An algebra on partitions

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ABSTRACT. — We endow the free **Z**-module on the set of all partitions with a filtered (commutative) ring structure denoted by P. Denote by LR the associated graded ring. The ring LR is well known. Its structure constants are given by the Littlewood-Richardson rule. As an abstract ring the new ring P is isomorphic to LR. The Murnaghan-Nakayama rules define finite difference operators on P whereas they act as derivations on LR. The ring P has also natural homomorphisms into $\mathbf{Q}[x]$ indexed by conjugacy classes of permutations of $\mathbf{N}$ with finite support or into $\mathbf{Q}$ indexed by partitions. The ring P has shown up in disguised form for instance in [**B**], Theorem of section 3.1 or Corollary 1 of section 3.4.

1. Symmetric groups

In this paragraph we recall some facts about the character-theory of the symmetric groupe S_n on n letters. Most proofs are omitted and can be found for instance in [**G**], [**JK**], [**M**], [**Ro**] or [**Sa**]. A nice exposition of character-theory for finite groups is given in [**Se**].

A *partition of content n* (or a partition of n) is a finite sequence of positive, decreasing integers $\alpha = (\alpha_1, \dots, \alpha_k)$ such that $\sum \alpha_i = n$. We denote by $|\alpha|$ the content of α. The integers α_i are the *parts* of the partition. For instance $(4, 2, 2, 1,)$ or $(4, 2^2, 1)$ for short, is a partition of content 9 with one part equal to 4, two parts equal to 2 and one part equal to 1. For a partition $\alpha = (\alpha_1, \dots, \alpha_k)$ we define the *transposed partition* $\alpha' = (\alpha'_1, \dots, \alpha'_s)$ by $\alpha'_i = \sharp\{\alpha_\ell \mid \alpha_\ell \geq i\}$.

Let S_n denote the symmetric group on n letters and let $\sigma \in S_n$ be a permutation with cycle lengths $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k$. We get hence a partition $\overline{\sigma} = (\sigma_1, \dots, \sigma_k)$ of $n = \sum \sigma_i$ which depends only on the conjugacy class of σ and which we call the *cycle type* of σ. The permutation $\sigma = (1345)(28)(69)(7) \in \sigma_9$ for instance gives the partition $\overline{\sigma} = (4, 2^2, 1)$ of 9. We have a bijection between conjugacy classes of S_n and partitions of n. Let α be a partition written exponentially as $\alpha = (n^{a_n}, (n-1)^{a_{n-1}}, \dots, 2^{a_2}, 1^{a_1})$ (ie. α has a_i parts equal to i for $i = 1, \dots, n$) and let σ_α be a permutation of cycle type α. The subgroup of all permutations in $\sigma_{|\alpha|}$ commuting with σ_α (the *commutant* of σ_α) is of order $n_\alpha = \prod_i i^{a_i} a_i!$. The conjugacy class of σ_α contains hence exactly $\frac{|\alpha|!}{n_\alpha}$ elements.

A partition $\pi = (\pi_1, \dots, \pi_p)$ can be represented graphically by a *Young diagram* by drawing a first row consisting of π_1 nodes or boxes, followed by a second row consisting of

π_2 nodes and so on. For instance

$$\begin{array}{l} XXXX \\ XX \\ XX \\ X \end{array}$$

is the Young diagram of the partition $(4, 2^2, 1)$ (with crosses instead of boxes) of content 9. The Young diagram of the transposed partition α' of α is obtained by reflecting the Young diagram of α through the diagonal.

In the sequel we identify partitions with the corresponding Young diagrams. The main result of representation theory for symmetric groups is the existence of a "natural" bijection between partitions (or Young diagrams) of n and irreducible representations of S_n which is constructive in some sense (you can write down matrices if you wish). The same holds obviously also at the level of characters where the Murnaghan-Nakayama rule gives recursive formulas.

A *character* of a finite group G is a function of the form $g \longmapsto \mathrm{trace}(\rho(g))$ where $\rho : G \to \mathrm{Aut}\, V$ is a representation of G on some finite dimensional complex vector space V (one can of course consider other fields than the complex numbers but for the sake of simplicity we will ignore these generalizations. A pleasant feature of the symmetric group is the fact that all its characters and representations can be defined over the integers). A character is *irreducible* if the representation ρ is irreducible, *i.e.* if only the trivial subspaces $\{0\}$ and V are invariant under the action of G.

A *generalized character* is an arbitrary linear combination of irreducible characters with integral coefficients (and a generalized character is a genuine character if and only if all its coefficients are ≥ 0). The set $Ch(G)$ of all generalized characters forms the *character-ring*. Addition comes from the direct sum of representations and multiplication from tensor products. The character ring has a natural inner product structure given by

$$\langle \eta, \zeta \rangle = \frac{1}{|G|} \sum_{g \in G} \eta(g) \overline{\zeta(g)} \, .$$

The main result of character theory for finite groups states that the set of irreducible characters is an orthonormal basis of $Ch(G) \otimes_{\mathbf{Z}} \mathbf{Q}$ with respect to this inner product and that the rank of $Ch(G)$ as a $\mathbf{Z}$-module equals the dimension of the center in the group algebra $\mathbf{C}[G]$ (which is the vector space of complex functions on G with product given by convolution). The center of $\mathbf{C}[G]$ is generated by characteristic functions of conjugacy classes. The rank of $Ch(G)$ equals hence the number of conjugacy classes in G. Irreducible characters of G correspond to minimal central idempotents of $\mathbf{C}[G]$ (and minimal central idempotents of $Ch(G) \otimes \mathbf{C}$ are characteristic functions of conjugacy classes in G). The ring $Ch(G)$ is however generally not isomorphic to the center of $\mathbf{Z}[G]$ (this can be seen for instance by computing minimal idempotents in $Ch(S_3) \otimes \mathbf{Q}$ and in the center of $\mathbf{Z}[S_3] \otimes \mathbf{Q} = \mathbf{Q}[S_3]$). There is in general no natural bijection between irreducible characters and conjugacy classes for finite groups. This corresponds to the well-known fact that finite abelian groups are isomorphic (via the Fourier transform) to their dual but such an isomorphism is not canonical.

For a subgroup $H \subset G$ and a representation $\rho : H \to \mathrm{Aut}\, V$ we get an *induced representation* $\tilde{\rho} : G \to \mathrm{Aut}\, W$ by setting $W = \mathbf{C}[G] \otimes_{\mathbf{C}[H]} V$ with the obvious action of G on W. The character of $\tilde{\rho}$ is the *induced character* from (H, ρ) to G. Related to induction is *restriction*: Given a character χ of a group G and a subgroup $H \subset G$, one gets a character of H by restricting χ to H. Induction and restriction extend linearly to character-rings. The Frobenius reciprocity formula shows that induction and restriction are adjoint with respect to the inner product structure on character rings.

We fix now a Young diagram α of content n. A *Young tableau* t_α is obtained by numbering the nodes of α from 1 to n. Consider also a fixed Young tableau $t = t_\alpha$. It gives rise to 2 particular subgroups in S_n. The horizontal subgroup H_t is the stabilizer of the row-sets in t and the vertical subgroup is the stabilizer of the column-sets in t. For the Young tableau

$$t = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & 8 & & \\ 9 & & & \end{array}$$

of $(4, 2^2, 1)$ we get for instance

$$\begin{array}{rcl} H_t & = & S_{\{1,2,3,4\}} \times S_{\{5,6\}} \times S_{\{7,8\}} \times S_{\{9\}} \\ V_t & = & S_{\{1,5,7,9\}} \times S_{\{2,6,8\}} \times S_{\{3\}} \times S_{\{4\}} \end{array}$$

where S_E denotes the symmetric group of the set E.

Symmetric groups have two 1-dimensional representations: The trivial representation and the signature (except for $n = 1$ where they coincide). Inducing the trivial character of H_t and the product of the signatures of V_t up to S_n yields two characters which contain a unique common irreducible character. Denote this character by $[\alpha]$ (it depends only on the Young diagram α, not on the particular Young tableau t_α). The main result of the character-theory of σ_n is that the characters $[\alpha]$ obtained in this way form a complete set of irreducible characters. For instance $[n]$ is the trivial character and $[1^n]$ is the signature.

Denote by LR the free $\mathbf{Z}$-module on the set of all partitions of all integers (we identify partitions, Young diagrams and corresponding irreducible characters) with the convention that the “empty character" $[\emptyset]$ corresponding to the empty partition $\emptyset$ of 0 denotes the unique “representation" of the empty “group" without elements.

The $\mathbf{Z}$-module LR has a graded ring structure. Indeed, define the product of 2 characters $[\alpha]$ and $[\beta]$ with associated representations $\rho_\alpha : S_m \to \mathrm{Aut}(V)$ and $\rho_\beta : S_n \to$

$\mathrm{Aut}(W)$ as the character obtained by inducing the "product representation" $\rho_\alpha \times \rho_\beta : S_m \times S_n \to V \otimes W$ from $S_m \times S_n$ up to S_{m+n} where $S_m \times S_n$ is embedded into S_{m+n} in the obvious way. Extending this product linearly gives a commutative ring structure on LR (associativity is easy to check).

The structure constants of LR are given by the Littlewood-Richardson rule (and this justifies the notation LR, see **[G]** who calls this ring the Littlewood-Richardson ring). The ring LR has a natural involutive automorphism given by $[\alpha] \to [\alpha']$ where α' is the transposed partition of α and which acts as the multiplication by the signature at the level of group characters. One endows the $\mathbf{Z}$-module LR with an inner product which turns the set of irreducible representations $[\alpha]$ into an orthonormal basis. The involution $[\alpha] \longmapsto [\alpha']$ is an isometry for this inner product. For a fixed irreducible representation $[\alpha]$, denote by M_α the linear operator defined by multiplication with $[\alpha]$ (this operator is not bounded). Its adjoint M_α^* is defined by

$$\langle M_\alpha^*[\beta], [\gamma]\rangle = \langle [\beta], M_\alpha[\gamma]\rangle = \langle [\beta], [\alpha] \cdot [\gamma]\rangle$$

and $M_\alpha^*[\beta]$ is the character of the so-called *skew diagram* $\beta \setminus \alpha$ (see **[Sa]** or **[JK]** for the definition).

In order to study LR further one introduces the countable set of variables $x_1, x_2, \ldots$ of degree 1. Consider the symmetric functions $e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$ (the r-th elementary symmetric function), $h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$ (the r-th complete symmetric function) and $p_r = \sum_i x_i^r$ (the r-th power sum) in the infinite set of variables $x_1, x_2, \ldots$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define $e_\lambda = \prod_i e_{\lambda_i}$, $\quad h_\lambda = \prod_i h_{\lambda_i}$ and $p_\lambda = \prod_i p_{\lambda_i}$. These functions form bases of the graded $\mathbf{Q}$-algebra Λ generated by homogeneous symmetric functions in $x_1, x_2, \ldots$. Still another basis is given by the *Schur functions* s_λ (see **M** or chapter 10 of **[Go]** for the slightly delicate definition). The ring Λ is endowed with an involutive graded isomorphism ω satisfying $\omega(e_r) = h_r,\ \omega(h_\lambda) = e_\lambda,\ \omega(p_r) = (-1)^{r-1} p_r$ and $\omega(s_\lambda) = s_{\lambda'}$ where λ' is the transposed partition of λ. We have also an inner product structure on Λ. An orthogonal basis is given by the Schur functions. For an irreducible character $[\alpha] \in Ch(\sigma_n)$ we define $\chi([\alpha]) = \sum_\pi n_\pi^{-1}([\alpha](\sigma_\pi))p_\pi = \frac{1}{n!}\sum_{\sigma \in S_n}([\alpha](\sigma))p_\pi$ and extend χ linearly to a map from $LR \otimes \mathbf{Q}$ to Λ. This map is an isomorphism of graded algebras. It is also an isometry between the inner products spaces $LR \otimes \mathbf{Q}$ and Λ. It sends an irreducible character $[\alpha]$ to the Schur function s_α, the characteristic function of a conjugacy class of cycle type α to n_α^{-1} times the product p_α of power sums, the character $[\alpha_1][\alpha_2]\cdots[\alpha_k] \in LR$ (which is the induced character of the trivial character on H_{t_α} up to $S_{|\alpha|}$) corresponds to h_α and the character $[1^{\alpha_1}]\cdots[1^{\alpha_k}] \in LR$ (which is the character induced from the signature on H_{t_α} up to $S_{|\alpha|}$) corresponds to e_α. The natural involution $[\alpha] \longmapsto [\alpha']$ of LR corresponds to the involution ω on Λ.

The main difficulties of the character theory for S_n arise from the fact that the base change from Schur functions to one of the bases $e_\alpha,\ h_\alpha$ or p_α is not completely straightforward. Computing the values of an irreducible character $[\alpha]$ amounts for instance to expressing the function s_α as a linear combination of the functions p_λ. Moreover the structure constants of the algebra Λ are very simple with respect to the bases $e_\lambda,\ h_\lambda$ or p_λ. They are quite involved (given by the Littlewood-Richardson rule) in terms of Schur functions.

Let us also remark that the free **Z**-modules on the bases $e_\lambda,\ h_\lambda$ and s_λ coincide (and are subrings of Λ isomorphic to LR via the map χ^{-1}). The free **Z**-module generated by the p_λ (which is also a ring) is different. The base change "matrix" from p_λ to one of the other bases is rational but not integral and one cannot avoid non-integral coefficients somewhere even by rescaling suitably the p_λ's.

Each node (i,j) of a Young diagram α defines a *hook* $H_{i,j}(\alpha)$ consisting of all nodes (i,j') with $j' \geq j$ and of all nodes (i',j) with $i' \geq i$. A *k-hook* is a hook with exactly k nodes. The *leglength* of a hook $H_{i,j}$ is the number of nodes (i',j) with $i' > i$ or equivalently the cardinality of the set $\{s > i | \alpha_s \geq j\}$.

Let α be a Young diagram of content n and let H be a k-hook of α. We denote by $\alpha - H$ the Young diagram of content $n-k$ obtained by erasing the nodes of H and by moving all nodes below H one node up and one node to the left. For instance the following shows first a diagram α with its 6-hook $H_{2,2}$ of leglength 3 followed by the diagram $\alpha - H_{2,2}$:

$$\alpha = \begin{matrix} X & X & X & X & X \\ X & O & O & O & \\ X & O & X & X & \\ X & O & X & & \\ X & O & X & & \\ X & & & & \end{matrix} \qquad \alpha - H = \begin{matrix} X & X & X & X & X \\ X & X & X & & \\ X & X & & & \\ X & X & & & \\ X & & & & \\ X & & & & \end{matrix}$$

Consider the linear operator $D_k : Ch(S_{n+k}) \rightarrow Ch_(S_n)$ defined by

$$D_k([\alpha]) = \sum_{H \text{ a } k-\text{hook of } \alpha} (-1)^{\text{leglength of } \alpha}[\alpha - H] .$$

The importance of the operator D_k stems from the

1.1. Murnaghan-Nakayama formula. — *Let $v \in Ch(S_n)$ be a generalized character of S_n and let $\sigma \in S_n$ be an element with cycle type $(\sigma_1, \ldots, \sigma_k)$. Then*

$$v(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \ldots, \sigma_k) = (D_{\sigma_i} v)(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_k)$$

where we write the cycle type of a permutation instead of the permutation. This allows a recursive computation of $v(\sigma)$ and shows that the operators D_i, D_j commute for all i, j.

The action of the operator D_k extends to the ring LR by setting $D_k[\alpha] = 0$ if $[\alpha]$ is an irreducible character of S_n with $n < k$.

1.2. Theorem. — *Denote by $[\alpha][\beta]$ the product of $[\alpha]$ and $[\beta]$ given by the Littlewood-Richardson rule (i.e. $[\alpha][\beta]$ is the character $[\alpha] \times [\beta]$ induced from $S_{|\alpha|} \times S_{|\beta|}$ up to $S_{|\alpha|+|\beta|}$). Then*

(i) $D_k([\alpha][\beta]) = (D_k[\alpha])[\beta] + [\alpha](D_k[\beta])$.

(ii) $D_k(D_j[\alpha]) = D_j(D_k[\alpha])$.

Proof. — It is enough to prove (i) for irreducible characters $[\alpha] \in Ch(S_m)$ and $[\beta] \in Ch(S_n)$. Let $\rho_\alpha : S_m \to \mathrm{Aut}\, V$ and $\rho_\beta : S_n \to \mathrm{Aut} W$ be corresponding representations. Let $\rho : S_{m+n} \to \mathrm{Aut}(U)$ with $U = \mathrm{C}[S_{m+n}] \otimes_{\rho_\alpha \otimes \rho_\beta (S_m \times S_n)} (V \otimes W)$ be the representation corresponding to $[\alpha][\beta]$. We have to compute the trace of $\rho(\sigma)$ for $\sigma \in S_{m+n}$.

Let $g_1, g_2, \ldots$ be representatives of the $\binom{m+n}{n}$ classes of $S_{m+n}/(S_m \times S_n)$. We have hence $U = \oplus_{i=1}^{\binom{n+m}{m}} g_i \otimes (V \otimes W)$ and a small computation shows that trace $\rho(\sigma)|_U =$ trace $\rho(\sigma)|_{\tilde{U}}$ where $\tilde{U} = \sum_{i,\, g_i^{-1}\sigma g_i \in S_m \times S_n} g_i \otimes (V \otimes W)$. Let $c_1, \ldots, c_\ell$ be the cycles of σ. Classes $g(S_m \times S_n)$ such that $g^{-1}\sigma g \in S_m \times S_n$ are in bijection with disjoint unions $A_i \cup B_i = \{c_1, \ldots, c_\ell\}$ such that the sum of lengths of cycles in A_i equals m. Denote by τ_i the permutation $\tau_i = \prod_{j \in A_i} c_j$ and identitfy τ_i with a permutation of S_m. In the same manner define $\tau'_i = \prod_{j \in B_i} c_j$. One has then

$$[\alpha][\beta](\sigma) = \sum_i \big([\alpha](\tau_i)\big)\big([\beta](\tau'_i)\big)$$

and (i) follows easily.

Assertion (ii) is obvious. QED

1.3. Remarks.

(i) Theorem 1.2 shows that $D_1,\ D_2, \ldots$ are commuting derivations of LR. The set of linear operators of the form $\sum_{\text{finite}} v(k) D_k, \quad v(k) \in \oplus_i (LR \otimes_{\mathbf{Z}} \mathbf{Q}) D_i$ is hence a Lie algebra with Lie-bracket given by $[vD_k, wD_\ell] = v \cdot (D_k w) D_\ell - w \cdot (D_\ell v) D_k$. The subspace generated by $1 \otimes D_1,\ 1 \otimes D_2, \ldots$ is a maximal abelian Lie-subalgebra. The derivation D_k corresponds via the isomorphism $\chi : LR \to \Lambda = \mathbf{Q}[p_1, p_2, \ldots]$ (where $p_i = \sum_j x_j^i$) to the derivation $k \frac{d}{dp_k}$ on Λ.

(ii) If $[\alpha] \in Ch(S_n)$, the character of $(D_1)^k[\alpha] \in Ch(S_{n-k})$ is the restriction of $[\alpha]$ to $S_{n-k} \subset S_n$ (for the obvious inclusion). For this reason D_1 is also called the *restriction operator.*

(iii) Applying $D_1^{|\alpha|+|\beta|}$ to $[\alpha][\beta]$ we get

$$\binom{|\alpha|+|\beta|}{|\alpha|} \dim[\alpha]\ \dim[\beta] = \dim([\alpha][\beta])$$

where $\dim[\alpha] = [\alpha](\mathrm{Id}_{S_{|\alpha|}})$ denotes the dimension of a (not necessarily irreducible) character of S_n. Of course, one can also prove this identity by an easy dimension count since $\binom{|\alpha|+|\beta|}{|\alpha|}$ is the index of $S_{|\alpha|} \times S_{|\beta|}$ in $S_{|\alpha|+|\beta|}$.

This identity implies that the linear application from LR into $\mathbf{Q}[x]$ defined by

$$[\alpha] \longmapsto \frac{\dim[\alpha]}{|\alpha|!} x^{|\alpha|}$$

is a homomorphism of graded algebras.

In order to get deeper insight into the ring LR it is usefull to characterize the minimal idempotents in $Ch(S_n)$. This is achieved by the following result:

1.4. Theorem. — *Set*

$$z_n = ([n]-[n-1,1]+[n-2,1^2]-[n-3,1^3]+\cdots+(-1)^{n-2}[2,1^{n-2}]+(-1)^{n-1}[1^n]) = \sum_{i=0}^{n-1}(-1)^i[n-i,1^i]$$

and define z_α *by* $z_\alpha = z_{\alpha_1}\cdots z_{\alpha_k}$ *for* $\alpha = (\alpha_1,\ldots,\alpha_k)$. *We have*

(i) $D_k z_n = k\delta_{k,n}$.

(ii) $z_\alpha(\sigma_\beta) = n_\alpha \delta_{\alpha,\beta} = \prod_i i^{a_i}(a_i)!\delta_{\alpha,\beta}$ *where* $\sigma_\beta \in S_{|\beta|}$ *is of cycle type* β *and where* $\alpha = (n^{a_n},\ldots,1^{a_1})$.

(iii) $LR \otimes_{\mathbf{Z}} \mathbf{Q} \sim \mathbf{Q}[z_1,z_2,z_3,\ldots]$ *(isomorphism of graded rings for* $\deg z_k = k$*).*

(iv) $\langle z_\alpha, z_\beta\rangle = n_\alpha\delta_{\alpha,\beta} = \prod_i i^{a_i}(a_i)!\delta_{\alpha,\beta}$ *where* $\alpha = (n^{a_n},\ldots,1^{a_1})$.

Proof.

(i) is a straightforward computation using the Murnaghan-Nakayama formula.

(ii) is a repeated application of the Murnaghan-Nakayama formula, of Theorem 1.2 and of assertion (i).

Assertion (ii) states that z_α is up to a scalar the characteristic function of the conjugacy class of $\sigma_\alpha \in S_{|\alpha|}$ where σ_α has cycle type $(n^{a_n},\ldots,1^{a_1})$. Assertion (iii) follows now.

For (iv), we remark that the scalar product $\langle \alpha, \beta \rangle$ is zero if $|\alpha| \neq |\beta|$. We suppose hence $n = |\alpha| = |\beta|$. Recall that $\frac{n!}{n_\alpha}$ is the number of elements in the conjugacy-class of σ_α. The inner product $\langle [\alpha], [\beta] \rangle$ of 2 characters $[\alpha], [\beta]$ in S_n is hence given by

$$\langle [\alpha], [\beta] \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} ([\alpha](\sigma))([\beta](\sigma)) = \frac{1}{n!} \sum_{\gamma, |\gamma|=n} \frac{n!}{n_\gamma} ([\alpha](\sigma_\gamma))([\beta](\sigma_\gamma))$$

(the last sum is over all partitions of content n) where $\sigma_\gamma \in S_n$ has cycle type γ). We get hence from (ii)

$$\langle z_\alpha, z_\beta \rangle = \frac{1}{n!} \frac{n!}{n_\alpha} n_\alpha^2 = n_\alpha = \prod_i i^{a_i} (a_i)!$$

if $\alpha = \beta$ and 0 otherwise. QED

1.5. Remarks.

(i) Assertion (ii) of this theorem states that the generalized character $z_\alpha = z_{\alpha_1} \cdots z_{\alpha_k}$ is equal to $n_\alpha = \prod i^{a_i}(a_i)!$ (where $\alpha = (\alpha_1, \ldots, \alpha_k) = (n^{a_n}, \ldots, 1^{a_1})$) times the characteristic function of the conjugacy class with cycle type α. The derivations D_k act in a particularly simple manner on them. Indeed, $D_k z_\alpha = \left(k \frac{d}{dz_k}\right) z_\alpha$ by (i) of Theorem 1.4. The Lie-algebra of Remark 1.3 (i) is hence simply the Lie-algebra of derivations with polynomial coefficients in the variables $z_1, z_2, \ldots$. The homomorphism of Remark 1.3 (iii) is the specialization $z_1 = x$, $z_2 = z_3 = \cdots = 0$.

(ii) Recall that we have an isomorphism $\chi : LR \otimes \mathbf{Q} \to \Lambda = \mathbf{Q}[p_1, p_2, \ldots]$ with $p_i = \sum_j x_i^j$. This isomorphism is defined by $\chi(z_\alpha) = p_\alpha$ where z_α is as in theorem 1.4 and where $p_\alpha = \prod p_i^{a_i}$ with $p_i = \sum_j x_j^i$. One checks now easily that the derivation D_k of LR corresponds indeed to the derivation $k \frac{d}{dp_k}$ of Λ.

2. A subring in $Ch(S_n)$.

We would like to study the subring of $Ch(S_n)$ generated by the trivial character $[n]$ and by the character $[n-1, 1]$ associated to the standard representation of S_n.

2.1. Theorem.

(i) The characters $[n-1, 1]^{\otimes 0}, \ldots, [n-1, 1]^{\otimes n-1}$ defined by $[n-1, 1]^{\otimes 0} = [n]$ and $[n-1, 1]^{\otimes k} = [n-1, 1]^{\otimes k-1} \otimes [n-1, 1]$ are a **Z***-basis of the subring X generated by $[n]$ and $[n-1, 1]$ in $Ch(S_n)$.*

(ii) For a fixed natural integer k, the squared norm $\langle [\alpha], [\alpha] \rangle$ of the character $\alpha = [n-1, 1]^{\otimes k}$ remains bounded when $n \to \infty$.

(iii) Every irreducible character $[\alpha_1, \alpha_2, \ldots, \alpha_a]$ with $\alpha_1 = n-k$ appears with positive multiplicity in $[n-1, 1]^{\otimes k}$.

(iv) No irreducible character $[\alpha_1, \alpha_2, \ldots, \alpha_a]$ with $\alpha_1 < n-k$ appears in $[n-1, 1]^{\otimes k}$.

Proof. — One has $[n](\sigma) = 1$ for all $\sigma \in S_n$ and $[n-1, 1](\sigma) = |\operatorname{Fix}(\sigma)| - 1$ where $\operatorname{Fix}(\sigma)$ is the set of fixed points of the permutation σ acting in the obvious way on the set $\{1, \ldots, n\}$. This shows that an element of the subring $X \subset Ch(S_n)$ generated by $[n]$ and $[n-1, 1]$ is constant on permutations having the same number of fixed points. Since the number of fixed points of a permutation belongs to the set $\{0, 1, 2, \ldots, n-3, n-2, n\}$ (no permutation can move exactly one point), the rank of X is at most n.

Let us consider the trivial representation $[n-k]$ of S_{n-k}. Denote by ψ_k the character of S_n obtained by inducing $[n-k]$ up to S_n. In order to compute the irreducible components of ψ_k it is enough to remark that the Young diagrams of ψ_k are obtained by adjoining k boxes in all possible licit ways (at the i-th step one must get a Young diagram of content $n-k+i$) to the Young diagram of $[n-k]$ consisting of a unique row made of $n-k$ boxes. For $k \leq [\frac{n}{2}]$ the character ψ_k equals

$$\psi_k = \sum_{\alpha,\ |\alpha|=n,\ \alpha_1 \geq n-k} \binom{k}{n-\alpha_1} \dim[\alpha_2, \ldots, \alpha_a]\,[\alpha]$$

where $\dim[\alpha_2, \ldots, \alpha_a]$ denotes the dimension of the character $[\alpha_2, \ldots, \alpha_a] \in Ch(S_{n-\alpha_1})$. The sum of squares of the multiplicities in ψ_k (which is the dimension of the commutant of $\psi_k(S_n)$ and equals the squared norm of ψ_k) turns out to be

$$\sum_{i=0}^{k} \binom{k}{i}^2 i! \quad \text{if } k \leq [\tfrac{n}{2}].$$

On the other hand, the character ψ_k is given by

$$\psi_k(\sigma) = k! \binom{|\operatorname{Fix}(\sigma)|}{k}.$$

Indeed, $\psi_k(\sigma) = \sharp\{gS_{n-k} \subset S_n \mid g^{-1}\sigma g \in S_{n-k}\}$ and there are exactly $k!\frac{|\operatorname{Fix}(\sigma)|}{k}$ such left classes in S_n/S_{n-k}. This implies that

$$\begin{aligned}\psi_k(\sigma) &= k! \binom{|\operatorname{Fix}(\sigma)|}{k} \\ &= ([n-1, 1]+[n]) \otimes [n-1, 1] \otimes ([n-1, 1]-[n]) \otimes \cdots \otimes ([n-1, 1]-(k-2)[n])(\sigma)\end{aligned}$$

which shows that the Z-module of X spanned by $\psi_0, \ldots, \psi_k$ is equal to the Z-module spanned by $[n-1, 1]^{\otimes 0}, \ldots, [n-1, 1]^{\otimes k}$. The base change matrix between this two bases is given by an integral upper triangular matrix (independent of n) with all diagonal coefficients equal to 1. This proves easily claims (ii),(iii) and (iv) since they are true for the characters ψ_k.

Observe that for $i < n$ the element $\psi_i \in Ch(S_n)$ contains the hook-shaped character $[n-i, 1^i]$ with multiplicity exactly 1 and it contains no hook-shaped character $[n-j, 1^j]$ with $j > i$. This holds also for $[n-1, 1]^{\otimes i}$ by induction on i. The fact that $[n-1, 1]^{\otimes 0}, \ldots, [n-1, 1]^{\otimes n-1}$ generate X as a Z-module follows now easily. QED

3. An algebra

Let $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_a]$ be a character of $S_{|\alpha|}$.

Consider an integer $n \geq 2|\alpha|$. Since $\alpha_1 \leq |\alpha| \leq n-|\alpha|$ we get an irreducible character $[n-|\alpha|, \alpha_1 \cdots \alpha_a]$ of S_n which we denote $[n-|\alpha|, \alpha]$. The Young diagram of $[n-|\alpha|, \alpha]$ is obtained by stacking a row of $(n-|\alpha|)$ nodes above the Young diagram of α.

3.1. Theorem.

(i) Let α and β be two partitions of two natural integers. Choose an integer n such that $n \geq 2(|\alpha| + |\beta|)$. Then there exist integers $\lambda_{\alpha\beta}^{\gamma} \geq 0$ independent of $n \geq 2(|\alpha| + |\beta|)$ such that

$$[n-|\alpha|, \alpha] \otimes [n-|\beta|, \beta] = \sum_{\gamma} \lambda_{\alpha\beta}^{\gamma}[n-|\gamma|, \gamma]$$

where the sum is over partitions γ of content $\leq |\alpha| + |\beta|$ and where $[n-|\gamma|, \gamma] \in Ch(S_n)$ is as above.

(ii) The leading terms are given by the Littlewood-Richardson rule, i.e. $\sum_{|\gamma|=|\alpha|+|\beta|} \lambda_{\alpha\beta}^{\gamma}[n-|\gamma|, \gamma]$ is the character of $[\alpha] \times [\beta]$ induced from $S_{|\alpha|} \times S_{|\beta|}$ to $S_{|\alpha|+|\beta|}$.

(iii) One has for all $n \geq 2(|\alpha| + |\beta|)$

$$\lambda_{\alpha\beta}^{\gamma} = \frac{1}{n!} \sum_{\sigma \in S_n} \Big([\alpha(n)](\sigma)\Big)\Big([\beta(n)](\sigma)\Big)\Big([\gamma(n)](\sigma)\Big)$$

and the integers $\lambda_{\alpha\beta}^{\gamma}$ are hence symmetric with respect to α, β and γ.

3.2. Examples. — We write $[\alpha] \circ [\beta] = \sum \lambda_{\alpha\beta}^{\gamma}[\gamma]$ instead of $[n-|\alpha|, \alpha] \otimes [n-|\beta|, \beta] = \sum_{\gamma} \lambda_{\alpha\beta}^{\gamma}[n-|\gamma|, \gamma]$ for $n \geq 2(|\alpha| + |\beta|)$. We have then

$$
\begin{array}{ll}
[\emptyset]\circ[\alpha] & =[\alpha] \quad \text{for all } \alpha \\
[1]\circ[1] & =[\emptyset]+[1]+[2]+[1^2] \\
[1]\circ[2] & =[1]+[2]+[1^2]+[3]+[2,1] \\
[1]\circ[1^2] & =[1]+[2]+[1^2]+[2,1]+[1^3] \\
[1]\circ[3] & =[2]+[3]+[2,1]+[4]+[3,1] \\
[1]\circ[2,1] & =[2]+[1^2]+[3]+2[2,1]+[1^3]+[3,1]+[2^2]+[2,1^2] \\
[1]\circ[1^3] & =[1^2]+[2,1]+[1^3]+[2,1^2]+[1^4] \\
[2]\circ[2] & =[\emptyset]+[1]+2[2]+[1^2]+[3]+2[2,1]+[1^3]+[4]+[3,1]+[2^2] \\
[2]\circ[1^2] & =[1]+[2]+2[1^2]+[3]+2[2,1]+[1^3]+[3,1]+[2,1^2] \\
[1^2]\circ[1^2] & =[\emptyset]+[1]+2[2]+[1^2]+[3]+2[2,1]+[1^3]+[2^2]+[2,1^2]+[1^4]
\end{array}
$$

Moreover

$$
\begin{array}{ll}
[1]\circ[k] & =[k-1]+[k]+[k-1,1]+[k+1]+[k,1] \\
[1]\circ[1^k] & =[1^{k-1}]+[2,1^{k-2}]+[1^k]+[2,1^{k-1}]+[1^{k+1}]
\end{array}
$$

for all $k \geq 2$.

We denote by P_n the free $\mathbf{Z}$-module on the set of all partitions of n.

3.3. Corollary. — *The free $\mathbf{Z}$-module $P=\oplus_{n\in\mathbf{N}}P_n$ on the set of partitions of natural integers is a commutative ring over $\mathbf{Z}$ for the product*

$$[\alpha]\circ[\beta]=\sum_\gamma \lambda^\gamma_{\alpha\beta}[\gamma]$$

where the structure constants $\lambda^\gamma_{\alpha\beta}$ are the integers defined in Theorem 3.1.

The proof of the Corollary is obvious.

Proof of Theorem 3.1.

(i) Let us write $[\alpha(n)]$ instead of $[n-|\alpha|,\alpha]$. We show first that there exist bounded functions $\lambda^\gamma_{\alpha\beta}(n)$ for $n\geq 2(|\alpha|+|\beta|)$ such that

$$[\alpha(n)]\otimes[\beta(n)]=\sum_\gamma \lambda^\gamma_{\alpha\beta}(n)[\gamma(n)]$$

where the sum is over partitions of integers $\leq |\alpha|+|\beta|$.

Theorem 2.1.(iii) shows that the representations $[\alpha(n)]$ and $[\beta(n)]$ are contained in the characters $[n-1,1]^{\otimes|\alpha|}$ and $[n-1,1]^{\otimes|\beta|}$. Hence $[\alpha(n)]\otimes[\beta(n)]$ is contained in $[n-1,1]^{\otimes(|\alpha|+|\beta|)}$ which has bounded norm by Theorem 2.1 (ii). The integers $\lambda^\gamma_{\alpha\beta}(n)$ are hence bounded and (iv) of Theorem 2.1 shows that $|\gamma|\leq|\alpha|+|\beta|$.

We prove now that the functions $\lambda^\gamma_{\alpha\beta}(n)$ are constant for $n\geq 2(|\alpha|+|\beta|)$.

The proof is by induction on $|\alpha|+|\beta|$.

If $|\alpha| + |\beta| = 0$ then α and β are both the empty partition of the integer 0 and one has obviously

$$[n] \otimes [n] = [n]$$

for all n which is equivalent to

$$[\emptyset(n)] \otimes [\emptyset(n)] = [\emptyset(n)] \,.$$

Let now α and β be two partitions such that $|\alpha| + |\beta| = k + 1$. Choose an integer $n > 2(k+1)$. We consider the restriction of

$$[\alpha(n)] \otimes [\beta(n)] = \sum_{\gamma} \lambda_{\alpha\beta}^{\gamma} [\gamma(n)]$$

to the group $S_{n-1} \subset S_n$. This is done by applying the restriction operator D_1 on both sides and since $[D_i(\delta(n))] = [\delta(n-i)] + [(D_i\delta)(n-i)]$ for $n \geq 2|\delta| + i$ we get

$$\begin{aligned}
&[\alpha(n-1)] \otimes [\beta(n-1)] + [(D_1\alpha)(n-1)] \otimes [\beta(n-1)] \\
&\qquad + [\alpha(n-1)] \otimes [(D_1\beta)(n-1)] + [(D_1\alpha)(n-1)] \otimes [(D_1\beta)(n-1)] \\
&\quad = \sum_{\gamma} \lambda_{\alpha\beta}^{\gamma}(n)[\gamma(n-1)] + \sum_{\gamma} \lambda_{\alpha_\beta}^{\gamma}(n)[(D_1\gamma)(n-1)] \,.
\end{aligned}$$

Since $|D_1\delta| = |\delta| - 1$ for $\delta \neq \emptyset$ induction implies that the term $\lambda_{\alpha\beta}^{\delta}[\delta(n-1)]$ with $|\delta| = |\alpha| + |\beta|$ is contained in $[\alpha(n-1)] \otimes [\beta(n-1)]$. This shows that $\lambda_{\alpha\beta}^{\delta}(n-1) = \lambda_{\alpha\beta}^{\delta}(n)$ for every δ of content $|\delta| = |\alpha| + |\beta|$.

Suppose now by (descending) induction on k that $\lambda_{\alpha\beta}^{\delta}(n)$ is constant for every δ of content $|\delta| > k$ and consider a partition η of content k. Induction on $|\alpha| + |\beta|$ shows that the multiplicity of $[\eta(n-1)]$ in

$$[(D_1\alpha)(n-1)] \otimes [\beta(n-1)] + [\alpha(n-1)] \otimes [(D_1\beta)(n-1)] + [(D_1\alpha)(n-1)] \otimes [(D_1\beta)(n-1)]$$

is independent of n (for $n > |\alpha| + |\beta|$ of course). Denote this multiplicity by ℓ_η.

The multiplicity of $[\eta(n-1)]$ in

$$\sum_{\gamma} \lambda_{\alpha\beta}^{\gamma}(n)[(D_1\gamma)(n-1)]$$

does not depend on n by induction on k. Denote it by r_η.

By equating coefficients we get

$$\lambda_{\alpha\beta}^{\eta}(n-1) + \ell_\eta = \lambda_{\alpha\beta}^{\eta}(n) + r_\eta \,.$$

If $\ell_\eta \neq r_\eta$ the integers $\lambda_{\alpha\beta}^{\eta}(n)$ cannot remain bounded for $n \to \infty$ and this contradicts (ii) of Theorem 2.1. The proof of (i) is now finished since the constants $\lambda_{\alpha\beta}^{\gamma}$ are non-negative integers.

Proof of (ii). — We recall that the operator D_k is defined on a Young diagram as the sum of diagrams obtained by removing k-hooks in all possible ways and by multiplying by (-1) if the removed k-hook has odd leglength. One gets for $n \geq 2(|\alpha| + |\beta|) + k$

$$\begin{aligned} D_k([\alpha(n)] \otimes [\beta(n)]) &= [\alpha(n-k)] \otimes [\beta(n-k)] + [(D_k\alpha)(n-k)] \otimes [\beta(n-k)] \\ &\quad + [\alpha(n-k)] \otimes [(D_k\beta)(n-k)] \\ &\quad + [(D_k\alpha)(n-k)] \otimes [(D_k\beta)(n-k)] \\ &= \sum_\gamma \lambda^\gamma_{\alpha\beta}[\gamma(n-k)] + \sum_\gamma \lambda^\gamma_{\alpha_\beta}[(D_k\gamma)(n-k)]\,. \end{aligned}$$

Since $[\alpha(n-k)] \otimes [\beta(n-k)] = \sum_\gamma \lambda^\gamma_{\alpha\beta}[\gamma(n-k)]$ we have

$$[D_k\alpha] \circ [\beta] + [\alpha] \circ [D_k\beta] + [D_k\alpha] \circ [D_k\beta] = D_k([\alpha] \circ [\beta])\,.$$

The terms of leading degree (the degree of an irreducible character is defined as its content) of the right-hand side have degree $|\alpha| + |\beta| - k$. Since the terms of $[D_k\alpha] \circ [D_k\beta]$ are of degree $\leq |\alpha| + |\beta| - 2k$ we have

$$(D_k([\alpha] \circ [\beta]))_{\text{leading}} = ([D_k\alpha] \circ [\beta] + [\alpha] \circ [D_k\beta])_{\text{leading}}\,.$$

This shows that we can calculate the leading terms of $[\alpha] \circ [\beta]$ using the Murnaghan-Nakayama rules in exactly the same way as the product $[\alpha][\beta]$ in the ring LR. The leading terms of $[\alpha] \circ [\beta]$ must hence be equal to the product $[\alpha][\beta]$ in LR.

The proof of (iii) is obvious. QED

3.4. Remarks.

(i) Theorem 3.1 (ii) states that LR is the associated graded ring of the filtered ring P.

(ii) By (iii) the computation of $[\alpha] \circ [\beta]$ can be done in the representation ring of S_n for any n with $n \geq 2(|\alpha| + |\beta|)$. One sees furthermore that $\lambda^\gamma_{\alpha\beta} = 0$ if $\gamma \notin [|\alpha| - |\beta|, |\alpha| + |\beta|]$.

(iii) One can easily check that $\lambda^\emptyset_{\alpha\beta} = \delta_{\alpha\beta}$.

Let us henceforth always write $[\alpha][\beta]$ for the Littlewood-Richardson product and $[\alpha] \circ [\beta]$ for the product in P.

Recall the definition of the generalized character $z_k = \sum_{i=0}^{k-1} (-1)^i [k-i, 1^i]$ and the definition of $z_\lambda = z_n^{\ell_n} z_{n-1}^{\ell_{n-1}} \cdots z_1^{\ell_1}$ where $\lambda = (n^{\ell_n} \cdots 1^{\ell_1})$ is a partition.

We define also

$$z_{\circ\lambda} = z_n^{\circ\ell_n} \circ z_{n-1}^{\circ\ell_{n-1}} \circ \cdots \circ z_1^{\circ\ell_1}$$

where $x^{\circ l}$ denotes the ℓ-th power in P.

3.5. Lemma.

(i) Identifying the underlying vector spaces of $LR \otimes \mathbf{Q}$ and $P \otimes \mathbf{Q}$ we have $z_k^{\circ l} = P_{k,l}(z_k) \in LR$ where $P_{k,l}$ is a monic polynomial of degree ℓ (powers in $P_{k,l}(z_k)$ are of course taken in LR).

(ii) $z_k^l = Q_{k,l}(z_k) \in P$ where $Q_{k,l}$ is a monic polynomial of degree ℓ (and powers are with respect to P on the right side).

Proof. — Let us prove by induction that $z_k^i \circ z_k^j$ is a monic polynomial of degree $i + j$. Since there is nothing to do if $i = 0$ or $j = 0$ we suppose $i, j > 0$. Since $D_\ell(a \circ b) = (D_\ell a) \circ b + (D_\ell a) \circ (D_\ell b) + a \circ (D_\ell b)$ in P we see that $D_\ell(z_k^i \circ z_k^j) = 0$ if $\ell \neq k$ and $D_k(z_k^i \circ z_k^j) = ki(z_k^{i-1} \circ z_k^j) + k^2 ij(z_k^{i-1} \circ z_k^{j-1}) + kj(z_k^i \circ z_k^{j-1})$. This shows that $z_k^i \circ z_k^j$ is a polynomial of degree $i + j$ in LR since D_ℓ acts as the derivation $\ell \frac{d}{dz_\ell}$ on LR. This polynomial is monic since highest degree terms in LR and P coincide. The proof of (i) follows now easily by induction on ℓ and (ii) follows from (i) by elementary linear algebra. QED

Denote by $\mathbf{Q}[z_k]$ the subring of $LR \otimes \mathbf{Q}$ generated by $1, z_k, z_k^2, \ldots$ and denote by $\mathbf{Q}[z_k]_\circ$ the subring of $P \otimes \mathbf{Q}$ generated by powers in P of z_k. Recall that LR and P are both rings on the free $\mathbf{Z}$-module generated by all partitions.

3.6. Proposition.

(i) $\mathbf{Q}[z_k] = \mathbf{Q}[z_k]_\circ$ as vector spaces of the $\mathbf{Q}$-vector space generated by on partitions.

(ii) Let $k \neq \ell$ be two natural integers. We have then for $A \in \mathbf{Q}[z_k]$ $(= \mathbf{Q}[z_k]_\circ)$ and $B \in \mathbf{Q}[z_\ell]$ $(= \mathbf{Q}[z_\ell]_\circ)$

$$AB = A \circ B$$

Proof.

(i) is a partial reformulation of Lemma 3.5.

For (ii) we remark that that $D_i(A \circ B) = (D_i A) \circ B + A \circ (D_i B)$ since at least one of the terms $D_i A$, $D_i B$ is zero. This shows that the Leibniz rules hold for both products AB and $A \circ B$. They differ hence by induction at most by a constant term. Since $\langle z_k^i, z_k^j \rangle = 0$ if $(i, j) \neq (0, 0)$ the constant terms on both sides are equal. QED

3.7. Theorem.

(i) The linear application

$$z_\lambda \longmapsto z_{\circ\lambda}$$

defines an isomorphism of the ring LR onto the ring P.

(ii) The operator D_k defined by the Murnaghan-Nakayama rules acts as the derivation $k\frac{d}{dz_k}$ on LR and as the finite difference operator

$$D_k\big(z_1^{\circ \ell_1}\circ\cdots\circ z_k^{\circ\ell_k}\circ\cdots\circ z_n^{\circ\ell_n}\big) = z_1^{\circ\ell_1}\circ\cdots\circ\big((z_k+k)^{\circ\ell_k}-z_k^{\circ\ell_k}\big)\circ\cdots\circ z_n^{\circ\ell_n}$$

($k = k[\emptyset]$ of course) on P.

Proof. — The application $z_\lambda \longmapsto z_{\circ\lambda}$ defines obviously a homomorphism. Bijectivity follows from the fact that the associated graded ring of P is the ring LR. We have yet to show that D_k acts as a finite difference operator on P. This follows from the fact that the linear application $p(x) \longmapsto p(x+k)-p(x)$ is the unique linear application $D : \mathbf{C}[x] \to \mathbf{C}[x]$ such that $\deg(Dp) = (\deg p) - 1$, $D(x) = k$ and $D(pq) = (Dp)q + (Dp)(Dq) + p(Dq)$ for all $p, q \in \mathbf{C}[x]$. QED

4. Homomorphisms

The ring P introduced in Corollary 3.3 exhibits many features of a character ring: Its elements can be evaluated on conjugacy classes of permutations of $\mathbf{N}$ with finite support (the support of a permutation is the complement of its fixed points).

4.1 Theorem. — *For each partition π there exists a ring-homomorphisme $\varphi_\pi : (P,\circ) \to \mathbf{Q}[x]$ with the following properties:*

(i) $\deg(\varphi_\pi([\alpha])) = |\alpha|$,

(ii) $\varphi_\emptyset(z_1) = x-1$ *and* $\varphi_\emptyset(z_k) = -1$ *if* $k \geq 2$,

(iii) $\varphi_{(\pi,1^k)}([\alpha]) = \varphi_\pi([\alpha])$ *for all* $k \geq 0$,

(iv) $\varphi_\pi(z_1)(x) = x - 1 - \sum_{i\geq 2} ip_i$ *and* $\varphi_\pi(z_k) = kp_k - 1$ *for* $k \geq 2$ *where* $\pi = (n^{p_n}\cdots 2^{p_2})$.

Proof. — The existence of φ_π is proved by induction on $|\pi|$. If $\pi = \emptyset$ set $\varphi_\emptyset([\alpha])(n) = \dim[n-|\alpha|,\alpha]$ for $n \geq 2|\alpha|$. For a partition $\alpha = (\alpha_1,\dots,\alpha_k)$ recall that the transposed partition $\alpha' = (\alpha'_1,\dots,\alpha'_s)$ of α is defined by $\alpha'_i = \sharp\{\alpha_\ell \mid \alpha_\ell \geq i\}$. The *Hook formula*

$$\dim[\alpha] = \frac{|\alpha|!}{\prod_{H_{i,j}\text{ hook of }\alpha} |H_{i,j}|}$$

shows that

$$\varphi_\emptyset([\alpha])(n) = \frac{n(n-1)\cdots(n-2|\alpha|+1)}{\prod_{k=1}^{|\alpha|}(n-|\alpha|-k+1+\alpha'_k)\prod_{H_{i,j}\text{ hook of }\alpha}|H_{i,j}|}$$
$$= \frac{n(n-1)\cdots(n-2|\alpha|+1)}{\prod_{k=1}^{|\alpha|}(n-|\alpha|-k+1+\alpha'_k)}\frac{\dim[\alpha])}{|\alpha|!}$$

is a polynomial of degree $|\alpha|$ in n. Set $\varphi_\emptyset([\alpha])$ equal to this polynomial. Since $\varphi_\emptyset([\alpha])(n)$ equals the dimension of the character $[n-|\alpha|,\alpha] \in Ch(S_n)$ for $n \geq 2|\alpha|$ one has

$$\varphi_\emptyset([\alpha]\circ[\beta])(n) = \varphi_\emptyset([\alpha])(n)\varphi_\emptyset([\beta])(n)$$

if $n \geq 2(|\alpha|+|\beta|)$. This shows that $\varphi_\emptyset([\alpha])\varphi_\emptyset([\beta]) = \varphi_\emptyset([\alpha]\circ[\beta])$ since equality holds for infinitely many evaluations of the polynomials $\varphi_\emptyset([\alpha])$, $\varphi_\emptyset([\beta])$ and $\varphi_\emptyset([\alpha]\circ[\beta])$.

For $\pi = (\pi_1, \pi_2, \pi_3, \dots)$ define $\varphi_\pi([\alpha])(x)$ by

$$\varphi_\pi([\alpha])(x) = \left(\varphi_{(\pi_2,\pi_3\cdots)}([\alpha]) + \varphi_{(\pi_2,\pi_3,\dots)}([D_{\pi_1}\alpha])\right)(x-\pi_1)\,.$$

This is a polynomial of degree $|\alpha|$ in n by induction on the content $|\pi|$ of π. The Murnaghan-Nakayama rules show that $\varphi_\pi([\alpha])(n) = [n-|\alpha|,\alpha](\sigma_{(\pi,1^{n-|\pi|})})$ for n big enough. Indeed, in order to evaluate $[n-|\alpha|,\alpha](\sigma_{(\pi,1^{n-|\pi|})})$ one gets

$$D_{\pi_1}([n-|\alpha|]) = [n-|\alpha|-\pi_1,\alpha] + [n-|\alpha|, D_{\pi_1}\alpha]$$

if $n-|\alpha|-\pi_1 \geq \alpha_1$. This implies in the same way as for $\varphi_\emptyset$ that φ_π is a homomorphism and finishes the proof of (i).

The proof of $\varphi_\emptyset(z_1) = x-1$ is obvious since $z_1 = [1]$ and $\varphi_\emptyset([1])(n) = \dim([n-1,1])$. Let us prove that $\varphi_\emptyset(z_k) = -1$ for $k \geq 2$. Let $z_k(n) \in Ch(S_n)$ (for $n \geq 2k$) denote the generalized character obtained by adjoining a first part equal to $n-k$ to all partitions involved in z_k. An easy computation shows that the restriction of this generalized character to S_{n-1} yields the generalized character $z_k(n-1)$ if $n > 2k$. This shows that $z_k(n)(\mathrm{Id}_{\mathrm{S_n}})$ is independent of n for $n \geq 2k$. The polynomial $\varphi_\emptyset(z_k)$ is thus constant. Applying the hook formula one sees that the dimensions of the irreducible representations in $z_k(n)$ are given by polynomials of degree k in n. A closer inspection shows that all these polynomials except the polynomial corresponding to $[n-k,1^k]$ have no constant term and the constant term of the dimension of $[n-k,1^k]$ equals $(-1)^k$. This character has multiplicity $(-1)^{k-1}$ in z_k and one gets the desired result.

The proof of (ii) follows from the equality $\varphi_\pi([\alpha])(n) = [n-|\alpha|,\alpha](\sigma_{(\pi,1^{n-|\pi|})})$ for n big enough.

Proof of (iv). Because of (iii) we can assume without loss of generality that π has the form $\pi = (n^{p_n} \cdots 2^{p_2})$. Using the definition one gets

$$\begin{aligned} & \varphi_{(n^{p_n}\dots 2^{p_2})}(z_1)(x) = \varphi_{(n^{p_n-1}(n-1)^{p_{n-1}}\dots 2^{p_2})}(z_1)(x-n) \\ = \quad & \varphi_{((n-1)^{p_{n-1}}\dots 2^{p_2})}(z_1)(x-np_n) = \cdots = \varphi_\emptyset(z_1)(x-\sum_{i\geq 2} ip_i) = x-1-\sum_{i\geq 2} ip_i\,. \end{aligned}$$

Similarly we have for $k \geq 2$

$$\begin{aligned} \varphi_{(n^{p_n}\dots 2^{p_2})}(z_k)(x) &= \varphi_{(k^{p_k})}(z_k)(x-\sum_{i\geq 2,\ i\neq k} ip_i) \\ &= \varphi_{(k^{p_k-1})}(z_k+k[\emptyset])(x-k-\sum_{i\geq 2,\ \to\neq k} ip_i) = \cdots \\ &= \varphi_\emptyset(z_k+kp_k[\emptyset])(x-\sum_{i\geq 2} ip_i) = -1+kp_k \end{aligned}$$

(the last equality follows from (ii) above) which ends the proof. QED

Let S_∞ be the group of permutations of **N** which have finite support (only a finite number of elements are not fixed). Conjugacy classes of S_∞ are in bijection with partitions having no parts equal to 1 (ie. with partitions π such that $\pi = (\pi_1, \dots, \pi_r)$ and $\pi_r \geq 2$). Let $\sigma_\pi \in S_\infty$ denote an element of cycle type π. Setting $[\alpha](\sigma_\pi) = \varphi_\pi([\alpha])$ we get hence a kind of "character ring" with values in $\mathbf{Q}[x]$ for S_∞. Indeed the previous theorem shows easily that $a(\sigma_\pi) = b(\sigma)$ for all σ in S_∞ implies $a = b$ (where $a, b \in P$) and $a(\sigma_\pi) = a(\sigma_\psi)$ for all $a \in P$ implies $\pi = \psi$.

An easy consequence of Theorem 4.1 is the following

4.2. Corollary. — *For each partition π the linear application*

$$[\alpha] \longmapsto \varphi_\pi([\alpha])(|\pi|)$$

defines a homomorphism of P into Z. These homomorphisms are all distinct and only 0 *is in the kernel of all these homomorphisms.*

Formulas giving the values of $\varphi_\pi(z_k)(|\pi|)$ for the generators $z_1, z_2, \dots$ can easily be retrieved from Theorem 4.1.

5. Some open problems

(1) Does there exist an analogue of the Littlewood-Richardson rules for the product $\circ$ of P?

(2) Is there a natural interpretation of the homomorphisms $\varphi_{[\alpha]}$ introduced in section 4? Is there a natural scalar product between them?

(3) The algebra LR carries also a Hopf-algebra structure (cf [**Z**]. Is there also a natural Hopf-algebra structure on P? If yes, is there a natural deformation from the structure on LR to the structure on P?

(4) Can this be generalized by replacing S_n by $\mathrm{GL}_n(\mathbf{F}_q)$ or by some other family of Lie groups over finite fields.

(5) The differential operators (or finite difference operators) D_k are related to the power sums $p_k = \sum_i x_i^k$ of $\mathbb{Q}[p_1, p_2, \dots]$. Have the differential operators $\frac{d}{de_k}$ or $\frac{d}{dh_k}$ (or the corresponding finite difference operators) any combinatorial meaning? It would perhaps be even more interesting to find differential operators associated to some Schur functions since these correspond to irreducible characters of S_n (and Schur functions are in some sense dual to the products of power sums).

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