Midconvex functions in locally compact groups

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Abstract

The theorems of Bernstein-Doetsch, and Ostrowski, concerning the continuity of midconvex functions are extended to open subsets of locally compact and root-apprximable topological groups.[1]

1 Introduction

In his original article [11], Jensen proved that a midconvex function f on an open interval $\Omega = (\alpha, \beta)$ of the real line $\mathbf{R}$ is continuous at a point $a \in \Omega$, provided it satisfies $\limsup_{x \to a} f(x) < +\infty$. As corollaries, f is continuous provided it is assumed to have finite lim sup at each point, and specially if it is assumed to be upper semi-continuous or to be bounded from above. Many authors have been then interested into the problem of continuity of functions satisfying the Jensen's functional inequation

$$2f(x) \leq f(x+h) + f(x-h) \tag{1}$$

where $f : \Omega \to \mathbf{R}$ is a real function defined on a convex subset Ω of some real vector space X on which a suitable topology is given, and the variables x and h are restricted only to the conditions $x, x+h, x-h \in \Omega$. These functions are called convex in [11], convex, convex(J), J-convex or midconvex in further literature. Well-known classical theorems guarantee the continuity of such an f, provided it is assuumed to be bounded from above on some rich subset of Ω, the word *rich* being essentially interpreted by topological or measure theoretical arguments. Very often Ω is assumed in plus to be open. We refer to the books [14, 18] and the references therein for historical notes and to the articles [13, 14] and their references for more recent developments. A comprehesive treatment of the subject with proofs may be found when $X = \mathbf{R}^n$ in Kuczma [14]. Among the recent generalizations, let us restrict ourselves to notice that in Kominek

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and Kuczma [13], the generalization consists essentially in the fact that the topology on the space X is semilinear, and in the article of Ng and Nikodem [16] the Bernstein-Doetsch theorem is extended to approximately midconvex functions in a semilinear topological space. There is extensions of these continuity theorems to other classes of functions, such as n-convex functions, see Ger [7, 8].

Concerning midconvex functions in topological groups, the theorems of Jensen and Blumberg-Sirpinski have been already proved in [6]. As this reference does not seem widely accessible, we shall recall, with some additional remarks, the definitions and the principal statements from [6] in the Preliminary section.

The object of the present paper is to extend the continuity theorems of Bernstein-Doetsch, and Ostrowsky by relaxing the convexity of Ω and by localizing h in (1) by assuming that $x+h$ and $x-h$ remain in a small neighborhood of x only. In such a setting, the underlying space X need no more have a vector space structure, but the structure of a topological group or semitopological group, or even semitopological semigroup will suffice. However, we shall restrict ourselves here to the case of *non discrete topological groups*, restriction that will be respected in whole the paper without express mentions.

The Jensen's theorem about upper semicontinuity remains valid (see the remark following the definition (5), while its theorem about boundedness from above is no more valid for locally midconvex functions as is shown by the elementary example consisting of the characteristic function of an open interval in $\mathbf{X} = \mathbf{R}$. This example may be interesting, because all discontinuous midconvex functions in the usual sens are non measurable. Other conditions are needed to imply the continuity of such a *locally midconvex* function. We are led to introduce as is done in [6], with slight modifications, the notions of locally and *globally midconvex* functions, as well as some intermediate ones, *sequentially midconvex* functions and *locally uniformly midconvex* functions. The usual midconvex functions, as used in [17, chapter 7], corresponds to our globally midconvex ones. Some of the continuity theorems may be proved in any topological group, while for others the existence of successive square roots with an approximation property is needed. These groups are called *root-approximabe topological groups.*

To be more precise, let G be a (non-discrete) topological group with multiplicative low and let Φ denote the filter of neighborhoods of its neutral element e. Consider an open set $\Omega \subset G$, and a function $f : \Omega \longrightarrow [-\infty, +\infty]$. To avoid ambiguities, it is not allowed that f takes respectively $-\infty, +\infty$ at any two symmetric points $ay, ay^{-1} \in \Omega$ with respect to a point $a \in \Omega$. Whenever G is locally compact, we suppose that it is endowed with a left invariant Haar measure $\mu > 0$.

The organization of the article is as follows: We give first in a Preliminary section the definitions, then we recall briefly finiteness and measurability results following [6]. The root-approximability and the Doetsch-Bernstein theorem will follow, and finally midconvex hull and Ostrowski's theorem will be treated in the last section. Our main results here are the following theorems:

Theorem 1. *A globally midconvex function in a midconvex open subset of a root-approximable topological group which is bounded from above in some neighborhood of a point, is everywhere continuous.*

This is the theorem of Bernstein-Doetsch [1] generalized.

Theorem 2. *Let Ω be a midconvex open subset in a root-approximable locally compact group G endowed with a left-invariant Haar measure μ, and let $f : \Omega \longrightarrow \mathbf{R}$ be a globally midconvex function. If f is bounded from above on a set $E \subset \Omega$ with $\mu(E) > 0$, then f must be continuous on Ω.*

This is the theorem of Ostrowski [17] generalized.

2 Preliminaries

With the notations and conventions precised in the introduction, we shall give some definitions, and recall some known facts about finiteness and measurability of midconvex functions.

2.1 Definitions

1) The function f is called *globally midconvex* in Ω if for every a, y such that $a, ay, ay^{-1} \in \Omega$, the *midconvex inequality*

$$2f(a) \leq f(ay) + f(ay^{-1}) \tag{2}$$

holds.

2) Similarly, f is said to be *locally midconvex* in Ω if for every $a \in \Omega$ there exists an open symmetric $V = V^{-1} \in \Phi$ such that the midconvex inequality (2) holds for every $y \in V$.

3) As an intermediate notion, f will be called *sequentially midconvex at* a, if it is locally midconvex in Ω and there exists a symmetric open $V \in \Phi$ such that $aV^2 \subset \Omega$, and for each $y \in V$, the following inequality holds:

$$2f(ay) \leq f(a) + f(ay^2). \tag{3}$$

A function which is sequentially midconvex at each point of Ω, is called sequentially midconvex in Ω.

4) We say that f is *locally uniformly midconvex* at a point a, if there exists a neighborhood $aV, V \in \Phi$, such that $aV^2 \subset \Omega$, and for each $y \in V$, the following condition is satisfied:

$$\forall z \in V, 2f(ay) \leq f(ayz^{-1}) + f(ayz). \tag{4}$$

5) Finally, as in the case of topological vector spaces, f is called *convex* if it is locally midconvex, upper semicontinuous, and $f(x) < +\infty$ for all $x \in \Omega$.

Remark. Taking $\liminf$ in (2) after bringing one of its terms from the right side to the left side, it is easily shown that a convex function is continuous ([5]), but the characteristic function χ_A, where A is any open subset with non-empty boundary is a locally midconvex function on $\Omega = G$ which is lower semicontinuous, discontinuous, and bounded. Taking $G = \mathbf{R}$ and $A = (0, 1)$, one has a very elementary discontinuous example.

6) A group G endowed with a topology $\mathcal{T}$ is said to be a *root-approximable group* if each $x \in G$ has a sequence of successive square roots converging to e, in other words to each $x \in G$ corresponds a sequence $(y_n)_{n \geq 0}$ of 2^n-th roots of x such that

$$lim_{n \to \infty} y_n = e, \qquad {y_n}^{2^n} = x, (n = 0, 1, 2, ...). \tag{5}$$

A *root-approximable topological group* is a topological group which is also root-approximable.

7) Let G be a root-approximable topological group. For any subset $E \subset G$, we define by induction a sequence of subsets $H_j(E)$ in the following way: $H_0(E) = E$,

and $H_j(E)$ is the *midpoints set* of $H_{j-1}(E)$, where by the midpoints set of a set F, we mean the set of all elements of G of the form xa, provided that $x \in F, xa^2 \in F$. The *midconvex hull* of E is then by definition the union of all $H_j(E)$, for $0 \leq j < \infty$.

8) A subset E of the topological group G is called *right midconvex* if for every $x, y \in E$, there exists a $z \in G$ such that $xz \in E$ and $xz^2 = y$.

Let us recall now some results from [6], concerning the finiteness and measurability of locally midconvex functions.

2.2 Finiteness

Being concerned with the continuity problem, under some additional hypothesis, we can restrict ourselves in the case of a locally compact group to finite valued locally midconvex functions. Let us observe first that we can have quite well pathological locally midconvex functions on $\Omega = G = \mathbf{R}$ such as

$$f_1 = \chi_A \times (+\infty), \quad f_2 = (1 - \chi_A) \times (+\infty), \tag{6}$$

where $A = (0, 1)$ and χ_A denotes the characteristic function of A (Notice that $f_1 + f_2$ is not locally midconvex). More precisely, we have the following known results ([10, chapter 4, theorem 9.1b], [6, 18]):

Proposition 1. *If G is locally compact, any locally midconvex function $f : \Omega \longrightarrow [-\infty, +\infty]$, satisfying $f(x) < +\infty$ for μ-almost all $x \in \Omega$ will satisfy $f(x) < +\infty$, for all $x \in \Omega$.*

Remark. If $f : \Omega \longrightarrow [-\infty, +\infty]$ is locally midconvex, then $f^{-1}(-\infty)$ is closed. It is not open in general. However, it follows from the definitions that in the case of sequentially midconvex functions, this set is open too, so it is one of the connected components of Ω. The following precise and obvious ressult is much stronger than the continuity of sequentially midconvex functions on their set of $-\infty$ ([6] Proposition 2).

Proposition 2. *If $f : \Omega \longrightarrow [-\infty, +\infty]$ is sequentially midconvex at a point $a \in \Omega$, and if $f(a) = -\infty$, then f is the constant $-\infty$ near a.*

Remark. The sequential midconvexity can not be replaced by local midconvexity in the above proposition. The function f_2 is a counter-example.

Concerning the continuity of sequentially midconvex functions, we may then restrict ourselelves here to *real valued* functions in open subsets of G. The precise theorem of Jensen may be generalized as follows:

Proposition 3.(Jensen's theorem) *Suppose that $a \in \Omega$, $f : \Omega \longrightarrow \mathbf{R}$ is sequentially midconvex at a. If*

$$\limsup_{x \to e} f(ax) < +\infty,$$

then f is continuous at a.

Corollary. *A sequentially midconvex function $f : \Omega \longrightarrow \mathbf{R}$ is continuous if and only if it is locally bounded from above.*

2.3 Measurability

The continuity of measurable usual midconvex functions in the classical setting of finite dimensional euclidean spaces, proved by Blumberg [2] and Sierpinski [19] is one of the most striking old results. It tells us that midconvex functions are very regular or very irregular. As observed above, the situation is not the same for local point of view. We have the following important theorem of Blumberg-Sierpinski:

Proposition 4. *Suppose that $a \in \Omega$, and $f : \Omega \longrightarrow \mathbf{R}$ is locally uniformly midconvex at a. If f is measurable, then f is continuous at a.*

Sketch of the proof.(following [6]): Let $W = W^{-1} \in \Phi$ be open with compact closure $\overline{W}$. The modular function Δ, has a positive minimum $m > 0$ on $\overline{W}$. It satisfies

$$\mu(Az) = \Delta(z)\ \mu(A)$$

for every $z \in G$, and μ-measurable set $A \subset G$, (see [20], p.39). One can choose (see Bourbaki [2]), $U, V \in \Phi$, such that $U^2 \subset V$, $V^2 \subset W$. The inequation (3) may be used. If f is not bounded from above near a, then for each integer $n \geq 1$, there exists a $y_n \in U$ such that $f(ay_n) > n$. Define $\varphi_n : U \longrightarrow V$ by putting $\varphi_n(x) = x^{-1}y_n$. It is allowed to assume V as small as to have the inequation (4). Given an arbitrary $x \in U$, put $z_n = \varphi_n(x)$, and substitute in the above inequality y_n, z_n respectively instead of y, z, to obtain:

$$2f(ay_n) \leq f(ax) + f(ay_n\varphi_n(x)).$$

Define for each $n \geq 1$,

$$A_n = \{x \in W | f(ax) > n\}, \quad B_n = \{x \in W | f(ax) \leq n\}.$$

Some calculations lead to

$$\mu(A_n) \geq \frac{\alpha m^2}{1+m^2},$$

a positive constant which is independent from n (where $\alpha = \mu(U)$). The sequence $(A_n)_{n\geq 1}$ satisfies

$$\mu(A_1) \leq \beta = \mu(W),$$

$$\forall n \geq 1, \qquad A_{n+1} \subset A_n.$$

The classical argument may be applied to obtain $\mu(\bigcap_{n\geq 1} A_n) \geq m\alpha^2/(1+m^2) > 0$. Hence, it is not empty, an obvious contradiction with the standard analysis of real numbers.

Corollary 1. *A real valued locally uniformly midconvex function in an open subset of any locally compact group is convex if and only if it is measurable with respect to the left-invariant Haar measure.*

Corollary 2.(Sierpinski's Theorem for Topological Groups) *If a real valued globally midconvex function in an open subset of a locally compact group is measurable with respect to its Haar measure, then it is continuous.*

3 Root-approximability and Bernstein-Doetsch Theorem

To generalize the Bernstein-Doetsch theorem, the notion of root-approximable groups is introduced. There are interesting examples and counter-examples among the classical groups. The verification will bring us however far from the scope of this paper and is left to the readers. The following lemma gives a convexity inequality with rational coefficients, proved by induction.

Lemma 1. *Suppose that* $f : \Omega \longrightarrow \mathbf{R}$ *is globally midconvex. Let* $x, a \in G$ *and* $n \in \mathbf{N}$ *be such that* $\{x, xa, xa^2, \cdots, xa^n\} \subset \Omega$. *Then for every integer* $m \leq n$, *we have*

$$f(xa^m) \leq (1 - m/n)f(x) + (m/n)f(xa^n). \tag{7}$$

Theorem 1. *Let G be a root-approximable topological group, and $\Omega \subset G$ midconvex. Suppose that $f : \Omega \longrightarrow \mathbf{R}$ is globally midconvex. If there exists a point $a \in \Omega$ and a neighborhood aV of a, $V \in \Phi$, such that f is bounded from above on aV, then f must be continuous in Ω.*

Proof. Suppose, without loss of generality, that $e \in \Omega$, and f satisfies the inequality $f \leq B$ for some fixed $V \in \Phi, V \subset \Omega$. In view of the proposition 3, it is sufficient to show that for every $y \in \Omega$, f is bounded from above on some neighborhood of y. Let y be an arbitrary element of Ω. By the hypothesis of root-approximability, y has a sequence of successive square roots $y^{1/2^n}$, with $y^{1/2^n} \longrightarrow e$. There exists an integer N such that $yy^{1/2^N} \in \Omega$. We can take a chain $(V_j)_{1 \leq j \leq 2^N+1}$ of symmetric open members of Φ with

$$V_{2^N+1} \subset V_{2^N} \subset \cdots \subset V_2 \subset V_1$$

and the properties:

(i) $V_1^{2^N+1} \subset V$;

(ii) $y^{-1/2^N} V_j \subset V_{j-1} y^{-1/2^N} \cap y^{-1/2^N} V_{j-1}$.

We shall prove that on the neighborhood $V_{2^N+1}y$ of y the function f satisfies an inequality of the form $f \leq C$, for some constant C

$$C = (1 - \frac{2^N}{2^N+1})B + \frac{2^N}{2^N+1} f(yy^{1/2^N}).$$

depending on B, y and N, all of them being fixed. Given $x \in V_{2^N+1}$, define $X = (y^{-1/2^N} x)^{2^N}$. The property (ii) above shows that there exists $x_1 \in V_{2^N}$ such that

$$y^{-1/2^N} x = x_1 y^{-1/2^N}.$$

Thus,

$$X = (x_1 y^{-1/2^N}) \cdots (x_1 y^{-1/2^N})$$

$$= x_1 (y^{-1/2^N} x_1) \cdots (y^{-1/2^N} x_1) y^{-1/2^N}$$

$$= x_1 (y^{-1/2^N} x_1)^{2^N - 1} y^{-1/2^N}.$$

This process will enable us to construct, for $1 \leq j \leq 2^N$, suitable $x_j \in V_{2^N+1-j}$ satisfying

$$X = x_1 \cdots x_j (y^{-1/2^N} x_j)^{2^N - j} y^{-j/2^N}.$$

For $j = 2^N$, we have

$$X = x_1 \cdots x_{2^N} y^{-1}$$

therefore $Xy \in V_1^{2^N}$ and by the property (i)

$$xXy \in V_1^{2^N+1} \subset V.$$

By the Lemma 1, we have

$$\begin{aligned} f(xy) &\leq (1 - \frac{2^N}{2^N+1}) f(xXy) + \frac{2^N}{2^N+1} f(yy^{1/2^N}) \\ &\leq (1 - \frac{2^N}{2^N+1}) B + \frac{2^N}{2^N+1} f(yy^{1/2^N}). \end{aligned}$$

The proof is complete.

4 Midconvex Hull and Ostrowski's Theorem

Lemma 2. *Let Ω be an open midconvex set in a root-approximable group G, $f : \Omega \longrightarrow$ **R** a globally midconvex function. Suppose that $E \subset \Omega$, and $f \leq C$ on E. Then $f \leq C$ on $H(E) \cap \Omega$.*

Proof. We can express an element of $H(M)$ of the form

$$x = x_1 a_{n_1}^{m_1} a_{n_2}^{m_2} \cdots a_{n_{k-1}}^{m_{k-1}}$$

with suitable $x_1 \in E, a_{n_j} \in G, x_2 = x_1 a_{n_1}^{2^{n_1}} \in E, \cdots$, and generally

$$x_1 a_{n_1}^{m_1} a_{n_2}^{m_2} \cdots a_{n_{j-1}}^{2^{n_{j-1}}} = x_j \in E.$$

An application of the Lemma 1 gives the following inequality:

$$f(x) \leq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

where α_j are calculated and cheked to be of the form adapted (*i.e. in binomial finite development*), and

$$\alpha_1 + \cdots + \alpha_k = 1.$$

We omit the details of this calculation. Therefore $f(x) \leq \alpha_1 C + \cdots + \alpha_k C = C$.

The proof is complete.

Remark. The Lemma 2 contains much more than what we need to prove the Ostrowski's theorem. Indeed, it is well-known that in a locally compact group G with the left Haar measure μ any set of the form $E^{-1}E$ with a measurable set E having positive measure $\mu(E) > 0$, contains a non empty $V \in \Phi$. It is not difficult to observe that then $(E^{-1}E)^{1/2}$ contains an open $W \in \Phi$. It follows that

$$EW \subset H_1(E).$$

Applying now the generalized Bernstein-Doetsch theorem as stated in our Theorem 1, we are in a position to complete the proof of our main result:

Theorem 2. *Let Ω be an open midconvex set in a root-approximable locally compact non discrete group G, μ a left invariant measure on G and $f : \Omega \longrightarrow \mathbf{R}$ a globally midconvex function. If f is bounded above on a set $E \subset \Omega$ with positive measure $\mu(E) > 0$, then f must be continuous everywhere on Ω.*

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