b) and (c) one has that $P_i \cup D_i$ is a set of $c - c_i$ pairwise distinct variables. By construction of $\mathcal{N}$ none of these appears in the minors lying in M_i. Thus $\operatorname{codim} J_i = c_i - 1 + c - c_i = c - 1$ for all $i = 1, \ldots, r$. □

Next we prove that J is *connected in codimension 1*. According to the definition given by Hartshorne [9], for an equidimensional ideal this property follows from the following condition:

$$\operatorname{codim}(J_k + \bigcap_{i=1}^{k-1} J_i) = \operatorname{codim} J_k + 1 \qquad \text{for } k = 1, \ldots, r \qquad (*)$$

Proposition 1.4. *The prime decomposition of J verifies condition $(*)$.*

Proof. Let $k \in \{1, \ldots, r\}$. We show that

$$J_k + \bigcap_{i=1}^{k-1} J_i = (P_k, D_{k-1}).$$

Let

$$\widetilde{\mathcal{B}}_{k-1} = \left(\; B_1 \; \| \; \ldots \; \| \; B_{k-1} \; \right).$$

For all $i = 1, \ldots, k-1$ let

$$\widetilde{J}_i = (M_i, P_i, \Delta_{i+1}, \ldots, \Delta_{k-1}),$$

i.e. the intersection of J_i with the subring generated by the entries of $\widetilde{\mathcal{B}}_{k-1}$. Note that

$$J_i = \widetilde{J}_i + (D_{k-1}) \qquad \text{for } i = 1, \dots, k-1.$$

Applying 1.1 to $\widetilde{\mathcal{B}}_{k-1}$ one deduces that the intersection $\widetilde{J} = \cap_{i=1}^{k-1} \widetilde{J}_1$ is generated by $M_1, \dots, M_{k-1}$ and the transversal products of $\widetilde{\mathcal{B}}_{k-1}$. In particular $\widetilde{J} \subseteq (P_k)$. It follows that

$$\begin{aligned} J_k + \bigcap_{i=1}^{k-1} J_i &= (M_k, P_k, D_k) + \widetilde{J} + (D_{k-1}) \\ &\subseteq (P_k, D_{k-1}), \end{aligned}$$

where we used properties (a) and (d). Since the opposite inclusion is obvious, this suffices to conclude. □

The ideal J has an another relevant property: for the details of the proof of the next results we again refer to [4] and [8].

Proposition 1.5. *Let J be the ideal defined above. Then*

$$\deg J = \operatorname{codim} J + 1.$$

Proof. The degree of a rational normal scroll X is equal to the number of columns of the associated matrix. The degree is the same for all cylinders over X. Hence

$$\deg J_i = c_i$$

for all $i = 1, \dots, r$, so that

$$\deg J = \sum_{i=1}^{r} \deg J_i = \sum_{i=1}^{r} c_i = c.$$

The claim then follows from 1.3. □

The following definition is due to Vasconcelos:

Definition 1.6. ([15], Def. 1) Let R be a finitely generated standard algebra over K. Let

$$A = K[z_1, \dots, z_\ell] \hookrightarrow R$$

be a Noether normalization of R, and assume that all the z_i are of degree 1. Let $\{b_1, b_2, \dots, b_s\}$ be a minimal set of homogeneous generators of R as an A-module. The number

$$r_A(R) = \max\{\deg b_i \mid i = 1, \dots, s\}$$

is called the *reduction number* of R with respect to A. The minimum of $r_A(R)$ taken over all possible Noether normalizations A of R is called the *(absolute) reduction number* of R.

From 1.4, 1.5 and [4], Th. 4.2 it follows

Proposition 1.7. *Let $\mathcal{N}$ be a barred matrix whose set of entries is a system $\underline{T}$ of variables over the field K. Let J be the ideal of $K[\underline{T}]$ associated to $\mathcal{N}$. Then $K[\underline{T}]$ is Cohen-Macaulay and*

$$r(K[\underline{T}]/J) = 1.$$

In the sequel we present an explicit construction of a Noether normalization of $K[\underline{T}]/J$. First of all we introduce the barred matrix $\mathcal{N}'$ obtained by replacing the ν-th small block of $\mathcal{N}$ with the column

$$\begin{pmatrix} T_{i_\nu+1} \\ T_{j_\nu} \end{pmatrix}.$$

We consider all sequences of the form

$$T_{i_{\nu_1}+1}, T_{j_{\nu_1}}, T_{j_{\nu_2}}, \dots, T_{j_{\nu_s}} \qquad (1 \leq \nu_1 < \nu_2 < \cdots < \nu_s \leq r)$$

which verify the following conditions:

(i) $T_{j_{\nu_k}} = T_{i_{\nu_{k+1}-1}+1}$, for all $k = 1, \dots, s$,

(ii) the sequence is maximal with respect to (i).

Note that T_{j_1} is the only entry of $\mathcal{N}$ that possibly does not appear in any of the above sequences. All the others occur one time exactly. We form the sum of the elements of each sequence:

$$\tau = T_{i_{\nu_1}+1} + \sum_{i=\nu_1}^{\nu_s} T_{j_i}.$$

Let $\mathcal{L}$ be the set whose elements are all these sums, T_{j_1} if it does not appear in any of these sums, and all variables not appearing in $\mathcal{N}$.

Example 1.8. In the ring $K[T_1, \dots, T_{11}]$ consider the barred matrix:

$$\mathcal{N} = \left(\begin{array}{ccc|cc||c|c} T_1 & T_2 & T_3 & T_5 & T_6 & T_7 & T_9 \\ T_2 & T_3 & T_4 & T_6 & T_7 & T_8 & T_{10} \end{array} \right).$$

Then

$$\mathcal{N}' = \left(\begin{array}{c|c|c|c} T_1 & T_5 & T_7 & T_9 \\ T_4 & T_7 & T_8 & T_{10} \end{array} \right).$$

In this case the elements of $\mathcal{L}$ are:

$$T_1 + T_7 + T_{10},\ T_5 + T_8,\ T_4,\ T_9, T_{11}$$

Remarks and Notations

(1) For every variable T_a appearing in $\mathcal{N}'$ we shall denote:
- by $\tau(T_a)$ the (unique) element of $\mathcal{L}$ containing T_a;
- by $\tau(T_a)^-$ the sum of all terms of $\tau(T_a)$ that precede T_a and appear in the same big block of $\mathcal{N}$;
- by $\tau(T_a)^{--}$ the sum of the remaining terms of $\tau(T_a)$ preceding T_a.

In a similar way we define $\tau(T_a)^+$ and $\tau(T_a)^{++}$.

(2) If a variable T_a only appears above (below) in $\mathcal{N}$, then it also appears in $\mathcal{N}'$, and it is the first (last) term of $\tau(T_a)$. This follows from condition (ii). Each of the remaining terms of $\tau(T_a)$ appears one time below and one time above. The variables not appearing in $\mathcal{N}'$ are those appearing one time below and one time above in the same small block of $\mathcal{N}$.

(3) Each of $\tau(T_a)^-$ and $\tau(T_a)^+$ contains one summand at most. More precisely we have that
- $\tau(T_a)^- \neq 0$ if and only if $T_a = T_{j_\nu}$ for some ν, and the ν-th small block is not the first left in a big block;
- $\tau(T_a)^+ \neq 0$ if and only if $T_a = T_{i_\nu+1}$ for some ν, and the ν-th small block is not the last right in a big block.

We are now ready to prove

Proposition 1.9. *Let A be the sub-K-algebra of $K[\underline{T}]/J$ generated by the images mod J of the elements of $\mathcal{L}$. Then the image mod J of the set of variables $\underline{T}$ generates $K[\underline{T}]/J$ as an A-module.*

PROOF. For the sake of simplicity we shall keep the notation T_a for the image of T_a mod J. This will not cause any confusion, since in this proof all equalities will be written in $K[\underline{T}]/J$. It suffices to show that, for all entries T_a and T_b of $\mathcal{N}$, the product T_aT_b is an element of $\sum_i AT_i$. We shall throughout suppose that there exist two indices μ, ν such that T_a appears in the μ-th column, T_b appears in the ν-th column of $\mathcal{N}$, and $\mu \leq \nu$.
We distinguish between several cases. For the sake of clearness, we structure our proof according to the *top-down* numbering.
1. T_a and T_b belong to different big blocks.
In this case we shall say that T_b *is right to* T_a or that T_a *is left to* T_b.
1.1. T_a only appears below in $\mathcal{N}$.
Then T_a also appears in $\mathcal{N}'$ and $\tau(T_a)^+ = \tau(T_a)^{++} = 0$.
1.1.1. T_b appears below in $\mathcal{N}$.
Since every term of $\tau(T_a)^{--}$ and $\tau(T_a)^-$ appears above in $\mathcal{N}'$, and is left to T_b, we have that $\tau(T_a)^{--}T_b = \tau(T_a)^-T_b = 0$. Hence

$$T_aT_b = \tau(T_a)T_b.$$

1.1.2. T_b only appears above in $\mathcal{N}$.
Then T_b certainly appears in $\mathcal{N}'$, and $\tau(T_b)^{--} = \tau(T_b)^- = 0$. Thus

$$T_aT_b = \tau(T_b)T_a - \tau(T_b)^{++}T_a - \tau(T_b)^+T_a.$$

Since every term in $\tau(T_b)^{++}$ and $\tau(T_b)^+$ appears below, and is right to T_a, we can apply the result in 1.1.1. to the last two summands.
1.2. T_a appears above in $\mathcal{N}$.
1.2.1. T_b appears below in $\mathcal{N}$.
Then $T_aT_b = 0$.
1.2.2. T_b only appears above in $\mathcal{N}$.
Then T_b certainly appears in $\mathcal{N}'$, and $\tau(T_b)^{--} = \tau(T_b)^- = 0$. Since all terms of $\tau(T_b)^+$ and $\tau(T_b)^{++}$ appear below, and are right to T_a, one has that $T_a\tau(T_b)^+ = T_a\tau(T_b)^{++} = 0$, hence

$$T_aT_b = T_a\tau(T_b).$$

2. T_a and T_b belong to the same big block $\mathcal{B}$, but to different small blocks.
The subcases 2.1, 2.2 and 2.3 correspond to three different steps of an algorithm. At each step we replace T_a and T_b by new entries $\bar{T}_a$ and $\bar{T}_b$ respectively, which also verify the assumption 2. Let s be the number of small blocks in $\mathcal{B}$. Suppose T_a and T_b appear in the α-th and in the β-th small block of $\mathcal{B}$ respectively ($\alpha < \beta$). Let $\bar{\alpha}$ and $\bar{\beta}$ be the indices of the small blocks of $\mathcal{B}$ occupied by $\bar{T}_a$ and $\bar{T}_b$ respectively.
2.1 T_a appears above and T_b appears below in $\mathcal{B}$.
Consider the following submatrix of $\mathcal{B}$:

$$\left(\begin{array}{cccccc|c|cccccc} \dots & T_a & T_{a'} & T_{a''} & \dots & \dots & \dots & T_d & \dots & \dots & T_{b''} & T_{b'} & \dots \\ \dots & T_{a'} & T_{a''} & \dots & \dots & T_c & \dots & \dots & \dots & T_{b''} & T_{b'} & T_b & \dots \end{array}\right).$$

One has that

$$0 = \begin{vmatrix} T_a & T_{b'} \\ T_{a'} & T_b \end{vmatrix} = T_aT_b - T_{a'}T_{b'},$$

and

$$0 = \begin{vmatrix} T_{a'} & T_{b''} \\ T_{a''} & T_{b'} \end{vmatrix} = T_{a'}T_{b'} - T_{a''}T_{b''}.$$

Finally

$$T_aT_b = T_{a''}T_{b''}.$$

Hence the problem is reduced to finding the required representation for $T_{a''}T_{b''}$. Take $\bar{T}_a = T_{a''}$, and $\bar{T}_b = T_{b''}$. If $T_{a''} = T_c$ apply 2.2, if $T_{b''} = T_d$, apply 2.3. Otherwise apply 2.1. once again.
2.2. T_a only appears below in $\mathcal{B}$.
In this case T_a appears in $\mathcal{N}'$ and $\tau(T_a)^+ = 0$. Hence

$$T_aT_b = \tau(T_a)T_b - \tau(T_a)^{--}T_b - \tau(T_a)^{++}T_b - \tau(T_a)^-T_b.$$

Note that all terms of $\tau(T_a)^{--}$ and $\tau(T_a)^{++}$ appear in a big block different from $\mathcal{B}$. Hence the required representation for the second and the third summand can be found according to 1. Thus it suffices to consider the last term. If $\alpha = 1$, then $\tau(T_a)^- = 0$ and we are done. Otherwise take $\bar{T}_a = \tau(T_a)^-$ and $\bar{T}_b = T_b$. Then $\bar{\alpha} = \alpha - 1$, and $\bar{\beta} = \beta$. Apply 2.1 or 2.3. to $\bar{T}_a\bar{T}_b$. This is possible, since $\tau(T_a)^-$ certainly appears above.

2.3 T_b only appears above in $\mathcal{B}$.

An argument similar to that developped in 2.2 permits us to reduce the problem to the product $T_a\tau(T_b)^+$. If $\beta = s$, then $\tau(T_b)^+ = 0$ and we are done. Otherwise take $\bar{T}_a = T_a$ and $\bar{T}_b = \tau(T_b)^+$. Then $\bar{\alpha} = \alpha$, and $\bar{\beta} = \beta + 1$. Apply 2.1 or 2.2 to $\bar{T}_a\bar{T}_b$.

It is clear that the above algorithm stops after a finite number of steps, when either T_a is the leftmost entry of the upper row of $\mathcal{B}$, or T_b is the rightmost entry of the lower row of $\mathcal{B}$.

3. T_a and T_b appear in the same small block B of $\mathcal{N}$.

3.1 T_a or T_b appears in $\mathcal{N}'$.

Suppose that T_a appears in $\mathcal{N}'$. Then

$$T_aT_b = \tau(T_a)T_b - \tau(T_a)^{--}T_b - \tau(T_a)^{++}T_b - \tau(T_a)^{-}T_b - \tau(T_a)^{+}T_b.$$

The four last terms are sums of products $T_{a'}T_b$, where $T_{a'}$ appears in a different small block with respect to T_b. Hence 1 or 2 can be applied to each of these products.

3.2 T_a and T_b do not appear in $\mathcal{N}'$.

First suppose $T_a = T_b$. Note that T_a is not the leftmost element of the first row of B. Hence the block B has the following minor:

$$0 = \begin{vmatrix} T_{a'} & T_a \\ T_a & T_{a''} \end{vmatrix} = T_{a'}T_{a''} - T_a^2.$$

Thus we may assume $T_a \neq T_b$. Then the block B is of one of the following forms:

(a) $$B = \begin{pmatrix} \dots & T_{a'} & T_a & T_b & \dots \\ \dots & T_a & T_b & T_{b'} & \dots \end{pmatrix}$$

or

(b) $$B = \begin{pmatrix} T_c & \dots & T_{a''} & T_a & \dots & T_b & T_{b''} & \dots & \\ \dots & T_{a''} & T_a & \dots & T_b & T_{b''} & \dots & T_d \end{pmatrix}.$$

In case (a) it holds $T_aT_b = T_{a'}T_{b'}$. If $T_{a'}$ is in the first column, then $T_{a'}$ appears in $\mathcal{N}'$, and we are back in 3.1. The same is true if $T_{b'}$ is in the last column. Otherwise we are in case (b). In this case $T_aT_b = T_{a''}T_{b''}$. Reapply this identity successively to move the first factor to the left, the second to the right. After a finite number of steps the we end up with a product where the first factor is T_c, or the second id T_d. Since T_c and T_d both appear in $\mathcal{N}'$, we are back in 3.1. □

Corollary 1.10. *If d is the cardinality of $\mathcal{L}$, then $d = n-c+1$ and $\dim K[\underline{T}]/J = d$. In particular, A is a Noether normalization of $K[\underline{T}]/J$.*

PROOF. From 1.3 it follows that

$$\dim R/J = n - c + 1,$$

where n is the number of variables. Thus it suffices to show that $d = n-c+1$. Now let V be the set of all variables appearing below in $\mathcal{N}$: its cardinality is obviously equal to the number c of columns. Let $\bar{V}$ denote its complementary set. Let F be the set of the first terms of the sums in $\mathcal{L}$. By construction it is clear that $F = \bar{V} \cup T_{j_1}$. But the cardinality of F is d. Hence $d = n-c+1$. ☐

2 Projective monomial varieties of codimension 2

Proposition 1.9 can be applied to the study of the special fibre of a projective monomial variety of codimension 2: see [5] for a complete and detailed presentation of this subject. We first quote the basic notions and the main results from [6].

Let R be the polynomial ring in $n+2$ indeterminates over the field K. Let I be an ideal minimally generated by τ elements $\{F_i\}_{1\leq i\leq\tau}$. The *Rees algebra* of I is defined to be the graded ring $R[It] = \oplus_{n\geq 0} I^n t^n$, and the *special fibre* is the quotient $R[It]/\mathcal{M}R[It]$, where $\mathcal{M}$ denotes the irrelevant maximal ideal of R. The dimension of the special fibre is called the *analytic spread* of I and is denoted by $\ell(I)$. We introduce τ independent variables over R, say $\underline{T} = \{T_i\}_{1\leq i\leq\tau}$, and we consider the ideal $\mathcal{J} = \ker\{R[\underline{T}] \to R[It] \to 0, T_i \to F_i t\}$. We obtain a presentation $R[It] \simeq R[\underline{T}]/\mathcal{J}$ of the Rees algebra, from which we can deduce a presentation of the special fibre: $R[It]/\mathcal{M}R[It] \simeq K[\underline{T}]/\widetilde{\mathcal{J}}$, where $\widetilde{\mathcal{J}}$ denotes the image of $\mathcal{J}$ modulo $\mathcal{M}R[\underline{T}]$.

The ideal J is called a *reduction* of I if $JI^n = I^{n+1}$ for some nonnegative integer n. Let $r_J(I)$ denote the least number n such that the above equality holds. Then the *(absolute) reduction number* of I is defined to be the minimum of $r_J(I)$ taken over all possible reductions J of I.

Let us place ourselves in the case where I is the defining ideal of a toric variety admitting the following parametrization:

$$x_1 = u_1^{a_1},\ x_2 = u_2^{a_2},\ \ldots,\ x_n = u_n^{a_n},\ y = u_1^{c_1}u_2^{c_2}\cdots u_n^{c_n},\ z = u_1^{b_1}u_2^{b_2}\cdots u_n^{b_n},$$

where, for all i, $1 \leq i \leq n$, a_i, b_i and c_i are nonnegative integers such that $a_i \neq 0$, $(b_i, c_i) \neq (0,0)$, and $(b_1,\ldots,b_n) \neq (0,\ldots,0)$, $(c_1,\ldots,c_n) \neq (0,\ldots,0)$. We call this a *projective monomial variety of codimension 2.*

It is known that in general $\operatorname{codim}(I) \leq \ell(I)$, and equality holds for I a prime ideal if and only if I is a complete intersection. In [6] the number $\ell(I)$

was determined for all defining ideals I of a codimension 2 monomial variety, in spite of the fact that no complete description of the ideal $\widetilde{\mathcal{J}}$ was known yet. The approach, indeed, is indirect: the computations are done on a subideal $\widetilde{\mathcal{A}} \subseteq \widetilde{\mathcal{J}}$, generated in degree two, called *reduced essential ideal.* It turns out that $\dim R[\underline{T}]/\widetilde{\mathcal{A}} = 3$ (cf. [5], Cor. 5.26.1). This immediately implies the following result:

Theorem 2.1. ([6], Th. 4.2). *The analytic spread of I is equal to two if I is an complete intersection, equal to three in all the remaining cases.*

2.1.1. The ideal $\widetilde{\mathcal{A}}$ is always generated by monomials and binomials, with the exception of a very particular case, that is treated separatedly ([5], Prop. 5.18.1). In this case $\tau = 4$ and

$$\widetilde{\mathcal{A}} = (T_0T_1 - T_2^2 - T_3^2) \subseteq K[T_0, T_1, T_2, T_3].$$

One easily sees that

$$K[T_0, T_1, T_2] \subseteq K[T_0, T_1, T_2, T_3]/\widetilde{\mathcal{A}}$$

is a Noether normalization and that the reduction number is 1. In the general case $\widetilde{\mathcal{A}}$ is the ideal associated to a barred matrix $\mathcal{N}$ whose big and small blocks coincide: see [5], [6] or [11] for an explicit construction. We shall use our results to prove that in the general case $\widetilde{\mathcal{A}} = \widetilde{\mathcal{J}}$, i.e. the special fibre of I is entirely generated by the quadratic relations. We need the following preliminary

Lemma 2.2. *Let R be a finitely generated K-algebra, J an ideal of R and $\varphi : R \to R/J$ the canonical epimorphism. Moreover let $A \subseteq R$ be a Noether normalization of R. If $\dim R = \dim R/J$, then A is a Noether normalization of R/J with respect to the restriction $\varphi|_A$.*

PROOF. We consider the A-module structure defined on R/J by $\varphi|_A$. Since φ is a finite A-homomorphism, the same is true for $\varphi|_A$. Thus it suffices to prove that $\varphi|_A$ is an injection. Now $\mathcal{K} := \ker(\varphi|_A) = \ker(\varphi) \cap A$, and the map $\varphi|_A$ induces a finite monomorphism of rings

$$A/\mathcal{K} \longrightarrow R/J.$$

Thus $\dim(R/J) = \dim(A/\mathcal{K})$. But the left-hand side is equal to $\dim R = \dim A$. Thus the preceding equality is only possible if $\mathcal{K} = 0$. □

Let us fix a monomial variety of codimension 2, with defining ideal I, which is not a complete intersection and does not belong to the particular case. We shall refer to the notation just introduced. Proposition 1.10 permits us to determine a Noether normalization A of the quotient $R[\underline{T}]/\widetilde{\mathcal{A}}$. Consider the composition of maps

$$A \hookrightarrow R[\underline{T}]/\widetilde{\mathcal{A}} \longrightarrow R[\underline{T}]/\widetilde{\mathcal{J}},$$

where the right map is the canonical epimorphism. The equality of dimensions and 2.2 imply that A is a Noether normalization of $R[\underline{T}]/\widetilde{\mathcal{J}}$ with respect to the composed map. Since $R[\underline{T}]/\widetilde{\mathcal{A}}$ is generated by linear forms as an A-module, the same is true for its quotient $R[\underline{T}]/\widetilde{\mathcal{J}}$.

For the rest of this section we consider the local ring $R_{\mathcal{M}}$. The notions of Rees algebra, special fibre and reduction naturally extend to this local ring. The above results apply to $R_{\mathcal{M}}$ and to the localized ideals and subrings. The following result is due to Vasconcelos:

Proposition 2.3. ([14], Prop. 5.1.3) *Let $(R, \mathcal{M}, K)$ be a local Noetherian ring with infinite residue field, I an ideal of R of analytic spread ℓ. Suppose*

$$A = K[z_1, \ldots, z_\ell] \hookrightarrow F(I)$$

is a Noether normalization of the special fibre $F(I)$, and assume that all the z_i are of degree 1. Furthermore let $\{b_1, \ldots, b_s\}$ be a minimal set of homogeneous generators of $F(I)$ as an A-module. For all $i = 1, \ldots, \ell$, let $a_i \in I$ be a lift of z_i. Then $J = (a_1, \ldots, a_\ell)$ is a (minimal) reduction of I and

$$r_J(I) = \max\{\deg b_i \mid i = 1, \ldots, s\}.$$

Putting the above results together one immediately obtains the following

Corollary 2.4. *With respect to the notation introduced in 2.1.1 it holds:*

$$r(I_{\mathcal{M}}) = 1.$$

Next consider the following result by D'Cruz, Raghavan and Verma:

Proposition 2.5. ([3], Cor. 2.2) *Let $(R, \mathcal{M})$ be a local Noetherian ring, let I be an ideal of R. If $r(I) = 1$, then $F(I)$ is Cohen-Macaulay with minimal multiplicity. Moreover the Hilbert function of $F(I)$ is*

$$H_{F(I)}(t) = \frac{1 + (\tau - \ell)t}{(1 - t)^\ell},$$

where τ denotes the minimal number of generators of I and ℓ the analytic spread of I.

The Cohen-Macaulayness of $F(I)$ and the formula for the Hilbert function in our case also follow from the results by Cortadellas-Zarzuela [1], Th. 4.2. and Cor. 5.7. Now we are able to conclude:

Proposition 2.6. *Let $I \subseteq K[x_1, \ldots, x_n, y, z]$ be the defining ideal of a projective monomial variety of codimension 2. Suppose I is not a complete intersection. Then the reduced essential ideal $\widetilde{\mathcal{A}}$ coincides with the presentation ideal $\widetilde{\mathcal{J}}$ of the special fibre $F(I)$.*

PROOF. In [5], p. 117 it is proven that the Hilbert function of $K[\underline{T}]/\tilde{\mathcal{A}}$ is

$$H_{\tilde{\mathcal{A}}}(t) = \frac{1+(\tau-3)t}{(1-t)^3}.$$

Since $\ell(I) = 3$, it follows from 2.5 that $F(I)$ and $K[\underline{T}]/\tilde{\mathcal{A}}$ have the same Hilbert function. But $F(I)$ is a quotient of $K[\underline{T}]/\tilde{\mathcal{A}}$. □

Note that 1.9 and 2.3 yield an explicit construction for a minimal reduction of any ideal $I_{\mathcal{M}}$. We perform such a construction in the next examples, where I is throughout an ideal of $K[x, y, z, w]$ defining a projective monomial curve in $\mathbf{P}^3$.

Examples 2.7. (a) Suppose I is minimally generated by 4 elements. Coudurier-Morales [2] call this a monomial curve of type I. A system of generators is given by

$$\begin{aligned} F_1 &= x^k w^l - y^n z^m \\ F_2 &= y^{\pi+n} w^{r-l} - x^{k-r'} z^{\sigma+m} \\ F_3 &= y^{\pi+2n} - x^{2k-r'} z^{\sigma} w^{2l-r} \\ F_4 &= x^{r'} y^{\pi} w^{r} - z^{\sigma+2m} \end{aligned}$$

where all the exponents are supposed to be nonnegative. The ideal $\widetilde{\mathcal{J}}$ is the ideal associated to the barred matrix

$$\mathcal{N} = \left(\begin{array}{c|c} T_3 & T_2 \\ T_2 & T_4 \end{array} \right).$$

Thus a Noether normalization of $K[\underline{T}]_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}$ is

$$A_{\mathcal{M}} = K[t_1, t_2, t_3 + t_4],$$

where t_i denotes the image of the variable T_i via the localization at $\mathcal{M}$. Let

$$J = (F_1, F_2, F_3 + F_4).$$

According to 2.3 the ideal $J_{\mathcal{M}}$ is a minimal reduction of $I_{\mathcal{M}}$. Moreover note that

$$\begin{aligned} F_1^2 &= F_1(F_1 + F_3) - F_1F_3, \\ F_3^2 &= F_3(F_1 + F_3) - F_1F_3, \\ \text{and} \qquad F_1F_3 &= x^{r'} w^{2l-r} F_2^2 - y^{\pi} z^{\sigma} F_0^2. \end{aligned}$$

Hence J is a minimal reduction of I not only locally at $\mathcal{M}$, but also globally. In particular we have that $r(I) = 1$. The next example shows that this direct passage from a local to a global reduction is not always possible: in general the generators of the subalgebra A do not yield a global reduction.

(b) Consider the ideal I generated by the following 6 binomials:

$$\begin{aligned} F_1 &= y^6 z^{38} - x^{13} w^{31} \\ F_2 &= y^{18} z^{25} - x^{15} w^{28} \\ F_3 &= y^{30} z^{12} - x^{17} w^{25} \\ F_4 &= y^{42} - z x^{19} w^{22} \\ F_5 &= z^{51} - y^6 x^{11} w^{34} \\ F_6 &= y^{12} w^3 - x^2 z^{13}. \end{aligned}$$

The barred matrix associated to $\mathcal{J}$ is

$$\mathcal{N} = \left(\begin{array}{c||cc||c} T_4 & T_3 & T_2 & T_1 \\ T_6 & T_2 & T_1 & T_5 \end{array} \right).$$

Then

$$\mathcal{N}' = \left(\begin{array}{c|c|c} T_4 & T_3 & T_1 \\ T_6 & T_1 & T_5 \end{array} \right),$$

whence one obtains $A_{\mathcal{M}} = K[t_1 + t_4, t_3 + t_5, t_6]$ as a Noether normalization of $K[\underline{T}]_{\mathcal{M}}/\widetilde{\mathcal{J}}_{\mathcal{M}}$. Let $J = (F_1 + F_4, F_3 + F_5, F_6)$. Then $J_{\mathcal{M}}$ is a reduction of $I_{\mathcal{M}}$, but J is not a reduction of I. One can show that it even holds $\operatorname{Rad}(J) \neq I$. A reduction of I is given by

$$J' = ((1 + y^6)F_1 + F_4, F_3 + F_5, F_6).$$

We are not able to give the general form of three elements generating a minimal reduction of the ideal I defining a projective monomial curve. The problem was solved by Morales-Simis (cf. [12], Prop. 3.1.2) for projective monomial curves lying on a quadric surface.

It was conjectured in [5] that the ideal of presentation of the Rees algebra $R[It]$ is generated by forms of degree two at most. Now we are able to answer the question. The crucial result is due to Huckaba-Huneke.

Theorem 2.8. ([10], Th. 2.9 and Th. 4.5) Let R be a Cohen-Macaulay local ring and I an ideal having height $d \geq 1$ and analytic spread $\ell(I) = d + 1$. Assume that the minimal primes of R/I all have the same height, and the associated primes of R/I have height at most $d + 1$. Assume also that I is generically a complete intersection and there exists a minimal reduction J of I such that $r_J(I_Q) \leq 1$ for every prime ideal $Q \subseteq I$ with $\operatorname{codim}(Q/I) = 1$. Finally assume that $\operatorname{depth}(R/I) \geq \dim(R/I) - 1$. Then the presentation ideal of the Rees algebra $R[It]$ of I is defined by elements of degree two at most.

Moreover Peeva-Sturmfels (cf. [13], Th. 2.3), showed that

$$\text{projdim}\,(R/I) \leq 2^{\text{codim}(I)} - 1.$$

In particular, if I is the defining ideal f a projective monomial variety of codimension 2, by Auslander-Buchsbaum it holds

$$\text{depth}(R/I) = \dim(R) - \text{projdim}\,(R/I) \geq \dim(R) - 3 = \dim(R/I) - 1.$$

In view of this inequality and 2.4 the ideal I fulfils all the assumptions of 2.8. This proves:

Theorem 2.9. *Let $I \subseteq R$ be the defining ideal of a projective monomial variety of codimension 2. Then the presentation ideal of the Rees algebra $R[It]$ of I is generated by forms of degree two at most.*

Note that Gimenez (cf. [5], Th. 6.3.1) already showed 2.6 under the hypothesis that 2.9 be true.

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