Optimal Regularity for $\overline{\partial}_b$ on CR manifolds

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Abstract. In this paper an explicit integral formula is derived for solutions of the tangential Cauchy-Riemann equations on CR q- concave manifolds, and best possible estimates are obtained.

0. Introduction

The aim of this paper is to prove the following theorem:

THEOREM 0.1 *Let M be a q-concave CR generic submanifold (cf.sect 1.2) of codimension k and of class $\mathcal{C}^{2+\ell}$(resp. $\mathcal{C}^{3+\ell}$) in $\mathbb{C}^n$ $(\ell \geq 0)$ and z_0 a point in M. Then there exist an open neighborhood $M_0 \subseteq M$ of z_0 and kernels $\mathcal{R}_r(\zeta, z)$, for $r = 0, \ldots, q-1, n-k-q, \ldots, n-k$, with the following properties,*

(i) For every domain $\Omega \subset\subset M_0$ with piecewise $\mathcal{C}^1$ boundary and every $\mathcal{C}^1$ $(0,r)$-form f on $\overline{\Omega}$ $(0 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k)$, we have

$$f = \overline{\partial}_b \int_\Omega f \wedge \mathcal{R}_{r-1} - \int_\Omega \overline{\partial}_b f \wedge \mathcal{R}_r + \int_{b\Omega} f \wedge \mathcal{R}_r$$

on Ω.

(ii) For every open set $\Omega \subset\subset M_0$ the integral operator $\int_\Omega \cdot \wedge \mathcal{R}_r$ is a bounded linear operator from $W^{\ell,\infty}_{0,r+1}(\Omega)$ (cf.sect 1.1) to $\mathcal{C}^{\ell+\frac{1}{2}}_{0,r}(\Omega)$ for $r \geq n-k-q$ (resp. $r \leq q-1$).

This theorem has the following interesting Corollary

COROLLARY 0.2 *Let M be a 1-concave CR generic $\mathcal{C}^{3+\ell}$-submanifold of a complex manifold. Let T be a distribution of order 0 on M. If $\overline{\partial}_b T$ is defined by a $\mathcal{C}^\ell$ $(0,1)$-form on M then T is defined by a $\mathcal{C}^{\ell+\frac{1}{2}}$- function on M.*

The importance of Corollary 0.2 lies in the fact that under the hypothesis of 1-concavity the tangential Cauchy-Riemann equation for (0,1)-currents cannot be solved locally (see [3]).

Corollary 0.2 improves a result which was obtained by the author in [5](see also a related result given by Henkin and Airapetjan in [2], theorem 1) where he proved $\frac{1}{2} - \epsilon$(resp. $\frac{1}{2^k} - \epsilon$)Hölder regularity for $\overline{\partial}_b$ when M is of class $\mathcal{C}^3$(resp.$\mathcal{C}^2$). We don't know how to avoid the loss of regularity in the $\mathcal{C}^2$-case even for hypersurfaces.

The study of the tangential Cauchy-Riemann equations with the use of explicit integral formulas was initiated by Henkin [11](see also [1] and [2]). For further references

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and results on CR manifolds we refer the reader to the survey of Henkin[12] and the book of Boggess[7].

It is known that a fundamental solution for the $\overline{\partial}_b$ operator on certain hypersurfaces (see Henkin [13], Harvey-Polking[10], Boggess[7],Fischer-Leiterer[9]) can be constructed as the jump of two kernels, obtained by applying to the usual Bochner-Martinelli-Koppelman kernel (BMK kernel) in $\mathbb{C}^n$, a solution operator for $\overline{\partial}$, once on the left and once on the right hand side of the hypersurface.

Solutions for such equations can be given by applying the generalized Koppelman (cf.section 1.3) to the BMK section and the barrier functions of the hypersurface as was done in [13], [10] and [7] or by using a homotopy operator for $\overline{\partial}$ of Grauert-Lieb-Henkin type as in [9].

Inspired by the definition of a hyperfunction of several variables, the present author generalized in [5] the construction of Fischer/Leiterer [9]to higher codimensional CR submanifolds by solving some $\overline{\partial}$ equations on certain wedges attached to such manifolds with the use of $\overline{\partial}$ homotopy operators from [16] and [17].

In this paper we shall show that such equations can also be solved up to some error terms by using the Koppelman lemma (see (2.2)) and the key idea in this work is to "deform" via this lemma those error terms into ones with vanishing coefficients for some bidegrees (see lemmas 2.2 and 2.3), the strict q-convexity plays here an important role.

We shall give two fundamental solutions to the tangential Cauchy-Riemann complex. The first one (cf.sect 2.1) does not yield sharp estimates for the solutions of $\overline{\partial}_b$ (when $k > 1$) but is a "necessary" step to construct the second one (cf. sect 2.2) corresponding to kernels $\mathcal{R}_r$. To derive the latter fundamental solution from the former, we shall use an idea of Henkin [13].

In [8] B.Fischer proved Theorem 0.1 and Corollary 0.2 for hypersurfaces by using a version of the first fundamental solution.

Recently, Polyakov [20] proved sharp estimates for global solutions of $\overline{\partial}_b$ on q-concave CR manifolds, in Lipschitz spaces of Stein [21].

POLYAKOV'S THEOREM *Let M be a q-concave CR generic C^4-submanifold in $\mathbb{C}^n$ with $q \geq 2$ and let M' be a relatively compact open subset of M. Then for any $r = 1, \ldots, q-1$ there exist linear operators*

$$R_r : L^s_{(0,r)}(M) \to \Gamma^{s,1}_{(0,r-1)}(M) \text{ and } H_r : L^s_{(0,r)}(M) \to L^s_{(0,r)}(M)$$

such that for any $s \in [1, \infty]$ R_r is bounded and H_r is compact and such that for any differential form $f \in \mathcal{C}^\infty_{(0,r)}(M)$ the following equality:

$$f(z) = \overline{\partial}_b R_r(f)(z) + R_{r+1}(\overline{\partial}_b f)(z) + H_r(f)(z)$$

holds for $z \in M'$.

Our method is quite different from that of Polyakov, and it is not clear how one can get an analogous result to Corollary 0.2 from Polyakov's theorem.

This paper is organized as follows. In section 1.2 we give the definition of a q-concave CR manifold and we define the $\overline{\partial}_b$ operator. In section 1.3 we recall the generalized

Koppelman lemma which plays a key role in the construction of our kernels. In section 1.4 we recall the construction of a barrier function and a Leray map for a hypersurface at a point where the Levi form has some positive eigenvalues. In section 1.5 we state some elementary facts from Algebraic Topology, which we shall use later. Section 2.1 is devoted to the construction of our first fundamental solution. In section 2.2 we construct our second fundamental solution and in section 3 we prove estimates for our kernels.

1. Preliminaries and notations

1.1. Let X be a complex manifold, and M a real submanifold of X .

Let f be a differential form of degree m defined on a domain $D \subseteq M$. Then we denote by $\|f(z)\|, z \in D$, the Riemannian norm of f at z (cf.[15], section 0.4), and we set

$$\|f\|_{0,D} = \sup_{z \in D} \|f(z)\|$$

and

$$\|f\|_{\alpha,D} = \|f\|_{0,D} + \sup_{\substack{z,\zeta \\ z \neq \zeta}} \frac{\|f(z) - f(\zeta)\|}{|\zeta - z|^{\alpha}}$$

for $0 < \alpha < 1$.

If $0 < \alpha < 1$, then a form on D is called α-Hölder continuous on D if

$$\|f\|_{\alpha,K} \leq \infty.$$

for all compact sets $K \subseteq D$.

If l is a non-negative integer and $0 < \alpha < 1$, then we say f is a $\mathcal{C}^{l+\alpha}$ form on D if f is of class $\mathcal{C}^l$ and all derivatives of order $\leq l$ of f are α-Hölder continuous on D.

$W_{\star}^{\ell,\infty}(D)$ is the space of all $\mathcal{C}^{\ell}$-forms such that all derivatives up to order ℓ belong to $L^{\infty}(D)$.

1.2. Let M be a real submanifold of class $\mathcal{C}^2$ in $\mathbb{C}^n$ defined by

$$M = \{z \in \Omega / \rho_1(z) = \cdots = \rho_k(z) = 0\} \quad 1 \leq k \leq n \tag{1.1}$$

where Ω is an open subset of $\mathbb{C}^n$ and the functions ρ_ν, $1 \leq \nu \leq k$, are real-valued functions of class $\mathcal{C}^2$ on Ω with the property $d\rho_1(z) \wedge \cdots \wedge d\rho_k(z) \neq 0$ for each $z \in M$.

We denote by $T_z^{\mathbb{C}}(M)$ the complex tangent space to M at the point $z \in M$ i.e.,

$$T_z^{\mathbb{C}}(M) = \{\zeta \in \mathbb{C}^n / \sum_{j=1}^{n} \frac{\partial \rho_\nu}{\partial z_j}(z)\zeta_j = 0, \nu = 1, \dots, k\}.$$

We have $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M) \geq n - k$. The submanifold M is called a Cauchy-Riemann manifold (CR-manifold) if the number $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M)$ does not depend on the point

point $z \in M$. M is said to be CR generic if for every $z \in M$, $\dim_{\mathbb{C}} T_z^{\mathbb{C}}(M) = n-k$, this is equivalent to :

$$\overline{\partial}\rho_1 \wedge \overline{\partial}\rho_2 \wedge \cdots \wedge \overline{\partial}\rho_k \neq 0 \text{ on } M. \tag{1.2}$$

If M is CR generic , we call M q-concave, $0 \leq q \leq \frac{n-k}{2}$, if for each $z \in M$ and every $x \in \mathbb{R}^k \setminus \{0\}$ the following hermitian form on $T_z^{\mathbb{C}}(M)$

$$\sum_{\alpha,\beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \overline{z}_\beta}(z)\zeta_\alpha \overline{\zeta}_\beta, \text{ where } \rho_x = x_1\rho_1 + \cdots x_k\rho_k$$

has at least q negative eigenvalues.

If M is CR generic then we denote by $\mathcal{C}^s_{p,r}(M)$ the space of differential forms of type (p,r) on M which are of class $\mathcal{C}^s$. Here, two forms f and g in $\mathcal{C}^s_{p,r}(M)$ are considered to be equal if and only if for each form $\varphi \in \mathcal{C}^\infty_{n-p,n-k-r}(\Omega)$ of compact support, we have

$$\int_M f \wedge \varphi = \int_M g \wedge \varphi.$$

We denote by $\mathcal{L}^{(-s)}_{p,r}(M)$ the dual space to $\mathcal{C}^s_{n-p,n-k-r}(M)$.

We define the tangential Cauchy-Riemann operator on forms in $\mathcal{L}^{(-s)}_{0,r}(M)$ as follows. If $u \in \mathcal{C}^s_{0,r}(M)$, $s \geq 1$, then u can be extended to a smooth form $\tilde{u} \in \mathcal{C}^s_{0,r}(\Omega)$ and we may set

$$\overline{\partial}_{\mathrm{b}} u := \overline{\partial}\tilde{u}|_M$$

It follows from the condition for equality of forms on M that this definition does not depend on the choice of the extended form $\tilde{u}$. In general, for for forms $u \in \mathcal{L}^{(-s)}_{0,r-1}(M)$ and $f \in \mathcal{L}^{(-s)}_{0,r}(M)$, by definition,

$$\overline{\partial}_{\mathrm{b}} u = f$$

will mean that for each form $\varphi \in \mathcal{C}^\infty_{n-p,n-k-r}(\Omega)$ of compact support we have

$$\int_M f \wedge \varphi = (-1)^r \int_M u \wedge \overline{\partial}\varphi.$$

1.3. The generalized Koppelman lemma. In this section we recall a formal identity (the generalized Koppelman lemma) which is essential for the construction of our kernels. The exterior calculus we use here was developed by Harvey -Polking in [10].

Let V be an open set of $\mathbb{C}^n \times \mathbb{C}^n$. Suppose $G : V \to \mathbb{C}^n$ is a $\mathcal{C}^\infty$ map. We write

$$G(\zeta, z) = (g_1(\zeta, z), \ldots, g_n(\zeta, z))$$

and we use the following notation

$$G(\zeta, z).(\zeta - z) = \sum_{j=1}^n g_j(\zeta, z)(\zeta_j - z_j)$$

$$G(\zeta,z).d(\zeta-z)=\sum_{j=1}^{n} g_j(\zeta,z)d(\zeta_j-z_j)$$

$$\overline{\partial}_{\zeta,z}G(\zeta,z).d(\zeta-z)=\sum_{j=1}^{n}\overline{\partial}_{\zeta,z}g_j(\zeta,z)d(\zeta_j-z_j)$$

where $\overline{\partial}_{\zeta,z}=\overline{\partial}_{\zeta}+\overline{\partial}_{z}$.

We define the Cauchy-Fantappie form ω^G by

$$\omega^G=\frac{G(\zeta,z).d(\zeta-z)}{G(\zeta,z).(\zeta-z)}$$

on the set where $G(\zeta,z).(\zeta-z)\neq 0$.

Given m such maps, G^j, $1\leq j\leq m$, we define the kernel

$$\Omega(G^1,\dots,G^m)=\omega^{G^1}\wedge\dots\wedge\omega^{G^m}\wedge\sum_{\alpha_1+\dots+\alpha_m=n-m}(\overline{\partial}_{\zeta,z}\omega^{G^1})^{\alpha_1}\wedge\dots\wedge(\overline{\partial}_{\zeta,z}\omega^{G^m})^{\alpha_m}$$

on the set where all the denominators are nonzero.

LEMME 1.1. (The generalized Koppelmann lemma)

$$\overline{\partial}_{\zeta,z}\Omega(G^1,\dots,G^m)=\sum_{j=1}^{m}(-1)^j\Omega(G^1,\dots,\hat{G^j},\dots,G^m)$$

on the set where the denominators are nonzero, the symbol $\hat{G^j}$ means that the term G^j is deleted.

The following lemma is useful for the estimation of the kernel defined above.

LEMME 1.2. *For $k\geq 0$*

$$\omega^G\wedge(\overline{\partial}_{\zeta,z}\omega^G)^k=\frac{G(\zeta,z).d(\zeta-z)}{G(\zeta,z).(\zeta-z))}\wedge\Big(\frac{\overline{\partial}_{\zeta,z}G.d(\zeta-z)}{G(\zeta,z).(\zeta-z)}\Big)^k$$

For a proof of these two lemmas we refer the reader to [10] or [7].

It follows from Lemma 1.2 that for $B=\overline{\zeta-z}$, $\Omega(B)$ is the classical Martinelli-Bochner Koppelman kernel in $\mathbb{C}^n$.

1.4. Barrier function. In this section, we shall construct a barrier function for a hypersurface at a point where the Levi form has some positive eigenvalues.

For a detailed proof of what follows we refer the reader to sect.3 in [16].

Let H be an oriented real hypersurface of class $\mathcal{C}^2$ in $\mathbb{C}^n$ defined by

$$H=\{z\in\Omega/\rho(z)=0\}$$

where Ω is an open subset of $\mathbb{C}^n$ and ρ is a real-valued function of class $\mathcal{C}^2$ on Ω with $d\rho(z)\neq 0$ for each $z\in H$.

Denote by $F(\cdot,\zeta)$ the Levi polynomial of ρ at a point $\zeta\in\Omega$, i.e.

$$F(z,\zeta)=2\sum_{j=1}^{n}\frac{\partial\rho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)-\sum_{j,k=1}^{n}\frac{\partial^2\rho(\zeta)}{\partial\zeta_j\partial\zeta_k}(\zeta_j-z_j)(\zeta_k-z_k)$$

$\zeta\in\Omega, z\in\mathbb{C}^n$.

Let $z^0\in H$ and T be the largest vector subspace of $\mathbb{C}^n$ such that the Levi form of ρ at z^0 is positive definite on T. Set $\mathbf{dimT=d}$ and suppose $d\geq 1$.

Denote by P the orthogonal projection from C^n onto T, and set $Q=I-P$. Then it follows from Taylor's theorem that there exist a number R and two positives constants A and α such that the following holds:

$$\operatorname{Re}F(z,\zeta)\geq\rho(\zeta)-\rho(z)+\alpha|\zeta-z|^2-A|Q(\zeta-z)|^2 \tag{1.3}$$

for $|z^0-\zeta|\leq R$ and $|z^0-z|\leq R$.

Since ρ is of class $\mathcal{C}^2$ on Ω , We can find $\mathcal{C}^\infty$ functions $a^{kj}(k,j=1,\dots,n)$ on a neighborhood U of z^0 such that

$$\left|a^{kj}(\zeta)-\frac{\partial^2\rho(\zeta)}{\partial\zeta_k\partial\zeta_j}\right|<\frac{\alpha}{2n^2}$$

for all $\zeta\in U$. And then we have

$$\left|\sum_{k,j=1}^{n}\Big(a^{kj}(\zeta)-\frac{\partial^2\rho(\zeta)}{\partial\zeta_k\partial\zeta_j}\Big)t_kt_j\right|\leq\frac{\alpha}{2}|t|^2$$

for all $\zeta\in U$ and $t\in\mathbb{C}^n$. Set

$$\widetilde{F}(\zeta,z)=2\sum_{j=1}^{n}\frac{\partial\rho(\zeta)}{\partial\zeta j}(\zeta_j-z_j)-\sum_{k,j=1}^{n}a^{kj}(\zeta)(\zeta_k-z_k)(\zeta_j-z_j)$$

for $(z,\zeta)\in\mathbb{C}^n\times U$.Then it follows from (1.3) that

$$\operatorname{Re}\widetilde{F}(\zeta,z)\geq\rho(\zeta)-\rho(z)+\frac{\alpha}{2}|\zeta-z|^2-A|Q(\zeta-z)|^2 \tag{1.4}$$

for $|z^0-\zeta|\leq R$ and $|z^0-z|\leq R$.

Denote by Q_{kj} the entries of the matrix Q i.e

$$Q=\big(Q_{kj}\big)_{k,j=1}^{n}\quad(k=\ \text{column index}).$$

We set for $(z,\zeta)\in\mathbb{C}^n\times U$

$$\begin{cases} g_j(\zeta,z) &=2\frac{\partial\rho(\zeta)}{\partial\zeta j}-\sum_{k=1}^{n}a^{kj}(\zeta)(\zeta_k-z_k)+A\sum_{k=1}^{n}\overline{Q_{kj}(\zeta_k-z_k)}\\ G(\zeta,z) &=\big(g_1(\zeta,z),,\dots,,g_n(\zeta,z)\big)\\ \Phi(\zeta,z) &=G(\zeta,z).(\zeta-z).\end{cases}$$

Since Q is an orthogonal projection, then we have

$$\Phi(\zeta,z)=\widetilde{F}(\zeta,z)+A|Q(\zeta-z)|^2$$

And then it follows from (1.4) that

$$\operatorname{Re}\Phi(\zeta,z) \geq \rho(\zeta) - \rho(z) + \frac{\alpha}{2}|\zeta - z|^2 \tag{1.5}$$

for $|z^0 - \zeta| \leq R$ and $|z^0 - z| \leq R$.

G is called a **Leray map** and Φ is called a **barrier function** of H(or ρ) at z^0.

DEFINITION— *A map f defined on some complex manifold X will be called k-holomorphic if, for each point $\xi \in X$, there exist holomorphic coordinates $h_1, \ldots, h_k$ in a neighborhood of ξ such that f is holomorphic with respect to $h_1, \ldots, h_k$.*

LEMMA 1.3. *For every fixed $\zeta \in U$, the map $G(\zeta, z)$ and the function Φ, defined above, are d-holomorphic in $z \in \mathbb{C}^n$.*

Proof. Choose complex linear coordinates $h_1, \ldots, h_n$ on $\mathbb{C}^n$ with

$$\{z \in \mathbb{C}^n : Q(z) = 0\} = \{z \in \mathbb{C}^n : h_{d+1}(z) = \cdots = h_n(z) = 0\}.$$

Then the map $\mathbb{C}^n \ni z \to \overline{Q(\zeta - z)}$ is independent of $h_1, \ldots, h_d$. This implies that $G(\zeta, .)$ is complex linear with respect to $h_1, \ldots, h_d$, and $\Phi(\zeta, .)$ is quadratic complex polynomial with respect to $h_1, \ldots, h_d$. □

1.5 Some Algebraic Topology. Let N be a positive integer.Then we call p-simplex, $1 \leq p \leq N$, every collection of p lineary independent vectors in $\mathbb{R}^N$.

We define S_p as the set of all finite formal linear combinations, with integer coefficients, of p-simplices.

Let $\sigma = [a_1, \ldots, a_p]$ be a p-simplex, then we set

$$\partial_j \sigma = [a_1, \ldots, \hat{a_j}, \ldots, a_p]$$

for $1 \leq j \leq p$ and

$$\partial \sigma = \sum_{j=1}^{p} (-1)^j \partial_j \sigma$$

(this definition holds also for any collection of p vectors). If $1 \leq j_1 \leq p \ldots 1 \leq j_r \leq p - r$, we define

$$\partial^r_{j_r \cdots j_1} \sigma = \partial_{j_r}(\partial^{r-1}_{j_{r-1} \cdots j_1} \sigma)$$

where $\partial^1_j \sigma = \partial_j \sigma$.

All of these operations can be extended by linearity to S_p.

If σ is a p-simplex defined as above then we define the barycenter of σ by

$$b(\sigma) = \frac{1}{p} \sum_{j=1}^{p} a_j.$$

Now we define the first barycentric subdivision of σ by the following

$$sd(\sigma) = (-1)^{p+1} \sum_{\substack{j_1, \ldots, j_{p-1} \\ 1 \leq j_i \leq p-i+1}} (-1)^{j_1 + \cdots + j_{p-1}} \Big[b(\sigma), b(\partial_{j_1} \sigma), \ldots, b(\partial^{p-1}_{j_{p-1} \cdots j_1} \sigma) \Big].$$

By linearity we can also define the first barycentric Subdivision of any element of S_p.

It is easy to see that

LEMMA 1.4. *If σ is an element of S_p, then*

$$sd(\partial\sigma) = \partial sd(\sigma)$$

The barycentric subdivision of higher order of an element σ of S_p is defined as follows, we set for $m \geq 2$

$$sd^m(\sigma) = sd(sd^{m-1}(\sigma)).$$

$sd^0(\sigma)$ and $sd^1(\sigma)$ are defined respectively as σ and $sd(\sigma)$.

The following lemma is basic in Algebraic Topology.

LEMMA 1.5. *Given a simplex σ, and given $\epsilon > 0$, there is an m such that each simplex of $sd^m\sigma$ has diameter less than ϵ.*

For a proof of this lemma, see for example [19].

Let $\sigma = [\nu_1, \ldots, \nu_p]$ and $\tau = [\mu_1, \ldots, \mu_r]$. We shall adopt the following notations

$$[\sigma, \tau] = [\sigma, \mu_1, \ldots, \mu_r] = [\nu_1, \ldots, \nu_p, \tau] = [\nu_1, \ldots, \nu_p, \mu_1, \ldots, \mu_r].$$

Now let σ be a p-simplex, $p \geq 2$, set

$$T(\sigma) = \Big[b(\sigma), \sigma\Big] + \sum_{\ell=1}^{p-2} \sum_{\substack{j_1, \ldots, j_\ell \\ 1 \leq j_i \leq p-i+1}} (-1)^{j_1 + \cdots + j_\ell} \Big[b(\sigma), b(\partial_{j_1}\sigma), \ldots, b(\partial^\ell_{j_\ell \cdots j_1}\sigma), \partial^\ell_{j_\ell \cdots j_1}\sigma\Big]$$

and extend T by linearity to S_p.

If τ is an element of S_1 then we set

$$T(\tau) = 0$$

PROPOSITION 1.6 *If σ is an element of S_p, $p \geq 2$, then*

$$\partial T(\sigma) + T(\partial\sigma) = sd(\sigma) - \sigma.$$

This proposition follows by a straightforward computation.

2. Fundamental solutions for $\overline{\partial}_{\mathrm{b}}$

In this section, we shall construct two fundamental solutions for the tangential Cauchy-Riemann Complex. The second solution will be derived from the first and will yield optimal Hölder estimates for $\overline{\partial}_{\mathrm{b}}$.

Let us begin by some notations.

2.0. Notations. Throughout this section M will denote a q-concave CR generic $\mathcal{C}^2$ submanifold of codimension k in $\mathbb{C}^n$.

$\mathcal{I}$ is the set of all subsets $I \subseteq \{\pm 1, \ldots, \pm k\}$ such that $|i| \neq |j|$ for all $i, j \in I$ with $i \neq j$.

For $I \in \mathcal{I}$, $|I|$ denotes the number of elements in I . We set

$$\Delta_{1\cdots|I|} = \{(\lambda_1, , \ldots, , \lambda_{|I|}) \in (\mathbb{R}^+)^{|I|} \ with \sum_{j=1}^{|I|} \lambda_j = 1\}$$

$\mathcal{I}(\ell), 1 \leq \ell \leq k$, is the set of all $I \in \mathcal{I}$ with $|I| = \ell$.

$\mathcal{I}'(\ell), 1 \leq \ell \leq k$, is the set of all $I \in \mathcal{I}(\ell)$ of the form $I = \{j_1, \ldots, j_\ell\}$ with $|j_\nu| = \nu$ for $\nu = 1, \ldots, \ell$.

If $I \in \mathcal{I}$ and $\nu \in \{1, \ldots, |I|\}$, then I_ν is the element with number ν in I after ordering I by modulus.We set $I(\hat{\nu}) = I \setminus \{I_\nu\}$.

If $I \in \mathcal{I}$, then

$$\operatorname{sgn} I := \begin{cases} 1 & \text{if the number of negative elements in } I \text{ is even} \\ -1 & \text{if the number of negative elements in } I \text{ is odd} \end{cases}$$

2.1. First fundamental solution for $\overline{\partial}_b$. In this section we shall construct our first fundamental solution for the tangential Cauchy-Riemann complex.

Let $z^0 \in M$, $U \subseteq \mathbb{C}^n$ be a neighborhood of z^0 and $\hat{\rho}_1, \ldots, \hat{\rho}_k : U \to \mathbb{R}$ be functions of class $\mathcal{C}^2$ such that :

$$M \cap U = \{\hat{\rho}_1 = \cdots = \hat{\rho}_k = 0\} \text{ and } \partial\hat{\rho}_1(z^0) \wedge \cdots \wedge \partial\hat{\rho}_k(z^0) \neq 0.$$

Since M is q-concave , it follows from lemma 3.1.1 in [1] that we can find a constant $C > 0$ such that the functions

$$\begin{aligned} \rho_j &:= \hat{\rho}_j + C \textstyle\sum_{\nu=1}^k \hat{\rho}_\nu^2 \quad (j = 1, \ldots, k) \\ \rho_j &:= -\hat{\rho}_{-j} + C \textstyle\sum_{\nu=1}^k \hat{\rho}_\nu^2 \quad (j = -1, \ldots, -k) \end{aligned}$$

have the following property : for each $I \in \mathcal{I}$ and every $\lambda \in \Delta_{1\cdots|I|}$ the Levi form of $\lambda_1 \rho_{I_1} + \cdots + \lambda_{|I|} \rho_{I_{|I|}}$ at z^0 has at least $q + k$ positive eigenvalues.

Let $(e_1, \ldots, e_k)$ be the canonical basis of $\mathbb{R}^k$, set $e_{-j} := -e_j$ for every $1 \leq j \leq k$.

Let $I = (j_1, \ldots, j_k)$ be in $\mathcal{I}'(k)$, set

$$\Delta_I = \{\sum_{i=1}^k \lambda_i e_{j_i} \ with \ \lambda_i \geq 0, \ all \ i, \ and \ \sum_{i=1}^k \lambda_i = 1\},$$

and for each $a = \sum_{i=1}^k \lambda_i e_{j_i}$, let G_a and Φ_a be respectively the Leray map and the barrier function at z^0 corresponding to $\rho_a = \lambda_1 \rho_{j_1} + \cdots + \lambda_k \rho_{j_k}$ (see sect. 1.4).

We call ρ_a(rep.ϕ_a) the defining function (resp.the barrier function) of M in the direction a.

Let $\sigma = [a^1, \ldots, a^p]$, $p \geq 1$, be a collection of p vectors , where $a^i \in \bigcup_{I \in \mathcal{I}'(k)} \Delta_I$, for every $1 \leq i \leq k$.

Then we define

$$\tilde{\Omega}[\sigma] := \Omega(G_{a^1}, \ldots, G_{a^p})$$

(cf sect 1.3) , and for every $0 \leq s \leq n$ and every $0 \leq r \leq n-p$, we define $\tilde{\Omega}_{s,r}[\sigma]$ as the piece of $\tilde{\Omega}[\sigma]$ which of type (s,r) in z.(cf sect 1.3).

If we denote by S_p the set of all finite formal linear combinations of such collections, with integer coefficients, and we extend $\tilde{\Omega}$ by linearity to S_p; then the generalized Koppelman lemma now becomes

LEMMA 2.1 *For every $\tau \in S_p$, we have*

$$\overline{\partial}_{\zeta,z}\tilde{\Omega}[\tau] = \tilde{\Omega}[\partial\tau]$$

outside the singularities.

Let $I = (j_1, \ldots, j_l)$ be in $\mathcal{I}'(l)$, $1 \leq l \leq k$ and $\sigma_I = [e_{j_1}, \ldots, e_{j_l}]$. Then by continuity of the Levi form, by lemma 1.3 and lemma 1.5, we can find a positive integer m independant of I and l such that for every simplexe $\tau = [a^1, \ldots, a^l]$ in $sd^m(\sigma_I)$, the Leray maps of $G_{a^1}, \ldots, G_{a^l}$ are $q+k$-holomorphic in the same directions in $z \in \mathbb{C}^n$. Therefore we have the following lemma.

LEMMA 2.2 *There is a positive integer m such that for every $I \in \mathcal{I}'(l)$, $1 \leq l \leq k$, any $s \geq 0$ and every $r \geq n-k-q+1$*

(i) $\tilde{\Omega}_{s,r}(sd^m(\sigma_I)) = 0$

(ii)$\overline{\partial}_z\tilde{\Omega}_{s,r-1}(sd^m(\sigma_I)) = 0$

on the set where all the denominators are non-zero.

Proof.– this follows by linearity from the fact that $\Omega(G_{a^1}, \ldots, G_{a^l}) = 0$, for every $[a^1, \ldots, a^l]$ in $sd^m(\sigma_I)$. The last statement is easy to prove , looking at the definition of Ω (see sect. 1.3), since $G_{a^1}, \ldots, G_{a^l}$ are $q+k$-holomorphic in the same directions in $z \in \mathbb{C}^n$

By the same arguments, we have $\tilde{\Omega}_{s,r}(\partial(sd^m(\sigma_I))) = \tilde{\Omega}_{s,r}(sd^m(\partial\sigma_I) = 0$, for all $r \geq n-k-q+1$,and from lemma 2.1, we have

$$\overline{\partial}_z\tilde{\Omega}_{s,r-1}(sd^m(\sigma_I)) = -\overline{\partial}_\zeta\tilde{\Omega}_{s,r}(sd^m(\sigma_I)) + \tilde{\Omega}_{s,r}(\partial(sd^m(\sigma_I))$$

which implies, taking into account (i), the statement (ii). □

Now let D be a neighborhood of z^0 such that for every $1 \leq l \leq k$, all $0 \leq i \leq m$ and every vertex a in $sd^i(\sigma_I)$, the barrier function Φ_a satisfies an inequality such (1.5) , for $\zeta, z \in D$. Set

$$M_0 := M \cap D,$$

and for $I \in \mathcal{I}$

$$\begin{aligned} D_I &:= \{\rho_{I_1} < 0\} \cap \cdots \cap \{\rho_{I_{|I|}} < 0\} \cap D, \\ D_I^* &:= \{\rho_{I_1} > 0\} \cap \cdots \cap \{\rho_{I_{|I|}} > 0\} \cap D. \\ S_I &:= \{\rho_{I_1} = \cdots = \rho_{I_{|I|}} = 0\} \cap D, \\ S_{\{j\}}^+ &:= \overline{D}_{\{j\}} \text{ for } j = \pm 1, \ldots, \pm k \\ S_I^+ &:= S_{I(|\widehat{I}|)} \cap \overline{D}_{\{I_{|I|}\}} \text{ if } I \in \mathcal{I} \text{ and } |I| \geq 2. \end{aligned}$$

We oriente these manifolds as follows :

$$\begin{array}{rl} D_I \text{ and } D_I^* & \text{as } \mathbb{C}^n \quad \forall I \in \mathcal{I} \\ S_{\{j\}}^+ & \text{as } D_{\{j\}} \text{ for } j = \pm 1, \ldots, \pm k \\ S_I & \text{as } \partial S_I^+ \quad I \in \mathcal{I} \\ S_I^+ & \text{as } S_{I(|\widehat{I}|)} \text{ for all } I \in \mathcal{I} \text{ such that } |I| \geq 2 \\ M_0 & \text{as } S_I \text{ where } I = \{1, \ldots, k\}. \end{array}$$

Fix $1 \leq l \leq k$ and $I \in \mathcal{I}'(l)$.
Let $B = \left(\overline{\zeta_1 - z_1}, \ldots, \overline{\zeta_n - z_n}\right)$ and define

$$\tilde{\Omega}_B[\tau] := \Omega(B, G_{\nu^1}, \ldots, G_{\nu^p}) \tag{2.1}$$

for any $\tau = [\nu^1, \ldots, \nu^p]$ in S_p, $p \geq 1$. Extend this operation, by linearity, to all elements of S_p.

Now by applying lemma 1.1 we get

$$\overline{\partial}_{\zeta,z} \tilde{\Omega}_B[\sigma_I] = -\tilde{\Omega}[\sigma_I] - \tilde{\Omega}_B[\partial \sigma_I] \tag{2.2}$$

(where $\tilde{\Omega}_B[\partial \sigma_I] := \Omega(B)$ if $|I| = 1$) for $z \in \overline{D_I}$ and $\zeta \in \overline{D_I^*}$, with $\zeta \neq z$.

Let $|I| \geq 2$ and T be defined as in sect. 1.5 and m an integer such that lemma 2.2 holds.

By applying lemma 1.4, lemma 2.1 , proposition 1.6 , we obtain

$$\overline{\partial}_{\zeta,z} \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\sigma_I))] = -\tilde{\Omega}[\sigma_I] - \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\partial \sigma_I))] + \tilde{\Omega}[sd^m(\sigma_I)] \tag{2.3}$$

for $z \in \overline{D_I}$ and $\zeta \in \overline{D_I^*}$, with $\zeta \neq z$.

Now define for $|I| \geq 2$

$$K^I(\zeta, z) = \tilde{\Omega}_B[\sigma_I](\zeta, z) - \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\sigma_I))](\zeta, z),$$

$$B^I(\zeta, z) = \sum_{\nu=1}^{|I|} (-1)^{\nu+1} K^{I(\hat{\nu})} = -\tilde{\Omega}_B[\partial \sigma_I](\zeta, z) + \sum_{i=0}^{m-1} \tilde{\Omega}[T(sd^i(\partial \sigma_I))](\zeta, z) \tag{2.4}$$

and set for $|I| = 1$

$$K^I(\zeta, z) = \tilde{\Omega}_B[\sigma_I](\zeta, z),$$

and

$$B^I(\zeta, z) = \Omega(B)(\zeta, z) \ (\ the Martinelli - Bochner - Koppelman\ kernel). \tag{2.5}$$

Then we have the following

LEMMA 2.3 *(i) For $z \in \overline{D_I}$ and $\zeta \in \overline{D_I^*}$, with $\zeta \neq z$, we have*

$$\overline{\partial}_{\zeta,z} K^I = B^I - \tilde{\Omega}[sd^m(\sigma_I)]$$

(ii) There exist a constant $\mathcal{C} > 0$ and a finite family $\{\gamma_1, \dots, \gamma_L\}$ of linearly independent families $\gamma_i = [\gamma_i^1, \dots, \gamma_i^{|I|}]$ in Δ_I such that

$$\|K^I(\zeta, z)\| \leq \mathcal{C} \sum_{i=1}^{L} \frac{1}{\prod_{j=1}^{|I|} |\Phi^{\gamma_i^j}(\zeta, z)| \, |\zeta - z|^{2n-2|I|-1}}$$

Proof – (i) is a consequence of (2.1) and (2.3). The estimate in (ii) is easy to see from the definition of Ω, by using lemma 1.2 and (1.5) (see the proof of Lemma 2.6). □

We shall now show that the kernel K^I has locally integrable coefficients on S_I in both variables ζ and z.

LEMMA 2.4 *Let $I \in \mathcal{I}$ and $(\gamma^1, \dots, \gamma^{|I|})$ be a family of linearly independent vectors in $\mathbb{R}^{|I|}$ and $z \in \overline{D}_I$ then there exists $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$*

$$\int_{\substack{\zeta \in S_I \\ |\zeta - z| < \varepsilon}} \frac{d\lambda(\zeta)}{\prod_{i=1}^{|I|} |\Phi_{\gamma^i}(\zeta, z)| |\zeta - z|^{2n-2|I|-1}} \leq C\varepsilon \big(1 + |\ell n \varepsilon|\big)^{|I|} \qquad (2.6)$$

$$\int_{\substack{\zeta \in S^+_{I \cup \{j\}} \\ |\zeta - z| < \varepsilon}} \frac{d\lambda(\zeta)}{\prod_{i=1}^{|I|} |\Phi_{\gamma^i}(\zeta, z)| |\zeta - z|^{2n-2|I|-1}} \leq C\varepsilon \big(1 + |\ell n \varepsilon|\big)^{|I|} \qquad (2.7)$$

for all $j \in \{\pm 1, \dots, \pm k\} \setminus I, |j| \leq |I|$

$$\int_{\substack{\zeta \in S^+_I \\ |\zeta - z| < \varepsilon}} \frac{d\lambda(\zeta)}{\prod_{i=1}^{|I|-1} |\Phi_{\gamma^i}(\zeta, z)| |\zeta - z|^{2n-2|I|+1}} \leq C\varepsilon \big(1 + |\ell n \varepsilon|\big)^{|I|-1} \qquad (2.8)$$

All of the above estimates hold if we integrate with respect to z instead of ζ

Proof – Since M is CR generic and $\gamma^1, \dots, \gamma^{|I|}$ are linearly independent, we can take $Im\Phi_{\gamma^1}(\cdot, z), \dots, Im\Phi_{\gamma^{|I|}}(\cdot, z)$ as coordinates on S_I and $S^+_{I \cup \{j\}}$, for $|z - \zeta| < \varepsilon$ with $\varepsilon > 0$ very small(cf. lemma 2.3 in [6]). Thus ,taking into account the following inequality (cf. (1.5))

$$|\Phi_{\gamma^i}(\zeta, z)| \leq C(|Im\Phi_{\gamma^i}(\zeta, z)| + |\zeta - z|^2)$$

we see that the left-side term in (2.6) (resp.(2.7)) is bounded by

$$\int_{\substack{X \in \mathbb{R}^{2n-|I|} \\ |X| < \varepsilon}} \frac{dX}{\prod_{i=1}^{|I|} \big(|X_i| + |X|^2\big) |X|^{2n-2|I|-1}} \leq C\varepsilon \big(1 + |\ell n \varepsilon|\big)^{|I|}.$$

(2.8) is proved likewise and the proof of (ii) is similar. □

DEFINITION 2.5 (i) Let $I \in \mathcal{I}$ and $j \in \{\pm 1, \dots, \pm k\} \setminus I, |j| \leq |I|$. It follows from lemma 2.3 and estimates (2.6) , (2.7), that the following operators are well defined and continuous :

$$\widehat{K}^I_{0,r} \quad : \mathcal{C}^0_*(S_I) \longrightarrow \mathcal{C}^0_{0,r}(\overline{D}_I) \cap \mathcal{C}^\infty_*(D_I)$$

and

$$\widehat{K}^{I,j}_{0,r} \quad : \mathcal{C}^0_*(S^+_{I\cup\{j\}}) \longrightarrow \mathcal{C}^0_{0,r}(\overline{D}_I) \cap \mathcal{C}^\infty_*(D_I)$$

where

$$\widehat{K}^I_{0,r}f(z) \quad := \int_{\zeta\in S_I} f(\zeta)\wedge K^I_{0,r}(\zeta,z), \quad z\in\overline{D}_I, f\in\mathcal{C}^0_*(S_I)$$

and

$$\widehat{K}^{I,j}_{0,r}f(z) \quad := \int_{\zeta\in S^+_{I\cup\{j\}}} f(\zeta)\wedge K^I_{0,r}(\zeta,z), \quad z\in\overline{D}_I, f\in\mathcal{C}^0_*(S^+_{I\cup\{j\}})$$

(ii) Let $I\in\mathcal{I}$, it follows from lemma 2.3 and estimate (2.8) that the operator defined by :

$$\widehat{B}^I_{0,r}f(z) := \int_{\zeta\in S^+_I} f(\zeta)\wedge B^I_{0,r}(\zeta,z), \quad z\in\overline{D}_I, f\in\mathcal{C}^0_*(S^+_I)$$

is continuous from $\mathcal{C}^0_*(S^+_I)$ into $\mathcal{C}^0_*(\overline{D}_I)\cap\mathcal{C}^\infty_*(D_I)$.

REMARK. (i) If $|I|\geq 2$ et $\nu\in\{1,\dots,|I|\}$ then $S^+_{I(\hat{\nu})\cup(I_\nu)} = S^+_I$ and therefore

$$\widehat{K}^{I(\hat{\nu}),I_\nu}f(z) = \int_{\xi\in S^+_I} f(\xi)\wedge K^{I(\hat{\nu})}_{0,r}(z,\xi) \text{ for } z\in\overline{D}_{I(\hat{\nu})} \text{ and } f\in\mathcal{C}^0_*(S^+_I)$$

and then from the definition of $B^I_{0,r}$, we obtain

$$\widehat{B}^I_{0,r}f(z) = \sum_{\nu=1}^{|I|}(-1)^{\nu+|I|}\widehat{K}^{I(\hat{\nu}),I_\nu}_{0,r}f(z). \tag{2.9}$$

LEMMA 2.6 *Let $I\in\mathcal{I}, n-k-q+1\leq r\leq n-k$ and $f\in\mathcal{C}^1_{0,r}(D)$ with compact support on D. Then we have the following equality in the sens of currents :*

$$\overline{\partial}\widehat{K}^I_{0,r-1}f + (-1)^{|I|+1}\widehat{K}^I_{0,r}\overline{\partial}f = (-1)^r\left[\overline{\partial}\widehat{B}^I_{0,r-1}f + (-1)^{|I|}\widehat{B}^I_{0,r}\overline{\partial}f\right] \tag{2.10}$$

on $\overline{D}_I$.

Proof – The following equality is true from lemma 2.3(i) and Stokes' formula : if $n-k-q+1\leq s\leq n-k, g\in C^1_{0,s+1}(D)$ with compact support on D and if $z\in D_I$ then :

$$\begin{aligned}\widehat{K}^I_{0,s}g(z) \quad = \quad & \int_{S^+_I}\overline{\partial}g\wedge K^I_{0,s}(\cdot,z) + (-1)^{s+1}\widehat{B}^I_{0,s}g(z) \\ & +(-1)^{|I|}\overline{\partial}\int_{S^+_I} g\wedge K^I_{0,s-1}(\cdot,z) + (-1)^s\int_{S^+_I} g\wedge\tilde{\Omega}_{0,s}(sd^m\sigma_I)(\cdot,z).\end{aligned} \tag{2.11}$$

Since the forms $\widehat{K}_{0,r-1}f, \widehat{K}_{0,r-1}\overline{\partial}f, \widehat{B}_{0,r-1}f$ and $\widehat{B}_{0,r}\overline{\partial}f$ are continuous on $\overline{D}_I$, it is enough to prove (2.10) on D_I (where these forms are smooth).

By setting $s=r$ and $g=\overline{\partial}f$ in (2.11) and using lemma 2.2 (i), we obtain

$$\widehat{K}^I_{0,r}\overline{\partial}f(z)=(-1)^{r+1}\widehat{B}^I_{0,r}\overline{\partial}f(z)+(-1)^{|I|}\overline{\partial}\int_{S_I^+}\overline{\partial}f\wedge K^I_{0,r-1}(\cdot,z) \tag{2.12}$$

for all $z\in D_I$.

If we set now $s=r-1$ and $g=f$ in (2.11), then we get

$$\begin{aligned}\widehat{K}^I_{0,r-1}f(z) \quad &= \quad \int_{S_I^+}\overline{\partial}f\wedge K^I_{0,r-1}(\cdot,z)+(-1)^r\widehat{B}^I_{0,r-1}f(z)\\ &+(-1)^{|I|}\overline{\partial}\int_{S_I^+}f\wedge K^I_{0,r-2}(\cdot,z)+(-1)^{r-1}\int_{S_I^+}f\wedge\tilde{\Omega}_{0,r-1}(sd^m\sigma_I)(\cdot,z).\end{aligned}$$

and then by lemma 2.2 (ii)

$$\overline{\partial}\widehat{K}^I_{0,r-1}f(z)=\overline{\partial}\int_{S_I^+}\overline{\partial}f\wedge K^I_{0,r-1}(\cdot,z)+(-1)^r\overline{\partial}\widehat{B}^I_{0,r-1}f(z) \tag{2.13}$$

for all $z\in D_I$.

(2.10) follows now from (2.12), (2.13) . □

Now define

$$K(\zeta,z):=\sum_{I\in\mathcal{I}'(k)}(sgnI)K^I(\zeta,z) \tag{2.14}$$

for $\zeta,z\in M_0$ with $\zeta\neq z$, and denote by $K_{s,r}$ the piece of K which is of type (s,r) in z.

From lemma 2.4, we see that the kernel K has locally integrable coefficients in both variables ζ and z.

Now by applying (2.10) k times , taking into account (2.9) and using the classical Martinelli-Bochner-Koppelman formula (see [5] or [6] for technical details) we obtain the following integral representation

Theorem 2.7– *Let $\Omega\subset\subset M_0$ of piecewise $\mathcal{C}^1$ boundary and f a $(0,r)$ $\mathcal{C}^1$ form on $\overline{\Omega}$ with $n-k-q+1\leq r\leq n-k$, then*

$$\begin{aligned}(-1)^{r(k+1)}f(z)=\int_{b\Omega}f(\zeta)\wedge K_{0,r}(\zeta,z)-\int_{\Omega}\overline{\partial}_b f(\zeta)\wedge K_{0,r}(\zeta,z)\\ +(-1)^{k+1}\overline{\partial}_b\int_{\Omega}f(\zeta)\wedge K_{0,r-1}(\zeta,z).\end{aligned}$$

And by a duality argument we obtain

Corollary 2.8– *Let $\Omega\subset\subset M_0$ of piecewise $\mathcal{C}^1$ boundary and f a $\mathcal{C}^1$ $(0,r)$-form on $\overline{\Omega}$ with $0\leq r\leq q-1$,then we have*

$$\begin{aligned}(-1)^{r(k+1)}f(\zeta)=\int_{z\in b\Omega}f(z)\wedge K_{n,n-k-r}(\zeta,z)-\int_{z\in\Omega}\overline{\partial}_b f(z)\wedge K_{n,n-k-r}(\zeta,z)+\\ +(-1)^{(k+1)}\overline{\partial}_b\int_{\Omega}f(z)\wedge K_{n,n-k-r+1}(\zeta,z),\end{aligned}$$

We say that K is a fundamental solution for $\overline{\partial}_{\mathrm{b}}$ on M_0.

2.2. Second fundamental solution for $\overline{\partial}_{\mathrm{b}}$. In this section, we shall construct our second fundamental solution for the tangential Cauchy-Riemann complex on M_0. This fundamental solution will be derived from the first one , by using an idea of Henkin [13](For more details, see [7]).

Let m be as in Lemma 2.2 and $\nu^* \in \bigcup_{I\in\mathcal{I}'(k)} \Delta_I$ such that

$$(*) \qquad \begin{cases} \text{For any } k\text{- simplex } \tau \text{ in } sd^m(\sigma_I)\text{, each collection of } k \text{ elements in} \\ [\nu^*, \tau] \text{ is a } k\text{-simplex.} \end{cases}$$

REMARK *The choice of such ν^* is very important for our optimal estimates.*

We adopt the following notation

$$[\nu^*, \sum_i c_i\sigma_i] = \sum_i c_i[\nu^*, \sigma_i]$$

for any element $\sum_i c_i\sigma_i$ in S_p.

Set

$$E(\zeta, z) = \sum_{I\in\mathcal{I}'(k)} sgnI\Big(\tilde{\Omega}_B[\nu^*, \sigma_I] + \sum_{i=0}^{m-1} \tilde{\Omega}[\nu^*, T(sd^i(\sigma_I))]\Big) \tag{2.15}$$

and

$$R(\zeta, z) = \sum_{I\in\mathcal{I}'(k)} (sgnI)\tilde{\Omega}[\nu^*, sd^m(\sigma_I)]. \tag{2.16}$$

Since

$$\sum_{I\in\mathcal{I}'(k)} (sgnI)\partial\sigma_I = 0,$$

then, by applying lemma 1.1, proposition 1.6 and (2.2), we obtain

$$\overline{\partial}_{\zeta,z}E(\zeta, z) = K(\zeta, z) - R(\zeta, z) \tag{2.17}$$

for $\zeta, z \in M_0$ with $\zeta \neq z$.

Now we claim that R is a fundamental solution for $\overline{\partial}_{\mathrm{b}}$ on M_0, this means that Theorem 2.7 holds also for the kernel R. To prove it, following Henkin [13], all we have to do is to show that the singularity of E is mild enough so that the identity (2.17) holds on all $M_0 \times M_0$ in the sens of distributions. For once this is done, our claim follows by applying $\overline{\partial}_{\zeta,z}$ to both sides of (2.17)and then using Theorem 2.7.

The proof of the first part of Theorem 0.1 will be complete by setting

DEFINITION 2.9

$$\mathcal{R}_r(\zeta, z) := \begin{cases} (-1)^{r(k+1)}R_{0,r}(\zeta, z) & \text{if } n-k-q \leq r \leq n-k \\ (-1)^{r(k+1)}R_{n,n-k-r}(z, \zeta) & \text{if } 0 \leq r \leq q-1 \ . \end{cases}$$

Now to realize our program, we follow the proof of Theorem 1 in [7](see chap.21).

First we need the following lemma

LEMMA 2.10– *Given $W \subset\subset M_0$, there is a positive constant C such that for each $\epsilon > 0$ and $z \in W$, we have*

(i)

$$\int_{\substack{\zeta\in M_0\\ |\zeta-z|\leq\epsilon}} \|K(\zeta,z)\| \, d\lambda(\zeta) \leq C\epsilon(1+|\ell n\epsilon|)^k$$

(ii)

$$\int_{\substack{\zeta\in M_0\\ |\zeta-z|\leq\epsilon}} \|R(\zeta,z)\| \, d\lambda(\zeta) \leq C\epsilon$$

(iii)

$$\int_{\substack{\zeta\in M_0\\ |\zeta-z|\leq\epsilon}} \|E(\zeta,z)\| \, d\lambda(\zeta) \leq C\epsilon^2(1+|\ell n\epsilon|)^k.$$

$(i\nu)$ All of the above inequalities hold if we integrate with respect to z instead of ζ.

Let us assume the lemma for the moment and show that equation (2.17) holds on all $M_0 \times M_0$.

For $\epsilon > 0$, choose a smooth function χ_ϵ on $M_0 \times M_0$ with the following properties

$$\chi_\epsilon(\zeta,z) = \begin{cases} 1 & \text{if } |\zeta - z| \geq \epsilon \\ 0 & \text{if } |\zeta - z| \leq \frac{\epsilon}{2}. \end{cases}$$

and for any first-order derivative $\mathcal{D}$,

$$|\mathcal{D}\{\chi_\epsilon\}| \leq \frac{\mathcal{C}}{\epsilon} \tag{2.18}$$

where $\mathcal{C}$ is a positive constant that is independent of ϵ.

Since χ_ϵ vanishes near the diagonal of $M_0 \times M_0$, we have from (2.17)

$$\overline{\partial}_{\zeta,z}\{\chi_\epsilon E\} = (\overline{\partial}_{\zeta,z}\chi_\epsilon) \wedge E + \chi_\epsilon(K - R) \tag{2.19}$$

on $M_0 \times M_0$. From Lemma 2.10, we have

$$\chi_\epsilon K \to K, \chi_\epsilon R \to R, \chi_\epsilon E \to E \textit{ and } \overline{\partial}_{\zeta,z}\{\chi_\epsilon E\} \to \overline{\partial}_{\zeta,z}E$$

in the sens of currents, as $\epsilon \to 0$

From part (iii) in lemma 2.10 and the estimate (2.18), we see that

$$(\overline{\partial}_{\zeta,z}\chi_\epsilon) \wedge E \to 0$$

as $\epsilon \to 0$, in the sens of currents. So, we obtain the desired result by letting $\epsilon \to 0$ in the equation (2.19).

PROOF OF LEMMA 2.10 Looking at the definitions of the kernels K, E and R (cf. (2.14),...,(2.16))and taking into account Lemma 1.2, we see that we have to estimate the following typical term

$$\frac{\mathcal{N}(\zeta,z)}{\prod_{i=1}^{k}(\Phi_{a^i}(\zeta,z))^{r_i}(\Phi_{a^0}(\zeta,z))^{r_0}(\Phi_{a^{k+1}}(\zeta,z))^{r_{k+1}}\ |\zeta-z|^{2s}} \tag{2.20}$$

where $a^1,\dots,a^k$ are linearly independent, $a^0=\sum_{i=1}^{k}x_ia^i$, $a^{k+1}=\sum_{i=1}^{k}y_ia^i$ and

$$r_i\geq 1,\ all\ 1\leq i\leq k;\ s,r_0,r_{k+1}\geq 0$$
$$s+\sum_{i=0}^{k+1}r_i=n.$$

For the kernel K, we have $r_{k+1}=0$ and
either $r_0=0$, $s\geq 1$ and the function $\mathcal{N}$ involves coefficients of the differential form

$$\Big(G_{a^1}.d(\zeta-z)\Big)\wedge\cdots\wedge\Big(G_{a^k}.d(\zeta-z)\Big)\wedge\Big(\overline{(\zeta-z)}.d(\zeta-z)\Big)$$

or $s=0$, $r_0\geq 1$ and the function $\mathcal{N}$ contains the coefficients of the term

$$\Big(G_{a^1}.d(\zeta-z)\Big)\wedge\cdots\wedge\Big(G_{a^k}.d(\zeta-z)\Big)\wedge\Big(G_{a^0}.d(\zeta-z)\Big)$$

Since

$$G_{a^0}(\zeta,z)=\sum_{i=1}^{k}x_iG_{a^i}+\mathcal{O}(|\zeta-z|)$$

we obtain in both cases

$$|\mathcal{N}(\zeta,z)|\leq C|\zeta-z| \tag{2.21}$$

for some positive constant C. Since M is CR generic and $a^1,\dots,a^k$ are linearly independent,
$Im\Phi_{a^1}(.,z),\dots,Im\Phi_{a^k}(.,z)$ can be taken as local coordinates on M_0(cf.Lemma 2.3 in [5]). Then in view of inequality (1.5),

$$\begin{aligned}\int_{\substack{\zeta\in M_0\\|\zeta-z|\leq\epsilon}}\|K(\zeta,z)\|\,d\lambda(\zeta)\quad&\leq\int_{\substack{X\in\mathbb{R}^{2n-k}\\|X|<\epsilon}}\frac{dX}{\prod_{j=1}^{k}\big(|X_j|+|X|^2\big)|X|^{2n-2k-1}}\\&\leq C\epsilon(1+|\ell n\epsilon|)^k.\end{aligned}$$

Now for the kernel E, we have $r_{k+1}\geq 1$ and
either $s=0$, $r_0\geq 1$ and the function $\mathcal{N}$ involves the coefficients of the term

$$\Big(G_{a^1}.d(\zeta-z)\Big)\wedge\cdots\wedge\Big(G_{a^k}.d(\zeta-z)\Big)\wedge\Big(G_{a^0}.d(\zeta-z)\Big)\wedge\Big(G_{a^{k+1}}.d(\zeta-z)\Big)$$

or $s\geq 1$, $r_0=0$ and the the function $\mathcal{N}$ contains the coefficients of the differential form

$$\Big(G_{a^1}.d(\zeta-z)\Big)\wedge\cdots\wedge\Big(G_{a^k}.d(\zeta-z)\Big)\wedge\Big(\overline{(\zeta-z)}.d(\zeta-z)\Big)\wedge\Big(G_{a^{k+1}}.d(\zeta-z)\Big)$$

By the same arguments as above, we obtain in this case

$$|\mathcal{N}(\zeta,z)| \leq C|\zeta - z|^2$$

for some positive constant C, and then

$$\begin{aligned}\int_{\substack{\zeta\in M_0\\|\zeta-z|\leq\epsilon}} \|E(\zeta,z)\|\, d\lambda(\zeta) &\leq \int_{\substack{X\in\mathbb{R}^{2n-k}\\|X|<\epsilon}} \frac{dX}{\prod_{j=1}^{k}\left(|X_j|+|X|^2\right)|X|^{2n-2k-2}}\\ &\leq C\epsilon^2(1+|\ln\epsilon|)^k.\end{aligned}$$

For the kernel R, we have $s=0$, $r_0=0$, $r_{k+1}\geq 1$ and every collection of k elements in $\{a^1\dots a^{k+1}\}$ is a family of linearly independent vectors (see condition $(*)$ and Remark 2.2.0). It is easy to see , just as above, that inequality (2.21) holds also in this case.

Now we use the following easily established inequality:if $0\leq\alpha_1,\dots,\alpha_k\leq\alpha_{k+1}$, then

$$\prod_{i=1}^{k+1}\alpha_i \geq \prod_{i=1}^{k}\alpha_i^{1+\frac{1}{k}} \tag{2.22}$$

If we use (2.22) with $\alpha_i=|\Phi_{a^i}|$, $1\leq i\leq k{+}1$ and (up to a permutation of $\{1,\dots,k{+}1\}$), we obtain by using local coordinates as above and inequality(1.5),

$$\begin{aligned}\int_{\substack{\zeta\in M_0\\|\zeta-z|\leq\epsilon}} \|R(\zeta,z)\|\, d\lambda(\zeta) &\leq \int_{\substack{X\in\mathbb{R}^{2n-k}\\|X|<\epsilon}} \frac{dX}{\prod_{j=1}^{k}\left(|X_j|+|X|^2\right)^{1+\frac{1}{k}}|X|^{2n-2k-3}}\\ &\leq C\epsilon.\end{aligned}$$

Thus the proof of $(i),(ii),(iii)$ in Lemma 2.10 is complete. $(i\nu)$ follows in the same way. □

3. End of proof of Theorem 0.1

In this section we shall prove $\mathcal{C}^{\ell+\frac{1}{2}}-$estimates. We first prove $\mathcal{C}^{\frac{1}{2}}-$estimates and then we derive $\mathcal{C}^{\ell+\frac{1}{2}}-$estimates by using similar arguments as in [18] and [8].

3.1 $\mathcal{C}^{\frac{1}{2}}-$Estimates Recall that the coefficients of the kernel $R(\zeta,z)$ have the form

$$\frac{\mathcal{N}(\zeta,z)}{\prod_{i=1}^{k+1}(\Phi_{a^i}(\zeta,z))^{r_i}}$$

where $a^1,\dots,a^{k+1}$ are vectors in $\mathbb{R}^k$ such that every subset of k elements in $\{a^1,\dots,a^{k+1}\}$ is a family of linearly independent vectors (condition $(*)$), the estimate (2.21) holds for $\mathcal{N}$ and

$$r_i\geq 1,\ all\ 1\leq i\leq k+1;\ \sum_{i=0}^{k+1} r_i = n.$$

We have

$$\int_{\zeta\in M_0}\|R(\zeta,z^1)-R(\zeta,z^2)\|\, d\lambda(\zeta)\leq J_1(z^1,z^2)+J_2(z^1,z^2)$$

where

$$J_1(z^1, z^2) := \int_{\substack{\zeta \in M_0 \\ |\zeta - z^1| \le |z^1 - z^2|^{\frac{1}{2}}}} \left(\|R(\zeta, z^1)\| + \|R(\zeta, z^2)\| \right) \, d\lambda(\zeta)$$

and

$$J_2(z^1, z^2) := \int_{\substack{\zeta \in M_0 \\ |\zeta - z^1| \ge |z^1 - z^2|^{\frac{1}{2}}}} \|R(\zeta, z^1) - R(\zeta, z^2)\| \, d\lambda(\zeta)$$

It follows from lemma 2.10 (ii) that

$$J_1(z^1, z^2) \le C|z^1 - z^2|^{\frac{1}{2}}$$

Now since $\mathcal{N}(\zeta, z)$ is smooth in z, it is not difficult to see by the same arguments as in the proof of Lemma 2.10 that

$$\begin{aligned} J_2(z^1, z^2) \quad &\le C|z^1 - z^2| \int_{\substack{X \in \mathbb{R}^{2n-k} \\ |X| \ge |z^1 - z^2|^{\frac{1}{2}}}} \frac{dX}{\left(|X_1| + |X|^2\right)^{2+\frac{1}{k}} \prod_{j=2}^k \left(|X_j| + |X|^2\right)^{1+\frac{1}{k}} |X|^{2n-2k-3}} \\ &\le C'|z^1 - z^2|^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\int_{\zeta \in M_0} \|R(\zeta, z^1) - R(\zeta, z^2)\| \, d\lambda(\zeta) \le C'|z^1 - z^2|^{\frac{1}{2}}. \tag{2.23}$$

Analogously we can show that

$$\int_{z \in M_0} \|R(\zeta^1, z) - R(\zeta^2, z)\| \, d\lambda(z) \le C|\zeta^1 - \zeta^2|^{\frac{1}{2}}. \tag{2.24}$$

under the hypothesis that M is of class $\mathcal{C}^3$. This is because $R(\zeta, z)$ involves second-order derivatives in ζ of the defining functions of M.

3.2. $\mathcal{C}^{\ell+\frac{1}{2}}$−Estimates. We assume that M is of class $\mathcal{C}^{\ell+2}$ ($\ell \ge 1$).

Let $a^1, \dots, a^k$ be linearly independent vectors in $\bigcup_{I \in \mathcal{I}'(k)} \Delta_I$ and $a^{k+1} = \sum_{i=1}^k y_i a^i$ with $y_i \ne 0$, all $1 \le i \le k$ (this means that every collection of k vectors in $\{a^1, \dots, a^{k+1}\}$ is a family of linearly independent vectors).

Denote by $\tilde{\rho}_i$(resp.ϕ_i) the defininig function (resp.the barrier function) of M in the direction a^i for $1 \le i \le k+1$.

$\mathcal{E}^j (j \ge 0)$will denote a smooth differential form on $M \times M$ vanishing of order j for $\zeta = z$. It is clear that

$$\phi_{k+1} = \sum_{i=1}^k y_i \phi_i + \mathcal{E}^2. \tag{2.25}$$

We need the following lemma.

LEMMA 3.2.1 *There exist $Y_1^\zeta, \dots, Y_k^\zeta$, tangential vector fields to M such that for every $\zeta \in M_0$ and every $1 \le i, j \le k$,*

$$Y_i^\zeta \phi_j(\zeta, \zeta) = \delta_{ij},$$

where δij is Kronecker's symbol.

Proof –Since M is CR generic and $a^1, \dots, a^k$ are linearly independent, we have

$$\partial\tilde{\rho}_1 \wedge \cdots \wedge \partial\tilde{\rho}_k \neq 0 \text{ on } M_0.$$

Then the matrix

$$A = \begin{pmatrix} < \partial\tilde{\rho}_1(\zeta), \partial\tilde{\rho}_1(\zeta) > & \cdots & < \partial\tilde{\rho}_k(\zeta), \partial\tilde{\rho}_1(\zeta) > \\ \vdots & & \vdots \\ < \partial\tilde{\rho}_1(\zeta), \partial\tilde{\rho}_k(\zeta) > & \cdots & < \partial\tilde{\rho}_k(\zeta), \partial\tilde{\rho}_k(\zeta) > \end{pmatrix}$$

is invertible for all $\zeta \in M_0$ (here $< ., . >$ denotes the Hermitian inner product), and there exist $\nu_1, \dots, \nu_k \in \{1, \dots, n\}$ such that the matrix

$$B = \begin{pmatrix} \frac{\partial\tilde{\rho}_1}{\partial\bar{\zeta}_{\nu_1}}(\zeta) & \cdots & \frac{\partial\tilde{\rho}_k}{\partial\bar{\zeta}_{\nu_1}}(\zeta) \\ \vdots & & \vdots \\ \frac{\partial\tilde{\rho}_1}{\partial\bar{\zeta}_{\nu_k}}(\zeta) & \cdots & \frac{\partial\tilde{\rho}_k}{\partial\bar{\zeta}_{\nu_k}}(\zeta) \end{pmatrix}$$

is also invertible for all $\zeta \in M_0$.

Set

$$Y_i^\zeta = \frac{1}{2}\sum_{j=1}^{k} \alpha_{ij}(\zeta) \sum_{\nu=1}^{n} \frac{\partial\tilde{\rho}_j}{\partial\bar{\zeta}_\nu}\frac{\partial}{\partial\zeta_\nu} - \frac{1}{2}\sum_{j=1}^{k} \beta_{ij}(\zeta)\frac{\partial}{\partial\bar{\zeta}_{\nu_j}}$$

where $\left[\alpha_{ij}(\zeta)\right] = A^{-1}$ and $\left[\beta_{ij}(\zeta)\right] = B^{-1}$.

Now it is easy to check that

$Y_i^\zeta \phi_j(\zeta, \zeta) = \delta_{ij}$ and $Y_i^\zeta \tilde{\rho}_j = 0$ for all $1 \le i, j \le k$. □

Let us introduce the following class of kernels for $\delta \ge 0$,

$$\mathcal{L}_\delta = \frac{\mathcal{E}^j}{\prod_{i=1}^{k+1}(\phi_i + \delta)^{r_i}}$$

where

$$2n - 1 - 2\sum_{i=1}^{k+1} r_i + j \ge 0$$

and

$$r_i \ge 1 \; for \; all \; 1 \le i \le k+1$$

REMARK 3.2.2 *Notice that the kernel R is a finite sum of kernels of type $\mathcal{L}_0$, and estimate (2.23) with estimate (2.24) hold, independently of δ, for kernels $\mathcal{L}_\delta$.*

If we denote by X^z a tangential vector field to M in z-variable and X^ζ the corresponding operator in ζ-coordinates , then we have the following

LEMMA 3.2.3— *Let $\delta > 0$, then we have*

$$X^z \mathcal{L}_\delta = -X^\zeta \mathcal{L}_\delta + \sum_{i=1}^{k} \frac{(X^z + X^\zeta)\phi_i}{Y_i^\zeta \phi_i} Y_i^\zeta(\mathcal{L}_\delta) + S_\delta.$$

where S_δ is a finite sum of kernels of type $\mathcal{L}_\delta$.

Proof – It is not difficult to see that the following facts are true:

(i) $(X^z + X^\zeta)\mathcal{E}^j$ is of type $\mathcal{E}^j$.

(ii) $(X^z + X^\zeta)\phi_i$ is of type $\mathcal{E}^1$.

(iii) $|Y_i^\zeta \phi_i(\zeta, z)| \geq C$ for $|\zeta - z| \leq \epsilon$ and $1 \leq i \leq k$ (see Lemma 3.2.1)

$(i\nu)$ If $i \neq j$ then $Y_i^\zeta \phi_j$ is of type $\mathcal{E}^1$ (cf. Lemma 3.2.1) .

(ν) $Y_i^\zeta \phi_{k+1} - y_i Y_i^\zeta \phi_i$ *is of type* $\mathcal{E}^1$ *for all* $1 \leq i \leq k$ (see (2.25) and Lemma 3.2.1).

(νi) $(X^z + X^\zeta)\phi_{k+1} - \sum_{i=1}^{k} y_i (X^z + X^\zeta)\phi_i$ is of type $\mathcal{E}^2$ (see (2.25)).

(νii) $(X^z + X^\zeta)(\frac{1}{\phi_i^{r_i}}) = \frac{(X^z + X^\zeta)\phi_i}{Y_i^\zeta \phi_i} Y_i^\zeta(\frac{1}{\phi_i^{r_i}})$.

The lemma follows now by a straightforward computation. □

Now let $\Omega \subset\subset M_0$ and $f \in W_\star^{\ell,\infty}(\Omega)$. Let $z_1 \in \Omega$ and χ a smooth compactly supported function on Ω such that

$$\chi(\zeta) = \begin{cases} 0 & \text{if } |\zeta - z_1| \geq \frac{\epsilon}{2} \\ 1 & \text{if } |\zeta - z_1| \leq \frac{\epsilon}{4}. \end{cases}$$

where ϵ is chosen so that (see Lemma 3.2.1)

$$|Y_i^\zeta \phi_i(\zeta, z)| \geq C \text{ for } |\zeta - z| \leq \epsilon \text{ and all } 1 \leq i \leq k.$$

Set $K := \{z \in \Omega / |z - z_1| \leq \frac{\epsilon}{4}\}$.

We write

$$\int_\Omega f(\zeta) \wedge R(\zeta, z) = \int_\Omega \chi(\zeta) f(\zeta) \wedge R(\zeta, z) + \int_\Omega (1 - \chi(\zeta)) f(\zeta) \wedge R(\zeta, z).$$

Let $J_1(f)$ denote the first integral in the right-hand side and $J_2(f)$ the second one.

Since $R(\zeta, z)$ is of class $\mathcal{C}^\infty$ in z for $\zeta \neq z$ then $J_2(f)$ is of class $\mathcal{C}^\infty$ on K.

By Remark 3.2.2 to estimate $J_1(f)$, it is enough to do so for $\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z)$.

We have

$$\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z) = \lim_{\delta \to 0} \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z).$$

By Lemma 3.2.3, we obtain from Stokes' theorem

$$\begin{aligned} X^z \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z) &= \pm \int_\Omega X^\zeta(\chi f) \wedge \mathcal{L}_\delta(\cdot, z) \\ &\pm \sum_{i=1}^k \int_\Omega Y_i^\zeta(\chi f) \wedge \frac{(X^z + X^\zeta)\phi_i}{Y_i^\zeta \phi_i} \mathcal{L}_\delta(\cdot, z) + \int_\Omega \chi f \wedge S_\delta(\cdot, z). \end{aligned}$$

where S_δ is a finite sum of kernels of type $\mathcal{L}_\delta$.

Now , if we apply $r \leq \ell$ derivatives , we can write

$$X_1^z \cdots X_r^z \int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z)$$

as a sum of terms

$$\int_\Omega \tilde{X}_1^\zeta \cdots \tilde{X}_j^\zeta(\chi f) \wedge \mathcal{L}_\delta(\cdot, z)$$

with $0 \leq j \leq r$.

This proves that all derivatives of $\int_\Omega \chi f \wedge \mathcal{L}_\delta(\cdot, z)$ up to ℓ converge uniformly on Ω.

And since

$$\| \int_\Omega \tilde{X}_1^\zeta \cdots \tilde{X}_j^\zeta(\chi f) \wedge \mathcal{L}_\delta(\cdot, z) \|_{\frac{1}{2}} \leq C \|f\|_{\mathcal{C}^\ell}$$

for $0 \leq j \leq \ell$, independently of δ (see Remark 3.2.2), we conclude that

$$\int_\Omega \chi f \wedge \mathcal{L}_0(\cdot, z) \text{ is of class } \mathcal{C}^{\ell+\frac{1}{2}} \text{ on } \Omega.$$

Thus $J_1(f)$ is of class $\mathcal{C}^{\ell+\frac{1}{2}}$ on Ω, and therefore

$$\int_\Omega f(\zeta) \wedge R(\zeta, z) \text{ is of class } \mathcal{C}^{\ell+\frac{1}{2}} \text{ on } K.$$

By noticing that $Y_i^z \Phi_j = -Y_i^\zeta \Phi_j + \mathcal{E}^1 \ for \ 1 \leq i, j \leq k$ one can show in the same way

$$\int_\Omega f(z) \wedge R(\zeta, z) \text{ is of class } \mathcal{C}^{\ell+\frac{1}{2}} \text{ on } K$$

provided M is of class $\mathcal{C}^{\ell+3}$ (see (2.24)). This completes the proof of the second part of Theorem 0.1 (cf. Definition 2.9)

For a proof of corllory 0.2, we refer the reader to [5] . □

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